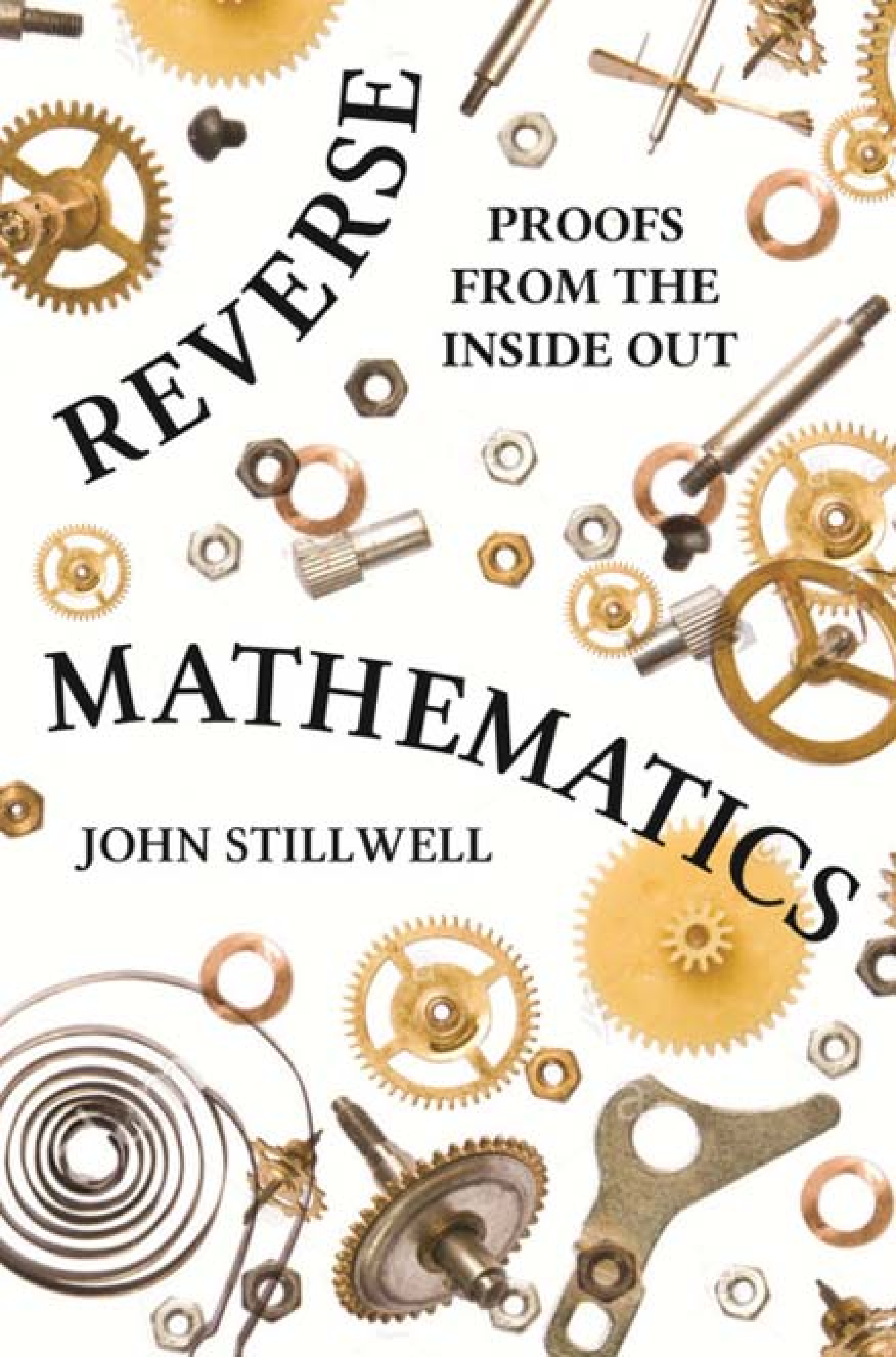
REVERSE
MATHEMATICS
PROOFS FROM THE INSIDE OUT
JOHN STILLWELL

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To Elaine

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Preface

This is a book about the foundations of mathematics—a topic once of interest to outstanding mathematicians, such as Dedekind, Poincaré, and Hilbert, but today sadly neglected. This neglect is unfortunate for several reasons:

- As mathematics splits into more and more specialties, the need for a unifying viewpoint becomes more acute.
- Foundations unify not only mathematics but also the neighboring disciplines of computer science and physics.
- Recent advances in mathematical logic throw new light on the foundations of analysis, and on the elusive concept of mathematical "depth."

This book aims at the last point in particular, by focusing on the topic of *reverse mathematics*.

As its name suggests, reverse mathematics looks at the concept of proof in the opposite to normal direction. Instead of seeking the consequences of given axioms, it seeks the axioms needed to prove given theorems. This is actually an old idea, at least in the foundations of geometry. From the time of Euclid until the nineteenth century it was a burning question whether the parallel axiom was needed to prove theorems such as the Pythagorean theorem. We review the history of the parallel axiom in chapter 1 of this book, as a case study in reverse mathematical ideas, together with the similar story of the axiom of choice in set theory.

Although both these axioms illustrate the idea of reverse mathematics, the subject as it is understood today lies mostly in a narrow but important region *between* geometry and set theory: the theory of real numbers, which is the foundation of calculus, analysis, and most of mathematical physics. (Reverse mathematics has also made interesting contributions to algebra, combinatorics, and topology which we mention more briefly.)

The real numbers, as we understand them today, emerged from nineteenth century efforts to *arithmetize* analysis and geometry. By building real numbers from sets of rational numbers (and hence, ultimately, from sets of natural numbers) it becomes possible to encode sequences of real numbers and arbitrary continuous functions—and hence most of the objects of analysis—by sets of natural numbers. We review the arithmetization of analysis, and also the basic theorems of analysis, in chapters 2 and 3. After this we are ready to ask: which *axioms* do we need to prove these basic theorems? The answer, roughly, is a set of axioms for the natural numbers (the *Peano axioms*) plus a suitable *set existence axiom*.

Now set existence axioms come in various *strengths*, depending on the strength of the theorems we wish to prove. The lowest useful strength turns out to be intimately related to the foundations of *computation*: it asserts the existence of computable sets. This in turn involves a study of the concept of computation, which merges with analysis because both have a common basis in arithmetic. After an informal introduction to computability in chapter 4 we develop a formal concept of computation, and its arithmetization, in chapter 5.

In chapters 6 and 7 we bring together the ideas of analysis, arithmetic, and computation in some axiom systems for analysis, known as RCA_0, WKL_0, and ACA_0. These systems, which are distinguished mainly by set existence axioms of increasing strength, between them prove most of the basic theorems of analysis. More remarkably, they sort the basic theorems into three levels because, once above the "base" level of RCA_0, most theorems are *equivalent* to the set existence axiom of the system that proves them. This makes each of these set existence axioms the "right axiom" in the sense articulated by Friedman (1975):

> When a theorem is proved from the right axioms, the axioms can be proved from the theorem.

We will see, for example, that RCA_0 can prove the intermediate value theorem; the defining axiom of WKL_0 is the right axiom to prove the Heine-Borel theorem and the extreme value theorem; and the defining axiom of ACA_0 is the right axiom to prove the Cauchy convergence criterion and the Bolzano-Weierstrass theorem.

Thus in reverse mathematics we meet the usual cast of characters from an introductory real analysis course, but in an entirely new story.

In chapter 8 we give some glimpses of the bigger picture of analysis, computation, and logic, which will hopefully prepare the reader for specialist treatments of reverse mathematics, notably Simpson (2009). The present book is very much for non-specialists—in some ways a sequel to my book *Elements of Mathematics. From Euclid to Gödel*. It develops computability and logic far enough to explain results that *Elements of Mathematics* could only mention, but the latter book is not a prerequisite for this one. Anyone at an upper undergraduate level with an interest in foundations should be able to approach the reverse mathematics in this book directly. The same goes, of course, for professional mathematicians who want to refresh their memory of foundations and to see how the subject has reinvented itself in recent times.

Acknowledgements. I thank Harvey Friedman for information on the history of reverse mathematics, Keita Yokoyama for his insights on topology, and two anonymous referees for many helpful comments and corrections. My wife Elaine as usual did sterling work in proofreading, and Vickie Kearn and her team at Princeton University Press were ever-helpful and meticulous in the production of the book.

John Stillwell
San Francisco, 24 November 2016

REVERSE MATHEMATICS

CHAPTER 1

Historical Introduction

The purpose of this introductory chapter is to prepare the reader's mind for *reverse mathematics*. As its name suggests, reverse mathematics seeks not theorems but the right axioms to prove theorems already known. The criterion for an axiom to be "right" was expressed by Friedman (1975) as follows:

> When the theorem is proved from the right axioms, the axioms can be proved from the theorem.

Reverse mathematics began as a technical field of mathematical logic, but its main ideas have precedents in the ancient field of geometry and the early twentieth-century field of set theory.

In geometry, the parallel axiom is the right axiom to prove many theorems of Euclidean geometry, such as the Pythagorean theorem. To see why, we need to separate the parallel axiom from the *base theory* of Euclid's other axioms, and show that the parallel axiom is not a theorem of the base theory. This was not achieved until 1868. It is easier to see that the base theory can prove the parallel axiom *equivalent* to many other theorems, including the Pythagorean theorem. This is the hallmark of a good base theory: what it cannot prove outright it can prove equivalent to the "right axioms."

Set theory offers a more modern example: a base theory called ZF, a theorem that ZF cannot prove (the well-ordering theorem) and the "right axiom" for proving it—the axiom of choice.

From these and similar examples we can guess at a base theory for analysis, and the "right axioms" for proving some of its well-known theorems.

1.1 EUCLID AND THE PARALLEL AXIOM

The search for the "right axioms" for mathematics began with Euclid, around 300 BCE, when he proposed axioms for what we now call *Euclidean geometry*. Euclid's axioms are now known to be incomplete; nevertheless, they outline a complete system, and they distinguish between really obvious "basic" axioms and a less obvious one that is crucial for obtaining the most important theorems. For historical commentary on the axioms, see Heath (1956).

The basic axioms say, for example, that there is a unique line through two distinct points and that lines are unbounded in length. Also basic, though expressed only vaguely by Euclid, are criteria for *congruence of triangles*, such as what we call the "side angle side" or SAS criterion: if two triangles agree in two sides and the included angle then they agree in all sides and all angles. (Likewise ASA: they agree if they agree in two angles and the side between them.)

Using the basic axioms it is possible to prove many theorems of a rather unsurprising kind. An example is the *isosceles triangle theorem*: if a triangle ABC has side AB = side AC then the angles at B and C are equal. However, the basic axioms fail to prove the signature theorem of Euclidean geometry, the *Pythagorean theorem*, illustrated by figure 1.1.

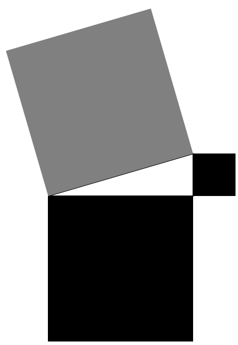

Figure 1.1 : The Pythagorean theorem

As everybody knows, the theorem says that the gray square is equal to the sum of the black squares, but the basic axioms cannot even prove the *existence* of squares. To prove the Pythagorean theorem, as Euclid realized, we need an axiom about infinity: the *parallel axiom*.

The Parallel Axiom

I call the parallel axiom an axiom about infinity because it is about lines that do not meet, *no matter how far they are extended*—and one of Euclid's basic axioms is that lines can be extended indefinitely. Thus parallelism cannot be "seen" unless we have the power to see to infinity, and Euclid preferred not to assume such a superhuman power. Instead, he gave a criterion for lines *not* to be parallel, since a meeting of lines can be "seen" a finite distance away.

Parallel axiom. If a line n falling on lines l and m (figure 1.2) makes angles α and β with $\alpha + \beta$ less than two right angles, then l and m meet on the side on which α and β occur.

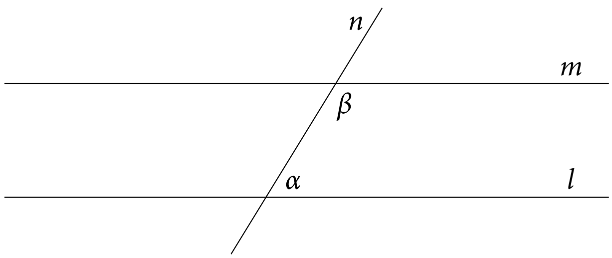

Figure 1.2 : Angles involved in the parallel axiom

It follows that if $\alpha + \beta$ equals two right angles (that is, a straight angle) then l and m do *not* meet. Because if they meet on one side (forming a triangle) they must meet on the other (forming a congruent triangle, by ASA), since there are angles α and β on both sides and one side in common (figure 1.3). This contradicts uniqueness of the line through any two points.

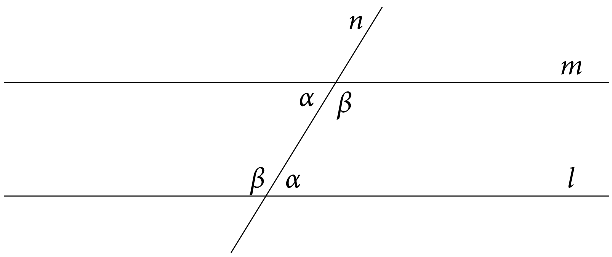

Figure 1.3 : Parallel lines