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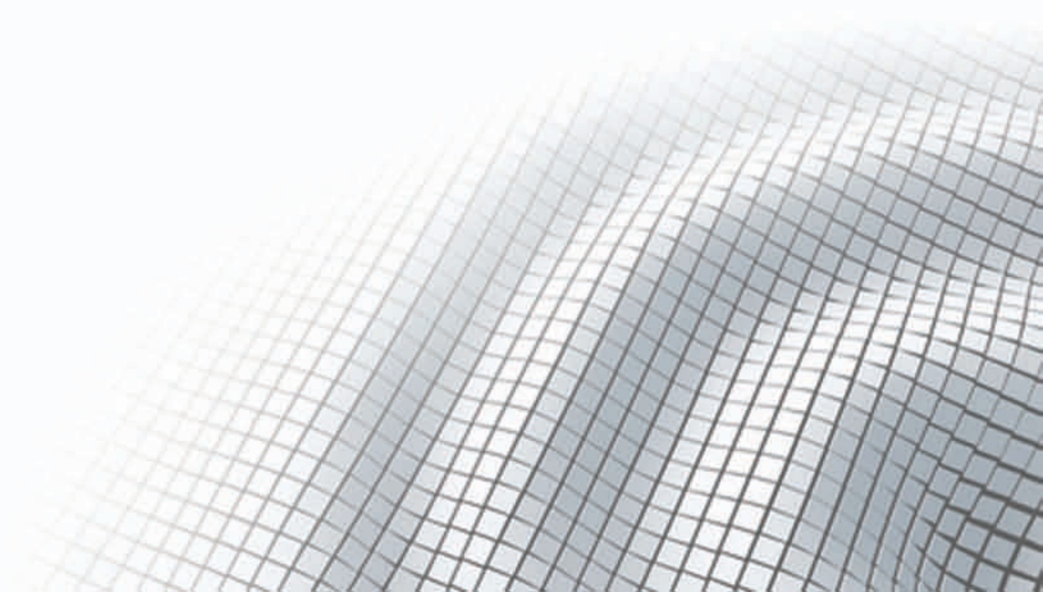
PROBLEM BOOK IN RELATIVITY AND GRAVITATION



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ALAN P. LIGHTMAN
WILLIAM H. PRESS
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SAUL A. TEUKOLSKY

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PREFACE

This book contains almost 500 problems and solutions in the fields of special relativity, general relativity, gravitation, relativistic astrophysics and cosmology. The collection is motivated by a simple premise: that the most important content of this field does not lie in its rigorous axiomatic development, nor, necessarily, in its intrinsic aesthetic beauty, but rather does lie in computable results, predictions, and models for phenomena in the real universe. Accordingly, we have aimed for problems whose statement is broadly understandable in physical terms and have tried to make their statement independent of notational conventions. We hope to awaken the reader's curiosity. ("Now how *would* one show that...?") We have steered clear of purely technical problems, found in texts, of the form "prove equation 17.4.38." In our solutions we also try to show the reader "good" ways to compute things, methods and tricks which can vastly reduce the labor of a plug-in and grind-away approach, but we also try to avoid the opposite pitfall of introducing too much confusing but powerful formalism for an easy problem. There is often a lot of leeway in this balance, and the reader should not be surprised if his solutions use a rather smaller (or larger) set of calculational tools.

The first five chapters of this book deal only with special relativity, and are designed for advanced undergraduates and graduate students in any course in modern physics, classical mechanics or electromagnetism. They are arranged roughly in order of increasing sophistication, beginning at about the easy level of *Spacetime Physics* by E. F. Taylor and J. A. Wheeler (Freeman, 1963); there are, however, both easy and difficult problems in each chapter.

The remainder of the book is aimed at the student in a course in general relativity and/or cosmology. The chapters cover aspects of metric geometry, the equations of Einstein's gravitation theory (and some competing theories), the effect of gravitation on other physical phenomena, and applications to a variety of experimental and astrophysical situations. A final chapter deals with some more formal topics whose applications are less direct.

Each chapter begins with an introductory note whose purpose is largely to define the notation used. These by no means constitute a complete or orderly presentation of the material covered in the chapter, but are intended to aid the student familiar with a notation different from ours. We assume that the reader has the benefit of one or more of the following texts (which we have used heavily):

C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, 1973) [cited in this book as "MTW"] .

S. Weinberg, *Gravitation and Cosmology* (Wiley, 1972) [cited in this book as "Weinberg"].

R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* 2nd ed. (McGraw-Hill, 1975).

We have also been influenced by the following texts or monographs:

Anderson, J. L., *Principles of Relativity Physics* (Academic Press, 1967).

Batygin, V. V., and Toptygin, I. N., *Problems in Electrodynamics* (Academic Press - Infosearch, 1964) .

Hawking, S. W., and Ellis, G. F. R., *The Large-Scale Structure of Space-Time* (Cambridge University Press, 1973) .

Landau, L. D., and Lifschitz, E. M., *The Classical Theory of Fields*, 3rd ed., (Addison-Wesley, 1971).

Peebles, P. J. E., *Physical Cosmology* (Princeton University Press, 1971).

Robertson, H. P., and Noonan, T. W., *Relativity and Cosmology* (Saunders, 1968).

Sexl, R. U., and Urbantke, H. K., *Gravitation and Kosmologie*, (Wiener Berichte über Gravitationstheorie, 1973).

We have cited the primary literature where appropriate.

We are pleased to express our appreciation to colleagues who have contributed original problems to this collection: Douglas Eardley, Charles W. Misner, Don Page, Bernard F. Schutz, and our friend and teacher, Kip S. Thorne.

We are also grateful to C. R. Alcock, B. C. Barrois, J. Conwell, H. B. French, K. S. Jancaitis, C. Jayaprakash, S. J. Kovacs and W. A. Russell for valuable help in improving the problems and solutions. Our thanks go to Steve Wilson for preparing most of the illustrations in this book. We acknowledge support from the Department of Physics at the California Institute of Technology while we were there. Of course, we are responsible for the errors which inevitably must be present in a book of this sort. We have tried particularly hard for problems and solutions which are *conceptually* free from error, but we also apologize in advance for the algebraic slips that the diligent reader will certainly find; we invite his corrections.

A. P. LIGHTMAN
W. H. PRESS
R. H. PRICE
S. A. TEUKOLSKY

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NOTATION

It is intended that this book be compatible with several different textbooks, each with its own system of notational conventions. Thus, no single notational system will be used exclusively in this book. In almost all instances, meanings will be clear from the context. The following is a list of the *usual* meanings of some frequently used symbols and conventions.

$\alpha, \beta, \mu, \nu \cdots$	Greek indices range over $0, 1, 2, 3$ and represent spacetime coordinates, components, etc.
$i, j, k \cdots$	Latin indices range over $1, 2, 3$ and represent coordinates etc. in 3-dimensional space
$e_\alpha, e_j \cdots$	Basis vectors
$\mathbf{A}$	(Any boldface symbol) a spacetime vector, tensor, or form
$\underset{\sim}{A}$	A 3-dimensional vector
$A^\mu, B^\alpha{}_\beta \cdots$	Tensor components
(A^0, A^1, A^2, A^3)	A vector represented by its components
$(A^0, \underset{\sim}{A})$	A vector represented by its time component and spatial part
$\hat{}$	(Caret) indicates unit vector, components in orthonormal basis
$d/d\lambda$	Occasionally used to represent a vector (see Introduction to Chapter 7)
$\mathbf{A}(f)$	A vector operating on a function $= A^\alpha f_{,\alpha}$

$\tilde{\boldsymbol{\omega}}$	A one-form
$\otimes$	Outer product, tensor product e.g. $\mathbf{A}\otimes\mathbf{B}$ has components $A^{\mu}B^{\nu}$
$\wedge$	Wedge product (see Introduction to Chapter 8)
∇	Covariant derivative operator (see Introduction to Chapter 7). Also used as in ordinary physics $\nabla\times=$ curl, $\nabla^2=$ Laplacian, etc.
$\nabla_{\mathbf{A}}$	Directional derivative (see Introduction to Chapter 7)
$D/d\lambda$	Covariant derivative along a curve (see Introduction to Chapter 7)
d	Gradient operator as in e.g. the one-form $\widetilde{df}$ (see introduction to Chapter 8)
$\mathfrak{L}$	Lie derivative (see Problem 8.13)
$\Gamma^{\alpha}{}_{\beta\gamma}$	Christoffel symbol (see Introduction to Chapter 7)
$\Box$	d'Alembertian operator $\equiv\nabla^2-\partial^2/\partial t^2$ in Special Relativity
,	Partial derivative
;	Covariant derivative (see Introduction to Chapter 7)
$R_{\alpha\beta\gamma\delta}$	Riemann tensor (see Introduction to Chapter 9)
$R_{\alpha\beta}$	Ricci tensor $\equiv R^{\gamma}{}_{\alpha\gamma\beta}$
R	Ricci scalar $\equiv R^{\alpha}{}_{\alpha}$. Also scale factor in Robertson-Walker metric.
$G_{\alpha\beta}$	Einstein tensor (see Introduction to Chapter 9)
$C_{\alpha\beta\gamma\delta}$	Weyl (conformal) tensor (see Introduction to Chapter 9)
K_{ij}	Extrinsic curvature tensor (see Introduction to Chapter 9)
τ	Proper time
c	Speed of light (usually taken as unity in the problems)
G	Gravitational constant (usually taken as unity in the problems)
u	4-velocity
a	4-acceleration $\equiv d\mathbf{u}/d\tau$
p or **P**	4-momentum

p or P	Pressure
$T^{\mu\nu}$	Stress-energy tensor (see Introduction to Chapter 5)
$F^{\mu\nu}$	Electromagnetic field tensor (see Introduction to Chapter 4)
J^{μ}	Current density (see Introduction to Chapter 4)
$J^{\mu\nu}$	Angular momentum tensor (see Problems 11.1, 11.2)
$\eta_{\mu\nu}$	Minkowski metric (see Introduction to Chapter 1)
$h_{\mu\nu}$	Metric perturbations (see Introduction to Chapter 13)
C.M.	Center of momentum frame, center of mass
ν, ω	Frequency in cycles per unit time, radians per unit time
γ	Lorentz factor $\equiv (1-v^2/c^2)^{-\frac{1}{2}}$, or photon symbol
$\Lambda^{\alpha}{}_{\beta}$	Lorentz transformation matrix
det	Determinant
Tr	Trace
$< \; >$	Average (as in $<E>$ = average energy)
$< \, , >$	Scalar combination of vector and one-form, as in $<\tilde{\omega}, \mathbf{A}>$ (see Introduction to Chapter 8)
[ ]	Antisymmetrization (see Problem 3.17) or commutator (see Introduction to Chapter 8) or discontinuity (as in Problem 21.9)
( )	Symmetrization (see Problem 3.17)
$\varepsilon^{\alpha\beta\gamma\delta}$	The totally antisymmetric tensor (see Problem 3.20)
$*$	Duality symbol (see Problem 3.25)
Re	Real part
Ω	Solid angle (as in $\int d\Omega$), angular velocity
$P^{\alpha\beta}$	Projection tensor (see Problems 5.18, 6.6)
θ	Expansion (see Problem 5.18)
$\sigma_{\alpha\beta}$	Shear (see Problem 5.18)
$\omega_{\alpha\beta}$	Rotation (see Problem 5.18)

Ɨ_{jk}	Reduced quadrupole tensor (see Introduction to Chapter 18)
H_0	Hubble constant
q_0	Deceleration parameter
$M_\odot, R_\odot, \cdots$	Mass, radius, $\cdots$ of sun
z	Redshift factor (see Problem 8.28, Introduction to Chapter 19)
$\mathcal{O}$	Order of magnitude
$\propto$	Proportional to (e.g., $r^3 \propto t^2$) or parallel vector to (e.g., $\mathbf{A} \propto \mathbf{B}$)

PROBLEMS

CHAPTER 1
SPECIAL-RELATIVISTIC KINEMATICS

The path of an observer through spacetime is called the worldline of that observer. The time measured by the observer's own clocks, called his proper time τ, is given by

$$-d\tau^2 \equiv ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \ ,$$

where t, x, y, z are the observer's (Minkowski) coordinates along his path. Here, and unless noted otherwise throughout this book, we use units in which c, the speed of light, is unity.

The 4-velocity $\mathbf{u}$, with components $(dt/d\tau, dx/d\tau, dy/d\tau, dz/d\tau)$, and 4-acceleration $\mathbf{a} \equiv d\mathbf{u}/d\tau$, components $(d^2t/d\tau^2, d^2x/d\tau^2, d^2y/d\tau^2, d^2z/d\tau^2)$, are defined on the worldline. The (contravariant) components of these or other 4-vectors are denoted u^α, a^β, A^γ, B^δ, etc., where a Greek index indicates any of the 4 components $t, x, y, z \equiv 0, 1, 2, 3$. Latin indices $i, j, k\cdots$ are used to indicate only the spatial components $x, y, z \equiv 1, 2, 3$.

The Einstein summation convention is used, that is, any repeated literal index is assumed to be summed over its range. For example,

$$\mathbf{V} = V^\mu \mathbf{e}_\mu$$

expresses a vector as a sum of contravariant components multiplied by basis vectors, $\mathbf{e}_0 \equiv (1,0,0,0)$, $\mathbf{e}_1 \equiv (0,1,0,0)$, etc.

The invariant dot product of two 4-vectors is, in Minkowski coordinates,

$$\mathbf{A}\cdot\mathbf{B} = -A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3 \ .$$

This can be written as $\mathbf{A}\cdot\mathbf{B} = A_\mu B^\mu$, where the numbers A_μ, called covariant components of $\mathbf{A}$, are defined by $A_\mu \equiv \eta_{\mu\nu}A^\nu$, or $A^\mu = \eta^{\mu\nu}A_\nu$

$$\eta_{\mu\nu} \equiv \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (\text{also} \equiv \eta^{\mu\nu}) \ .$$

Vectors are called spacelike, timelike, or null, according to whether their square $\mathbf{v}\cdot\mathbf{v}$ is positive, negative, or zero. 4-velocities are always timelike.

Two Lorentz frames may differ by a relative 3-velocity $\underset{\sim}{v}$ or by a spatial rotation, or by a combination of relative velocity and rotation. If t, x, y, z are the coordinates of one frame, then the coordinates in a different frame are usually written t', x', y', z'. Similarly, vector components in the primed frame are written $A^{\mu'}$, $B_{\nu'}$, etc., and its basis vectors are $e_{\mu'}$. The basis vectors and the components of vectors in Lorentz frames are related by

$$e_{\mu'} = \Lambda^{\alpha}_{\mu'} e_{\alpha} \,, \qquad V_{\mu'} = \Lambda^{\alpha}_{\mu'} V_{\alpha}$$

$$V^{\mu'} = \Lambda^{\mu'}_{\alpha} V^{\alpha} \qquad (\Lambda^{\mu'}_{\alpha} \equiv \text{matrix inverse of } \Lambda^{\alpha}_{\mu'})$$

where the Λ's are Lorentz transformation matrices. Of special interest are the "boost" transformations involving changes in velocity with no rotation. For a primed frame with velocity β in the x-direction,

$$\Lambda^{\mu'}_{\nu} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \gamma \equiv (1-\beta^2)^{-\frac{1}{2}} \ .$$

The velocity between two frames is sometimes parameterized by $\theta \equiv \tanh^{-1}\beta$ ("the rapidity parameter").

A particle of rest mass m and 4-velocity $\mathbf{u}$ has 4-momentum $\mathbf{p} \equiv m\mathbf{u}$. If $m = 0$ (photons), $\mathbf{p}$ is defined by its components in the frame of any observer $p^0 \equiv$ photon energy, $p^i \equiv \underset{\sim}{p} =$ photon 3-momentum.

Problem 1.1. The 4-velocity $\mathbf{u}$ corresponds to 3-velocity $\underset{\sim}{v}$. Express:

(a) u^0 in terms of $|\underset{\sim}{v}|$

(b) $u^j (j=1,2,3)$ in terms of $\underset{\sim}{v}$

(c) u^0 in terms of u^j

(d) $d/d\tau$ in terms of d/dt and $\underset{\sim}{v}$

(e) v^j in terms of u^j

(f) $|\underset{\sim}{v}|$ in terms of u^0.

Problem 1.2. Find the matrix for the Lorentz transformation consisting of a boost v_x in the x-direction followed by a boost v_y in the y-direction. Show that the boosts performed in the reverse order would give a different transformation.

Problem 1.3. If two frames move with 3-velocities $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$, show that their relative velocity is given by

$$v^2 = \frac{(\underset{\sim}{v}_1 - \underset{\sim}{v}_2)^2 - (\underset{\sim}{v}_1 \times \underset{\sim}{v}_2)^2}{(1 - \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2)^2} .$$

Problem 1.4. A cart rolls on a long table with velocity β. A smaller cart rolls on the first cart in the same direction with velocity β relative to the first cart. A third cart rolls on the second cart in the same direction with relative velocity β, and so on up to n carts. What is the velocity v_n of the nth cart in the frame of the table? What does v_n tend to as $n \to \infty$?

Problem 1.5. A distant camera snaps a photograph of a speeding bullet (velocity v) with length b in its rest frame. Behind the bullet and parallel to its path is a meter stick, at rest with respect to the camera. The direction to the camera is an angle α from the direction of the bullet's velocity. What will be the *apparent length* of the bullet as seen in the photo? (i.e. How much of the meter stick is hidden?).

Problem 1.6. Tachyons are hypothetical particles whose velocity is faster than light. Suppose that a tachyon transmitter emits particles of a constant velocity $u > c$ in its rest frame. If a tachyonic message is sent to an observer at rest at a distance L, how much time will elapse before a tachyonic reply can be received? How much time will elapse if the distant observer is moving directly away at velocity v, and is at a distance L at the instant he receives the message and replies? (Show that for $u > [1+(1-v^2)^{\frac{1}{2}}]/v$ the reply can be received before the signal is sent!)

Problem 1.7. Frame S' moves with velocity $\underset{\sim}{v}$ relative to frame S. A rod in frame S' makes an angle θ' with respect to the forward direction of motion. What is this angle θ as measured in S?

Problem 1.8. Frame S' moves with velocity $\underset{\sim}{\beta}$ relative to frame S. A bullet in frame S' is fired with velocity $\underset{\sim}{v}'$ at an angle θ' with respect to the forward direction of motion. What is this angle θ as measured in S? What if the bullet is a photon?

Problem 1.9. Suppose that an observer at rest with respect to the fixed distant stars sees an isotropic distribution of stars. That is, in any solid angle $d\Omega$ he sees $dN = N(d\Omega/4\pi)$ stars, where N is the total number of stars he can see.

Suppose now that another observer (whose rest frame is S') is moving at a relativistic velocity β in the e_x direction. What is the distribution of stars seen by this observer? Specifically, what is the distribution function $P(\theta', \phi')$ such that the number of stars seen by this observer in his solid angle $d\Omega'$ is $P(\theta', \phi')\, d\Omega'$? Check to see that $\int_{\text{sphere}} P(\theta', \phi') d\Omega' = N$, and check that $P(\theta', \phi') \to \frac{N}{4\pi}$ as $\beta \to 0$. Where will the observer see the stars "bunch up"?

Problem 1.10. Show that $A = 3^{\frac{1}{2}} e_t + 2^{\frac{1}{2}} e_x$ is a unit timelike vector in special relativity. Show that the angle between A and e_t is not real.

Problem 1.11. Two rings rotate with equal and opposite angular velocity ω about a common center. Suppose Adam rides on one ring and Eve on the other, and that at some moment they pass each other and their clocks agree. At the moment they pass, Eve sees Adam's clock running more slowly, so she expects to be ahead the next time they meet. But Adam expects just the reverse. What really happens? Can you reconcile this with Adam's (or Eve's) observations?

Problem 1.12. Define an imaginary coordinate $w = it$. Show that a rotation of angle θ in the x_i, w plane $(i=1,2,3)$, where θ is a pure imaginary number, corresponds to a pure Lorentz boost in t, x, y, z coordinates. How is the boost velocity v related to the angle θ?

Problem 1.13. Show that the curve

$$\begin{aligned} x &= \int r \cos\theta \cos\phi \, d\lambda \\ y &= \int r \cos\theta \sin\phi \, d\lambda \\ z &= \int r \sin\theta \, d\lambda \\ t &= \int r \, d\lambda \quad , \end{aligned}$$

where r, θ and ϕ are arbitrary functions of λ, is a null curve in special relativity. Under what conditions is it a null geodesic?

Problem 1.14. Show that an observer's 4-acceleration $du^\alpha/d\tau$ has only 3 independent components, and give the relation of these to the 3 components of ordinary acceleration that he would measure with a Newtonian accelerometer in his local frame.

Problem 1.15. Write the magnitude of the acceleration measured in the observer's frame as an invariant.

Problem 1.16. A particle moves with 3-velocity $\underset{\sim}{u}$ and 3-acceleration $\underset{\sim}{a}$ as seen by an inertial observer $\mathcal{O}$. Another inertial observer $\mathcal{O}'$ has 3-velocity $\underset{\sim}{v}$ relative to $\mathcal{O}$. Show that the components of acceleration of the particle parallel and perpendicular to $\underset{\sim}{v}$ as measured by $\mathcal{O}'$ are

$$\underset{\sim}{a}'_{\parallel} = \frac{(1-v^2)^{3/2}}{(1-\underset{\sim}{v}\cdot\underset{\sim}{u})^3}\,\underset{\sim}{a}_{\parallel}$$

$$\underset{\sim}{a}'_{\perp} = \frac{(1-v^2)}{(1-\underset{\sim}{v}\cdot\underset{\sim}{u})^3}\,[\underset{\sim}{a}_{\perp} - \underset{\sim}{v}\times(\underset{\sim}{a}\times\underset{\sim}{u})] \ .$$

Problem 1.17. An observer experiences a uniform acceleration in the x direction, of magnitude g. Define a coordinate system $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ for him in the following way: (i) Let the observer be at $\bar{x} = \bar{y} = \bar{z} = 0$ and let $\bar{t}$ be his proper time. (ii) Let his hyperplanes of simultaneity agree with the hyperplanes of simultaneity of an instantaneously comoving inertial frame. (iii) Let the other "coordinate stationary observers" (for whom $\bar{x}, \bar{y}, \bar{z}$ are constant) move in such a way that they are always at rest with respect to the observer on the hyperplanes of simultaneity. At $t = 0$ label all spatial points with the same labels as the momentarily comoving inertial system $t = 0$, x, y, z.

Give the coordinate transformation between t, x, y, z and $\bar{t}, \bar{x}, \bar{y}, \bar{z}$. Show that coordinate stationary clocks cannot remain synchronized.

Problem 1.18. A mirror moves perpendicular to its plane with a velocity v. With what angle to the normal is a ray of light reflected, if it is incident at an angle θ? What is the change in the frequency of the light?

Problem 1.19. A mirror is moving parallel to its plane. Show that the angle of incidence of a photon equals the angle of reflection.

Problem 1.20. A particle of rest mass m and 4-momentum $\mathbf{p}$ is examined by an observer with 4-velocity $\mathbf{u}$. Show that:

(a) the energy he measures is $E = -\mathbf{p} \cdot \mathbf{u}$;

(b) the rest mass he attributes to the particle is $m^2 = -\mathbf{p} \cdot \mathbf{p}$;

(c) the momentum he measures has magnitude $|\underset{\sim}{p}| = [(\mathbf{p} \cdot \mathbf{u})^2 + \mathbf{p} \cdot \mathbf{p}]^{\frac{1}{2}}$;

(d) the ordinary velocity v he measures has magnitude

$$|\underset{\sim}{v}| = \left[1 + \frac{\mathbf{p} \cdot \mathbf{p}}{(\mathbf{p} \cdot \mathbf{u})^2}\right]^{\frac{1}{2}}$$

(e) the 4-vector v, whose components in the observer's Lorentz frame are

$$v^0 = 0, \quad v^j = (dx^j/dt)_{\text{particle}} = \text{ordinary velocity} ,$$

is given by $v = -\mathbf{u} - \dfrac{\mathbf{p}}{\mathbf{p} \cdot \mathbf{u}}$.

Problem 1.21. An iron nucleus emits a Mössbauer gamma ray with frequency ν_0 as measured in its own rest frame. The nucleus is traveling with velocity $\underset{\sim}{\beta}$ with respect to some inertial observer. What frequency does the observer measure when the gamma ray reaches him? Express the answer in terms of $\underset{\sim}{\beta}$, ν_0, and the unit vector $\underset{\sim}{n}$ pointing towards the nucleus at the time it emitted the γ-ray, as measured by the observer.

Problem 1.22. An observer receives light from a source of light which is moving with a velocity $\underset{\sim}{v}$; the angle between $\underset{\sim}{v}$ and the line between observer and source is θ at the time the light is emitted. If the observer sees no net redshift or blueshift, what is θ in terms of $|\underset{\sim}{v}|$?

Problem 1.23. Suppose in some inertial frame S a photon has 4-momentum components

$$p^0 = p^x = E, \quad p^y = p^z = 0 \ .$$

There is a special class of Lorentz transformations – called the "little group of p" – which leave the components of p unchanged, e.g. a pure rotation through an angle α in the y-z plane

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} E \\ E \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} E \\ E \\ 0 \\ 0 \end{bmatrix}$$

is such a transformation. Find a sequence of pure boosts and pure rotations whose product is *not* a pure rotation in the y-z plane, but *is* in the little group of p.

Problem 1.24. Two giant frogs are captured, imprisoned in a large metal cylinder, and placed on an airplane. While in flight, the storage doors accidentally open and the cylinder containing the frogs falls out. Sensing something amiss, the frogs decide to try to break out. Centering themselves in the cylinder, they push off from each other and slam simultaneously into the ends of the cylinder. They instantly push off from the ends and shoot across the cylinder past each other into the opposite ends. This continues until the cylinder hits the ground. Consider how this looks from some other inertial frame, falling at another speed. In this frame, the frogs do *not* hit the ends of the cylinder simultaneously, so the cylinder jerks back and forth about its mean speed β. The cylinder, however, was at rest in one inertial frame. Does this mean that one inertial frame can jerk back and forth with respect to another?

Problem 1.25. Let J_x, J_y, J_z be infinitesimal rotation operators defined so that $1 + iJ_j\theta/2$ is a rotation by a small angle θ around the j-axis. Let K_x, K_y, K_z be infinitesimal boost operators defined so that $1 + iK_jv/2$ is a boost by a small velocity v in the j-direction. Show that the following relationships, and all their cyclic permutations, are true:

$$[J_x, J_y] = 2iJ_z$$
$$[J_x, K_y] = 2iK_z$$
$$[K_x, K_y] = -2iJ_z \ .$$

Find a representation of the Lorentz group in terms of Pauli spin matrices σ_x, σ_y, σ_z, and the unit matrix.

Problem 1.26. Two successive, arbitrary pure Lorentz boosts $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ are equivalent to a pure boost $\underset{\sim}{v}_3$ followed by a pure rotation $\theta\underset{\sim}{n}$, where n is a unit vector. Find the magnitude of θ in terms of $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ and show that $\underset{\sim}{n} \cdot \underset{\sim}{v}_3 = 0$.

Problem 1.27. Show that any proper (non time-reversing, non parity-reversing) homogeneous Lorentz transformation leaves fixed at least one null direction.

Problem 1.28. What is the least number of pure boosts which generate an arbitrary Lorentz transformation? Note: This is a difficult problem!

CHAPTER 2
SPECIAL-RELATIVISTIC DYNAMICS

In our laboratory frame, a particle with 4-momentum $\mathbf{p}$ has total energy $E = p^0$ and 3-momentum $\underset{\sim}{p} = p^i$. If the particle has a nonzero rest mass m, the 4-momentum, 4-velocity $\mathbf{u}$ and 3-velocity $\underset{\sim}{v}$ are related by

$$\mathbf{p} = m\mathbf{u} = m(\gamma, \gamma\underset{\sim}{v}), \qquad \gamma \equiv (1-v^2)^{-\frac{1}{2}} ,$$

so $E = \gamma m$, $\underset{\sim}{p} = \gamma m \underset{\sim}{v}$. The square of a particle's 4-momentum, an invariant in all frames, is

$$\mathbf{p}\cdot\mathbf{p} = -E^2 + \underset{\sim}{p}^2 = -m^2 .$$

The kinetic energy of a particle is $T \equiv E - m$.

The fundamental dynamical law for particle interactions is that in any frame the vector sum of the 4-momenta of all particles is a conserved constant in time.

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Problem 2.1. (Compton scattering.) A photon of wavelength λ hits a stationary electron (mass m_e) and comes off with wavelength λ' at an angle θ. Derive the expression

$$\lambda' - \lambda = (h/m_e)(1-\cos\theta) .$$

Problem 2.2.

(a) When a photon scatters off a charged particle which is moving with a speed very nearly that of light, the photon is said to have undergone an inverse Compton scattering. Consider an inverse Compton scattering in which a charged particle of rest mass m and total mass-energy (as seen in the lab frame) $E \gg m$, collides head-on with a photon of frequency $\nu\,(h\nu \ll m)$. What is the maximum energy the particle can transfer to the photon?

(b) If space is filled with black-body radiation of temperature 3^0K and contains cosmic ray protons of energies up to 10^{20} eV, how much energy can a proton of energy 10^{20} eV transfer to a 3^0K photon?

Problem 2.3. Show that it is impossible for an isolated free electron to absorb or emit a photon.

Problem 2.4. A particle of rest mass m_1 and velocity $\underset{\sim}{v}_1$ collides with a stationary particle of rest mass m_2 and is absorbed by it. Find the rest mass m and velocity $\underset{\sim}{v}$ of the resultant compound system.

Problem 2.5. The beta-decay of a neutron is isotropic in the rest frame of the neutron, with the velocity of the emitted electron $v_e = 0.77$. If the neutron is moving with velocity β through the laboratory, what values of the electron's laboratory momentum vector $\underset{\sim}{P}$ are possible?

Problem 2.6. Evaluate the "available energy" of two different proton-proton scattering experiments. The first is of the conventional type, where a beam of protons is accelerated to 30 GeV and allowed to strike a target (liquid hydrogen, for example). In the second, two separate beams of protons are accelerated to 15 GeV each, then directed toward each other and allowed to collide. Evaluate the total energy of two colliding protons in the center of momentum frame for each experiment. To what energy would a beam in the first type of experiment have to be accelerated to match the CM energy of the 15 GeV protons in the second experiment?

Problem 2.7. A particle of rest mass m collides elastically with a stationary particle of equal mass. The incident particle has kinetic energy T_0. What is its kinetic energy after the collision, if the scattering angle is θ?

Problem 2.8. Calculate the threshold energy of a nucleon N for it to undergo the reaction

$$\gamma + N \to N + \pi$$

where γ represents a photon of temperature 3^0K. Assume the collision is head-on; take the photon energy to be $\sim kT$; $m_N = 940$ MeV; $m_\pi = 140$ MeV. (This effect probably produces a cut-off in the cosmic-ray spectrum at this threshold energy.)

Problem 2.9. Consider the reaction $\pi^+ + n \to K^+ + \Lambda^0$. The rest masses of the particles are $m_\pi = 140$ MeV, $m_n = 940$ MeV, $m_K = 494$ MeV, $m_\Lambda = 1115$ MeV. What is the threshold kinetic energy of the π to create a K at an angle of 90^0 in the lab in which the n is at rest?

Problem 2.10. Consider the reaction $A \to B + C$ (with particle masses m_A, m_B, m_C).

(a) If A is at rest in the lab frame, show that in the lab frame particle B has energy $E_B = (m_A^2 + m_B^2 - m_C^2)/2m_A$.

(b) An atom of mass M at rest decays to a state of rest energy $M-\delta$ by emitting a photon of energy $h\nu$. Show that $h\nu < \delta$. In the Mössbauer effect, why is $h\nu = \delta$?

(c) If A decays while moving in the lab frame, find the relation between the angle at which B comes off, and the energies of A and B.

Problem 2.11. Consider the reaction $1+2 \to 3+4$. The lab frame is defined to be the one in which $\underset{\sim}{P}_2 = 0$. The C.M. frame is defined to be the one in which $\underset{\sim}{P}_1^{\text{C.M.}} + \underset{\sim}{P}_2^{\text{C.M.}} = 0$. Show that:

(a) $E_{\text{total}}^{\text{C.M.}} = (m_1{}^2 + m_2{}^2 + 2m_2E_1)^{\frac{1}{2}}$

(b) $E_1^{\text{C.M.}} = [(E_{\text{total}}^{\text{C.M.}})^2 + m_1{}^2 - m_2{}^2]/2E_{\text{total}}^{\text{C.M.}}$

(c) $P_1^{\text{C.M.}} = m_2P_1/E_{\text{total}}^{\text{C.M.}}$

(d) $\gamma_{\text{C.M.}} = (E_1 + m_2)/E_{\text{total}}^{\text{C.M.}}$ ($v_{\text{C.M.}} \equiv$ velocity of C.M. in lab frame, and $\gamma_{\text{C.M.}} \equiv (1 - v^2{}_{\text{C.M.}})^{-\frac{1}{2}}$.)

(e) $v_{\text{C.M.}} = P_1/(E_1 + m_2)$.

Problem 2.12. Consider the elastic collision of a particle of mass m_1 with a stationary particle of mass $m_2 < m_1$. Let θ_{max} be the maximum scattering angle of m_1. In nonrelativistic calculations, $\sin\theta_{max} = m_2/m_1$. Prove that this result also holds relativistically.

Problem 2.13.

(a) If a rocket has engines that give it a constant acceleration of 1 g (relative to its instantaneous inertial frame, of course), and the rocket starts from rest near the earth, how far from the earth (as measured in the earth's frame) will the rocket be in 40 years as measured on the earth? How far after 40 years as measured in the rocket?

(b) Compute the proper time for the occupants of a rocket ship to travel the 30,000 light years from the Earth to the center of the galaxy. Assume they maintain an acceleration of 1 g for half the trip and decelerate at 1 g for the remaining half.

(c) What fraction of the initial mass of the rocket can be payload in part (b)? Assume an ideal rocket that converts rest mass into radiation and ejects all of the radiation out of the back with 100% efficiency and perfect collimation.

Problem 2.14. What is the maximum energy one could get out of a fixed frequency electron cyclotron with accelerating potential V.

Problem 2.15. A new force field $F^\mu(x^\nu)$ is discovered which induces a 4-acceleration $a^\mu \equiv du^\mu/d\tau = m^{-1}F^\mu(x^\nu)$ on a particle of mass m, at position x^ν. Notice that F^μ does not depend on u^ν. Show that this force is not consistent with special relativity.

CHAPTER 3

SPECIAL-RELATIVISTIC COORDINATE TRANSFORMATIONS, INVARIANTS AND TENSORS

Spacetime in special relativity can be described by more general (curvilinear) coordinates than "inertial" or Minkowski coordinates, e.g. coordinates $x^{\mu'}$,

$$x^{\mu'} = f^{\mu}(x^{\nu})$$

where x^{ν} are Minkowski coordinates, and f^{μ} four arbitrary functions. One then shows that the basis vectors and components of vectors in the new coordinates are related to the old by

$$e_{\alpha'} = \frac{\partial x^{\mu}}{\partial x^{\alpha'}} e_{\mu} , \qquad e_{\mu} = \frac{\partial x^{\alpha'}}{\partial x^{\mu}} e_{\alpha'}$$

$$V^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^{\mu}} V^{\mu} , \qquad V^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\alpha'}} V^{\alpha'}$$

$$V_{\alpha'} = \frac{\partial x^{\mu}}{\partial x^{\alpha'}} V_{\mu} , \qquad V_{\mu} = \frac{\partial x^{\alpha'}}{\partial x^{\mu}} V_{\alpha'} \quad .$$

In other words, the transformation matrix $\Lambda^{\mu'}_{\alpha} \equiv \partial x^{\mu'}/\partial x^{\alpha}$ replaces the less general Lorentz matrices (which are applicable only for transformations between two systems of Minkowski coordinates).

In general coordinates, the relation $\mathbf{A} \cdot \mathbf{B} = A_{\mu}B^{\mu}$ still holds, but we no longer have $A_{\mu} = \eta_{\mu\nu}A^{\mu}$. Rather, corresponding to every coordinate system there is a metric tensor with components $g_{\alpha\beta}$, such that $ds^2 = g_{\alpha\beta}\,dx^{\alpha}\,dx^{\beta}$, which leads to $A_{\mu} = g_{\mu\nu}A^{\nu}$ and therefore $\mathbf{A}\cdot\mathbf{B} = g_{\mu\nu}A^{\mu}B^{\nu}$. Note also $A^{\mu} = g^{\mu\nu}A_{\nu}$ where $g^{\mu\nu}$ is the matrix inverse of $g_{\mu\nu}$.

Various formal definitions of a *tensor* are possible. Here, it suffices to say that it is a geometrical object which, like a vector, has components whose numerical values are different in different coordinate systems. A tensor has 4^n components, where n is its rank (number of "slots" or indices for components). Slots may be contravariant or covariant; examples: $T^{\mu\nu}$, $F_{\mu\nu}$, $R^{\alpha}{}_{\beta\gamma\delta}$, $G_{\mu}{}^{\nu}$. Tensors transform with one matrix for each slot, e.g.

$$G_{\mu'}{}^{\nu'} = \Lambda^{\alpha}{}_{\mu'}\Lambda^{\nu'}{}_{\beta}G_{\alpha}{}^{\beta} .$$

Tensors may be "contracted" (a covariant and a contravariant index summed) or multiplied as a "direct product" with other tensors or with themselves to form new tensors, e.g.

$$Q_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} , \quad A^{\mu} = G^{\mu}{}_{\nu}B^{\nu} , \quad F_{\mu\nu} = A_{\mu}B_{\nu} .$$

A special case is contraction with the metric tensor, where the same symbol is usually used for the result as in the relation of covariant and contravariant vectors, $F^{\mu}{}_{\nu} = g_{\nu\alpha}F^{\mu\alpha}$. A tensor expression with no free indices, e.g. $F_{\mu\nu}A^{\mu}A^{\nu}$ or $R_{\beta\gamma}F^{\beta\gamma}$ or $A^{\alpha}B^{\beta}g_{\alpha\beta}$, is a scalar and is an invariant number in all frames. The analog of the index free notation $\mathbf{A}$ for a vector A^{μ} is to write, e.g. $\mathbf{T}$ for a tensor $T^{\mu\nu}$. In both cases the existence of covariant or contravariant slots must be deduced from the context.

In index free notation, $\otimes$ represents the direct product, e.g. $F^{\mu\nu}A^{\rho}$ is written $\mathbf{F}\otimes\mathbf{A}$; the contracted product is written with a dot, e.g. $\mathbf{F}\cdot\mathbf{A}$ for $F^{\mu\alpha}A_{\alpha}$.

We denote partial derivatives by a comma, e.g. $f_{,\alpha} \equiv \partial f/\partial x^{\alpha}$.

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Problem 3.1.

(I) If 2 events are separated by a spacelike interval, show that

(a) there exists a Lorentz frame in which they are simultaneous, and

(b) in no Lorentz frame do they occur at the same point.

(II) If 2 events are separated by a timelike interval, show that

(a) there exists a frame in which they happen at the same point, and

(b) in no Lorentz frame are they simultaneous.

Problem 3.2. Find 4 linearly independent null vectors in Minkowski space. Can you find 4 which are orthogonal?

Problem 3.3. Show that the only non-spacelike vectors orthogonal to a given nonzero null vector are multiples of it.

Problem 3.4. Show that the sum of two vectors can be spacelike, null, or timelike, independently of whether the two vectors are spacelike, null, or timelike.

Problem 3.5. Show that the cross-sectional area of a parallel beam of light is invariant under Lorentz transformations.

Problem 3.6. Show that $\sum_\mu D^{\mu\mu}$ and $\sum_\mu D_{\mu\mu}$ are not invariant under coordinate transformations, but that $\sum_\mu D_\mu{}^\mu$ *is* invariant. (Take $\mathbf{D}$ to be a tensor defined by its components $D^{\mu\nu}$.)

Problem 3.7. $F^{\alpha\beta}$ is antisymmetric on its two indices. Show that

$$F_\mu{}^\alpha{}_{,\nu} F^\nu{}_\alpha = -F_{\mu\alpha,\beta} F^{\alpha\beta} \; .$$

Problem 3.8. In a coordinate system with coordinates x^μ, the invariant line element is $ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta$. If the coordinates are transformed $x^\mu \to \bar{x}^\mu$, show that the line element is $ds^2 = g_{\bar{\mu}\bar{\nu}} d\bar{x}^\mu d\bar{x}^\nu$, and express $g_{\bar{\mu}\bar{\nu}}$ in terms of the partial derivatives $\partial x^\mu / \partial \bar{x}^\nu$. For two arbitrary 4-vectors $\mathbf{U}$ and $\mathbf{V}$, show that

$$\mathbf{U} \cdot \mathbf{V} = U^\alpha V^\beta \eta_{\alpha\beta} = U^{\bar{\alpha}} V^{\bar{\beta}} g_{\bar{\alpha}\bar{\beta}} \; .$$

Problem 3.9. Show that the determinant of the metric tensor $g \equiv \det(g_{\mu\nu})$ is not a scalar.

Problem 3.10. If Λ^α_β and $\tilde{\Lambda}^\alpha_\beta$ are two matrices which transform the components of a tensor from one coordinate basis to another, show that the matrix $\Lambda^\alpha_\gamma \tilde{\Lambda}^\gamma_\beta$ is also a coordinate transformation.

Problem 3.11. You are given a tensor $K^{\alpha\beta}$. How can you test whether it is a direct product of two vectors $K^{\alpha\beta} = A^{\alpha}B^{\beta}$? Can you express the test in coordinate-free language?

Problem 3.12. Prove that the general second-rank tensor in n-dimensions cannot be represented as a simple direct product of two vectors, but *can* be expressed as a sum over many such products.

Problem 3.13. A two index "object" $X^{\mu\nu}$ is defined by the "direct sum" of two vectors $X^{\mu\nu} = A^{\mu} + B^{\nu}$. Is $X^{\mu\nu}$ a tensor? Is there a transformation law to take X to a new coordinate system, i.e. to obtain $X^{\hat{\mu}\hat{\nu}}$ from $X^{\mu\nu}$?

Problem 3.14. Show that a second rank tensor F which is antisymmetric in one coordinate frame $(F_{\mu\nu} = -F_{\nu\mu})$ is antisymmetric in all frames. Show that the contravariant components are also antisymmetric $(F^{\mu\nu} = -F^{\nu\mu})$. Show that symmetry is also coordinate invariant.

Problem 3.15. Let $A_{\mu\nu}$ be an antisymmetric tensor so that $A_{\mu\nu} = -A_{\nu\mu}$; and let $S^{\mu\nu}$ be a symmetric tensor so that $S^{\mu\nu} = S^{\nu\mu}$. Show that $A_{\mu\nu}S^{\mu\nu} = 0$. Establish the following two identities for any arbitrary tensor $V_{\mu\nu}$:

$$V^{\mu\nu}A_{\mu\nu} = \frac{1}{2}(V^{\mu\nu} - V^{\nu\mu})A_{\mu\nu}, \qquad V^{\mu\nu}S_{\mu\nu} = \frac{1}{2}(V^{\mu\nu} + V^{\nu\mu})S_{\mu\nu} .$$

Problem 3.16.

(a) In an n-dimensional metric space, how many independent components are there for an r-rank tensor $T^{\alpha\beta\cdots}$ with no symmetries?

(b) If the tensor is symmetric on s of its indices, how many independent components are there?

(c) If the tensor is antisymmetric on a of its indices, how many independent components are there?

Problem 3.17. We define the meaning of square and round brackets enclosing a set of indices as follows:

$$V_{(\alpha_1,\cdots,\alpha_p)} \equiv \frac{1}{p!}\Sigma V_{\alpha_{\pi_1}\cdots\alpha_{\pi_p}}; \qquad V_{[\alpha_1,\cdots\alpha_p]} \equiv \frac{1}{p!}\Sigma(-1)^\pi V_{\alpha_{\pi_1}\cdots\alpha_{\pi_p}} .$$

Here the sum is taken over all permutations π of the numbers $1, 2, \cdots, p$ and $(-1)^\pi$ is $+1$ or -1 depending on whether the permutation is even or odd. The quantity V may have other indices, not shown here, besides the set of p indices $\alpha_1, \alpha_2, \cdots, \alpha_p$, but only this set of indices is affected by the operations described here. The numbers $\pi_1, \pi_2, \cdots, \pi_p$ are the numbers $1, 2, \cdots, p$ rearranged according to the permutation π. Thus for example $V_{(\alpha_1\alpha_2)} \equiv \frac{1}{2}(V_{\alpha_1\alpha_2} + V_{\alpha_2\alpha_1})$ or equivalently $V_{(\mu\nu)} = \frac{1}{2}(V_{\mu\nu} + V_{\nu\mu})$.

(a) If F is antisymmetric and T is symmetric, apply these definitions to give explicit formulas for the following: $V_{[\mu\nu]}$, $F_{[\mu\nu]}$, $F_{(\mu\nu)}$, $T_{[\mu\nu]}$, $T_{(\mu\nu)}$, $V_{[\alpha\beta\gamma]}$, $T_{(\alpha\beta,\gamma)}$, $F_{[\alpha\beta,\gamma]}$.

(b) Establish the following formulae: $V_{((\alpha_1\cdots\alpha_p))} = V_{(\alpha_1\cdots\alpha_p)}$; $V_{[[\alpha_1\cdots\alpha_p]]} = V_{[\alpha_1\cdots\alpha_p]}$; $V_{(\alpha_1\cdots[\alpha_\ell\alpha_m]\cdots\alpha_p)} = 0$; $V_{[\alpha_1\cdots[\alpha_\ell\alpha_m]\cdots\alpha_p]} = V_{[\alpha_1\cdots\alpha_\ell\alpha_m\cdots\alpha_p]}$.

(c) Use these notations to show that $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ implies $F_{\alpha\beta,\nu} + F_{\beta\nu,\alpha} + F_{\nu\alpha,\beta} = 0$. (Half of Maxwell's equations!)

Problem 3.18. Show for any two-index tensor X, that $X_{\alpha\beta} = X_{(\alpha\beta)} + X_{[\alpha\beta]}$ where ( ) and [ ] denote symmetrization and antisymmetrization, respectively. Show that in general

$$Y_{\alpha\beta\gamma} \neq Y_{(\alpha\beta\gamma)} + Y_{[\alpha\beta\gamma]} .$$

Problem 3.19. Prove that the Kronecker delta, $\delta^\mu{}_\nu$, is a tensor.

Problem 3.20. Prove that, except for scaling by a constant, there is a unique tensor $\varepsilon_{\alpha\beta\gamma\delta}$ which is totally antisymmetric on all its 4 indices. The usual choice is to take $\varepsilon_{0123} = 1$ in Minkowski coordinates. What are the components of ε in a general coordinate frame, with metric $g_{\mu\nu}$?

Problem 3.21. In an orthonormal frame, show that

$$\varepsilon_{\alpha\beta\gamma\delta} = -\,\varepsilon^{\alpha\beta\gamma\delta} .$$

What is the analogous relation in a general coordinate frame with metric $g_{\mu\nu}$?

Problem 3.22. Evaluate $\varepsilon_{\mu\nu\rho\sigma}\,\varepsilon^{\mu\nu\rho\sigma}$.

Problem 3.23. Show that for any tensor $A^{\alpha}{}_{\beta}$

$$\varepsilon_{\alpha\beta\gamma\delta}\, A^{\alpha}{}_{\mu}\, A^{\beta}{}_{\nu}\, A^{\gamma}{}_{\lambda}\, A^{\delta}{}_{\sigma} = \varepsilon_{\mu\nu\lambda\sigma} \det \|A^{\alpha}{}_{\beta}\|$$

where $\|A^{\alpha}{}_{\beta}\|$ is the matrix of the components $A^{\alpha}{}_{\beta}$.

Problem 3.24. Show that four vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$, are linearly independent if and only if $\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}\wedge\mathbf{x} \neq 0$. Show that in this case $\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}\wedge\mathbf{x}$ is proportional to the totally antisymmetric tensor ε. (The "wedge" product is defined as the antisymmetrized direct product, e.g. $\mathbf{u}\wedge\mathbf{v} = \mathbf{u}\otimes\mathbf{v} - \mathbf{v}\otimes\mathbf{u}$.)

Problem 3.25. Let $\mathbf{F}$ be an antisymmetric second-rank tensor with components $F^{\mu\nu}$. From $\mathbf{F}$ construct another second-rank, anti-symmetric tensor, $*\mathbf{F}$, called the dual of $\mathbf{F}$, as follows

$$*\mathbf{F} = \tfrac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}\, \mathbf{e}_{\mu}\otimes\mathbf{e}_{\nu} .$$

Show that $*(*\mathbf{F}) = -\mathbf{F}$.

Problem 3.26. Show that

$$V_{\sigma}V^{\sigma} = -\frac{1}{3!}\,(*V)_{\alpha\beta\gamma}\,(*V)^{\alpha\beta\gamma} .$$

Problem 3.27. The tensor $\delta^{\mu\cdots\lambda}_{\rho\cdots\sigma}$ is defined by

$$\delta^{\mu\cdots\lambda}_{\rho\cdots\sigma} \equiv \det\begin{bmatrix} \delta^{\mu}_{\rho} & \cdots & \delta^{\lambda}_{\rho} \\ \vdots & & \vdots \\ \delta^{\mu}_{\sigma} & \cdots & \delta^{\lambda}_{\sigma} \end{bmatrix} .$$

Show that if there are more than 4 upper (or lower) indices, the tensor identically vanishes.

Problem 3.28. Show that $\delta^{\mu\nu}_{\lambda\kappa} = -\frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\,\varepsilon_{\lambda\kappa\rho\sigma}$, and generalize to $\delta^{\mu\cdots\nu}_{\lambda\cdots\kappa}$ of other ranks.

Problem 3.29. Show that if the antisymmetric tensor $p^{\alpha\beta}$ is a bivector (i.e. $p^{\alpha\beta} = A^{[\alpha}B^{\beta]}$) then

$$p^{\alpha\beta}\,p^{\gamma\delta} + p^{\alpha\gamma}\,p^{\delta\beta} + p^{\alpha\delta}\,p^{\beta\gamma} = 0$$

(the Plücker relations).

Problem 3.30. In 4-space define the 3-dimensional volume element in a hypersurface $x^{\alpha} = x^{\alpha}(a,b,c)$ by $d^3\Sigma_{\mu} = (1/3!)\,\varepsilon_{\mu\alpha\beta\gamma}\,da\,db\,dc\,[\partial(x^{\alpha},x^{\beta},x^{\gamma})/\partial(a,b,c)]$, where the last factor is a 3×3 Jacobian determinant. Compute the components of $d^3\Sigma_{\mu}$ for a space-like hypersurface $x^0 = \text{constant}$, parameterized by $x^1 = a$, $x^2 = b$, $x^3 = c$.

Problem 3.31. Show that the invariant proper volume element in 4-dimensional space is given by

$$dV = (-g)^{\frac{1}{2}}\,d^4x$$

where $d^4x = dxdydzdt$ in the coordinate system of the metric $g_{\mu\nu}$.

Problem 3.32. Show that the proper 3-volume element of an observer with 4-velocity $\mathbf{u}$ is $d^3V = (-g)^{\frac{1}{2}}\,u^0 d^3x$, and show that this is a scalar invariant.

Problem 3.33. What is the invariant volume element of contravariant momentum d^4P for 4-dimensional momentum space? What is the invariant 3-volume "on the mass shell", i.e. when the constraint $(-\mathbf{P}\cdot\mathbf{P})^{\frac{1}{2}} = m$ is imposed?

Problem 3.34. A group of N particles is seen to occupy a volume $dx\,dy\,dz\,dP^x\,dP^y\,dP^z$ in 6-dimensional phase space, so that the number density of particles in phase space $\mathfrak{N}$ is given by

$$N = \mathfrak{N}\, dx\, dy\, dz\, dP^x\, dP^y\, dP^z \quad .$$

Show that $\mathfrak{N}$ is a Lorentz invariant, i.e. that all observers will compute the same numerical value for $\mathfrak{N}$.

Problem 3.35. A vector field $J^\alpha(x^\mu)$ satisfies $J^\alpha{}_{,\alpha} = 0$ and J^α falls off faster than r^{-2} at large distances from the origin. (a) Show that $\int J^0 d^3x$ is constant in time. (b) Show that the integral is a scalar, i.e. $\int J^0 d^3x = \int J^{0'} d^3x'$.

CHAPTER 4
ELECTROMAGNETISM

The electromagnetic field is described relativistically by the anti-symmetric electromagnetic field tensor (Maxwell tensor) $F^{\mu\nu}$. In any Lorentz frame the components of $F^{\mu\nu}$ are related to the electric and magnetic field strengths, $\underset{\sim}{E}$ and $\underset{\sim}{B}$, in that frame by

$$F^{\mu\nu} = \begin{bmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{bmatrix} .$$

Here μ is the row index and ν the column index. Maxwell's equations can be written

$$F^{\mu\nu}{}_{,\nu} = 4\pi J^{\mu}$$

$$F_{\alpha\beta,\gamma} + F_{\gamma\alpha,\beta} + F_{\beta\gamma,\alpha} = 0 ,$$

where $J^{\mu} = (\rho, \underset{\sim}{J})$ is the 4-current density. The Lorentz force law is

$$dp^{\mu}/d\tau = eF^{\mu\nu}u_{\nu}$$

for a particle of charge e, 4-momentum p and 4-velocity u.

The energy density $\mathcal{E} = (E^2 + B^2)/8\pi$, the Poynting energy flux $\underset{\sim}{S} = (\underset{\sim}{E} \times \underset{\sim}{B})/4\pi$, and the 3-dimensional stress-tensor

$$T^{ij} = [-(E^iE^j + B^iB^j) + \frac{1}{2}\delta^{ij}(E^2 + B^2)]/4\pi$$

are combined to form the electromagnetic stress-energy tensor

$$T^{\mu\nu} = (F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta})/4\pi .$$

Problem 4.1. Find the magnetic field $\underset{\sim}{B}$ from a current I in an infinitely long straight wire, by appropriate Lorentz transformations and superpositions of the electric field of an infinitely long straight charge distribution.

Problem 4.2. For electric and magnetic fields, show that $B^2 - E^2$ and $\underset{\sim}{E} \cdot \underset{\sim}{B}$ are invariant under changes of coordinates and Lorentz transformations. Are there any invariants which are not merely algebraic combinations of these two?

Problem 4.3. A particular electromagnetic field has its E field at an angle θ_0 to its $\underset{\sim}{B}$ field, and θ_0 is invariant to all observers. What is the value of θ_0?

Problem 4.4. Show that $\mathcal{E}^2 - |\underset{\sim}{S}|^2$ is a Lorentz invariant of the electromagnetic field, where $\mathcal{E}$ is the energy density and $\underset{\sim}{S}$ the Poynting flux.

Problem 4.5. Prove that except when $(\underset{\sim}{B} \cdot \underset{\sim}{E})^2 + (B^2 - E^2)^2 = 0$, there is a Lorentz transformation which will make $\underset{\sim}{E}$ and $\underset{\sim}{B}$ parallel $(\underset{\sim}{E}' \times \underset{\sim}{B}' = 0)$. [Hint: Try $\underset{\sim}{v} = \alpha(\underset{\sim}{E} \times \underset{\sim}{B})$ for some α.]

Problem 4.6. Suppose that $\underset{\sim}{E} \cdot \underset{\sim}{B} = 0$. Show that there is a Lorentz transformation which makes $\underset{\sim}{E} = 0$ if $B^2 - E^2 > 0$, or one that makes $\underset{\sim}{B} = 0$ if $B^2 - E^2 < 0$. What if $B^2 - E^2 = 0$ in addition to $\underset{\sim}{E} \cdot \underset{\sim}{B} = 0$?

Problem 4.7. A collection of charged particles of charges e_i has 3-velocities $\underset{\sim}{v}_i$ and trajectories $\underset{\sim}{x} = \underset{\sim}{z}_i(t)$. The 4-current has components $J^0 = \sum_i e_i \delta^3[\underset{\sim}{x} - \underset{\sim}{z}_i(t)]$; $J^i = \sum_k e_k v^i \delta^3[\underset{\sim}{x} - \underset{\sim}{z}_k(t)]$. Show that this can be written $J^\mu = \sum_k \int e_k \delta^4[x^\alpha - z^\alpha{}_k(\tau)]\, u_k^\mu \, d\tau$ where u_k^μ is the 4-velocity of particle k.

Problem 4.8. Show by explicit examination of components that the equations

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \qquad F^{\alpha\beta}{}_{,\beta} = 4\pi J^\alpha$$

reduce to Maxwell's equations:

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0, \quad \dot{\underset{\sim}{B}} + \underset{\sim}{\nabla} \times \underset{\sim}{E} = 0, \quad \underset{\sim}{\nabla} \cdot \underset{\sim}{E} = 4\pi\rho ,$$

$$\dot{\underset{\sim}{E}} - \underset{\sim}{\nabla} \times \underset{\sim}{B} = -4\pi \underset{\sim}{J} .$$

Problem 4.9. If $F^{\mu\nu}$ is the electromagnetic tensor, show that Maxwell's equations in vacuum can be written as $F^{\mu\nu}{}_{,\nu} = 0$ and $*F^{\mu\nu}{}_{,\nu} = 0$. [Here, $*F^{\mu\nu}$ is the dual of $F^{\mu\nu}$; see Problem 3.25.]

Problem 4.10. Write out the $\mu = 0$ component of the Lorentz force equation $du^{\mu}/d\tau = (e/m)F^{\mu\beta}u_{\beta}$ expressing $F^{\mu\nu}$ in terms of E_i and B_i, to obtain

$$dP^0/dt = e\underset{\sim}{v} \cdot \underset{\sim}{E} .$$

Problem 4.11. From the spatial components of the Lorentz 4-force equation, find an equation for $d\underset{\sim}{P}/dt$ in terms of $\underset{\sim}{E}$ and $\underset{\sim}{B}$. (Here $\underset{\sim}{P}$ is the spatial part of P).

Problem 4.12. A particle of charge q and mass m is coasting through the lab with velocity $v\underset{\sim}{e}_x$ when it encounters a constant $\underset{\sim}{E}$ field in the y-direction. Find $y(x)$, the shape of the particle's subsequent motion.

Problem 4.13. A particle of charge q, mass m, moves in a circular orbit of radius R in a uniform B field $B\underset{\sim}{e}_z$. (a) Find B in terms of R, q, m and ω, the angular frequency. (b) The speed of the particle is constant since the B field can do no work on the particle. An observer moving at velocity $\beta\underset{\sim}{e}_x$, however, does not see the speed as constant. What is $u^{0'}$ measured by this observer? (c) Calculate $du^{0'}/d\tau$ and thus $dP^{0'}/d\tau$. Explain how the energy of the particle can change since the B field does no work on it.

Problem 4.14. A small test particle (mass m, positive charge q) makes circular orbits around a "fixed" (i.e. very massive) body of positive charge Q. A uniform magnetic field $\underset{\sim}{B}$ perpendicular to the orbital plane serves to keep the particle in orbit. In the inertial frame in which the central body is at rest, the test charge is seen to circle in the plane perpendicular to the B field with an angular frequency ω. What is the charge to mass ratio of the test particle in terms of ω, R, B, Q?

Problem 4.15. Show that the stress-energy tensor for the electromagnetic field is divergenceless (i.e. $T^{\mu\nu}{}_{,\nu} = 0$) in the absence of charge sources.

Problem 4.16. Show that the stress-energy tensor for the electromagnetic field has zero trace.

Problem 4.17. If $T^{\mu\nu}$ is the stress-energy tensor of the electromagnetic field, show that

$$T^{\mu}{}_{\alpha} T^{\alpha}{}_{\nu} = \delta^{\mu}{}_{\nu} [(E^2 - B^2)^2 + (2\underset{\sim}{E} \cdot \underset{\sim}{B})^2]/(8\pi)^2 \ .$$

Problem 4.18. Write Ohm's law $\underset{\sim}{J} = \sigma \underset{\sim}{E}$ invariantly in terms of J^{μ}, $F^{\mu\nu}$, σ and u^{μ} (the 4-velocity of the conducting element).

Problem 4.19. Derive the Lorentz force law for a charged particle from the action $\int J^{\mu} A_{\mu} d^4x - m \int d\tau$, where J^{μ} is the 4-current, A_{μ} the vector potential, and $d\tau^2 \equiv -\eta_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta}$.

Problem 4.20.

(a) Show that $\underset{\sim}{E} \to -\underset{\sim}{B}$ and $\underset{\sim}{B} \to \underset{\sim}{E}$ under the "duality transformation" $F \to *F$.

(b) Show that if F is a solution of the free-space Maxwell equations, so is $*F$ and also $e^{*\alpha}F \equiv F \cos\alpha + *F \sin\alpha$ for arbitrary α. ($F \to e^{*\alpha}F$ is called a "duality rotation".)

Problem 4.21. If one believes that esthetics should be an important consideration in physical laws, then by symmetry Maxwell's laws should read

$$F^{\mu\nu}{}_{,\nu} = 4\pi J^{\mu}$$

$$*F^{\mu\nu}{}_{,\nu} = 4\pi K^{\mu} \ .$$

What would the significance of K be?

Problem 4.22. In Minkowski spacetime, there is an electromagnetic current $J^{\mu}(x^{\nu})$. Show that the solution to Maxwell's equation is

$$F^{\mu\nu}(x^\alpha) = \frac{4}{\pi i}\int \frac{r^{[\mu}J^{\nu]}d^4\tilde{x}}{(r_\sigma r^\sigma)^2}$$

where $r^\beta \equiv \tilde{x}^\beta - x^\beta$. (Start by finding a Green's function of $\Box A^\mu = -4\pi J^\mu$.) How are the retarded boundary conditions specified?

Problem 4.23. Find the equation for the convective time rate of change of a magnetic field which is "frozen in" a perfectly conducting fluid, in terms of the expansion, shear, and rotation of the fluid. (See Problem 5.18 for definitions of these quantities.)

CHAPTER 5
MATTER AND RADIATION

A proper description of the energy, momentum and stress of a relativistic fluid or field uses the symmetric tensor $\mathbf{T}$, the stress-energy tensor (also called the energy-momentum tensor). The components of this tensor in the Lorentz frame of an observer are related to the measurements made by that observer in the following way:

$T^{00} \equiv$ density of mass-energy (often denoted ρ).
$T^{0j} = T^{j0} \equiv$ j-component of momentum-density
$\qquad = $ j-component of energy-flux
$T^{ij} \equiv$ components of the ordinary stress tensor (e.g. T^{xx} = x-component of pressure).

If $T^{\mu\nu}$ describes *all* fields, fluids, particles etc. present in a system, the interrelation of momentum flow and energy change is summarized by the *equations of motion*:

$$T^{\mu\nu}{}_{,\nu} = 0 \ .$$

The basic concepts of relativistic thermodynamics and hydrodynamics which follow from this are developed in the problems.

With a view to developments later in the book, several problems in this chapter use covariant differentiation, denoted by a semicolon. The reader not yet familiar with this may replace all semicolons by commas (partial differentiation in Minkowski coordinates). Also, the ∇ notation is introduced; e.g. $\nabla\mathbf{S}$ for $S^{\alpha}{}_{;\beta}$, ∇f for $f_{,\alpha}$, $\nabla\cdot\mathbf{T}$ for $T^{\mu\nu}{}_{;\nu}$, etc.

Problem 5.1. Calculate the nonzero components in an inertial frame S of the stress-energy tensor for the following systems:

(a) A group of particles all moving with the same velocity $\underset{\sim}{\beta} = \beta\, \underset{\sim}{e}_x$ as seen in S. Let the rest-mass density of these particles be ρ_0 as measured in their comoving frame. Assume a high density of particles and treat them in the continuum approximation.

(b) A ring of N similar particles of mass m rotating counter-clockwise in the x-y plane about some point fixed in S at a radius a and angular velocity ω. (The width of the ring is much less than a.) Do not include the stress-energy of whatever forces keep them in orbit. Assume N is large enough that one can treat the particles as being continuously distributed.

(c) Two such rings of particles, one rotating clockwise, the other counter-clockwise, at the same radius a. The particles do not collide or interact with each other in any way.

Problem 5.2. What is the stress energy of a gas with a proper number density (i.e. number density as measured in the local rest frame of the gas) N of noninteracting particles of mass m, if the particles all have the same speed v but move isotropically? (Do *not* assume v << c.)

Problem 5.3. In the rest frame of a perfect fluid its stress energy tensor, in terms of mass-energy density ρ and pressure p, is the diagonal tensor

$$T^{\mu\nu} = \begin{bmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{bmatrix}.$$

If a fluid element of proper density and pressure, ρ and p is moving with 4-velocity $\mathbf{u}$, what is its stress-energy?

Problem 5.4. Find the stress-energy tensor for a uniform magnetic field. What is the average stress-energy if the B field is static but "chaotic" i.e. the direction of the B field varies, and is isotropic on the average?

Problem 5.5. A rod has cross sectional area A and mass per unit length μ. Write down the stress-energy tensor inside the rod when the rod is under a tension F. (Assume that the tension is uniformly distributed over the cross section.)

Problem 5.6. A rope of mass per unit length μ has a static breaking strength F. What is the maximum F can be without violating the "weak" energy condition that T^{00} should be positive to all observers? How close is steel cable to this theoretical maximum strength?

Problem 5.7. An infinitesimally thin rod of length $2a$ has a point mass m at each of its ends. The center of the rod is fixed in the laboratory and the rod rotates about this point with a relativistic angular velocity ω. (i.e. $\omega \ell$ is comparable with c). Assume the rod is massless. What is $T^{\mu\nu}$ for the rod and particle system?

Problem 5.8. A parallel plate capacitor consists of two large plates of area A, perpendicular to the x-direction, separated by a small distance d. The capacitor is charged so that a uniform electric field of magnitude E is present between the plates; fringe effects at the edge of the plate can be neglected. The "electrostatic mass" of this capacitor is $E^2 Ad/8\pi$ in the rest frame of the capacitor. Show that the electrostatic energy is *smaller* if the capacitor is moving in the x-direction! Consider now that the plates must be held apart. Let the plates be held apart by an ideal gas of proper density ρ_0. Show that the *total* energy (electrostatic + gas) of the capacitor increases with velocity in the x-direction in precisely the same manner that the energy of a point mass does.

Problem 5.9. Consider a system of discrete particles of charge q_i and mass m_i interacting through electromagnetic forces. From the explicit expression for $T^{\mu\nu}$ of the particles show that the total $T^{\mu\nu}$ (particles plus field) is conserved, i.e. that $T^{\mu\nu}{}_{,\nu} = 0$.

Problem 5.10. The specific intensity I_ν of radiation measures the intensity of radiation at a particular frequency ν in a particular direction. It is defined as the flux per unit frequency interval, per unit solid angle. Show that I_ν/ν^3 is a Lorentz invariant.

Problem 5.11. A star emits radiation isotropically in its own rest frame, with luminosity L (energy per unit time). At a particular instant, as measured from the earth, the star is at a distance R, and is moving with a velocity v which makes an angle θ with respect to the direction from the earth to the star. What is the flux of radiation (energy per unit time per unit area) seen by an observer on the earth in terms of R, v and θ evaluated at the instant the radiation was emitted?

Problem 5.12. Consider a spherical particle of mass m which scatters all electromagnetic radiation incident on it, isotropically in its rest frame. Let A be the effective cross sectional area of the particle. Find the equation of motion of the particle in a constant radiation field of intensity S (energy per time per area), and solve it for the case of a particle initially at rest. (Poynting-Robertson effect).

Problem 5.13. A thermally-conducting black sphere with a thermometer attached moves with velocity v through a black body radiation field of temperature T_0. What does the thermometer read?

Problem 5.14. In an electron gas of temperature $T \ll m_e c^2/k$ a photon of energy $E \ll m_e c^2$ undergoes collisions and is Compton scattered. Show that, to lowest order in E and T the average energy lost by a photon in a collision is

$$\langle \Delta E \rangle = (E/m_e c^2)(E - 4kT) \ .$$

Problem 5.15. Show that in special relativity the stress-energy of an isolated physical system of finite extent obeys the tensor virial theorem

$$\int T^{ij}\, d^3x = \frac{1}{2}\frac{d^2}{dt^2}\int T^{00} x^i x^j d^3x \ .$$

Problem 5.16. Show that the stress-energy tensor $T^{\mu\nu}$ has a timelike eigenvector if and only if there is a physical observer who sees no net energy flux in any direction. What is the significance of the eigenvalue?

Problem 5.17.

(a) Consider a stressed medium which moves through a particular inertial frame with velocity $|\underset{\sim}{v}| << 1$. Show that to first order in the velocity, the spatial components of the momentum density are

$$g^j = m^{jk} v^k ,$$

where m^{jk}, the "inertial mass per unit volume" is

$$m^{jk} = T^{0'0'}\delta^{jk} + T^{j'k'}$$

in terms of $T^{\mu'\nu'}$, the components of the stress-energy in the rest frame of the medium. What is m^{jk} for a perfect fluid?

(b) Consider an isolated, stressed body at rest and in equilibrium $(T^{\alpha\beta}{}_{,0} = 0)$ in the laboratory frame. Show that its total inertial mass, defined by

$$M^{ij} \equiv \int_{\substack{\text{stressed}\\ \text{body}}} m^{ij}\, dx\, dy\, dz$$

is isotropic and equals the rest mass of the body, i.e. show that

$$M^{ij} = \delta^{ij} \int T^{00}\, dx\, dy\, dz \ .$$

Problem 5.18. If $\mathbf{u}$ is the 4-velocity of a fluid show that $\nabla \mathbf{u}$ can be decomposed as

$$u_{\alpha;\beta} = \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{3}\theta P_{\alpha\beta} - a_\alpha u_\beta ,$$

where $\mathbf{a}$ is the "4-acceleration" of the fluid

$$a_\alpha \equiv u_{\alpha;\beta} u^\beta ,$$

θ is the "expansion" of the fluid world lines

$$\theta \equiv \nabla \cdot \mathbf{u} = u^{\alpha}{}_{;\alpha} ,$$

$\omega_{\alpha\beta}$ is the "rotation 2-form" of the fluid, and $\sigma_{\alpha\beta}$ is the "shear tensor"

$$\omega_{\alpha\beta} \equiv \frac{1}{2} (u_{\alpha;\mu} P^{\mu}{}_{\beta} - u_{\beta;\mu} P^{\mu}{}_{\alpha}) ,$$

$$\sigma_{\alpha\beta} \equiv \frac{1}{2} (u_{\alpha;\mu} P^{\mu}{}_{\beta} + u_{\beta;\mu} P^{\mu}{}_{\alpha}) - \frac{1}{3} \theta P_{\alpha\beta} .$$

Here $\mathbf{P}$ is the projection tensor

$$P_{\alpha\beta} \equiv g_{\alpha\beta} + u_{\alpha} u_{\beta}$$

which projects a vector onto the 3-surface perpendicular to $\mathbf{u}$.

Problem 5.19. Write the first law of thermodynamics for a relativistic fluid. (i.e. Write the law of conservation of mass-energy for a fluid element.)

Problem 5.20. Use the equations of motion $(T^{\mu\nu}{}_{;\nu} = 0)$ to show that the flow of a perfect fluid is isentropic.

Problem 5.21. For a perfect fluid with equation of state $\rho = \rho(n)$ (where n = baryon density) show that $T^{\mu}{}_{\mu}$, the trace of the stress-energy tensor is negative if and only if

$$d \log \rho / d \log n < 4/3 .$$

Problem 5.22. Show that the velocity of sound v_s in a relativistic perfect fluid is given by
$$v_s^2 = \partial p / \partial \rho |_{s = \text{constant}} .$$

For a high temperature relativistic gas with an equation of state $\rho \approx 3P$ (essentially that for a photon gas) show that $v_s \approx 1/\sqrt{3}$.

Problem 5.23. The velocity of sound in a fluid is $v_s^2 = \partial p / \partial \rho |_{s = \text{constant}}$. Show that $v_s^2 = \Gamma_1 p / (\rho + p)$ where Γ_1 is the adiabatic index

$$\Gamma_1 = \partial \log p / \partial \log n |_{s = \text{constant}} .$$

Problem 5.24. What is the speed of sound in an ideal Fermi gas at zero temperature?

Problem 5.25. A relativistic wind tunnel is to be fed from a tank of perfect adiabatic compressed gas. Suppose the gas has an equation of state $p \propto n^{\gamma}$ with γ constant, and the speed of sound in the tank is a. What is the largest wind velocity v_{max} which can be obtained? (No gravitational forces; isentropic flow.)

Problem 5.26. An idealized description of heat flow in a fluid uses the heat flux 4-vector q with components in the fluid rest frame $q^0 = 0$, q^j = (energy per unit time crossing a unit surface perpendicular to e_j, in the positive j direction). What is the stress-energy tensor associated with the heat flow?

Problem 5.27. Let s, n and q be respectively entropy per baryon, number density of baryons, and heat flux, all measured in the proper frame of the fluid. In this proper frame q is purely spatial. Let S be the entropy density-flux 4-vector. Show that

$$S = nsu + q/T \ ,$$

where u is the 4-velocity of the fluid rest frame.

Problem 5.28. A fluid is "perfect" except for admitting some heat conduction, described by a heat flow 4-vector q. Calculate the local rate of entropy generation $\nabla \cdot S$.

Problem 5.29. In a uniformly accelerating system, show that the condition for thermal equilibrium is not $T = \text{constant} = T_0$ but rather is

$$T = T_0 \exp(-\underset{\sim}{a} \cdot \underset{\sim}{x})$$

where $\underset{\sim}{x}$ is coordinate position in the accelerating frame.

Problem 5.30. The stress energy tensor of a viscous fluid is

$$T^{\alpha\beta} = \rho u^{\alpha} u^{\beta} + p P^{\alpha\beta} - 2\eta\sigma^{\alpha\beta} - \zeta\theta P^{\alpha\beta} \ .$$

Here η and ζ are respectively the coefficients of shear and bulk viscosity. The definitions of $\sigma^{\alpha\beta}$, θ, $P^{\alpha\beta}$ are those of Problem 5.18. The pressure and density are p and ρ. Show that the viscous terms lead to the production of entropy at a rate

$$S^{\alpha}{}_{;\alpha} = (\zeta\theta^2 + 2\eta\,\sigma_{\alpha\beta}\,\sigma^{\alpha\beta})/T$$

where T is the temperature of the fluid. (Hint: First show that $S^{\alpha}{}_{;\alpha} = [d\rho/d\tau + \theta(\rho+p)]/T$ for a fluid without heat flow, then differentiate $\rho u^{\beta} = -T^{\alpha\beta}u_{\alpha}$ to get $d\rho/d\tau$.)

Problem 5.31. From the stress-energy

$$T^{\alpha\beta} = \rho u^{\alpha}u^{\beta} + pP^{\alpha\beta} - 2\eta\sigma^{\alpha\beta} - \zeta\theta P^{\alpha\beta} ,$$

show that the equations of motion derived from $T^{\alpha\beta}{}_{,\beta} = 0$ reduce to the Navier-Stokes equations in the nonrelativistic limit.

Problem 5.32. As in non-relativistic thermodynamics, one defines the specific heat of a gas at constant volume and constant pressure by

$$c_v = T\,\frac{ds}{dT}\bigg|_n \qquad c_p = T\,\frac{ds}{dT}\bigg|_p \ .$$

For a perfect Maxwell-Boltzmann gas, show that $c_p = c_v + k$. (Here k = Boltzmann's constant.) Show that the adiabatic index

$$\Gamma_1 \equiv \frac{\partial \log p}{\partial \log n}\bigg|_s$$

is equal to the ratio of specific heats,

$$\gamma \equiv c_p/c_v \ .$$

Problem 5.33. For a perfect Maxwell-Boltzmann gas, show that if γ is approximately constant in some regime of interest, then $p = Kn^{\gamma}$ and $\rho = mn + Kn^{\gamma}/(\gamma-1)$ under adiabatic conditions. (K = constant, m = mass of particles.)

Problem 5.34. The invariant equilibrium distribution function of a relativistic gas is

$$\mathfrak{N}(p^\alpha, x^\alpha) \equiv \frac{dN}{d^3x\, d^3P} = \frac{(2J+1)/h^3}{\exp[-\frac{P\cdot u}{kT} - \theta] - \varepsilon} .$$

Here J = spin of particles, h = Planck's constant, $\mathbf{u}$ = mean 4-velocity of gas, $\varepsilon = 1, 0$ or -1 for Bose-Einstein, Maxwell-Boltzmann or Fermi-Dirac statistics respectively. The parameter θ is independent of $\mathbf{P}$. The first two moments of $\mathfrak{N}$ are

$$J^\mu \equiv \int \mathfrak{N} P^\mu \frac{d^3P}{(-\mathbf{P}\cdot\mathbf{u})} , \qquad T^{\mu\nu} \equiv \int \mathfrak{N} P^\mu P^\nu \frac{d^3P}{(-\mathbf{P}\cdot\mathbf{u})} .$$

Since $\mathbf{u}$ is the only free vector, these integrals must have the form

$$J^\mu = nu^\mu , \qquad T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu} .$$

(This is the kinetic-theory *definition* of n, ρ, p.)

(a) Obtain 1-dimensional integrals for n, ρ and p.

(b) Show that $dp = (\rho+p)/T\, dT + nkT\, d\theta$.

(c) Use the first law of thermodynamics to identify $kT\theta$ as the chemical potential $\mu = (\rho+p)/n - Ts$.

(d) Show that for a Maxwell-Boltzmann gas, $p = nkT$ for all T.

(e) Show that for a Maxwell-Boltzmann gas, $\rho = n(m + \frac{3}{2}kT)$ is an approximation valid only for $kT << m$. Find the exact expression for ρ/n. What is ρ/n for $kT >> m$? (Here m is the mass of a gas particle.)

Problem 5.35. For a perfect Maxwell-Boltzmann gas, find $\gamma(T)$, the ratio of specific heats, as a function of temperature.

CHAPTER 6
METRICS

Metric geometry, geometry specified by a distance formula $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$, is the foundation for general relativity and for most of the remaining chapters in this book. The most important metric is of course a spacetime metric, formally a metric which can locally be transformed to the Minkowski metric, i.e. for every point P in the spacetime, there is some coordinate transformation which makes $g_{\alpha\beta} = \eta_{\alpha\beta}$ at P.

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Problem 6.1.

(a) Prove that the 2-dimensional metric space described by

$$ds^2 = dv^2 - v^2 du^2 \qquad (1)$$

is just the flat 2-dimensional Minkowski space usually described by

$$ds^2 = dx^2 - dt^2 \ . \qquad (2)$$

Do this by finding the coordinate transformations $x(v, u)$ and $t(v, u)$ which take the metric (2) into the form (1).

(b) For an unaccelerated particle, show that the component of the 4-momentum P_u is constant, but that P_v is not.

Problem 6.2. Show that the line element

$$ds^2 = R^2[d\alpha^2 + \sin^2\alpha\, (d\theta^2 + \sin^2\theta\, d\phi^2)]$$

represents a hypersphere of radius R in Euclidean 4-space, i.e. a locus of points a distance R from a given point.

Problem 6.3. The metric for the surface of a globe of the Earth is

$$ds^2 = a^2(d\lambda^2 + \cos^2\lambda\, d\phi^2)$$

where λ is the latitude and ϕ is the longitude. The metric of a flat map of the world, with Cartesian coordinates x and y is $ds^2 = dx^2 + dy^2$; however we are not interested in *this* geometry, but in that of the globe it represents. What is the metric of the globe expressed in x and y coordinates for (a) a cylindrical projection, and (b) a stereographic projection map of the world?

Problem 6.4. Mercator's projection is defined as follows: The map coordinates are rectangular Cartesian coordinates (x, y) such that a straight line on the map is a line of constant compass bearing on the globe.

(a) Show that the map is defined by $x = \phi$, $y = \log\cot\frac{1}{2}\theta$, where (θ, ϕ) are the polar coordinates of a point on the globe.

(b) What is the metric of the globe in (x, y) coordinates?

(c) Show that the great circles are given by $\sinh y = \alpha \sin(x+\beta)$ (except for the special cases $y = 0$ or $x = \text{constant}$).

Problem 6.5. A space purports to be 3-dimensional, with coordinates x, y, z and the metric

$$ds^2 = dx^2 + dy^2 + dz^2 - \left(\frac{3}{13}\,dx + \frac{4}{13}\,dy + \frac{12}{13}\,dz\right)^2 .$$

Show that it is really a two-dimensional space, and find two new coordinates ζ and η for which the line element takes the form

$$ds^2 = d\zeta^2 + d\eta^2 .$$

Problem 6.6. Show that a contraction of a vector $\mathbf{V}$ with the "projection tensor" $\mathbf{P} \equiv \mathbf{g} + \mathbf{u}\otimes\mathbf{u}$ projects $\mathbf{V}$ into the 3-surface orthogonal to the 4-velocity vector $\mathbf{u}$. If $\mathbf{n}$ is a unit spacelike vector show that

$$\mathbf{P} \equiv \mathbf{g} - \mathbf{n}\otimes\mathbf{n}$$

is the corresponding projection operator. Show that there is no unique projection operator orthogonal to a null vector.

Problem 6.7. Show that a conformal transformation of a metric, i.e. $g_{\alpha\beta} \to f(x^\mu) g_{\alpha\beta}$ for an arbitrary function f, preserves all angles. (Figure out how to define angles!) Show that all null curves remain null curves.

Problem 6.8. One can put a metric on the velocity space of a particle by defining the distance between two nearby velocities as their relative velocity. Show that the metric can be written in the form

$$ds^2 = d\chi^2 + \sinh^2\chi\,(d\theta^2 + \sin^2\theta d\phi^2) \ .$$

where the magnitude of the velocity is $v = \tanh \chi$.

Problem 6.9. A manifold which has the topology of a 2-sphere has – in a neighborhood of $\theta = \frac{1}{2}$, $\chi = 0$ – the metric

$$ds^2 = d\theta^2 + (\theta - \theta^3)^2\, d\chi^2 \ .$$

The manifold has precisely *one* point which is not locally flat, and that point is a "conical" singularity. Show that there are two different maximal analytic extensions of the metric, i.e. that there are two different ways of extending the metric, and satisfying the condition that there is only one conical singularity. Note that this shows that a metric in a local coordinate patch does not always pin down the global nature of the manifold. (Hint: consider the periodicity of the χ coordinate.)

Problem 6.10. Find the most general form for a spacetime metric that is (spatially) spherically symmetric.

CHAPTER 7

COVARIANT DIFFERENTIATION AND GEODESIC CURVES

The partial derivatives of a vector or tensor with respect to the coordinates of a space (e.g. $A^{\mu}{}_{,\nu}$ or $Q^{\alpha\beta\cdots}{}_{\gamma\delta\cdots,\nu}$) are not themselves components of a tensor. Rather, the curvilinearity of the coordinates (optional in flat space, but inevitable in a curved space) must be taken into account, leading to the idea of covariant differentiation.

The tensor formed by differentiating a tensor $\mathbf{Q}$ with components $Q^{\alpha\beta\cdots}{}_{\gamma\delta\cdots}$ is denoted $\nabla\mathbf{Q}$ and has components denoted

$$\begin{aligned} Q^{\alpha\beta\cdots}{}_{\gamma\delta\cdots;\sigma} \equiv\ & Q^{\alpha\beta\cdots}{}_{\gamma\delta\cdots,\sigma} + \Gamma^{\alpha}{}_{\nu\sigma} Q^{\nu\beta\cdots}{}_{\gamma\delta\cdots} \\ & + \Gamma^{\beta}{}_{\nu\sigma} Q^{\alpha\nu\cdots}{}_{\gamma\delta\cdots} + \cdots - \Gamma^{\nu}{}_{\gamma\sigma} Q^{\alpha\beta\cdots}{}_{\nu\delta\cdots} \\ & - \Gamma^{\nu}{}_{\delta\sigma} Q^{\alpha\beta\cdots}{}_{\gamma\nu\cdots} \end{aligned}$$

where there is one "correction" term for every index of $\mathbf{Q}$. The Γ's are called Christoffel symbols or [affine] connection coefficients. In a coordibasis they are related to the partial derivatives of the metric by

$$\Gamma^{\alpha}{}_{\beta\gamma} = g^{\alpha\mu}\Gamma_{\mu\beta\gamma} = \frac{1}{2} g^{\alpha\mu}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu})$$

(the first equality *defines* $\Gamma_{\mu\beta\gamma}$). The Γ's are sets of numbers, but they are not components of a tensor. (They do not transform like a tensor.)

A covariant derivative which is dotted into a vector is called a directional derivative:
$$(\nabla\mathbf{Q})\cdot\mathbf{u} \equiv \nabla_{\mathbf{u}}\mathbf{Q} \equiv Q^{\alpha\beta\cdots}{}_{\gamma\delta\cdots;\nu}\, u^{\nu} .$$

If the vector $\mathbf{u}$ is tangent to a curve parameterized by λ, one sometimes writes $\mathbf{u} = d/d\lambda$ for $u^{\alpha} = dx^{\alpha}/d\lambda$ and

$$\nabla_{\mathbf{u}} \mathbf{Q} \equiv \frac{D\mathbf{Q}}{d\lambda} .$$

If the vector happens to be a basis vector, one writes

$$\nabla_{\mathbf{e}_\alpha} \mathbf{Q} \equiv \nabla_\alpha \mathbf{Q} .$$

In terms of the basis vectors, the connection coefficients can be written

$$\nabla_\beta \mathbf{e}_\alpha = \Gamma^\mu{}_{\alpha\beta} \mathbf{e}_\mu \quad \text{or} \quad \Gamma_{\mu\alpha\beta} = \mathbf{e}_\mu \cdot \nabla_\beta \mathbf{e}_\alpha .$$

The covariant derivative operator ∇ obeys all of the nice rules expected of a derivative operator, except that in curved space $\nabla_{\mathbf{u}} \nabla_{\mathbf{v}} \neq \nabla_{\mathbf{v}} \nabla_{\mathbf{u}}$ (see Chapter 9).

If $\mathbf{u}$ is the tangent vector to a curve, a tensor $\mathbf{Q}$ is said to be parallel propagated along the curve if

$$\nabla_{\mathbf{u}} \mathbf{Q} = 0 .$$

If the tangent vector is itself parallel propagated,

$$\nabla_{\mathbf{u}} \mathbf{u} = 0$$

(tangent vector "covariantly constant") the curve is a geodesic, the generalization of a straight line in flat space. If $x^\alpha(\lambda)$ is the geodesic (with $u^\alpha = dx^\alpha/d\lambda$) then the components of this *geodesic equation* are

$$0 = (\nabla_{\mathbf{u}} \mathbf{u})^\mu = \frac{du^\mu}{d\lambda} + u^\alpha u^\beta \Gamma^\mu{}_{\alpha\beta} .$$

Here λ must be an affine parameter along the curve; for non-null curves this means λ must be proportional to proper length.

If a curve is timelike, $\mathbf{u}$ is its tangent vector, and $\mathbf{a} \equiv \nabla_{\mathbf{u}} \mathbf{u} = D\mathbf{u}/d\tau$, then a vector $\mathbf{V}$ is said to be Fermi-Walker transported along $\mathbf{u}$ if

$$\nabla_{\mathbf{u}} \mathbf{V} = (\mathbf{u} \otimes \mathbf{a} - \mathbf{a} \otimes \mathbf{u}) \cdot \mathbf{V} .$$

Problem 7.1. Show that the connection coefficients $\Gamma^{\alpha}{}_{\beta\gamma}$ do *not* obey the tensor transformation law.

Problem 7.2. For a 2-dimensional flat, Euclidean space described by polar coordinates r, θ, assume that the geodesics are the usual straight lines.

(a) Find the connection coefficients $\Gamma^{\alpha}{}_{\beta\gamma}$, using your knowledge of these geodesics, and the geodesic equation

$$\frac{d^2x^{\mu}}{ds^2} + \frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds}\Gamma^{\mu}{}_{\alpha\beta} = 0 \ .$$

(b) Next, in the coordinates x, y which are related to r, θ in the usual way, take the covariant structure to be given by $\Gamma^{x}{}_{xx} = \Gamma^{x}{}_{xy} = \cdots = 0$. Using the transformation law for connection coefficients find the connection coefficients in the r, θ coordinates.

(c) Finally, from the line element $ds^2 = dr^2 + r^2 d\theta^2$ find the Christoffel symbols, in the usual way, as derivatives of the metric coefficients $g_{\mu\nu}$. (All three methods, of course, must give the same Christoffel symbols.)

Problem 7.3. Consider the familiar metric space

$$ds^2 = dr^2 + r^2 d\theta^2 \ .$$

(a) Write the 2 equations that result from the geodesic equation, and show that the following are first integrals of these equations:

$$r^2 \frac{d\theta}{ds} = R_0 = \text{constant}$$

$$\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 = 1 \ .$$

(b) Use the results in (a) to get a first-order differential equation for $r(\theta)$. [That is: Eliminate s as a parameter and replace by θ].

(c) Using the fact that the metric space is just flat 2-dimensional

Euclidean space, write down the general equation for a straight line in r, θ coordinates, and show that the straight line satisfies the equation in (b).

Problem 7.4. For the 2-dimensional metric $ds^2 = (dx^2 - dt^2)/t^2$, find all connection coefficients $\Gamma_{\alpha\beta\gamma}$, and find all timelike geodesic curves.

Problem 7.5. Show that the metric tensor is covariantly constant.

Problem 7.6. For a diagonal metric, prove that (in a coordinate frame) the Christoffel symbols are given by

$$\Gamma^{\mu}{}_{\nu\lambda} = 0\,, \qquad \Gamma^{\mu}{}_{\lambda\lambda} = -\frac{1}{2g_{\mu\mu}}\frac{\partial g_{\lambda\lambda}}{\partial x^{\mu}}$$

$$\Gamma^{\mu}{}_{\mu\lambda} = \frac{\partial}{\partial x^{\lambda}}\left(\log\left(|g_{\mu\mu}|\right)^{\frac{1}{2}}\right), \qquad \Gamma^{\mu}{}_{\mu\mu} = \frac{\partial}{\partial x^{\mu}}\left(\log\left(|g_{\mu\mu}|\right)^{\frac{1}{2}}\right).$$

Here $\mu \neq \nu \neq \lambda$ and there is no summation over repeated indices.

Problem 7.7. Prove the following indentities:

(a) $g_{\alpha\beta,\gamma} = \Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma}$.

(b) $g_{\alpha\mu}g^{\mu\beta}{}_{,\gamma} = -\, g^{\mu\beta}g_{\alpha\mu,\gamma}$.

(c) $g^{\alpha\beta}{}_{,\gamma} = -\,\Gamma^{\alpha}{}_{\mu\gamma}g^{\mu\beta} - \Gamma^{\beta}{}_{\mu\gamma}\,g^{\mu\alpha}$.

(d) $g_{,\alpha} = -\,gg_{\beta\gamma}\,g^{\beta\gamma}{}_{,\alpha} = gg^{\beta\gamma}g_{\beta\gamma,\alpha}$.

(e) $\Gamma^{\alpha}{}_{\alpha\beta} = (\log|g|^{\frac{1}{2}})_{,\beta}$ in a coordinate frame.

(f) $g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu} = -\dfrac{1}{|g|^{\frac{1}{2}}}\,(g^{\alpha\nu}|g|^{\frac{1}{2}})_{,\nu}$ in a coordinate frame .

(g) $A^{\alpha}{}_{;\alpha} = \dfrac{1}{|g|^{\frac{1}{2}}}\,(|g|^{\frac{1}{2}}A^{\alpha})_{,\alpha}$ in a coordinate frame.

(h) $A_{\alpha}{}^{\beta}{}_{;\beta} = \dfrac{1}{|g|^{\frac{1}{2}}}\,(|g|^{\frac{1}{2}}A_{\alpha}{}^{\beta})_{,\beta} - \Gamma^{\lambda}{}_{\alpha\mu}A_{\lambda}{}^{\mu}$ in a coordinate frame .

(i) $A^{\alpha\beta}{}_{;\beta} = \dfrac{1}{|g|^{\frac{1}{2}}}\,(|g|^{\frac{1}{2}}A^{\alpha\beta})_{,\beta}$ in a coordinate frame, if $A^{\alpha\beta}$ is antisymmetric.

(j) $\Box S = S_{;\alpha}{}^{;\alpha} = \dfrac{1}{|g|^{\frac{1}{2}}}\,(|g|^{\frac{1}{2}}g^{\alpha\beta}S_{,\beta})_{,\alpha}$ in a coordinate frame.

Problem 7.8. Let $A \equiv \det(A_{\mu\nu})$ where $A_{\mu\nu}$ is a second rank tensor. Show that A is not a scalar. (i.e. Show that its value changes under coordinate transformations.) Since A is not a scalar one cannot define $A_{;\alpha} = A_{,\alpha}$. How *should* $A_{;\alpha}$ be defined (in terms of $A_{,\alpha}$ and A)?

Problem 7.9. If a geodesic is timelike at a given point P, show that it is timelike everywhere along its length, and similarly for spacelike or null, geodesics.

Problem 7.10. Derive the geodesic equation from the definition of a geodesic as a curve of extremal length.

Problem 7.11. An affine parameter λ is one for which the equation of geodesic motion has the form

$$\frac{dx^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0 \ .$$

Show that all affine parameters are related by linear transformations with constant coefficients.

Problem 7.12. Show that in flat spacetime, the conservation law for the 4-momentum of a freely moving particle can be written $\nabla_{\mathbf{p}} \mathbf{p} = 0$. Show that particles with nonzero rest mass move along timelike geodesics.

Problem 7.13. Suppose the coordinate x^1 is a cyclic coordinate, i.e. the metric functions $g_{\alpha\beta}$ are independent of x^1. If $\mathbf{p}$ is the momentum of an unaccelerated particle, show that the component p_1 is constant along the particle world line.

Problem 7.14. Prove the general relativity version of Fermat's principle: In any static metric $(g_{0j} = g_{\alpha\beta,0} = 0)$, consider all null curves between two points in space, $x^j = a^j$ and $x^j = b^j$. Each such null curve $x^j(t)$ requires a particular coordinate time Δt to get from a^j to b^j. Show that the curves of extremal time Δt are null geodesics of the spacetime.

Problem 7.15.

(a) Show that the *geodesics* of the velocity space metric defined in Problem 6.8 are paths of minimum fuel use for a rocket ship changing its velocity.

(b) A rocket ship in interstellar space with velocity $\underset{\sim}{V}_1$ (with respect to the earth) changes its velocity to a new velocity $\underset{\sim}{V}_2$, in a manner that uses up the least fuel. What is the ship's smallest velocity relative to earth during the change?

Problem 7.16. On the surface of a two-sphere, $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$, the vector $\mathbf{A}$ is equal to $\mathbf{e}_\theta$ at $\theta = \theta_0$, $\phi = 0$. What is $\mathbf{A}$ after it is parallel transported around the circle $\theta = \theta_0$? What is the magnitude of $\mathbf{A}$?

Problem 7.17. Consider an observer with 4-velocity $\mathbf{u}$ who transports his four basis vectors $\mathbf{e}_\alpha$ along with him according to the transport law $\nabla_{\mathbf{u}}\mathbf{e}_\alpha = A_\alpha{}^\beta \mathbf{e}_\beta$. What is the most general form of $A_\alpha{}^\beta$ if:

(i) the basis vectors are to be orthonormal?

(ii) in addition $\mathbf{e}_{\hat{0}} = \mathbf{u}$? (i.e. the frame is his rest-frame).

(iii) in addition the spatial vectors are to be nonrotating? (i.e. He sees a freely-falling particle move with no Coriolis forces.)

Problem 7.18. Show that the scalar product of two vectors is not altered as they are both Fermi-Walker transported along a curve $\mathcal{C}$.

Problem 7.19. Show that Fermi-Walker transport along a *geodesic* curve is the same as parallel transport.

Problem 7.20. Write the following expressions in index-free notation:

(a) $U_{\alpha;\beta}\, U^\beta U^\alpha$ (b) $V^\alpha{}_{;\beta}\, U^\beta - U^\alpha{}_{;\beta}\, V^\beta$

(c) $T_{\alpha\beta;\gamma}\, V^\alpha W^\beta U^\gamma$ (d) $W^{\alpha;\beta}\, V_{\beta;\gamma}\, U^\gamma$

(e) $W^\alpha{}_{;\gamma\beta}\, U^\gamma U^\beta + W^\alpha{}_{;\gamma}\, U^\gamma{}_{;\beta} U^\beta - U^\alpha{}_{;\beta}\, W^\beta{}_{;\gamma}\, U^\gamma$.

Problem 7.21. Show that the paths of light rays in a static, isotropic spacetime can be described by taking the space to have a certain spatially varying "index of refraction" $n(x^j)$. What is n in terms of $g_{\alpha\beta}$? Assume $g_{\alpha\beta}$ has the form

$$ds^2 = g_{00}dt^2 - f(dx_1^2 + dx_2^2 + dx_3^2).$$

Problem 7.22. An inebriated astronaut pulses his rocket, firing in a random direction for each pulse. As measured in a momentarily comoving frame, each pulse corresponds to a velocity boost of $\Delta v \ll c$. Find the probability distribution of his resultant velocity after n boosts, where n is a very large number. Show that the drunk astronaut achieves highly relativistic velocities less efficiently than a sober astronaut (who fires his rocket always in the same direction), and takes on the average $3c/\Delta v$ times as many pulses to achieve the same velocity.

Problem 7.23.

(a) Suppose a vector field $\mathbf{k}$ is orthogonal to a family of hypersurfaces ("hypersurface-orthogonal"). Show that this implies $k_{[\mu;\nu}k_{\lambda]} = 0$.

(b) What is the geometric interpretation if $k_{[\mu;\nu]}$ also vanishes?

Problem 7.24. Prove that any congruence of null curves that is hypersurface-orthogonal consists of null geodesics.

Problem 7.25. Show that the variational principle

$$\delta \int (g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta)\, ds = 0$$

gives the same geodesics as the defining property for geodesics

$$\delta \int (g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta)^{\frac{1}{2}}\, ds = 0$$

when s is proper length (*not* an arbitrary parameterization) and $\dot{x} \equiv dx^\alpha/ds$. If $y \equiv (g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta)^{\frac{1}{2}}$, show that

$$\delta \int F(y)\, ds = 0$$

gives the same geodesics for *any* monotonic function $F(y)$.

CHAPTER 8
DIFFERENTIAL GEOMETRY: FURTHER CONCEPTS

A vector $\mathbf{B}$ is related to its contravariant components B^μ by

$$\mathbf{B} = B^\mu \mathrm{e}_\mu \ ,$$

where the e_μ are basis vectors. The *covariant* components B_μ represent the same vector, but represent it as a different type of "vector", called a one-form. (Loosely, one-forms are often called "covariant vectors".) For one-forms the analog of the above equation is

$$\tilde{\mathbf{B}} = B_\mu \tilde{\boldsymbol{\omega}}^\mu \ ,$$

where $\sim$ indicates a one-form and $\tilde{\boldsymbol{\omega}}^\mu$ are basis one-forms with covariant components $(1,0,0,0)$, $(0,1,0,0)$, etc. For an arbitrary tensor $\mathbf{T}$, with components $T_{\alpha\beta\cdots}{}^{\gamma\delta\cdots}$,

$$\mathbf{T} = T_{\alpha\beta\cdots}{}^{\gamma\delta\cdots}\, \tilde{\boldsymbol{\omega}}^\alpha \otimes \tilde{\boldsymbol{\omega}}^\beta \otimes \cdots \otimes \mathrm{e}_\gamma \otimes \mathrm{e}_\delta \otimes \cdots \ .$$

The scalar product of two vectors, or of two one-forms, involves the metric tensor, and is denoted by a "dot":

$$\mathbf{A}\cdot\mathbf{B} = g_{\mu\nu} A^\mu B^\nu$$

$$\tilde{\mathbf{A}}\cdot\tilde{\mathbf{B}} = g^{\mu\nu} A_\mu B_\nu \ .$$

(Here $g^{\mu\nu}$ is the matrix inverse of $g_{\mu\nu}$.) The scalar product of a vector with a one-form does not involve the metric, only a summation over an index. This is sometimes distinguished notationally as

$$\tilde{\mathbf{B}}\cdot\mathbf{A} \equiv \langle \tilde{\mathbf{B}}, \mathbf{A} \rangle \equiv B_\mu A^\mu$$