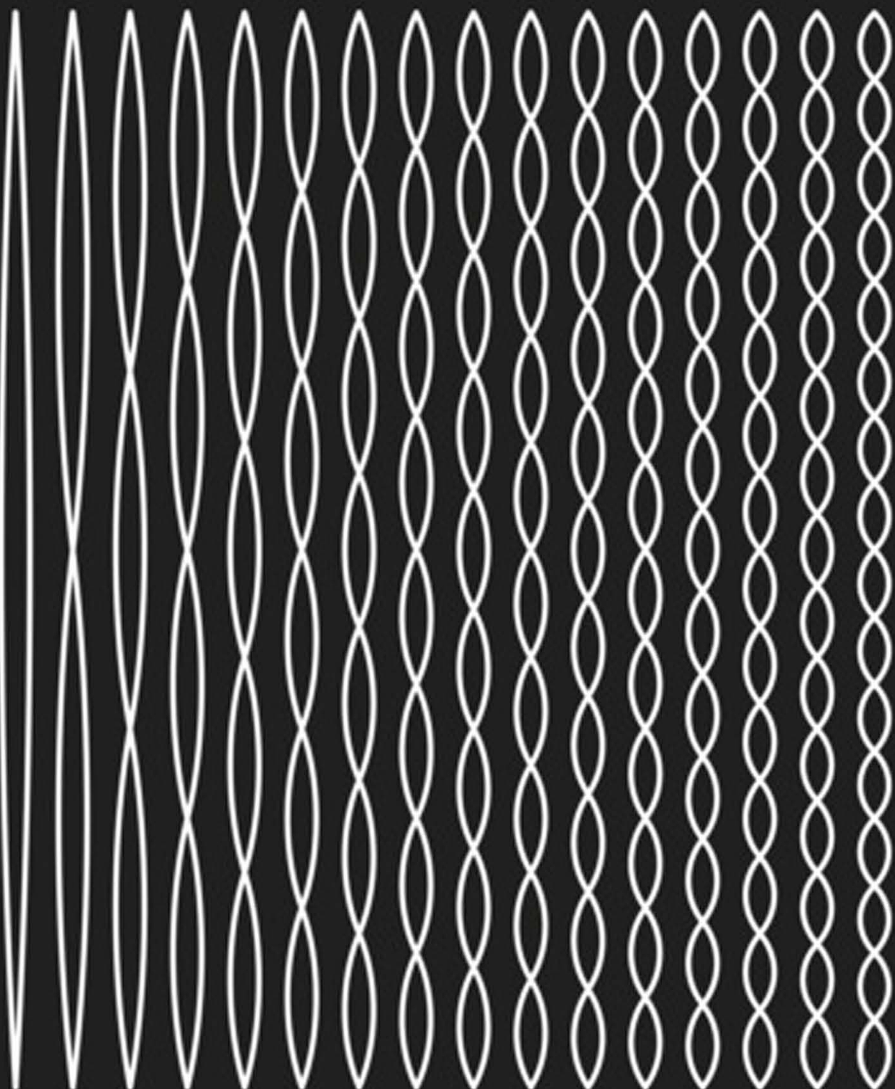
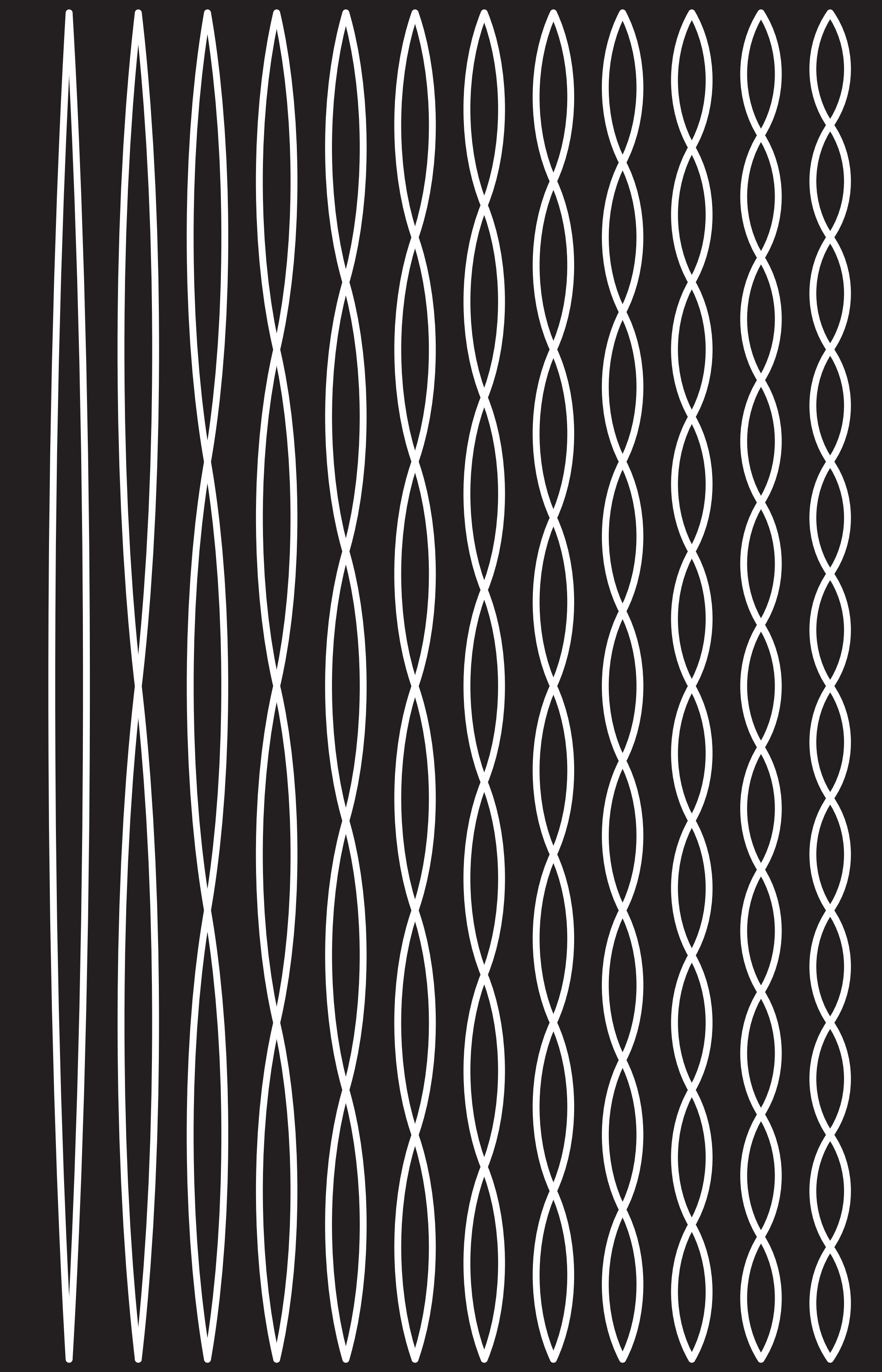
Music by the Numbers



FROM PYTHAGORAS
TO SCHOENBERG

Eli Maor

MUSIC BY THE NUMBERS



Music by the Numbers

FROM PYTHAGORAS TO SCHOENBERG

Eli Maor

PRINCETON UNIVERSITY PRESS
PRINCETON AND OXFORD

Published by Princeton University Press
41 William Street, Princeton, New Jersey 08540

In the United Kingdom: Princeton University Press
6 Oxford Street, Woodstock, Oxfordshire OX20 1TR

press.princeton.edu

Visit chenalexander.com/harmonics for an interactive diagram of the harmonic series by Alexander Chen

First paperback printing, 2020
Paperback ISBN 9780691202969

The Library of Congress has cataloged the cloth edition as follows:
Names: Maor, Eli, author.
Title: Music by the numbers : from Pythagoras to Schoenberg / Eli Maor.
Description: Princeton : Princeton University Press, [2018] | Includes bibliographical references and index.
Identifiers: LCCN 2017031417 | ISBN 9780691176901 (hardcover : alk paper)
Subjects: LCSH: Music—Mathematics. | Music—Acoustics and physics.
Classification: LCC ML3805 .M3 2018 | DDC 781 .2—dc23
LC record available at https://lccn.loc.gov/2017031417

Editorial: Vickie Kearn and Lauren Bucca
Production Editorial: Deborah Tegarden
Text and Jacket/Cover Design: Chris Ferrante
Production: Jacquie Poirier
Publicity: Sarah Henning-Stout and Lucy Zhou
Copyeditor: Carole Schwager

British Library Cataloging-in-Publication Data is available

This book has been composed in New Century Schoolbook and Gotham

Printed in the United States of America

In memory of my grandfather,
Karl Stiefel (1881–1947),
who instilled in me an interest
in science and love of music.

CONTENTS

PREFACE

THE GREAT COMPOSER Igor Stravinsky once said, “Musical form is close to mathematics—not perhaps to mathematics itself, but certainly to something like mathematical thinking and relationships.” Indeed, numerous writers have commented on the supposed affinity between mathematics and music, citing the fact that many scientists enjoy listening to music or actually practice it; Albert Einstein and his iconic violin immediately come to mind, but there were many, many others.

This may be true, but the relations between the two disciplines were never truly symmetric. Yes, there are many similarities between the two. For example, mathematics and music both depend on an efficient system of notation—a set of written symbols that convey a precise, unambiguous meaning to its practitioners (although in music this is augmented by a large assortment of verbal terms to indicate the more emotional aspects of playing). It is also interesting to note that the two systems started to evolve roughly around the same time, beginning about 1000 CE, although in mathematics this system of notation continues to evolve even today as new branches of the discipline come into being.

Mathematics and music also share many terms. Take, for example, the word *harmonic*. As an adjective it means “pleasing to the ear”; as a noun, it refers to the series of higher overtones that accompany the sound of practically all musical instruments. Now this word is almost as common in mathematics as it is in music; the two-volume *Encyclopedic Dictionary of Mathematics* lists no fewer than twenty usages of the word, including *harmonic mean*, *harmonic series*, *harmonic analysis*, and *harmonic functions*. Other examples of common terms are inversion (of a musical interval; of a point with respect to a circle), *root* (of a musical chord; of a number or an equation), *progression* (of notes; of numbers), and *series* (in music, Arnold Schoenberg's twelve-tone system of composition; in mathematics, an infinite sum of terms).

Over the past twenty-five hundred years, music has been a great source of inspiration to mathematicians, who found in it a perennial source of outstanding problems to keep their minds busy. Perhaps the most famous of these is the problem of the vibrating string, a subject that pitted against each other some of the greatest mathematicians of the eighteenth century in a debate that lasted well over fifty years and that would ultimately lead to the development of post-calculus mathematics. The arithmetic, geometric, and harmonic means $A = \frac{a+b}{2}$, $G = \sqrt{ab}$, and $H = \frac{2ab}{a+b}$ of two positive numbers a and b most likely originated with the ratios 2:1, 3:2, and 4:3 of the octave, fifth, and fourth—the Pythagorean perfect consonances—as the adjective "harmonic" alludes to. And the branch of number theory dealing with continued fractions may have had its origin in the quest to find the best numerical ratios for the various musical intervals of the scale.

But did mathematics have a similar influence on music? Mathematics has, of course, much to say about the more technical aspects of music, such as the tuning of musical instruments or the design of acoustically satisfying concert halls. But as to its influence on music as an *art*, it was, with a few notable exceptions, rather limited; the two disciplines simply followed their own separate ways. Typical of the disconnect is Leonhard Euler's extensive treatise on music theory (see chapter 4), of which it was said that "it contained too much geometry for musicians, and too much music for geometers."

//

I grew up in a home that loved European culture—literature, art, and music. Neither of my parents was musically trained, but my mother, who was an artist, admired Mozart; the radio was always tuned to the classical music station while she was at her desk, painting beautiful flowers. So Mozart was part of my childhood, both his music and the many stories my mother told me about him. One day she took me to a movie about Mozart's life. That was decades before Peter Shaffer's fictional *Amadeus* made the headlines. I remember crying at the scene of Mozart's final hours, lying on his deathbed while dictating the notes of his unfinished *Requiem* to his student Süssmayr.

But it was my maternal grandfather who instilled in me a lifelong interest in both science and music. He and my grandmother left Germany for Israel (then Palestine) in 1938 when life for Jews under the Nazi regime became intolerable. I have a photo of him playing his violin for me when I was about five years old (shown on the dedication page).[1] On the back side, my mother—who took the picture—wrote the name of the song he played for me on that day: *Guter Mond, du gehst so stille* (lovely moon, you sail by so silently), a traditional German lullaby.[2] That was the first live performance I attended, and I still remember it quite clearly. Then one day my grandfather told me that he must part with his violin—he desperately needed the money. I was in tears.[3]

And then there was the physics book from which he studied when he attended the gymnasium (high school). It was published in 1897 and came with hundreds of beautifully engraved illustrations; what's more, it reported on the latest developments in physics, including the discovery of x-rays (then known as Röntgen rays) and their potential benefit to medicine. My grandfather

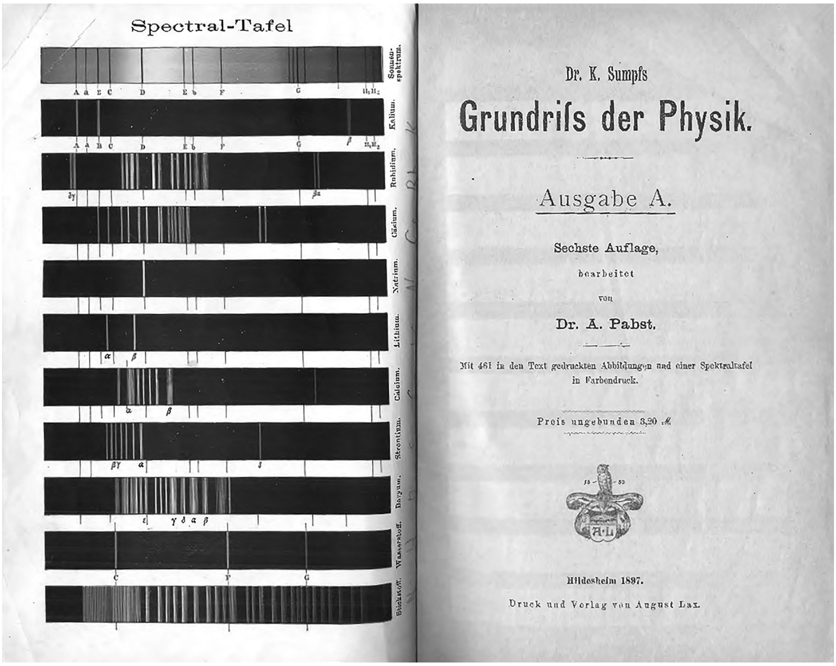


FIGURE P.1. Title page and frontispiece of *Grundrifs der Physik* (*Fundamentals of Physics*) by D. K. Sumpfs (Hildesheim, 1897).

must have studied that book thoroughly, as his handwritten annotations appear on nearly every page. We sat together for hours, going over various subjects which he explained to me—my earliest introduction to science. I still have that book, and I treasure it immensely (see figure P.1).

In the 1940s, when the dark clouds of war hung over the world, my parents hosted occasional evenings of classical music at their home in Tel Aviv, where a mechanical turntable—a gramophone—spun Bakelite-made records 78 rpm. How much I cherished those occasions! The gramophone had to be wound manually with a large crank—popularly known as a manuela—that allowed the machine to function for about ten minutes, just long enough to play the two sides of one disk. If you failed to rewind it in time, the turntable would slow down, and with it the rhythm and pitch of the music. A forty-minute Beethoven symphony took up five or six of those records, stored in the sleeves of a large ornate book that looked like an old-fashioned photo album (the modern word "album" for a collection of songs likely comes from those physical albums). Each of those tomes was as heavy as a thousand-page calculus textbook! Heaven forbid if one of the records should slip out of its sleeve and break up on the floor. But the greatest concern about playing those disks was the stylus, or needle. You were supposed to change it every dozen or so hours of playing, lest it become blunt and damage the grooves. These needles were made of chrome, and during the war their supply was severely limited. Soon, however, a substitute became available—wooden needles! Needless to say (no pun intended), the sound of those records was quite scratchy, but they were my introduction to classical music.

//

"Every intelligent musician should be familiar with the physical laws which underline his art," says Clarence G. Hamilton in his charming little book, *Sound and Its Relation to Music*, published in 1912. Ignoring for a moment the slightly condescending tenor of his statement (note that he addressed it only to male musicians, which was of course in line with the social norms of the time), it is true that very few among classical composers were directly involved in the mathematics or physics of their profession. Among the few who were, two names stand out: Jean-Philippe Rameau (1683–1764), who wrote an extensive treatise on acoustics, and Giuseppe Tartini (1692–1770), who discovered what are now known as combination tones (see chapter 5). In our own time things have changed somewhat, and several composers have tried—with varying degrees of success—to base their music on mathematical principles. Foremost among them was Schoenberg (1874–1951), whose serial music will be the subject of chapters 9 and 10. I should also mention Iannis Xenakis (1922–2001) and Karlheinz Stockhausen (1928–2007). The former, having been trained as a civil engineer and architect before turning to music, and used stochastic principles in his work; his scores often look like an assortment of graphs and lines rather than the notes and staves of a traditional musical score. Their works were initially received with much enthusiasm by avant-garde audiences, but whether they will become part of the mainstream oeuvre of classical music remains to be seen.

This, then, is the story of the relations between two great disciplines that have so much in common yet have always kept a respectable distance between them. It is by no means a comprehensive history of the subject, nor is it a textbook on the mathematics and physics of music, of which there already exist many good books. Rather, I attempted to survey the musical-mathematical affinity from a historical perspective, highlighting not only the facts but also the people behind them—the scientists, inventors, composers, and occasional eccentrics. I have not shied away from expressing my own thoughts on several issues with which some readers may disagree, such as the emotional attributes—greatly exaggerated, in my view—that are often associated with musical key designations. The book is intended for the general reader with an interest in mathematics, music, and science; no mathematical background is assumed beyond high school algebra and trigonometry, but a basic training in musical notation will be advantageous.

In the end, though, the attempts to relate mathematics to music are inherently limited by the contradictory goals of the two disciplines: mathematics—and science in general—aims at our intellect, our capacity to analyze abstract patterns and relations in an objective, logical manner, while music strives to touch our hearts, to awaken our emotions to its sounds, its rhythms, and its temporal and aural patterns. To quote the sign that greets visitors to the Musical Instrument Museum (MIM) in Phoenix, Arizona: *Music is the language of the soul.*

//

Any discussion of an interdisciplinary subject like this one inevitably will touch upon several adjacent fields. It goes without saying that the laws of physics play a role in music, but so does astronomy—from the Pythagorean belief that the planetary orbits are governed by the laws of musical harmony, to the discovery in the late nineteenth century of resonances among the orbits of planets and their satellites, resonances that often bear the ratios of common musical intervals (see chapter 12). We might also mention the recent detection of sound waves in the vast space between galaxies, waves with a specific wavelength and musical pitch (see Sidebar C)—perhaps the ultimate reincarnation of the age-old allegorical *Music of the Spheres*.

In any case, the clear distinction we draw today between mathematics and the physical sciences—and more broadly, the humanities—was not the common practice in years bygone; in fact, most of the great minds of classical science up until the early nineteenth century considered themselves as much mathematicians as philosophers, physicists, and natural scientists. They felt at home in a wide range of disciplines, which they collectively regarded as a quest to understand the workings of nature. And that included music.

//

A note about references: to avoid repetition, books that are referred to in the text and appear also in the bibliography are identified by the author's name and book title only. I hardly need to mention that all terms related to music, as well as the biographies of numerous composers, are covered at length in the exhaustive, twenty-nine-volume *The New Grove Dictionary of Music and Musicians* (Macmillan, 2001, now also available on the internet at www.oxfordmusiconline.com). Excellent biographies of numerous mathematicians can be found at the website of the School of Mathematics and Statistics of the University of St. Andrews, Scotland (www-groups.dcs.st-and.ac.uk/~history/Indexes/HistoryTopics.html).

I am deeply indebted to my friends and colleagues David Andrea Anati, Wilbur Hoppe, Robert Langer, and Michael Sterling for the many discussions we have had on questions relating to mathematics and music. I also wish to thank the anonymous reviewers who read the manuscript and offered their useful comments and suggestions. A big hug goes to Vickie Kearn, my trusted editor for nearly twenty years at Princeton University Press, for her invaluable advice and guidance at all stages of writing this book. I am also deeply indebted to the staff at Princeton University Press for their tireless care in handling the various phases of the production of this book. In particular, I would like to thank Debbie Tegarden, who supervised the book's entire production process and was of enormous help in selecting its many illustrations, to Lauren Bucca, who took upon herself to obtain many of the permissions needed for these illustrations, to Chris Ferrante for his text and jacket designs, and to Dimitri Karetnikov, Meghan Kanabay, Elizabeth Blazejewski, and Jacquie Poirier for their collective handling of the graphics, digital files and typesetting, to Carole Schwager for her copyediting of the manuscript, and to Jodi Beder for her help in transcribing Schoenberg's tone row in chapter 10. My grandson Richard Maor helped me greatly in obtaining good images of some of the objects shown in the illustrations, and I owe him many thanks. And last but not least, it is to my dear wife Dalia that I owe my greatest gratitude for encouraging me to see this book come to fruition and for her meticulous proofreading before it went to the printer. Without her this book would have never seen the light of day. Thank you all!

NOTES

1. I am grateful to my sister, Shulamit Nathansohn, who discovered the photo among hundreds we inherited from our parents.
2. Thanks to the internet I was able to rediscover that song, a good seventy-five years after my grandfather played it for me. You can hear it sung by Julien Neel at www.youtube.com/watch?v=7C_yFytWKlU, or follow the notes and lyrics at www.labbe.de/liederbaum/index.asp?themaid=25&titelid=438.
3. I still have the tuning fork with which he tuned his violin. It has the inscription A on one stem; and although rusted, it still produces the note A with its correct frequency, 440 hz.