The Mathematics of Shock Reflection-Diffraction and von Neumann's Conjectures

Gui-Qiang G. Chen
Mikhail Feldman

ANNALS OF MATHEMATICS STUDIES

Annals of Mathematics Studies
Number 197

The Mathematics of Shock Reflection-Diffraction and von Neumann's Conjectures

Gui-Qiang G. Chen
Mikhail Feldman

PRINCETON UNIVERSITY PRESS
PRINCETON AND OXFORD
2018

Published by Princeton University Press, 41 William Street,
Princeton, New Jersey 08540

In the United Kingdom: Princeton University Press, 6 Oxford Street,
Woodstock, Oxfordshire OX20 1TR

press.princeton.edu

Library of Congress Cataloging-in-Publication Data

Names: Chen, Gui-Qiang, 1963– | Feldman, Mikhail, 1960–
Title: The mathematics of shock reflection-diffraction and von Neumann's conjectures / Gui-Qiang G. Chen and Mikhail Feldman.
Description: Princeton : Princeton University Press, 2017. | Series: Annals of mathematics studies ; number 197 | Includes bibliographical references and index.
Identifiers: LCCN 2017008667| ISBN 9780691160542 (hardcover : alk. paper) | ISBN 9780691160559 (pbk. : alk. paper)
Subjects: LCSH: Shock waves–Diffraction. | Shock waves–Free boundaries. | Shock waves–Mathematics. | von Neumann conjectures. | Analysis of PDEs.
Classification: LCC QC168.85.S45 C44 2017 | DDC 531/.1133–dc23 LC record available at https://lccn.loc.gov/2017008667

British Library Cataloging-in-Publication Data is available

This book has been composed in LaTeX. [sjp]

The publisher would like to acknowledge the authors of this volume for providing the camera-ready copy from which this book was printed.

Printed on acid-free paper ∞

10 9 8 7 6 5 4 3 2 1

Contents

Preface

The purpose of this research monograph is to survey some recent developments in the analysis of shock reflection-diffraction, to present our original mathematical proofs of von Neumann's conjectures for potential flow, to collect most of the related results and new techniques in the analysis of partial differential equations (PDEs) achieved in the last decades, and to discuss a set of fundamental open problems relevant to the directions of future research in this and related areas.

Shock waves are fundamental in nature, especially in high-speed fluid flows. Shocks are generated by supersonic or near-sonic aircraft, explosions, solar wind, and other natural processes. They are governed by the Euler equations for compressible fluids or their variants, generally in the form of nonlinear conservation laws – nonlinear PDEs of divergence form. The Euler equations describing the motion of a perfect fluid were first formulated by Euler [112, 113, 114] in 1752 (based in part on the earlier work of Bernoulli [15]), and were among the first PDEs for describing physical processes to be written down.

When a shock hits an obstacle (steady or flying), shock reflection-diffraction configurations take shape. One of the most fundamental research directions in mathematical fluid dynamics is the analysis of shock reflection-diffraction by wedges, with focus on the wave patterns of the reflection-diffraction configurations formed around the wedge. The complexity of such configurations was first reported by Ernst Mach [206] in 1878, who observed two patterns of shock reflection-diffraction configurations that are now named the Regular Reflection (RR) and the Mach Reflection (MR). The subject remained dormant until the 1940s when von Neumann [267, 268, 269], as well as other mathematical and experimental scientists, began extensive research on shock reflection-diffraction phenomena, owing to their fundamental importance in applications. It has since been found that the phenomena are much more complicated than what Mach originally observed, and various other patterns of shock reflection-diffraction configurations may occur. On the other hand, the shock reflection-diffraction configurations are core configurations in the structure of global entropy solutions of the two-dimensional Riemann problem, while the Riemann solutions themselves are local building blocks and determine local structures, global attractors, and large-time asymptotic states of general entropy solutions of multidimensional hyperbolic systems of conservation laws. In this sense, we have to understand the shock reflection-diffraction configurations, in order to understand fully the global entropy solutions of multidimensional hyperbolic systems of conservation laws.

Diverse patterns of shock reflection-diffraction configurations have attracted many asymptotic/numerical analysts since the middle of the 20th century. However, most of the fundamental issues involved, such as the structure and transition criteria of the different patterns, have not been understood. This is partially because physical and numerical experiments are hampered by various difficulties and have not yielded clear transition criteria between the different patterns. In light of this, a natural approach for understanding fully the shock reflection-diffraction configurations, especially with regard to the transition criteria, is via rigorous mathematical analysis. To achieve this, it is essential to establish the global existence, regularity, and structural stability of shock reflection-diffraction configurations: That is the main topic of this book.

Mathematical analysis of shock reflection-diffraction configurations involves dealing with several core difficulties in the analysis of nonlinear PDEs. These include nonlinear PDEs of mixed hyperbolic-elliptic type, nonlinear degenerate elliptic PDEs, nonlinear degenerate hyperbolic PDEs, free boundary problems for nonlinear degenerate PDEs, and corner singularities (especially when free boundaries meet the fixed boundaries), among others. These difficulties also arise in many further fundamental problems in continuum mechanics, differential geometry, mathematical physics, materials science, and other areas, including transonic flow problems, isometric embedding problems, and phase transition problems. Therefore, any progress in solving these problems requires new mathematical ideas, approaches, and techniques, all of which will both be very helpful for solving other problems with similar difficulties and open up new research directions.

Our efforts in the analysis of shock reflection-diffraction configurations for potential flow started 18 years ago when both of us were at Northwestern University, USA. We soon realized that the first step to achieving our goal should be to develop new free boundary techniques for multidimensional transonic shocks, along with other analytical techniques for nonlinear degenerate elliptic PDEs. After about two years of struggle, we developed such techniques, and these were published in [49] in 2003 and subsequent papers [42, 50, 51, 53]. With this groundwork, we first succeeded in developing a rigorous mathematical approach to establish the global existence and stability of regular shock reflection-diffraction solutions for large-angle wedges in [52] in 2005, the complete version of which was published electronically in 2006 and in print form in [54] in 2010. Since 2005, we have continued our efforts to solve von Neumann's sonic conjecture (*i.e.*, the existence of global regular reflection-diffraction solutions up to the sonic wedge angle with the supersonic reflection-diffraction configuration, containing a transonic reflected-diffracted shock), as well as von Neumann's detachment conjecture (*i.e.*, the necessary and sufficient condition for the existence of global regular reflection-diffraction solutions, even beyond the sonic angle, up to the detachment angle with the subsonic reflection-diffraction configuration, containing a transonic reflected-diffracted shock) (*cf.* [55, 57]). The results of these efforts were announced in [56, 58], and their detailed proofs constitute the main part of this book.

Some efforts have also been made by several groups of researchers on related models, including the unsteady small disturbance equation (USD), the pressure gradient equations, and the nonlinear wave system, as well as for some partial results for the potential flow equation and the full Euler equations. For the sake of completeness, we have made remarks and notes about these contributions throughout the book, and have tried to collect a detailed list of appropriate references in the bibliography.

Based on these results, along with our recent results on von Neumann's conjectures for potential flow, mathematical understanding of shock reflection-diffraction, especially for the global regular reflection-diffraction configurations, has reached a new height, and several new mathematical approaches and techniques have been developed. Moreover, new research opportunities and many new, challenging, and important problems have arisen during this exploration. Given these developments, we feel that it is the right time to publish this research monograph.

During the process of assembling this work, we have received persistent encouragement and invaluable suggestions from many leading mathematicians and scientists, especially John Ball, Luis Caffarelli, Alexander Chorin, Demetrios Christodoulou, Peter Constantin, Constantine Dafermos, Emmanuele DiBenedetto, Xiaxi Ding, Weinan E, Björn Engquist, Lawrence Craig Evans, Charles Fefferman, Edward Fraenkel, James Glimm, Helge Holden, Jiaxing Hong, Carlos Kenig, Sergiu Klainerman, Peter D. Lax, Tatsien Li, Fanhua Lin, Andrew Majda, Cathleen Morawetz, Luis Nirenberg, Benoît Perthame, Richard Schoen, Henrik Shahgholian, Yakov Sinai, Joel Smoller, John Toland, Neil Trudinger, and Juan Luis Vázquez. The materials presented herein contain direct and indirect contributions from many leading experts – teachers, colleagues, collaborators, and students alike, including Myoungjean Bae, Sunčica Canić, Yi Chao, Jun Chen, Shuxing Chen, Volker Elling, Beixiang Fang, Jingchen Hu, Feimin Huang, John Hunter, Katarina Jegdić, Siran Li, Tianhong Li, Yachun Li, Gary Lieberman, Tai-Ping Liu, Barbara Keyfitz, Eun Heui Kim, Jie Kuang, Stefano Marchesani, Ho Cheung Pang, Matthew Rigby, Matthew Schrecker, Denis Serre, Wancheng Sheng, Marshall Slemrod, Eitan Tadmor, Dehua Wang, Tian-Yi Wang, Yaguang Wang, Wei Xiang, Zhouping Xin, Hairong Yuan, Tong Zhang, Yongqian Zhang, Yuxi Zheng, and Dianwen Zhu, among others. We are grateful to all of them.

A significant portion of this work was done while the authors attended the Spring 2011 Program "*Free Boundary Problems*: *Theory and Applications*" at the Mathematical Sciences Research Institute in Berkeley, California, USA, and the 2014 Program "*Free Boundary Problems and Related Topics*" at the Isaac Newton Institute for Mathematical Sciences in Cambridge, UK. A part of the work was also supported by Keble College, University of Oxford, and a UK EPSRC Science and Innovation Award to the Oxford Centre for Nonlinear PDE (EP/E035027/1) when Mikhail Feldman visited Oxford in 2010.

The work of Gui-Qiang G. Chen was supported in part by the National Science Foundation under Grants DMS-0935967 and DMS-0807551, a UK EP-

SRC Science and Innovation Award to the Oxford Centre for Nonlinear PDE (EP/E035027/1), a UK EPSRC Award to the EPSRC Centre for Doctoral Training in PDEs (EP/L015811/1), the National Natural Science Foundation of China (under joint project Grant 10728101), and the Royal Society–Wolfson Research Merit Award (UK). The work of Mikhail Feldman was supported in part by the National Science Foundation under Grants DMS-0800245, DMS-1101260, and DMS-1401490, the Vilas Award from the University of Wisconsin-Madison, and the Simons Foundation via the Simons Fellows Program. Kurt Ballstadt deserves our special thanks for his effective assistance during the preparation of the manuscript. We are indebted to Princeton University Press, especially Vickie Kearn (Executive Editor) and Betsy Blumenthal and Lauren Bucca (Editorial Assistants), for their professional assistance.

Finally, we remark in passing that further supplementary materials to this research monograph will be posted at:

http://people.maths.ox.ac.uk/chengq/books/Monograph-CF-17/index.html
https://www.math.wisc.edu/~feldman/Monograph-CF-17/monograph.html

Part I

Shock Reflection-Diffraction, Nonlinear Conservation Laws of Mixed Type, and von Neumann's Conjectures

Chapter One

Shock Reflection-Diffraction, Nonlinear Partial Differential Equations of Mixed Type, and Free Boundary Problems

Shock waves are steep fronts that propagate in compressible fluids when convection dominates diffusion. They are fundamental in nature, especially in high-speed fluid flows. Examples include transonic shocks around supersonic or near-sonic flying bodies (such as aircraft), transonic and/or supersonic shocks formed by supersonic flows impinging onto solid wedges, bow shocks created by solar wind in space, blast waves caused by explosions, and other shocks generated by natural processes. Such shocks are governed by the Euler equations for compressible fluids or their variants, generally in the form of nonlinear conservation laws – nonlinear partial differential equations (PDEs) of divergence form. When a shock hits an obstacle (steady or flying), shock reflection-diffraction phenomena occur. One of the most fundamental research directions in mathematical fluid mechanics is the analysis of shock reflection-diffraction by wedges; see Ben-Dor [12], Courant-Friedrichs [99], von Neumann [267, 268, 269], and the references cited therein. When a plane shock hits a two-dimensional wedge head-on (*cf.* Fig. 1.1), it experiences a reflection-diffraction process; a fundamental question arisen is then what types of wave patterns of shock reflection-diffraction configurations may be formed around the wedge.

An archetypal system of PDEs describing shock waves in fluid mechanics, widely used in aerodynamics, is that of the Euler equations for potential flow (*cf.* [16, 95, 99, 139, 146, 221]). The Euler equations for describing the motion of a perfect fluid were first formulated by Euler [112, 113, 114] in 1752, based in part on the earlier work of D. Bernoulli [15], and were among the first PDEs for describing physical processes to be written down. The n-dimensional Euler equations for potential flow consist of the conservation law of mass and the Bernoulli law for the density and velocity potential (ρ, Φ):

$$\begin{cases} \partial_t \rho + \mathrm{div}_{\mathbf{x}}(\rho \nabla_{\mathbf{x}} \Phi) = 0, \\ \partial_t \Phi + \dfrac{1}{2}|\nabla_{\mathbf{x}} \Phi|^2 + h(\rho) = B_0, \end{cases} \tag{1.1}$$

where $\mathbf{x} \in \mathbb{R}^n$, B_0 is the Bernoulli constant determined by the incoming flow

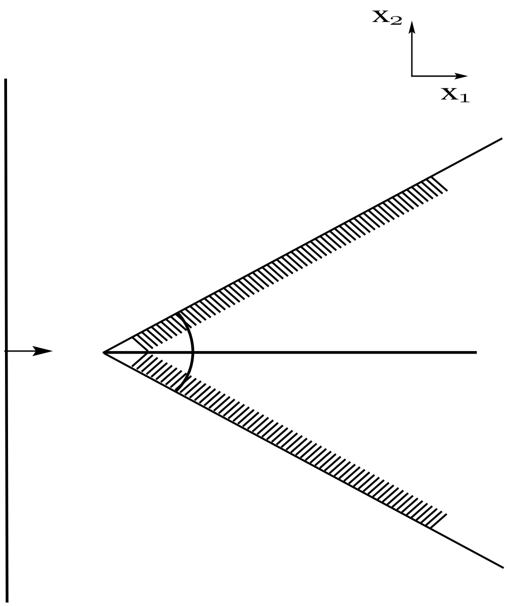


Figure 1.1: A plane shock hits a two-dimensional wedge in $\mathbb{R}^2$ head-on

and/or boundary conditions,

$$h'(\rho) = \frac{p'(\rho)}{\rho} = \frac{c^2(\rho)}{\rho},$$

and $c(\rho) = \sqrt{p'(\rho)}$ is the sonic speed (*i.e.*, the speed of sound).

The first equation in (1.1) is a transport-type equation for density ρ for a given $\nabla_{\mathbf{x}}\Phi$, while the second equation is the Hamilton-Jacobi equation for the velocity potential Φ coupling with density ρ through function $h(\rho)$.

For polytropic gases,

$$p(\rho) = \kappa\rho^\gamma, \qquad c^2(\rho) = \kappa\gamma\rho^{\gamma-1}, \qquad \gamma > 1, \ \kappa > 0.$$

Without loss of generality, we may choose $\kappa = \frac{1}{\gamma}$ so that

$$h(\rho) = \frac{\rho^{\gamma-1} - 1}{\gamma - 1}, \qquad c^2(\rho) = \rho^{\gamma-1}. \tag{1.2}$$

This can be achieved by noting that (1.1) is invariant under scaling:

$$(t, \mathbf{x}, B_0) \mapsto (\alpha^2 t, \alpha\mathbf{x}, \alpha^{-2}B_0)$$

with $\alpha^2 = \kappa\gamma$. In particular, Case $\gamma = 1$ can be considered as the limit of $\gamma \to 1+$ in (1.2):

$$h(\rho) = \ln\rho, \qquad c(\rho) = 1. \tag{1.3}$$

Henceforth, we will focus only on Case $\gamma > 1$, since Case $\gamma = 1$ can be handled similarly by making appropriate changes in the formulas so that the results of the main theorems for $\gamma > 1$ (below) also hold for $\gamma = 1$.

From the Bernoulli law, the second equation in (1.1), we have

$$\rho(\partial_t\Phi, |\nabla_{\mathbf{x}}\Phi|^2) = h^{-1}(B_0 - (\partial_t\Phi + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2)). \tag{1.4}$$

Then system (1.1) can be rewritten as the following time-dependent potential flow equation of second order:

$$\partial_t\rho(\partial_t\Phi, |\nabla_{\mathbf{x}}\Phi|^2) + \nabla_{\mathbf{x}} \cdot \big(\rho(\partial_t\Phi, |\nabla_{\mathbf{x}}\Phi|^2)\nabla_{\mathbf{x}}\Phi\big) = 0 \tag{1.5}$$

with $\rho(\partial_t\Phi, |\nabla_{\mathbf{x}}\Phi|^2)$ determined by (1.4). Equation (1.5) is a nonlinear wave equation of second order. Notice that equation (1.5) is invariant under a symmetry group formed of space-time dilations.

For a steady solution $\Phi = \varphi(\mathbf{x})$, *i.e.*, $\partial_t\Phi = 0$, we obtain the celebrated steady potential flow equation, especially in aerodynamics (*cf.* [16, 95, 99]):

$$\nabla_{\mathbf{x}} \cdot \big(\rho(|\nabla_{\mathbf{x}}\varphi|^2)\nabla_{\mathbf{x}}\varphi\big) = 0, \tag{1.6}$$

which is a second-order nonlinear PDE of mixed elliptic-hyperbolic type. This is a simpler case of the nonlinear PDE of mixed type for self-similar solutions, as shown in (1.12)–(1.13) later.

When the effects of vortex sheets and the deviation of vorticity become significant, the full Euler equations are required. The full Euler equations for compressible fluids in $\mathbb{R}^{n+1}_+ = \mathbb{R}_+ \times \mathbb{R}^n, t \in \mathbb{R}_+ := (0,\infty)$ and $\mathbf{x} \in \mathbb{R}^n$, are of the following form:

$$\begin{cases} \partial_t\,\rho + \nabla_{\mathbf{x}} \cdot (\rho\mathbf{v}) = 0, \\ \partial_t(\rho\mathbf{v}) + \nabla_{\mathbf{x}} \cdot (\rho\mathbf{v} \otimes \mathbf{v}) + \nabla_{\mathbf{x}} p = 0, \\ \partial_t\big(\rho(\frac{1}{2}|\mathbf{v}|^2 + e)\big) + \nabla_{\mathbf{x}} \cdot \big(\rho\mathbf{v}(\frac{1}{2}|\mathbf{v}|^2 + e + \frac{p}{\rho})\big) = 0, \end{cases} \tag{1.7}$$

where ρ is the density, $\mathbf{v} \in \mathbb{R}^n$ the fluid velocity, p the pressure, and e the internal energy. Two other important thermodynamic variables are temperature θ and entropy S. Here, $\mathbf{a} \otimes \mathbf{b}$ denotes the tensor product of vectors $\mathbf{a}$ and $\mathbf{b}$.

Choose (ρ, S) as the independent thermodynamical variables. Then the constitutive relations can be written as $(e, p, \theta) = (e(\rho, S), p(\rho, S), \theta(\rho, S))$, governed by

$$\theta dS = de + pd\tau = de - \frac{p}{\rho^2}d\rho,$$

as introduced by Gibbs [129].

For a polytropic gas,

$$p = (\gamma - 1)\rho e, \qquad e = c_v\theta, \qquad \gamma = 1 + \frac{R}{c_v}, \tag{1.8}$$

or equivalently,

$$p = p(\rho, S) = \kappa \rho^{\gamma} e^{S/c_v}, \qquad e = e(\rho, S) = \frac{\kappa}{\gamma - 1}\rho^{\gamma-1} e^{S/c_v}, \tag{1.9}$$

where $R > 0$ may be taken to be the universal gas constant divided by the effective molecular weight of the particular gas, $c_v > 0$ is the specific heat at constant volume, $\gamma > 1$ is the adiabatic exponent, and $\kappa > 0$ may be chosen as any constant through scaling.

The full Euler equations in the general form presented here were originally derived by Euler [112, 113, 114] for mass, Cauchy [29, 30] for linear and angular momentum, and Kirchhoff [165] for energy.

The nonlinear equations (1.5) and (1.7) fit into the general form of hyperbolic conservation laws:

$$\partial_t \mathbf{A}(\partial_t \mathbf{u}, \nabla_{\mathbf{x}} \mathbf{u}, \mathbf{u}) + \nabla_{\mathbf{x}} \cdot \mathbf{B}(\partial_t \mathbf{u}, \nabla_{\mathbf{x}} \mathbf{u}, \mathbf{u}) = 0, \tag{1.10}$$

or

$$\partial_t \mathbf{u} + \nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{u}) = 0, \qquad \mathbf{u} \in \mathbb{R}^m, \ \mathbf{x} \in \mathbb{R}^n, \tag{1.11}$$

where $\mathbf{A} : \mathbb{R}^m \times \mathbb{R}^{n\times m} \times \mathbb{R}^m \mapsto \mathbb{R}^m$, $\mathbf{B} : \mathbb{R}^m \times \mathbb{R}^{n\times m} \times \mathbb{R}^m \mapsto (\mathbb{R}^m)^n$, and $\mathbf{f} : \mathbb{R}^m \mapsto (\mathbb{R}^m)^n$ are nonlinear mappings. Besides (1.5) and (1.7), most of the nonlinear PDEs arising from physical or engineering science can also be formulated in accordance with form (1.10) or (1.11), or their variants. Moreover, the second-order form (1.10) of hyperbolic conservation laws can be reformulated as a first-order system (1.11). The hyperbolicity of system (1.11) requires that, for all $\boldsymbol{\xi} \in \mathbb{S}^{n-1}$, matrix $[\boldsymbol{\xi} \cdot \nabla_{\mathbf{u}} \mathbf{f}(\mathbf{u})]_{m\times m}$ have m real eigenvalues $\lambda_j(\mathbf{u}, \boldsymbol{\xi}), j = 1, 2, \cdots, m$, and be diagonalizable. See Lax [171], Glimm-Majda [139], and Majda [210].

The complexity of shock reflection-diffraction configurations was first reported in 1878 by Ernst Mach [206], who observed two patterns of shock reflection-diffraction configurations that are now named the Regular Reflection (RR: two-shock configuration; see Fig. 1.2) and the Simple Mach Reflection (SMR: three-shock and one-vortex-sheet configuration; see Fig. 1.3); see also [12, 167, 228]. The problem remained dormant until the 1940s when von Neumann [267, 268, 269], as well as other mathematical/experimental scientists, began extensive research on shock reflection-diffraction phenomena, owing to their fundamental importance in various applications (see von Neumann [267, 268] and Ben-Dor [12]; see also [11, 132, 152, 160, 166, 205, 248, 249] and the references cited therein).

It has since been found that there are more complexity and variety of shock reflection-diffraction configurations than what Mach originally observed: The Mach reflection can be further divided into more specific sub-patterns, and many other patterns of shock reflection-diffraction configurations may occur, for example, the Double Mach Reflection (see Fig. 1.4), the von Neumann Reflection, and the Guderley Reflection; see also [12, 99, 139, 143, 159, 243, 257, 258, 259, 263, 267, 268] and the references cited therein.

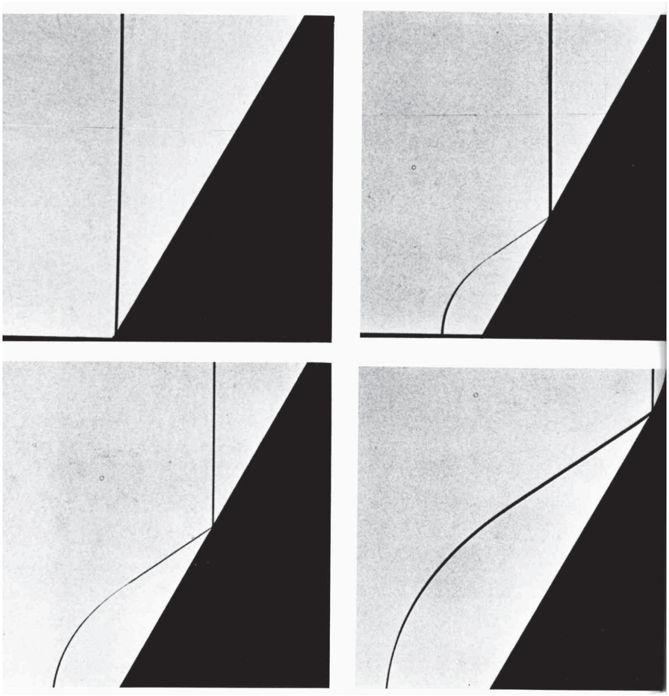


Figure 1.2: Regular Reflection for large-angle wedges. From Van Dyke [263, pp. 142].

The fundamental scientific issues arising from all of this are

(i) *The structure of shock reflection-diffraction configurations*;

(ii) *The transition criteria between the different patterns of shock reflection-diffraction configurations*;

(iii) *The dependence of the patterns upon the physical parameters such as the wedge angle* θ_{w}*, the incident-shock Mach number* M_I *(a measure of the strength of the shock), and the adiabatic exponent* $\gamma \geq 1$.

Careful asymptotic analysis has been made for various reflection-diffraction configurations in Lighthill [199, 200], Keller-Blank [162], Hunter-Keller [158], and Morawetz [221], as well as in [128, 148, 155, 255, 267, 268] and the references cited therein; see also Glimm-Majda [139]. Large or small scale numerical simulations have also been made; *e.g.*, [12, 139], [104, 105, 149, 170, 232, 240], and [133, 134, 135, 160, 273] (see also the references cited therein).

On the other hand, most of the fundamental issues for shock reflection-diffraction phenomena have not been understood, especially the global structure

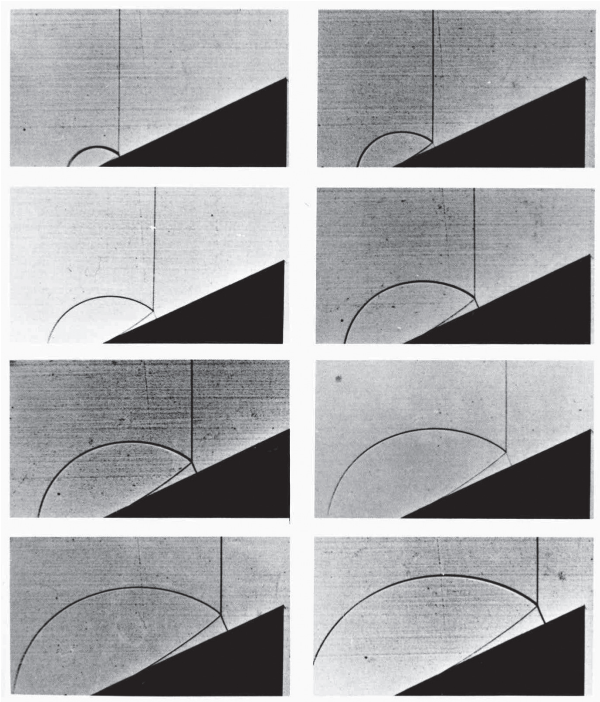


Figure 1.3: Simple Mach Reflection when the wedge angle becomes small. From Van Dyke [263, pp. 143].

and transition between the different patterns of shock reflection-diffraction configurations. This is partially because physical and numerical experiments are hampered by various difficulties and have not thusfar yielded clear transition criteria between the different patterns. In particular, numerical dissipation or physical viscosity smears the shocks and causes the boundary layers that interact with the reflection-diffraction configurations and may cause spurious Mach steams; *cf.* Woodward-Colella [273]. Furthermore, some different patterns occur in which the wedge angles are only fractions of a degree apart; a resolution has challenged even sophisticated modern numerical and laboratory experiments. For this reason, it is almost impossible to distinguish experimentally between the sonic and detachment criteria, as was pointed out by Ben-Dor in [12] (also *cf.* Chapter 7 below). On account of this, a natural approach to understand fully the shock reflection-diffraction configurations, especially the transition criteria, is via rigorous mathematical analysis. To carry out this analysis, it is essential to establish first the global existence, regularity, and structural stability of shock reflection-diffraction configurations: That is the main topic of this book.

Furthermore, the shock reflection-diffraction configurations are core configurations in the structure of global entropy solutions of the two-dimensional Rie-

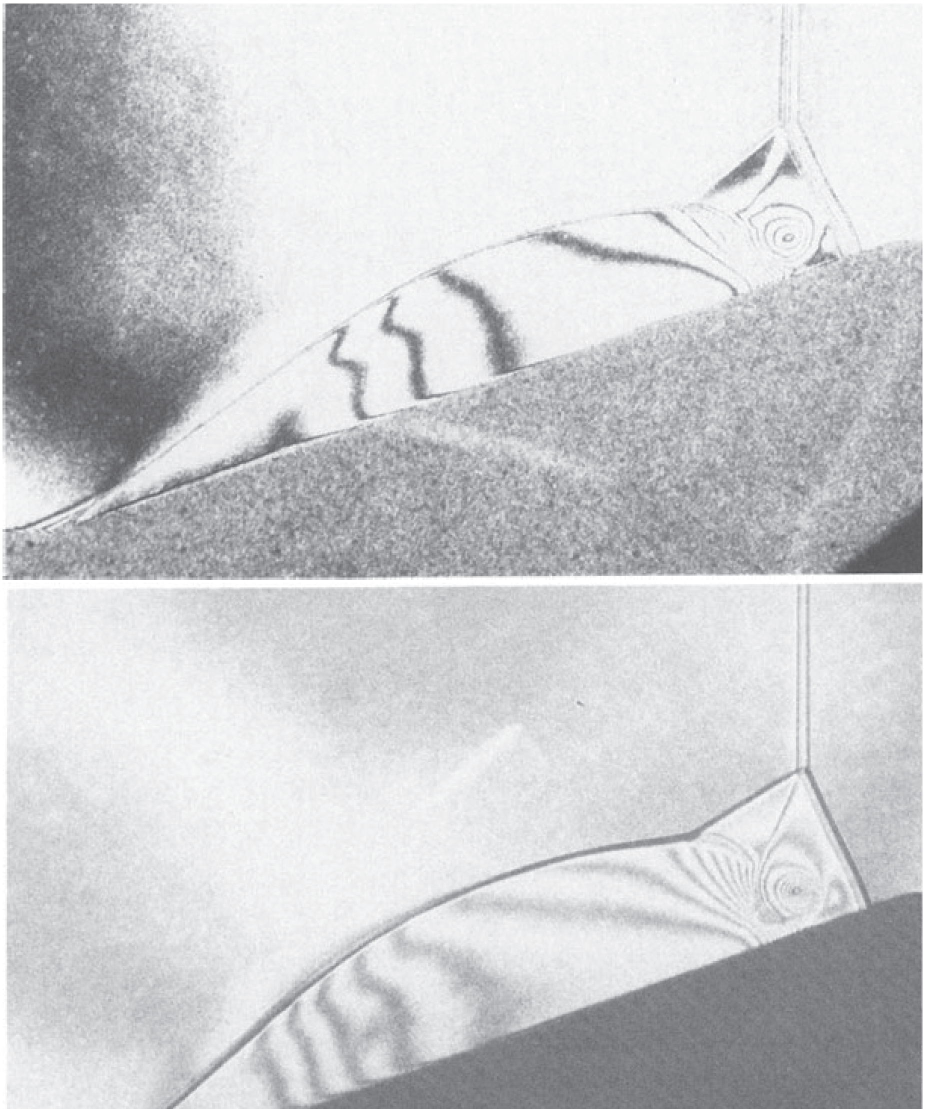


Figure 1.4: Double Mach Reflection when the wedge angle becomes even smaller. From Ben-Dor [12, pp. 67].

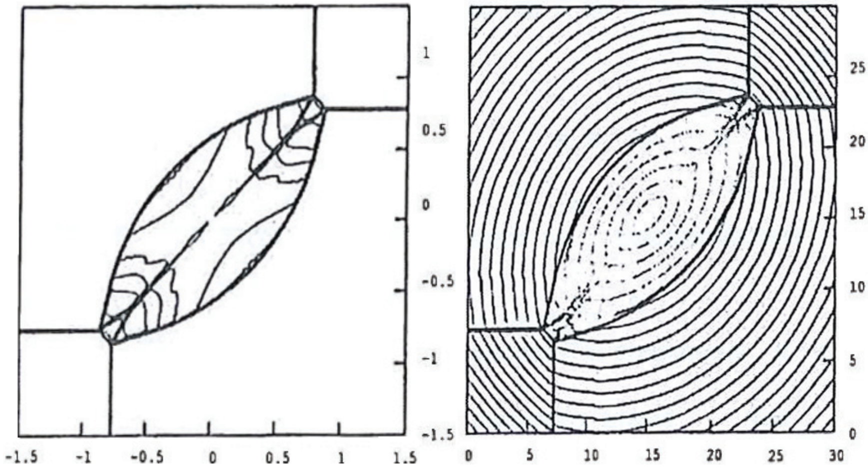


Figure 1.5: Riemann solutions: Simple Mach Reflection; see [33]

mann problem for hyperbolic conservation laws (see Figs. 1.5–1.6), while the Riemann solutions are building blocks and determine local structures, global attractors, and large-time asymptotic states of general entropy solutions of multidimensional hyperbolic systems of conservation laws (see [31]–[35], [138, 139, 169, 175, 181, 233, 235, 236, 286], and the references cited therein). Consequently, we have to understand the shock reflection-diffraction configurations in order to fully understand global entropy solutions of the multidimensional hyperbolic systems of conservation laws.

Mathematically, the analysis of shock reflection-diffraction configurations involves several core difficulties that we have to face for the mathematical theory of nonlinear PDEs:

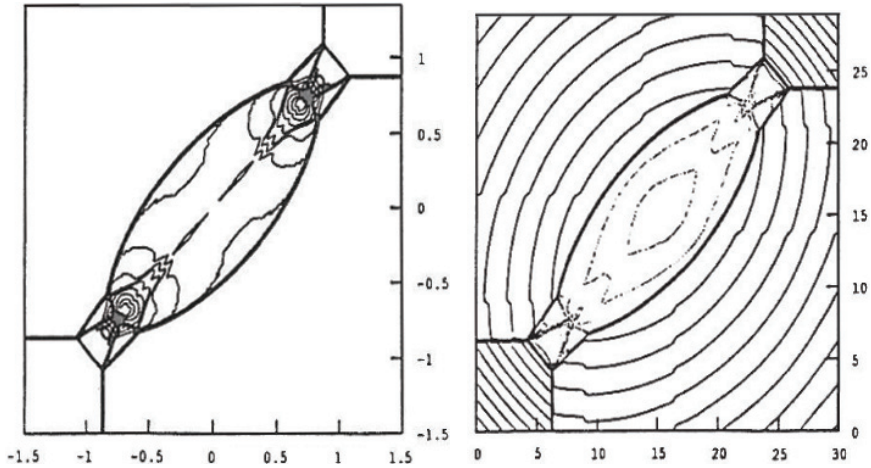
(i) Nonlinear PDEs of Mixed Elliptic-Hyperbolic Type: The first is that the underlying nonlinear PDEs change type from hyperbolic to elliptic in the shock reflection-diffraction configurations, so that the nonlinear PDEs are of mixed hyperbolic-elliptic type.

This can be seen as follows: Since both the system and the initial-boundary conditions admit a symmetry group formed of space-time dilations, we seek self-similar solutions of the problem:

$$\rho(t,\mathbf{x}) = \rho(\boldsymbol{\xi}), \quad \Phi(t,\mathbf{x}) = t\phi(\boldsymbol{\xi}),$$

depending only upon $\boldsymbol{\xi} = \frac{\mathbf{x}}{t} \in \mathbb{R}^2$. For the Euler equation (1.5) for potential flow, the corresponding pseudo-potential function $\varphi(\boldsymbol{\xi}) = \phi(\boldsymbol{\xi}) - \frac{|\boldsymbol{\xi}|^2}{2}$ satisfies the following potential flow equation of second order:

$$\operatorname{div}\big(\rho(|D\varphi|^2,\varphi)D\varphi\big) + 2\rho(|D\varphi|^2,\varphi) = 0 \tag{1.12}$$



Density contour curves Self-Mach number contour curves

Figure 1.6: Riemann solutions: Double Mach reflection; see [33]

with

$$\rho(|D\varphi|^2,\varphi)=\big(\rho_0^{\gamma-1}-(\gamma-1)(\varphi+\frac{1}{2}|D\varphi|^2)\big)^{\frac{1}{\gamma-1}}, \tag{1.13}$$

where div and D represent the divergence and the gradient, respectively, with respect to the self-similar variables $\boldsymbol{\xi}=(\xi_1,\xi_2)$, that is, $D:=(D_1,D_2)=(D_{\xi_1},D_{\xi_2})$. Then the sonic speed becomes:

$$c=c(|D\varphi|^2,\varphi,\rho_0^{\gamma-1})=\big(\rho_0^{\gamma-1}-(\gamma-1)(\frac{1}{2}|D\varphi|^2+\varphi)\big)^{\frac{1}{2}}. \tag{1.14}$$

Equation (1.12) can be written in the following non-divergence form of nonlinear PDE of second order:

$$\sum_{i,j=1}^{2}a_{ij}(\varphi,D\varphi)D_{ij}\varphi=f(\varphi,D\varphi), \tag{1.15}$$

where $[a_{ij}(\varphi,D\varphi)]_{1\le i,j\le 2}$ is a symmetric matrix and $D_{ij}=D_iD_j, i,j=1,2$. The type of equation that (1.12) or (1.15) is depends on the values of solution φ and its gradient $D\varphi$. More precisely, equation (1.15) is elliptic on a solution φ when the two eigenvalues $\lambda_j(\varphi,D\varphi), j=1,2$, of the symmetric matrix $[a_{ij}(\varphi,D\varphi)]$ have the same sign on φ:

$$\lambda_1(\varphi,D\varphi)\lambda_2(\varphi,D\varphi)>0. \tag{1.16}$$

Correspondingly, equation (1.15) is (strictly) hyperbolic on a solution φ if the two eigenvalues of the matrix have the opposite signs on φ:

$$\lambda_1(\varphi,D\varphi)\lambda_2(\varphi,D\varphi)<0. \tag{1.17}$$

The more complicated case is that of the mixed elliptic-hyperbolic type for which $\lambda_1(\varphi, D\varphi)\lambda_2(\varphi, D\varphi)$ changes its sign when the values of φ and $D\varphi$ change in the physical domain under consideration.

In particular, equation (1.12) is a *nonlinear second-order conservation law of mixed elliptic-hyperbolic type.* It is *elliptic* if

$$|D\varphi| < c(|D\varphi|^2, \varphi, \rho_0^{\gamma-1}), \tag{1.18}$$

and *hyperbolic* if

$$|D\varphi| > c(|D\varphi|^2, \varphi, \rho_0^{\gamma-1}). \tag{1.19}$$

The types normally change with $\boldsymbol{\xi}$ from hyperbolic in the far field to elliptic around the wedge vertex, which is the case that the corresponding physical velocity $\nabla_{\mathbf{x}}\Phi$ is bounded.

Similarly, for the full Euler equations, the corresponding self-similar solutions are governed by a nonlinear system of conservation laws of composite-mixed hyperbolic-elliptic type, as shown in (18.3.1) in Chapter 18.

Such nonlinear PDEs of mixed type also arise naturally in many other fundamental problems in continuum physics, differential geometry, elasticity, relativity, calculus of variations, and related areas.

Classical fundamental linear PDEs of mixed elliptic-hyperbolic type include the following:

The Lavrentyev-Bitsadze equation for an unknown function $u(x, y)$:

$$u_{xx} + \mathrm{sign}(x)u_{yy} = 0. \tag{1.20}$$

This becomes the wave equation (hyperbolic) in half-plane $x < 0$ and the Laplace equation (elliptic) in half-plane $x > 0$, and changes the type from elliptic to hyperbolic via a jump discontinuous coefficient $\mathrm{sign}(x)$.

The Keldysh equation for an unknown function $u(x, y)$:

$$xu_{xx} + u_{yy} = 0. \tag{1.21}$$

This is hyperbolic in half-plane $x < 0$, elliptic in half-plane $x > 0$, and degenerates on line $x = 0$. This equation is of parabolic degeneracy in domain $x \leq 0$, for which the two characteristic families are quadratic parabolas lying in half-plane $x < 0$ and tangential at contact points to the degenerate line $x = 0$. Its degeneracy is also determined by the classical elliptic or hyperbolic Euler-Poisson-Darboux equation:

$$u_{\tau\tau} \pm u_{yy} + \frac{\beta}{\tau}u_\tau = 0 \tag{1.22}$$

with $\beta = -\frac{1}{4}$, where $\tau = \frac{1}{2}|x|^{\frac{1}{2}}$, and signs "$\pm$" in (1.22) are determined by the corresponding half-planes $\pm x > 0$.

The Tricomi equation for an unknown function $u(x, y)$:

$$u_{xx} + xu_{yy} = 0. \tag{1.23}$$

This is hyperbolic when $x < 0$, elliptic when $x > 0$, and degenerates on line $x = 0$. This equation is of hyperbolic degeneracy in domain $x \leq 0$, for which the two characteristic families coincide *perpendicularly* to line $x = 0$. Its degeneracy is also determined by the classical elliptic or hyperbolic Euler-Poisson-Darboux equation (1.22) with $\beta = \frac{1}{3}$, where $\tau = \frac{2}{3}|x|^{\frac{3}{2}}$.

For linear PDEs of mixed elliptic-hyperbolic type such as (1.20)–(1.23), the transition boundary between the elliptic and hyperbolic phases is known *a priori*. One of the classical approaches to the study of such mixed-type linear equations is the fundamental solution approach, since the optimal regularity and/or singularities of solutions near the transition boundary are determined by the fundamental solution (see [17, 37, 39, 41, 275, 278]).

For nonlinear PDEs of mixed elliptic-hyperbolic type such as (1.12), the transition boundary between the elliptic and hyperbolic phases is *a priori* unknown, so that most of the classical approaches, especially the fundamental solution approach, no longer work. New ideas, approaches, and techniques are in great demand for both theoretical and numerical analysis.

(ii) Free Boundary Problems: Following the discussion in **(i)**, above, the analysis of shock reflection-diffraction configurations can be reduced to the analysis of a *free boundary problem*, as we will show in §2.4, in which the reflected-diffracted shock, defined as the transition boundary from the hyperbolic to elliptic phase, is a free boundary that cannot be determined prior to the determination of the solution.

The subject of free boundary problems has its origin in the study of the Stefan problem, which models the melting of ice (*cf.* Stefan [250]). In that problem, the moving-in-time boundary between water and ice is not known *a priori*, but is determined by the solution of the problem. More generally, free boundary problems are concerned with sharp transitions in the variables involved in the problems, such as the change in the temperature between water and ice in the Stefan problem, and the changes in the velocity and density across the shock wave in the shock reflection-diffraction configurations. Mathematically, this rapid transition is simplified to be seen as occurring infinitely fast across a curve or surface of discontinuity or constraint in the PDEs governing the physical or other processes under consideration. The location of these curves and surfaces, called free boundaries, is required to be determined in the process of solving the free boundary problem. Free boundaries subdivide the domain into subdomains in which the governing equations (usually PDEs) are satisfied. On the free boundaries, the free boundary conditions, derived from the models, are prescribed. The number of conditions on the free boundary is such that the PDE governing the problem, combined with the free boundary conditions, allows us to determine both the location of the free boundary and the solution in the whole domain. That is, more conditions are required on the free boundary than in the case of the fixed boundary value problem for the same PDEs in a fixed domain. Great progress has been made on free boundary problems for linear PDEs. Further developments, especially in terms of solving such problems for nonlinear PDEs of mixed type, ask for new mathematical approaches and techniques. For a better sense of these, see Chen-Shahgholian-Vázquez [67] and the references cited therein.

(iii) Estimates of Solutions to Nonlinear Degenerate PDEs: The third difficulty concerns the degeneracies that are along the sonic arc, since the sonic arc is another transition boundary from the hyperbolic to elliptic phase in the shock reflection-diffraction configurations, for which the corresponding nonlinear PDE becomes a nonlinear degenerate hyperbolic equation on its one side and a nonlinear degenerate elliptic equation on the other side; both of these degenerate on the sonic arc. Also, unlike the reflected-diffracted shock, the sonic arc is not a free boundary; its location is explicitly known. In order to construct a global regular reflection-diffraction configuration, we need to determine the unknown velocity potential in the subsonic (elliptic) domain such that the reflected-diffracted shock and the sonic arc are parts of its boundary. Thus, we can view our problem as a free boundary problem for an elliptic equation of second order with ellipticity degenerating along a part of the fixed boundary. Moreover, the solution should satisfy two Rankine-Hugoniot conditions on the transition boundary of the elliptic region, which includes both the shock and the sonic arc. While this over-determinacy gives the correct number of free boundary conditions on the shock, the situation is different on the sonic arc that is a fixed boundary. Normally, only one condition may be prescribed for the elliptic problem. Therefore, we have to prove that the other condition is also satisfied on the sonic arc by the solution. To achieve this, we exploit the detailed structure of the elliptic degeneracy of the nonlinear PDE to make careful estimates of the solution near the sonic arc in the properly weighted and scaled $C^{2,\alpha}$–spaces, for which the nonlinearity plays a crucial role.

(iv) Corner Singularities: Further difficulties include the singularities of solutions at the corner formed by the reflected-diffracted shock (free boundary) and the sonic arc (degenerate elliptic curve), at the wedge vertex, as well as at the corner between the reflected shock and the wedge at the reflection point for the transition from the supersonic to subsonic regular reflection-diffraction configurations when the wedge angle decreases. For the latter, it requires uniform *a priori* estimates for the solutions as the sonic arc shrinks to a point; the degenerate ellipticity then changes to the uniform ellipticity when the wedge angle decreases across the sonic angle up to the detachment angle, as described in §2.4–§2.6.

These difficulties also arise in many further fundamental problems in continuum physics (fluid/solid), differential geometry, mathematical physics, materials science, and other areas, such as transonic flow problems, isometric embedding problems, and phase transition problems; see [9, 10, 16, 93, 68, 69, 95, 99, 139, 147, 168, 181, 220, 270, 286] and the references cited therein. Therefore, any progress in shock reflection-diffraction analysis requires new mathematical ideas, approaches, and techniques, all of which will be very useful for solving other problems with similar difficulties and open up new research opportunities.

We focus mainly on the mathematics of shock reflection-diffraction and von Neumann's conjectures for potential flow, as well as offering (in Parts I–IV) new analysis to overcome the associated difficulties. The mathematical approaches and techniques developed here will be useful in tackling other nonlinear problems involving similar difficulties. One of the recent examples of this is the Prandtl-Meyer problem for supersonic flow impinging onto solid wedges, another longstanding open problem in mathematical fluid mechanics, which has been treated in Bae-Chen-Feldman [5, 6].

In Part I, we state our main results and give an overview of the main steps of their proofs.

In Part II, we present some relevant results for nonlinear elliptic equations of second order (for which the structural conditions and some regularity of coefficients are not required), convenient for applications in the rest of the book, and study the existence and regularity of solutions of certain boundary value problems in the domains of appropriate structure for an equation with ellipticity degenerating on a part of the boundary, which include the boundary value problems used in the construction of the iteration map in the later chapters. We also present basic properties of the self-similar potential flow equation, with focus on the two-dimensional case.

In Part III, we first focus on von Neumann's sonic conjecture – that is, the conjecture concerning the existence of regular reflection-diffraction solutions up to the sonic angle, with a supersonic shock reflection-diffraction configuration containing a transonic reflected-diffracted shock, and then provide its whole detailed proof and related analysis. We treat this first on account of the fact that the presentation in this case is both foundational and relatively simpler than that in the case beyond the sonic angle.

Once the analysis for the sonic conjecture is done, we present, in Part IV, our proof of von Neumann's detachment conjecture – that is, the conjecture concerning the existence of regular reflection-diffraction solutions, even beyond the sonic angle up to the detachment angle, with a subsonic shock reflection-diffraction configuration containing a transonic reflected-diffracted shock. This is more technically involved. To achieve it, we make the whole iteration again, starting from the normal reflection when the wedge angle is $\frac{\pi}{2}$, and prove the results for both the supersonic and subsonic regular reflection-diffraction configurations by going over the previous arguments with the necessary additions (instead of writing all the details of the proof up to the detachment angle from the beginning). We present the proof in this way to make it more readable.

In Part V, we present the mathematical formulation of the shock reflection-diffraction problem for the full Euler equations and uncover the role of the potential flow equation for the shock reflection-diffraction even in the realm of the full Euler equations. We also discuss further connections and their roles in developing new mathematical ideas, techniques, and approaches for solving further open problems in related scientific areas.

Chapter Two

Mathematical Formulations and Main Theorems

In this chapter, we first analyze the potential flow equation (1.5) and its planar shock-front solutions, and then formulate the shock reflection-diffraction problem into an initial-boundary value problem. Next we employ the self-similarity of the problem to reformulate the initial-boundary value problem into a boundary value problem in the self-similar coordinates. To solve von Neumann's conjectures, we further reformulate the boundary value problem into a free boundary problem for a nonlinear second-order conservation law of mixed hyperbolic-elliptic type. Finally, we present the main theorems for the existence, regularity, and stability of regular reflection-diffraction solutions of the free boundary problem.

2.1 THE POTENTIAL FLOW EQUATION

The time-dependent potential flow equation of second order for the velocity potential Φ takes the form of (1.5) with $\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi)$ determined by (1.4), which is a nonlinear wave equation.

Definition 2.1.1. *A function $\Phi \in W^{1,1}_{\rm loc}(\mathbb{R}_+ \times \mathbb{R}^2)$ is called a weak solution of equation* (1.5) *in a domain $\mathcal{D} \subset \mathbb{R}_+ \times \mathbb{R}^2$ if Φ satisfies the following properties*:

(i) $B_0 - \big(\partial_t\Phi + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2\big) \geq h(0+)$ *a.e. in* $\mathcal{D}$;

(ii) $(\rho(\partial_t\Phi, |\nabla_{\mathbf{x}}\Phi|^2), \rho(\partial_t\Phi, |\nabla_{\mathbf{x}}\Phi|^2)|\nabla_{\mathbf{x}}\Phi|) \in (L^1_{\rm loc}(\mathcal{D}))^2$;

(iii) *For every $\zeta \in C^\infty_c(\mathcal{D})$,*

$$\int_{\mathcal{D}} \Big(\rho(\partial_t\Phi, |\nabla_{\mathbf{x}}\Phi|^2)\partial_t\zeta + \rho(\partial_t\Phi, |\nabla_{\mathbf{x}}\Phi|^2)\nabla_{\mathbf{x}}\Phi\cdot\nabla_{\mathbf{x}}\zeta\Big)\, d\mathbf{x}dt = 0.$$

In the study of a piecewise smooth weak solution of (1.5) with jump for $(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi)$ across an oriented surface $\mathcal{S}$ with unit normal $\mathbf{n} = (n_t, \mathbf{n}_{\mathbf{x}}), \mathbf{n}_{\mathbf{x}} = (n_1, n_2)$, in the $(t, \mathbf{x})$–coordinates, the requirement of the weak solution of (1.5) in the sense of Definition 2.1.1 yields the *Rankine-Hugoniot jump condition* across $\mathcal{S}$:

$$[\rho(\partial_t\Phi, |\nabla_{\mathbf{x}}\Phi|^2)]n_t + [\rho(\partial_t\Phi, |\nabla_{\mathbf{x}}\Phi|^2)\nabla_{\mathbf{x}}\Phi]\cdot\mathbf{n}_{\mathbf{x}} = 0, \tag{2.1.1}$$

where the square bracket, $[w]$, denotes the jump of quantity w across the oriented surface $\mathcal{S}$; that is, assuming that $\mathcal{S}$ subdivides $\mathcal{D}$ into subregions $\mathcal{D}^+$ and $\mathcal{D}^-$

so that, for every $(t, \mathbf{x}) \in \mathcal{S}$, there exists $\varepsilon > 0$ such that $(t, \mathbf{x}) \pm s\mathbf{n} \in \mathcal{D}^{\pm}$ if $s \in (0, \varepsilon)$, define

$$[w](t, \mathbf{x}) := \lim_{\substack{(\tau, \mathbf{y}) \to (t, \mathbf{x}) \\ (\tau, \mathbf{y}) \in \mathcal{D}^{+}}} w(\tau, \mathbf{y}) - \lim_{\substack{(\tau, \mathbf{y}) \to (t, \mathbf{x}) \\ (\tau, \mathbf{y}) \in \mathcal{D}^{-}}} w(\tau, \mathbf{y}).$$

Notice that $\Phi \in W^{1,1}$ is required in Definition 2.1.1, which implies the continuity of Φ across a shock-front $\mathcal{S}$ for piecewise smooth solutions:

$$[\Phi]_{\mathcal{S}} = 0. \tag{2.1.2}$$

In fact, the continuity of Φ in (2.1.2) can also be derived for the piecewise smooth solution Φ (without assumption $\Phi \in W^{1,1}$) by requiring that Φ keep the validity of the equations:

$$\nabla_{\mathbf{x}} \times \mathbf{v} = 0, \quad \partial_t \mathbf{v} = \nabla_{\mathbf{x}}(\partial_t \Phi) \tag{2.1.3}$$

in the sense of distributions. This is tantamount to requiring that

$$(\partial_t \Phi, \mathbf{v}) = (\partial_t \Phi, \nabla_{\mathbf{x}} \Phi). \tag{2.1.4}$$

The condition on $\mathbf{v}$ is that

$$(\partial_t, \nabla_{\mathbf{x}}) \times (\partial_t \Phi, \mathbf{v}) = 0,$$

which is equivalent to (2.1.3). By definition, the equations in (2.1.4) for piecewise smooth solutions are understood as

$$\iint \Phi \partial_{x_i} \psi \, dt d\mathbf{x} = -\iint v_i \psi \, dt d\mathbf{x}, \qquad i = 1, 2,$$

for any test function $\psi \in C_0^\infty((0, \infty) \times \mathbb{R}^2)$. Using the Gauss-Green formula in the two regions of continuity of $(\partial_t \Phi, \nabla_{\mathbf{x}} \Phi)$ separated by $\mathcal{S}$ in the standard fashion, we obtain

$$\int_{\mathcal{S}} [\Phi] \psi \, d\sigma = 0,$$

where $d\sigma$ is the surface measure on $\mathcal{S}$. This implies the continuity of Φ across a shock-front $\mathcal{S}$ in (2.1.2).

The discontinuity $\mathcal{S}$ of $(\partial_t \Phi, \nabla_{\mathbf{x}} \Phi)$ is called a *shock* if Φ further satisfies the physical *entropy condition*: The corresponding density function $\rho(\partial_t \Phi, \nabla_{\mathbf{x}} \Phi)$ increases across $\mathcal{S}$ in the relative flow direction with respect to $\mathcal{S}$ (*cf.* [94, 99]).

Definition 2.1.2. *Let Φ be a piecewise smooth weak solution of* (1.5) *with jump for $(\partial_t \Phi, \nabla_{\mathbf{x}} \Phi)$ across an oriented surface $\mathcal{S}$. The discontinuity $\mathcal{S}$ of $(\partial_t \Phi, \nabla_{\mathbf{x}} \Phi)$ is called a shock if Φ further satisfies the physical entropy condition: The corresponding density function $\rho(\partial_t \Phi, \nabla_{\mathbf{x}} \Phi)$ increases across $\mathcal{S}$ in the relative flow direction with respect to $\mathcal{S}$* (*cf.* [94, 99]).

The jump condition in (2.1.1) from the conservation of mass and the continuity of Φ in (2.1.2) are the conditions that are actually used in practice, especially in aerodynamics, resulting from the *Rankine-Hugoniot conditions* for the time-dependent potential flow equation (1.5). The empirical evidence for this is that entropy solutions of (1.1) or (1.5) are fairly close to the corresponding entropy solutions of the full Euler equations, provided that the strengths of shock-fronts are small, the curvatures of shock-fronts are not too large, and the amount of vorticity is small in the region of interest. In fact, for the solutions containing a weak shock, especially in aerodynamic applications, the potential flow equation (1.5) and the full Euler flow model (1.7) match each other well up to the third order of the shock strength. Furthermore, we will show in Chapter 18 that, for the shock reflection-diffraction problem, the Euler equations (1.1) for potential flow are actually an *exact* match in an important region of the shock reflection-diffraction configurations to the full Euler equations (1.7). See also Bers [16], Glimm-Majda [139], and Morawetz [220, 221, 222].

Planar shock-front solutions are special piecewise smooth solutions given by the explicit formulae:

$$\Phi = \begin{cases} \Phi^+, & x_1 > st, \\ \Phi^-, & x_1 < st \end{cases} \tag{2.1.5}$$

with

$$\Phi^\pm = a_0^\pm t + u^\pm x_1 + v^\pm x_2. \tag{2.1.6}$$

Then the continuity condition (2.1.2) of Φ across the shock-front implies

$$[v] = 0, \qquad [a_0] + s[u] = 0. \tag{2.1.7}$$

The jump condition (2.1.1) yields

$$s[\rho] = [\rho u], \tag{2.1.8}$$

since $\mathbf{n} = \frac{1}{\sqrt{s^2+1}}(-s, 1, 0)$.

The relation between ρ and Φ via the Bernoulli law is

$$[a_0 + \frac{1}{2}u^2] + \frac{1}{\gamma - 1}[\rho^{\gamma-1}] = 0, \tag{2.1.9}$$

where we have used (1.2) for polytropic gases. From now on, we focus on $\gamma > 1$.

Combining (2.1.7)–(2.1.9), we conclude

$$\begin{cases} [u] = -\sqrt{\dfrac{2[\rho][\rho^{\gamma-1}]}{(\gamma-1)(\rho^+ + \rho^-)}}, \\ [a_0] = -\dfrac{[\rho u][u]}{[\rho]}, \\ s = \dfrac{[\rho u]}{[\rho]}. \end{cases} \tag{2.1.10}$$

This implies that the shock speed s is

$$s = u_+ + \rho_- \sqrt{\frac{2[\rho^{\gamma-1}]}{(\gamma-1)[\rho^2]}}. \tag{2.1.11}$$

The entropy condition is

$$\rho_+ < \rho_- \qquad \text{if } u^{\pm} > 0. \tag{2.1.12}$$

2.2 MATHEMATICAL PROBLEMS FOR SHOCK REFLECTION-DIFFRACTION

When a plane shock in the $(t, \mathbf{x})$–coordinates, $t \in \mathbb{R}_+ := [0, \infty), \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, with left state $(\rho, \nabla_{\mathbf{x}}\Phi) = (\rho_1, u_1, 0)$ and right state $(\rho_0, 0, 0), u_1 > 0$, $\rho_0 < \rho_1$, hits a symmetric wedge

$$W := \{\mathbf{x} \,:\, |x_2| < x_1 \tan\theta_{\mathrm{w}}, x_1 > 0\}$$

head-on (see Fig. 1.1), it experiences a reflection-diffraction process. Then system (1.1) in $\mathbb{R}_+ \times (\mathbb{R}^2 \setminus W)$ becomes

$$\begin{cases} \partial_t \rho + \mathrm{div}_{\mathbf{x}}(\rho \nabla_{\mathbf{x}}\Phi) = 0, \\ \partial_t \Phi + \dfrac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2 + \dfrac{\rho^{\gamma-1} - \rho_0^{\gamma-1}}{\gamma - 1} = 0, \end{cases} \tag{2.2.1}$$

where we have used the Bernoulli constant $B_0 = \frac{\rho_0^{\gamma-1}-1}{\gamma-1}$ determined by the right state $(\rho_0, 0, 0)$. From (2.1.10), we find that $u_1 > 0$ is uniquely determined by (ρ_0, ρ_1) and $\gamma > 1$:

$$u_1 = \sqrt{\frac{2(\rho_1 - \rho_0)(\rho_1^{\gamma-1} - \rho_0^{\gamma-1})}{(\gamma-1)(\rho_1+\rho_0)}} > 0, \tag{2.2.2}$$

where we have used that $\rho_0 < \rho_1$.

Then the shock reflection-diffraction problem can be formulated as the following problem:

Problem 2.2.1 (Initial-Boundary Value Problem). *Seek a solution of system* (2.2.1) *for* $B_0 = \frac{\rho_0^{\gamma-1}-1}{\gamma-1}$ *with the initial condition at* $t = 0$:

$$(\rho, \Phi)|_{t=0} = \begin{cases} (\rho_0, 0) & \textit{for } |x_2| > x_1 \tan\theta_{\mathrm{w}},\ x_1 > 0, \\ (\rho_1, u_1 x_1) & \textit{for } x_1 < 0, \end{cases} \tag{2.2.3}$$

and the slip boundary condition along the wedge boundary ∂W:

$$\nabla_{\mathbf{x}}\Phi \cdot \boldsymbol{\nu}|_{\mathbb{R}_+ \times \partial W} = 0, \tag{2.2.4}$$

where $\boldsymbol{\nu}$ *is the exterior unit normal to* ∂W *(see Fig.* 2.1*).*

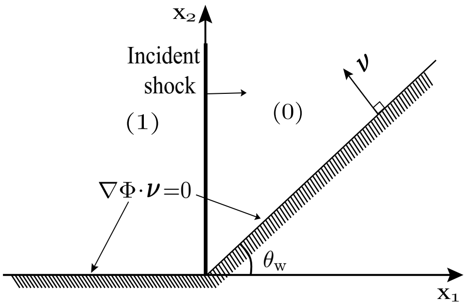


Figure 2.1: Initial-boundary value problem

Notice that the initial-boundary value problem (**Problem 2.2.1**) is invariant under the self-similar scaling:

$$(t,\mathbf{x})\mapsto(\alpha t,\alpha\mathbf{x}),\quad (\rho,\Phi)\mapsto(\rho,\frac{\Phi}{\alpha})\qquad \text{for}\quad \alpha\neq 0. \tag{2.2.5}$$

That is, if $(\rho,\Phi)(t,\mathbf{x})$ satisfy (2.2.1)–(2.2.4), so do $(\rho,\frac{\Phi}{\alpha})(\alpha t,\alpha\mathbf{x})$ for any constant $\alpha\neq 0$.

Therefore, we seek self-similar solutions with the following form:

$$\rho(t,\mathbf{x})=\rho(\boldsymbol{\xi}),\quad \Phi(t,\mathbf{x})=t\,\phi(\boldsymbol{\xi})\qquad \text{for}\quad \boldsymbol{\xi}=(\xi_1,\xi_2)=\frac{\mathbf{x}}{t}. \tag{2.2.6}$$

We then see that the pseudo-potential function $\varphi=\phi-\frac{|\boldsymbol{\xi}|^2}{2}$ satisfies the following Euler equations for self-similar solutions:

$$\begin{cases} \operatorname{div}(\rho\,D\varphi)+2\rho=0,\\ (\gamma-1)(\frac{1}{2}|D\varphi|^2+\varphi)+\rho^{\gamma-1}=\rho_0^{\gamma-1}, \end{cases} \tag{2.2.7}$$

where div and D represent the divergence and the gradient, respectively, with respect to the self-similar variables $\boldsymbol{\xi}$.

This implies that the pseudo-potential function $\varphi(\boldsymbol{\xi})$ is governed by the following potential flow equation of second order:

$$\operatorname{div}\big(\rho(|D\varphi|^2,\varphi)D\varphi\big)+2\rho(|D\varphi|^2,\varphi)=0 \tag{2.2.8}$$

with

$$\rho(|D\varphi|^2,\varphi)=\big(\rho_0^{\gamma-1}-(\gamma-1)(\varphi+\frac{1}{2}|D\varphi|^2)\big)^{\frac{1}{\gamma-1}}. \tag{2.2.9}$$

We consider (2.2.8) with (2.2.9) for functions φ satisfying

$$\rho_0^{\gamma-1}-(\gamma-1)\big(\varphi+\frac{1}{2}|D\varphi|^2\big)\geq 0. \tag{2.2.10}$$

Definition 2.2.2. *A function* $\varphi \in W^{1,1}_{\rm loc}(\Omega)$ *is called a weak solution of equation* (2.2.8) *in domain* Ω *if* φ *satisfies* (2.2.10) *and the following properties*:

(i) *For* $\rho(|D\varphi|^2, \varphi)$ *determined by* (2.2.9),

$$(\rho(|D\varphi|^2, \varphi), \rho(|D\varphi|^2, \varphi)|D\varphi|) \in (L^1_{loc}(\overline{\Omega}))^2;$$

(ii) *For every* $\zeta \in C^\infty_c(\Omega)$,

$$\int_\Omega \Big(\rho(|D\varphi|^2, \varphi)D\varphi \cdot D\zeta - 2\rho(|D\varphi|^2, \varphi)\zeta\Big)\, d\boldsymbol{\xi} = 0.$$

We will also use the non-divergence form of equation (2.2.8) for $\phi = \varphi + \frac{|\boldsymbol{\xi}|^2}{2}$:

$$(c^2 - \varphi^2_{\xi_1})\phi_{\xi_1\xi_1} - 2\varphi_{\xi_1}\varphi_{\xi_2}\phi_{\xi_1\xi_2} + (c^2 - \varphi^2_{\xi_2})\phi_{\xi_2\xi_2} = 0, \tag{2.2.11}$$

where the sonic speed $c = c(|D\varphi|^2, \varphi, \rho_0^{\gamma-1})$ is determined by (1.14).

Equation (2.2.8) or (2.2.11) is a nonlinear second-order PDE of mixed elliptic-hyperbolic type. It is elliptic if and only if (1.18) holds, which is equivalent to the following condition:

$$|D\varphi| < c_*(\varphi, \rho_0, \gamma) := \sqrt{\frac{2}{\gamma+1}\big(\rho_0^{\gamma-1} - (\gamma-1)\varphi\big)}. \tag{2.2.12}$$

Throughout the rest of this book, for simplicity, we drop term "pseudo" and simply call φ as a *potential function* and $D\varphi$ as a *velocity*, respectively, when no confusion arises.

Shocks are discontinuities in the velocity functions $D\varphi$. That is, if Ω^+ and $\Omega^- := \Omega \setminus \overline{\Omega^+}$ are two non-empty open subsets of $\Omega \subset \mathbb{R}^2$, and $\mathcal{S} := \partial\Omega^+ \cap \Omega$ is a C^1–curve where $D\varphi$ has a jump, then $\varphi \in W^{1,1}_{\rm loc}(\Omega) \cap C^1(\overline{\Omega^\pm}) \cap C^2(\Omega^\pm)$ is a global weak solution of (2.2.8) in Ω in the sense of Definition 2.2.2 if and only if φ is in $W^{1,\infty}_{\rm loc}(\Omega)$ and satisfies equation (2.2.8) in $\Omega^\pm$ and the Rankine-Hugoniot condition on $\mathcal{S}$:

$$\big[\rho(|D\varphi|^2, \varphi)D\varphi \cdot \boldsymbol{\nu}\big]_{\mathcal{S}} = 0. \tag{2.2.13}$$

Note that the condition that $\varphi \in W^{1,\infty}_{\rm loc}(\Omega)$ implies the continuity of φ across shock $\mathcal{S}$:

$$[\varphi]_{\mathcal{S}} = 0. \tag{2.2.14}$$

The plane incident shock solution in the $(t, \mathbf{x})$–coordinates with the left and right states:

$$(\rho, \nabla_{\mathbf{x}}\Phi) = (\rho_0, 0, 0),\ \ (\rho_1, u_1, 0)$$

corresponds to a continuous weak solution φ of (2.2.8) in the self-similar coordinates $\boldsymbol{\xi}$ with the following form:

$$\varphi = \begin{cases} \varphi_0 & \text{for } \xi_1 > \xi_1^0, \\ \varphi_1 & \text{for } \xi_1 < \xi_1^0, \end{cases} \tag{2.2.15}$$

where

$$\varphi_0(\boldsymbol{\xi}) = -\frac{|\boldsymbol{\xi}|^2}{2}, \tag{2.2.16}$$

$$\varphi_1(\boldsymbol{\xi}) = -\frac{|\boldsymbol{\xi}|^2}{2} + u_1(\xi_1 - \xi_1^0), \tag{2.2.17}$$

and

$$\xi_1^0 = \rho_1 \sqrt{\frac{2(\rho_1^{\gamma-1} - \rho_0^{\gamma-1})}{(\gamma-1)(\rho_1^2 - \rho_0^2)}} = \frac{\rho_1 u_1}{\rho_1 - \rho_0} > 0 \tag{2.2.18}$$

is the location of the incident shock in the $\boldsymbol{\xi}$–coordinates, uniquely determined by (ρ_0, ρ_1, γ) through (2.2.13), which is obtained from (2.1.11) and (2.2.2) owing to the fact that $\xi_1^0 = s$ here. Since the problem is symmetric with respect to the ξ_1–axis, it suffices to consider the problem in half-plane $\xi_2 > 0$ outside the half-wedge:

$$\Lambda := \{\boldsymbol{\xi} \;:\; \xi_1 \in \mathbb{R},\, \xi_2 > \max(\xi_1 \tan\theta_{\mathrm{w}},\, 0)\}. \tag{2.2.19}$$

Then the initial-boundary value problem (2.2.1)–(2.2.4) in the $(t, \mathbf{x})$–coordinates can be formulated as a boundary value problem in the self-similar coordinates $\boldsymbol{\xi}$.

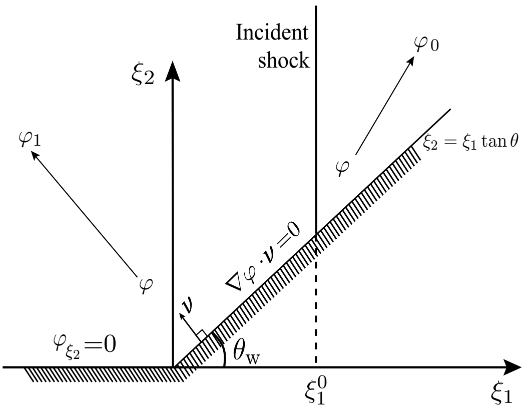


Figure 2.2: Boundary value problem

Problem 2.2.3 (Boundary Value Problem; see Fig. 2.2)**.** *Seek a solution φ of equation* (2.2.8) *in the self-similar domain Λ with the slip boundary condition*:

$$D\varphi \cdot \boldsymbol{\nu}|_{\partial\Lambda} = 0 \tag{2.2.20}$$

and the asymptotic boundary condition at infinity:

$$\varphi \to \bar{\varphi} = \begin{cases} \varphi_0 & \text{for } \boldsymbol{\xi} \in \Lambda,\ \xi_1 > \xi_1^0, \\ \varphi_1 & \text{for } \boldsymbol{\xi} \in \Lambda,\ \xi_1 < \xi_1^0, \end{cases} \qquad \text{when } |\boldsymbol{\xi}| \to \infty, \tag{2.2.21}$$

where (2.2.21) *holds in the sense that* $\lim_{R\to\infty} \|\varphi - \overline{\varphi}\|_{C^{0,1}(\Lambda\setminus B_R(\mathbf{0}))} = 0$.

This is a boundary value problem for the second-order nonlinear conservation law (2.2.8) of mixed elliptic-hyperbolic type in an unbounded domain. The main feature of this boundary value problem is that $D\varphi$ has a jump at $\xi_1 = \xi_1^0$ at infinity, which is not conventional, coupling with the wedge corner for the domain. The solutions with complicated patterns of wave configurations as observed experimentally should be the global solutions of this boundary value problem: **Problem 2.2.3**.

2.3 WEAK SOLUTIONS OF PROBLEM 2.2.1 AND PROBLEM 2.2.3

Note that the boundary condition (2.2.20) for **Problem 2.2.3** implies

$$\rho D\varphi \cdot \boldsymbol{\nu}|_{\partial\Lambda} = 0. \tag{2.3.1}$$

Conditions (2.2.20) and (2.3.1) are equivalent if $\rho \neq 0$. Since $\rho \neq 0$ for the solutions under consideration, we use condition (2.3.1) instead of (2.2.20) in the definition of weak solutions of **Problem 2.2.3**.

Similarly, we write the boundary condition (2.2.4) for **Problem 2.2.1** as

$$\rho \nabla_{\mathbf{x}}\Phi \cdot \boldsymbol{\nu}|_{\mathbb{R}_+\times\partial W} = 0. \tag{2.3.2}$$

Condition (2.3.1) is the conormal condition for equation (2.2.8). Also, (2.3.2) is the spatial conormal condition for equation (1.5). This yields the following definitions:

Definition 2.3.1 (Weak Solutions of **Problem 2.2.1**). *A function*

$$\Phi \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+ \times (\mathbb{R}^2 \setminus W))$$

is called a weak solution of **Problem 2.2.1** *if* Φ *satisfies the following properties*:

(i) $B_0 - \big(\partial_t\Phi + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2\big) \geq h(0+)$ *a.e. in* $\mathbb{R}_+ \times (\mathbb{R}^2 \setminus W)$;

(ii) *For* $\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi)$ *determined by* (1.4),

$$(\rho(\partial_t\Phi, |\nabla_{\mathbf{x}}\Phi|^2), \rho(\partial_t\Phi, |\nabla_{\mathbf{x}}\Phi|^2)|\nabla_{\mathbf{x}}\Phi|) \in (L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}^2 \setminus W}))^2;$$

(iii) *For every* $\zeta \in C_c^\infty(\overline{\mathbb{R}_+} \times \mathbb{R}^2)$,

$$\int_0^\infty \int_{\mathbb{R}^2\setminus W} \Big(\rho(\partial_t\Phi, |\nabla_{\mathbf{x}}\Phi|^2)\partial_t\zeta + \rho(\partial_t\Phi, |\nabla_{\mathbf{x}}\Phi|^2)\nabla\Phi\cdot\nabla\zeta\Big)\, d\mathbf{x}dt$$
$$+ \int_{\mathbb{R}^2\setminus W} \rho|_{t=0}\zeta(0,\mathbf{x})d\mathbf{x} = 0,$$

where

$$\rho|_{t=0} = \begin{cases} \rho_0 & \text{for } |x_2| > x_1 \tan\theta_{\mathrm{w}},\ x_1 > 0, \\ \rho_1 & \text{for } x_1 < 0. \end{cases}$$

Remark 2.3.2. *Since ζ does not need to be zero on ∂W, the integral identity in Definition* 2.3.1 *is a weak form of equation* (1.5) *and the boundary condition* (2.3.2).

Definition 2.3.3 (Weak solutions of **Problem 2.2.3**). *A function $\varphi \in W^{1,1}_{\mathrm{loc}}(\Lambda)$ is called a weak solution of* **Problem 2.2.3** *if φ satisfies* (2.2.21) *and the following properties*:

(i) $\rho_0^{\gamma-1} - (\gamma-1)\left(\varphi + \frac{1}{2}|D\varphi|^2\right) \geq 0$ *a.e. in* Λ;

(ii) *For $\rho(|D\varphi|^2, \varphi)$ determined by* (2.2.9),

$$(\rho(|D\varphi|^2, \varphi), \rho(|D\varphi|^2, \varphi)|D\varphi|) \in (L^1_{loc}(\overline{\Lambda}))^2;$$

(iii) *For every $\zeta \in C_c^\infty(\mathbb{R}^2)$,*

$$\int_\Lambda \Big(\rho(|D\varphi|^2, \varphi) D\varphi \cdot D\zeta - 2\rho(|D\varphi|^2, \varphi)\zeta\Big)\, d\boldsymbol{\xi} = 0.$$

Remark 2.3.4. *Since ζ does not need to be zero on $\partial\Lambda$, the integral identity in Definition* 2.3.3 *is a weak form of equation* (2.2.8) *and the boundary condition* (2.3.1).

Remark 2.3.5. *From Definition* 2.3.3, *we observe the following fact*: *If $B \subset \mathbb{R}^2$ is an open set, and φ is a weak solution of* **Problem 2.2.3** *satisfying $\varphi \in C^2(B \cap \Lambda) \cap C^1(B \cap \overline{\Lambda})$, then φ satisfies equation* (2.2.8) *in the classical sense in $B \cap \Lambda$, the boundary condition* (2.3.1) *on $B \cap \partial\Lambda \setminus \{\mathbf{0}\}$, and $D\varphi(\mathbf{0}) = \mathbf{0}$.*

2.4 STRUCTURE OF SOLUTIONS: REGULAR REFLECTION-DIFFRACTION CONFIGURATIONS

We now discuss the structure of solutions φ of **Problem 2.2.3** corresponding to shock reflection-diffraction.

Since φ_1 does not satisfy the slip boundary condition (2.2.20), the solution must differ from φ_1 in $\{\xi_1 < \xi_1^0\} \cap \Lambda$ so that a shock diffraction by the wedge occurs. We now describe two of the most important configurations: the supersonic and subsonic regular reflection-diffraction configurations, as shown in Fig. 2.3 and Fig. 2.4, respectively. From now on, we will refer to these two configurations as a *supersonic reflection configuration* and a *subsonic reflection configuration* respectively, whose corresponding solutions are called the *supersonic reflection solution* and the *subsonic reflection solution* respectively, when no confusion arises.

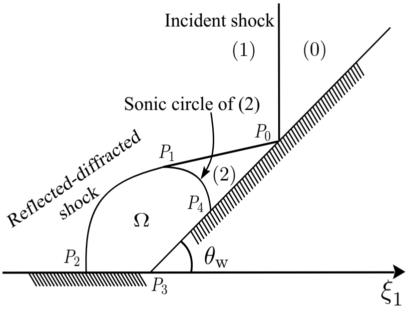


Figure 2.3: Supersonic regular reflection-diffraction configuration

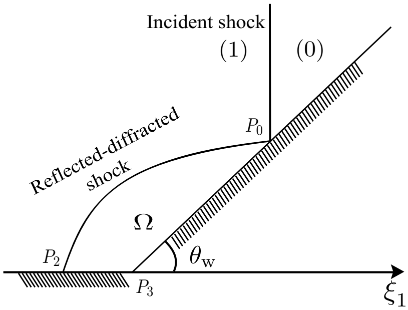


Figure 2.4: Subsonic regular reflection-diffraction configuration

In Figs. 2.3 and 2.4, the vertical line is the incident shock $\mathcal{S}_0 = \{\xi_1 = \xi_1^0\}$ that hits the wedge at point $P_0 = (\xi_1^0, \xi_1^0 \tan\theta_{\rm w})$, and state (0) and state (1), ahead of and behind $\mathcal{S}_0$, are given by φ_0 and φ_1 defined in (2.2.16) and (2.2.17), respectively. Thus, we only need to describe the solution in subregion $P_0P_2P_3$ between the wedge and the reflected-diffracted shock. The solution is expected to be C^1 in $P_0P_2P_3$. Now we describe its structure. Below, φ denotes the potential of the solution in $P_0P_2P_3$, while we use the uniform states φ_0 and φ_1 to describe the solution outside $P_0P_2P_3$, ahead of and behind the incident shock $\mathcal{S}_1$, respectively. In particular, $D\varphi(P_0)$ denotes the limit at P_0 of the gradient of the solution in $P_0P_2P_3$.

2.4.1 Definition of state (2)

Since φ is C^1 in region $P_0P_2P_3$, it should satisfy the boundary condition $D\varphi\cdot\boldsymbol{\nu} = 0$ on the wedge boundary P_0P_3 including the endpoints, as well as the Rankine-Hugoniot conditions (2.2.13)–(2.2.14) at P_0 across the reflected shock separating φ from φ_1. Let

$$(u_2, v_2) := D\varphi(P_0) + \boldsymbol{\xi}_{P_0}.$$

Then $v_2 = u_2 \tan\theta_{\rm w}$ by $D\varphi\cdot\boldsymbol{\nu} = 0$. Moreover, using (2.2.17), in addition to the previous properties, we see by a direct calculation that the uniform state with the pseudo-potential:

$$\varphi_2(\boldsymbol{\xi}) = -\frac{|\boldsymbol{\xi}|^2}{2} + u_2(\xi_1 - \xi_1^0) + u_2\tan\theta_{\rm w}(\xi_2 - \xi_1^0\tan\theta_{\rm w}), \tag{2.4.1}$$

called *state* (2), satisfies the boundary condition on the wedge boundary:

$$D\varphi_2\cdot\boldsymbol{\nu} = 0 \qquad \text{on } \partial\Lambda \cap \{\xi_1 = \xi_2\cot\theta_{\rm w}\}, \tag{2.4.2}$$

and the Rankine-Hugoniot conditions (2.2.13) on the flat shock $\mathcal{S}_1$ determined by (2.2.14):

$$\mathcal{S}_1 := \{\varphi_1 = \varphi_2\} \tag{2.4.3}$$

which passes through P_0 between states (1) and (2).

We note that the constant velocity $u_2 > 0$ is determined by $(\rho_0, \rho_1, \gamma, \theta_{\mathrm{w}})$ from the algebraic equation expressing (2.2.13) for φ_1 and φ_2 across $\mathcal{S}_1$, where we have used (2.2.2) to eliminate u_1 from the list of parameters and have noted that $\boldsymbol{\nu}_{\mathcal{S}_1} = \frac{(u_1-u_2,-u_2\tan\theta_{\mathrm{w}})}{|(u_1-u_2,-u_2\tan\theta_{\mathrm{w}})|}$.

Thus, state (2) is defined by the following requirements: It is a uniform state with pseudo-potential $\varphi_2(\boldsymbol{\xi})$ such that $D\varphi_2 \cdot \boldsymbol{\nu} = 0$ on the wedge boundary, and the Rankine-Hugoniot conditions (2.2.13)–(2.2.14) for φ_1 and φ_2 hold at P_0. As we will discuss in §2.5, such a state (2) exists for any wedge angle $\theta_{\mathrm{w}} \in [\theta_{\mathrm{w}}^{\mathrm{d}}, \frac{\pi}{2}]$, for some $\theta_{\mathrm{w}}^{\mathrm{d}} = \theta_{\mathrm{w}}^{\mathrm{d}}(\rho_0, \rho_1, \gamma) > 0$. From the discussion above, it is apparent that the existence of state (2) is a necessary condition for the existence of regular reflection-diffraction configurations as shown in Figs. 2.3–2.4.

From now on, we fix the data, *i.e.*, parameters (ρ_0, ρ_1, γ). Thus, the parameters of state (2) depend only on θ_{w}.

State (2) can be either pseudo-subsonic or pseudo-supersonic at P_0. This determines the subsonic or supersonic type of regular reflection-diffraction configurations, as shown in Figs. 2.3–2.4.

We note that the uniform state (2) is pseudo-subsonic within its sonic circle with center $\mathcal{O}_2 = (u_2, u_2\tan\theta_{\mathrm{w}})$ and radius $c_2 = \rho_2^{(\gamma-1)/2} > 0$, the sonic speed of state (2), and that φ_2 is pseudo-supersonic outside this circle.

Thus, if state (2) is pseudo-supersonic at P_0, P_0 lies outside the sonic circle $B_{c_2}(\mathcal{O}_2)$ of state (2). It can be shown (see §7.5) that line $\mathcal{S}_1$ intersects $\partial B_{c_2}(\mathcal{O}_2)$ at two points and, denoting by P_1 the point that is closer to P_0, we find that P_1 lies in Λ and segment P_0P_1 lies in Λ and outside of $B_{c_2}(u_2, v_2)$; see Fig. 2.3. Denote by P_4 the point of intersection of $\partial B_{c_2}(\mathcal{O}_2)$ with the wedge boundary $\{\xi_2 = \xi_1\tan\theta_{\mathrm{w}},\ \xi_1 > 0\}$ such that arc P_1P_4 lies between $\mathcal{S}_1$ and $\{\xi_2 = \xi_1\tan\theta_{\mathrm{w}},\ \xi_1 > 0\}$.

2.4.2 Supersonic regular reflection-diffraction configurations

This is the case when state (2) is supersonic at P_0.

The supersonic reflection configuration as shown in Fig. 2.3 consists of three uniform states: (0), (1), (2), plus a non-uniform state in domain $\Omega = P_1P_2P_3P_4$. As described above, the solution is equal to state (0) and state (1) ahead of and behind the incident shock $\mathcal{S}_0$, away from subregion $P_0P_2P_3$. The solution is equal to state (2) in subregion $P_0P_1P_4$. Note that state (2) is supersonic in $P_0P_1P_4$.

The non-uniform state in Ω is subsonic, *i.e.*, the potential flow equation (2.2.8) for φ is elliptic in Ω.

We denote the boundary parts of Ω by

$$\Gamma_{\mathrm{shock}} := P_1P_2,\ \ \Gamma_{\mathrm{sym}} := P_2P_3,\ \ \Gamma_{\mathrm{wedge}} := P_3P_4,\ \ \Gamma_{\mathrm{sonic}} := P_1P_4, \tag{2.4.4}$$

where Γ_{shock} is the curved part of the reflected shock, Γ_{sonic} is the sonic arc, and Γ_{wedge} (the wedge boundary) and Γ_{sym} are the straight segments, respectively.

Note that the curved part of the reflected shock Γ_{shock} separates the supersonic flow outside Ω from the subsonic flow in Ω, *i.e.*, Γ_{shock} is a *transonic* shock.

2.4.3 Subsonic regular reflection-diffraction configurations

This is the case when state (2) is subsonic or sonic at P_0.

The subsonic reflection configuration as shown in Fig. 2.4 consists of two uniform states – (0) and (1) – in the regions described above, and a non-uniform state in domain $\Omega = P_0P_2P_3$. The non-uniform state in Ω is subsonic, *i.e.*, the potential flow equation (2.2.8) for φ is elliptic in Ω. Moreover, solution φ in Ω matches with φ_2 at P_0 as follows:

$$\varphi(P_0) = \varphi_2(P_0), \qquad D\varphi(P_0) = D\varphi_2(P_0).$$

The boundary parts of Ω in this case are

$$\Gamma_{\text{shock}} := P_0P_2, \;\; \Gamma_{\text{sym}} := P_2P_3, \;\; \Gamma_{\text{wedge}} := P_0P_3. \tag{2.4.5}$$

Similar to the previous case, Γ_{shock} is a transonic shock. We unify the notations with supersonic reflection configurations by introducing points P_1 and P_4 for subsonic reflection configurations via setting

$$P_1 := P_0, \quad P_4 := P_0, \quad \overline{\Gamma_{\text{sonic}}} := \{P_0\}. \tag{2.4.6}$$

Note that, with this convention, (2.4.5) coincides with (2.4.4).

In Part III, we develop approaches, techniques, and related analysis to establish the global existence of a supersonic reflection configuration up to the sonic angle, or the critical angle in the attached case (defined in §2.6).

In Part IV, we develop the theory further to establish the global existence of regular reflection-diffraction configurations up to the detachment angle, or the critical angle in the attached case. In particular, this will imply the existence of both supersonic and subsonic reflection configurations.

2.5 EXISTENCE OF STATE (2) AND CONTINUOUS DEPENDENCE ON THE PARAMETERS

We note that state (2), the uniform state (2.4.1), satisfies (2.4.2) and the Rankine-Hugoniot condition with state (1) on $\mathcal{S}_1 = \{\varphi_1 = \varphi_2\}$ as defined in (2.4.3):

$$\rho_2 D\varphi_2 \cdot \boldsymbol{\nu} = \rho_1 D\varphi_1 \cdot \boldsymbol{\nu} \qquad \text{on } \mathcal{S}_1, \tag{2.5.1}$$

where $\boldsymbol{\nu}$ is the unit normal on $\mathcal{S}_1$.

From the regular reflection-diffraction configurations as described in §2.4.2–§2.4.3, the existence of state (2) is a necessary condition for the existence of such a solution. We note that $\mathcal{S}_1$, defined in (2.4.3), is a straight line, which is concluded from the explicit expressions of $\varphi_j, j = 1, 2$, and the fact that $(u_1, 0) \neq (u_2, v_2)$. The last statement holds since φ_1 does not satisfy (2.4.2).

State (2), (u_2, v_2) in (2.4.1), is obtained as a solution of the algebraic system involving the slope of $\mathcal{S}_1$ (*i.e.*, the direction of $\boldsymbol{\nu}$) and the equality in (2.5.1); see §7.4 below.

This algebraic system has solutions for some but not all $\theta_{\rm w} \in (0, \frac{\pi}{2})$. More precisely, there exist the sonic angle $\theta_{\rm w}^{\rm s}$ and the detachment angle $\theta_{\rm w}^{\rm d}$ satisfying

$$0 < \theta_{\rm w}^{\rm d} < \theta_{\rm w}^{\rm s} < \frac{\pi}{2}$$

such that there are two states (2), weak and strong with $\rho_2^{\rm wk} < \rho_2^{\rm sg}$, for all $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, but $\rho_2^{\rm wk} = \rho_2^{\rm sg}$ at $\theta_{\rm w} = \theta_{\rm w}^{\rm d}$. Moreover, the strong state (2) is always subsonic at the reflection point $P_0(\theta_{\rm w})$, while the weak state (2) is:

(i) supersonic at the reflection point $P_0(\theta_{\rm w})$ for $\theta_{\rm w} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$;

(ii) sonic at $P_0(\theta_{\rm w})$ for $\theta_{\rm w} = \theta_{\rm w}^{\rm s}$;

(iii) subsonic at $P_0(\theta_{\rm w})$ for $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \hat{\theta}_{\rm w}^{\rm s})$, for some $\hat{\theta}_{\rm w}^{\rm s} \in (\theta_{\rm w}^{\rm d}, \theta_{\rm w}^{\rm s}]$.

Moreover, the weak state (2)$= (u_2, v_2)$ depends continuously on $\theta_{\rm w}$ in $[\theta_{\rm w}^{\rm d}, \frac{\pi}{2}]$. For details of this, see Theorem 7.1.1 in Chapter 7.

As for the weak and strong states for each $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, there has been a long debate to determine which one is physical for the local theory; see Courant-Friedrichs [99], Ben-Dor [12], and the references cited therein. It has been conjectured that the strong reflection-diffraction configuration is non-physical. Indeed, when the wedge angle $\theta_{\rm w}$ tends to $\frac{\pi}{2}$, the weak reflection-diffraction configuration tends to the unique normal reflection as proved in Chen-Feldman [54]; however, the strong reflection-diffraction configuration does not (see Chapter 7 below).

In the existence results of regular reflection-diffraction solutions below, we always use the *weak* state (2).

2.6 VON NEUMANN'S CONJECTURES, PROBLEM 2.6.1 (FREE BOUNDARY PROBLEM), AND MAIN THEOREMS

If the weak state (2) is supersonic, on which equation (2.2.8) is hyperbolic, the propagation speeds of the solution are finite, and state (2) is completely determined by the local information: state (1), state (0), and the location of point P_0. That is, any information from the region of shock reflection-diffraction, such as the disturbance at corner P_3, cannot travel towards the reflection point P_0. However, if the weak state (2) is subsonic, on which equation (2.2.8) is elliptic, the information can reach P_0 and interact with it, potentially creating a new type of shock reflection-diffraction configurations. This argument motivated the conjecture by von Neumann in [267, 268], which can be formulated as follows:

von Neumann's Sonic Conjecture: *There exists a supersonic regular reflection-diffraction configuration when* $\theta_{\mathrm{w}} \in (\theta_{\mathrm{w}}^{\mathrm{s}}, \frac{\pi}{2})$, *i.e., the supersonicity of the weak state* (2) *at* $P_0(\theta_{\mathrm{w}})$ *implies the existence of a supersonic regular reflection-diffraction configuration to* **Problem 2.2.3** *as shown in Fig.* 2.3.

Another conjecture states that the global regular reflection-diffraction configuration is possible whenever the local regular reflection at the reflection point P_0 is possible, even beyond the sonic angle $\theta_{\mathrm{w}}^{\mathrm{s}}$ up to the detachment angle $\theta_{\mathrm{w}}^{\mathrm{d}}$:

von Neumann's Detachment Conjecture: *There exists a global regular reflection-diffraction configuration for any wedge angle* $\theta_{\mathrm{w}} \in (\theta_{\mathrm{w}}^{\mathrm{d}}, \frac{\pi}{2})$, *i.e., the existence of state* (2) *implies the existence of a regular reflection-diffraction configuration to* **Problem 2.2.3**. *Moreover, the type* (*subsonic or supersonic*) *of the reflection-diffraction configuration is determined by the type of the weak state* (2) *at* $P_0(\theta_{\mathrm{w}})$, *as shown in Figs.* 2.3–2.4.

It is clear that the supersonic and subsonic reflection configurations are not possible without a local two-shock configuration at the reflection point on the wedge, so that it is the necessary criterion for the existence of supersonic and subsonic reflection configurations.

There has been a long debate in the literature whether there still exists a global regular reflection-diffraction solution beyond the sonic angle $\theta_{\mathrm{w}}^{\mathrm{s}}$ up to the detachment angle $\theta_{\mathrm{w}}^{\mathrm{d}}$; see Ben-Dor [12] and the references cited therein. As shown in Fig. 18.7 for the full Euler case, the difference on the physical parameters between the sonic conjecture and the detachment conjecture is only fractions of a degree apart in terms of the wedge angles; a resolution has challenged even sophisticated modern numerical and laboratory experiments. In Part IV (Chapters 15–17), we rigorously prove the global existence of regular reflection-diffraction configurations, beyond the sonic angle up to the detachment angle. This indicates that the necessary criterion is also sufficient for the existence of supersonic and subsonic reflection configurations, at least for potential flow.

To solve von Neumann's conjectures, we reformulate **Problem 2.2.3** into the following free boundary problem:

Problem 2.6.1 (Free Boundary Problem). *For* $\theta_{\mathrm{w}} \in (\theta_{\mathrm{w}}^{\mathrm{d}}, \frac{\pi}{2})$, *find a free boundary* (*curved reflected-diffracted shock*) Γ_{shock} *in region* $\Lambda \cap \{\xi_1 < \xi_{1P_1}\}$ (*where we use* (2.4.6) *for subsonic reflections*) *and a function* φ *defined in region* Ω *as shown in Figs.* 2.3–2.4 *such that* φ *satisfies*:

(i) *Equation* (2.2.8) *in* Ω;

(ii) $\varphi = \varphi_1$ *and* $\rho D\varphi \cdot \boldsymbol{\nu} = \rho_1 D\varphi_1 \cdot \boldsymbol{\nu}$ *on the free boundary* Γ_{shock} *separating the elliptic phase from the hyperbolic phase*;

(iii) $\varphi = \varphi_2$ *and* $D\varphi = D\varphi_2$ *on* Γ_{sonic} *for the supersonic reflection configuration as shown in Fig.* 2.3 *and at* P_0 *for the subsonic reflection configuration as shown in Fig.* 2.4;

(iv) $D\varphi \cdot \boldsymbol{\nu} = 0$ *on* $\Gamma_{\text{wedge}} \cup \Gamma_{\text{sym}}$,

where $\boldsymbol{\nu}$ *is the interior unit normal to* Ω *on* $\Gamma_{\text{shock}} \cup \Gamma_{\text{wedge}} \cup \Gamma_{\text{sym}}$.

We remark that condition (iii) is equivalent to the Rankine-Hugoniot conditions for φ across Γ_{sonic}. The sonic arc Γ_{sonic} is a weak discontinuity of φ (which is different from a strong discontinuity such as Γ_{shock}); that is, if the state from one side is sonic, the Rankine-Hugoniot conditions require the gradient of φ to be continuous across Γ_{sonic}.

Furthermore, since condition (ii) is the Rankine-Hugoniot conditions for φ across Γ_{shock}, we can extend solution φ of **Problem 2.6.1** from Ω to Λ so that the extended function (still denoted) φ is a weak solution of **Problem 2.2.3** (at least when Γ_{shock} and φ are sufficiently regular). More specifically,

Definition 2.6.2. *Let* φ *be a solution of* **Problem** 2.6.1 *in region* Ω. *Define the extension of* φ *from* Ω *to* Λ *by setting:*

$$\varphi = \begin{cases} \varphi_0 & \text{for } \xi_1 > \xi_1^0 \text{ and } \xi_2 > \xi_1 \tan\theta_{\text{w}}, \\ \varphi_1 & \text{for } \xi_1 < \xi_1^0 \text{ and above curve } P_0P_1P_2, \\ \varphi_2 & \text{in region } P_0P_1P_4, \end{cases} \tag{2.6.1}$$

where we have used the notational convention (2.4.6) *for subsonic reflections. In particular, for subsonic reflections, region* $P_0P_1P_4$ *is one point, and curve* $P_0P_1P_2$ *is* P_0P_2. *See Figs.* 2.3 *and* 2.4.

We note that ξ_1^0 used in (2.6.1) is the location of the incident shock; *cf.* (2.2.15) and (2.2.18). Also, the extension by (2.6.1) is well-defined because of the requirement $\Gamma_{\text{shock}} \subset \Lambda \cap \{\xi_1 < \xi_{1P_1}\}$ in **Problem 2.6.1**.

From now on, using Definition 2.6.2, we consider solutions φ of **Problem 2.6.1** to be defined in Λ.

It turns out that another key obstacle for establishing the existence of regular reflection-diffraction configurations is the additional possibility that, for some wedge angle $\theta_{\text{w}}^{\text{c}} \in (\theta_{\text{w}}^{\text{d}}, \frac{\pi}{2})$, the reflected-diffracted shock P_0P_2 may strike the wedge vertex P_3, an additional sub-type of regular reflection-diffraction configurations in which the reflected-diffracted shock is attached to the wedge vertex P_3, *i.e.*, $P_2 = P_3$. Indeed, in such a case, we establish the existence of such a global solution of regular reflection-diffraction configurations as shown in Figs. 2.5–2.6 for any wedge angle $\theta_{\text{w}} \in (\theta_{\text{w}}^{\text{c}}, \frac{\pi}{2})$.

Observe that some experimental results (*cf.* [263, Fig. 238, Page 144]) suggest that solutions with an attached shock to the wedge vertex may exist for the Mach reflection case. We are not aware of experimental or numerical evidence of the existence of regular reflection-diffraction configurations with an attached shock to the wedge vertex. However, it is possible that such solutions may exist, as shown in Figs. 2.5–2.6. Thus, it is not surprising that two different cases on the parameters of the initial data in **Problem 2.2.1** are considered separately in our study.

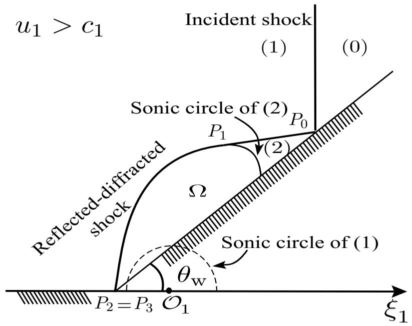


Figure 2.5: Attached supersonic regular reflection-diffraction configuration

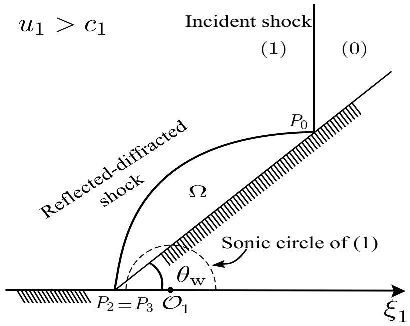


Figure 2.6: Attached subsonic regular reflection-diffraction configuration

We show that the solutions with an attached shock do not exist when the initial data of **Problem 2.2.1**, equivalently parameters (ρ_0, ρ_1, γ) in **Problem 2.6.1** which also define u_1 by (2.2.2), satisfy $u_1 \le c_1$. Moreover, in this case, the regular reflection-diffraction solution of **Problem 2.6.1** exists for each $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, as von Neumann conjectured [267, 268]. In the other case, $u_1 > c_1$, we assert the existence of a regular reflection-diffraction configuration for **Problem 2.6.1** for any $\theta_{\rm w} \in (\theta_{\rm w}^{\rm c}, \frac{\pi}{2})$, where either $\theta_{\rm w}^{\rm c} = \theta_{\rm w}^{\rm d}$ or $\theta_{\rm w}^{\rm c} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$ is the *hitting* wedge angle in the sense that a solution with $P_2 = P_3$ exists for $\theta_{\rm w} = \theta_{\rm w}^{\rm c}$.

We also note that both cases $u_1 \le c_1$ and $u_1 > c_1$ exist, for the corresponding parameters (ρ_0, ρ_1, γ), where $\rho_1 > \rho_0$ by the entropy condition on $\mathcal{S}_1$. Indeed, since $c_1 = \rho_1^{\frac{\gamma-1}{2}}$, and u_1 is a function of (ρ_0, ρ_1) for fixed $\gamma > 1$, determined by (2.2.2), then, for each $\rho_0 > 0$, there exists $\rho_1^* > \rho_0$ such that

(i) $u_1 < c_1$ for any $\rho_1 \in (\rho_0, \rho_1^*)$;

(ii) $u_1 = c_1$ for $\rho_1 = \rho_1^*$;

(iii) $u_1 > c_1$ for any $\rho_1 \in (\rho_1^*, \infty)$.

This is verified via a straightforward but lengthy calculation by both noting that (2.2.2) implies that $u_1 = 0 < c_1$ for $\rho_1 = \rho_0$ and showing that $\partial_{\rho_1}(u_1 - c_1) \ge C(\rho_0) > 0$ for $\rho_1 > \rho_0$.

Therefore, Case $u_1 \le c_1$ (resp. Case $u_1 > c_1$) corresponds to the weaker (resp. stronger) incident shocks.

In Parts II–III, we focus on von Neumann's sonic conjecture, that is, the existence of supersonic reflection configurations up to the sonic angle $\theta_{\rm w}^{\rm s}$ for $\theta_{\rm w} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$, or the critical angle $\theta_{\rm w}^{\rm c} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$ for $\theta_{\rm w} \in (\theta_{\rm w}^{\rm c}, \frac{\pi}{2})$. We establish two existence theorems: Theorem 2.6.3, which corresponds to the case of a relatively weaker incident shock, and Theorem 2.6.5, which corresponds to the case of a relatively stronger incident shock. We also establish a regularity theorem, Theorem 2.6.6, for supersonic reflection solutions. We stress that, in what follows, φ_2 always denotes the *weak* state (2). Furthermore, in all of the theorems below, we always assume that $\rho_1 > \rho_0 > 0$ and $\gamma > 1$.

Theorem 2.6.3 (Existence of Supersonic Reflection Configurations for $u_1 \leq c_1$)**.** *Consider all* (ρ_0, ρ_1, γ) *such that* $u_1 \leq c_1$. *Then there is* $\alpha = \alpha(\rho_0, \rho_1, \gamma) \in (0, \frac{1}{2})$ *so that, when* $\theta_{\rm w} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$, *there exists a solution* φ *of* **Problem 2.6.1** *such that*

$$\Phi(t, \mathbf{x}) = t\,\varphi(\frac{\mathbf{x}}{t}) + \frac{|\mathbf{x}|^2}{2t} \qquad \textit{for } \frac{\mathbf{x}}{t} \in \Lambda,\ t > 0$$

with

$$\rho(t, \mathbf{x}) = \Big(\rho_0^{\gamma-1} - (\gamma-1)\big(\Phi_t + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2\big)\Big)^{\frac{1}{\gamma-1}}$$

is a global weak solution of **Problem 2.2.1** *in the sense of Definition* 2.3.3, *which satisfies the entropy condition* (*cf. Definition* 2.1.2). *Furthermore,*

(a) $\varphi \in C^\infty(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$;

(b) φ *has structure* (2.6.1) *in* $\Lambda \setminus \overline{\Omega}$;

(c) φ *is* $C^{1,1}$ *across part* $\Gamma_{\rm sonic}$ *of the sonic circle including endpoints* P_1 *and* P_4;

(d) *The reflected-diffracted shock* $P_0P_1P_2$ *is* $C^{2,\beta}$ *up to its endpoints for any* $\beta \in [0, \frac{1}{2})$ *and* C^∞ *except* P_1;

(e) *The relative interior of the reflected-diffracted shock* $P_0P_1P_2$ *lies in* $\{\xi_2 > \xi_1 \tan\theta_{\rm w},\ \xi_2 > 0\}$, *i.e., the domain bounded by the wedge and the symmetry line* $\{\xi_2 = 0\}$.

Moreover, φ *satisfies the following properties:*

(i) *Equation* (2.2.8) *is strictly elliptic in* $\overline{\Omega} \setminus \overline{\Gamma_{\rm sonic}}$:

$$|D\varphi| < c(|D\varphi|^2, \varphi) \qquad \textit{in } \overline{\Omega} \setminus \overline{\Gamma_{\rm sonic}}; \tag{2.6.2}$$

(ii) *In* Ω,

$$\varphi_2 \leq \varphi \leq \varphi_1. \tag{2.6.3}$$

See Fig. 2.3.

Remark 2.6.4. *In fact,* φ *in Theorem* 2.6.3 *is an admissible solution in the sense of Definition* 8.1.1 *below so that* φ *satisfies the further conditions listed in Definition* 8.1.1.

Now we address Case $u_1 > c_1$. In this case, the results of Theorem 2.6.3 hold for any wedge angle $\theta_{\rm w}$ from $\frac{\pi}{2}$ until either $\theta_{\rm w}^{\rm s}$ or $\theta_{\rm w}^{\rm c} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$ when the shock hits the wedge vertex P_3.

Theorem 2.6.5 (Existence of Supersonic Reflection Configurations when $u_1 > c_1$). *Consider all (ρ_0, ρ_1, γ) such that $u_1 > c_1$. There are $\theta_{\rm w}^{\rm c} \in [\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$ and $\alpha \in (0, \frac{1}{2})$ depending only on (ρ_0, ρ_1, γ) so that the results of Theorem 2.6.3 hold for each wedge angle $\theta_{\rm w} \in (\theta_{\rm w}^{\rm c}, \frac{\pi}{2})$. If $\theta_{\rm w}^{\rm c} > \theta_{\rm w}^{\rm s}$, then, for the wedge angle $\theta_{\rm w} = \theta_{\rm w}^{\rm c}$, there exists an attached solution as shown in Fig. 2.5 (i.e., a solution φ of **Problem 2.6.1** with the properties as those in Theorem 2.6.3 except that $P_3 = P_2$) with the regularity:*

$$\varphi \in C^\infty(\Omega) \cap C^{2,\alpha}(\overline{\Omega} \setminus (\Gamma_{\rm sonic} \cup \{P_2\})) \cap C^{1,1}(\overline{\Omega} \setminus \{P_2\}) \cap C^{0,1}(\overline{\Omega}),$$

and the reflected-diffracted shock $P_0P_1P_2$ is Lipschitz up to its endpoints, $C^{2,\beta}$ for any $\beta \in [0, \frac{1}{2})$ except point P_2, and C^∞ except points P_1 and P_2.

Since solution φ of **Problem 2.6.1** constructed in Theorems 2.6.3 and 2.6.5 satisfies the $C^{1,1}$–continuity across $\overline{P_1P_4}$, (2.6.2)–(2.6.3), and further estimates including (11.2.23)–(11.2.24), (11.2.38)–(11.2.40), (11.4.38)–(11.4.39), and (11.5.2), as well as Propositions 11.2.8 and 11.4.6, then the regularity results of Theorem 14.2.8 and Corollary 14.2.11 apply. More precisely, we have

Theorem 2.6.6 (Regularity of Solutions up to $\Gamma_{\rm sonic}$). *Any solution φ in Theorems 2.6.3 and 2.6.5 satisfies the following:*

(i) *φ is $C^{2,\alpha}$ up to the sonic arc $\overline{\Gamma_{\rm sonic}}$ away from point P_1 for any $\alpha \in (0,1)$. That is, for any $\alpha \in (0,1)$ and any given $\boldsymbol{\xi}^0 \in \overline{\Gamma_{\rm sonic}} \setminus \{P_1\}$, there exist $C < \infty$ depending only on $(\rho_0, \rho_1, \gamma, \alpha)$ and $\mathrm{dist}(\boldsymbol{\xi}^0, \Gamma_{\rm shock})$, and $d > 0$ depending only on (ρ_0, ρ_1, γ) and $\mathrm{dist}(\boldsymbol{\xi}^0, \Gamma_{\rm shock})$ such that*

$$\|\varphi\|_{2,\alpha;\overline{B_d(\boldsymbol{\xi}^0)\cap\Omega}} \le C.$$

(ii) *For any $\boldsymbol{\xi}^0 \in \overline{\Gamma_{\rm sonic}} \setminus \{P_1\}$,*

$$\lim_{\substack{\boldsymbol{\xi}\to\boldsymbol{\xi}^0 \\ \boldsymbol{\xi}\in\Omega}} (D_{rr}\varphi - D_{rr}\varphi_2) = \frac{1}{\gamma+1},$$

where (r,θ) are the polar coordinates with the center at (u_2, v_2).

(iii) *$D^2\varphi$ has a jump across $\overline{\Gamma_{\rm sonic}}$: For any $\boldsymbol{\xi}^0 \in \overline{\Gamma_{\rm sonic}} \setminus \{P_1\}$,*

$$\lim_{\substack{\boldsymbol{\xi}\to\boldsymbol{\xi}^0 \\ \boldsymbol{\xi}\in\Omega}} D_{rr}\varphi \;-\; \lim_{\substack{\boldsymbol{\xi}\to\boldsymbol{\xi}^0 \\ \boldsymbol{\xi}\in\Lambda\setminus\Omega}} D_{rr}\varphi \;=\; \frac{1}{\gamma+1}.$$

(iv) *$\lim_{\substack{\boldsymbol{\xi}\to P_1 \\ \boldsymbol{\xi}\in\Omega}} D^2\varphi$ does not exist.*

From Chapter 4 to Chapter 14, we develop approaches, techniques, and related analysis to complete the proofs of these theorems in detail, which provide a solution to **von Neumann's Sonic Conjecture**. We give an overview of these techniques in Chapter 3.

In Part IV, we extend Theorems 2.6.3 and 2.6.5 beyond the sonic angle to include the wedge angles $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \theta_{\rm w}^{\rm s}]$. Therefore, we establish the global existence of regular reflection-diffraction configurations for any wedge angle between $\frac{\pi}{2}$ and the detachment angle $\theta_{\rm w}^{\rm d}$, or the critical angle $\theta_{\rm w}^{\rm c}$ in the attached case. As in Theorems 2.6.3 and 2.6.5, we need to analyze two separate cases: Case $u_1 \le c_1$ and Case $u_1 > c_1$, since the reflected-diffracted shock may hit the wedge vertex in the latter case. Below, we use the notations introduced in §2.4.2–§2.4.3, and φ_2 denotes the *weak* state (2).

The theorems that follow assert the global existence of regular reflection-diffraction configurations for any wedge angle between $\frac{\pi}{2}$ and the detachment angle $\theta_{\rm w}^{\rm d}$, or the critical angle $\theta_{\rm w}^{\rm c}$ in the attached case. The type of regular reflection-diffraction configuration for $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$ is supersonic if $|D\varphi_2(P_0)| > c_2$ and subsonic if $|D\varphi_2(P_0)| \le c_2$. In particular, the regular reflection-diffraction configuration is supersonic for all $\theta_{\rm w} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$, as we have presented in Theorems 2.6.3 and 2.6.5. Also, as we have discussed in (iii) of §2.5 and will prove in Theorem 7.1.1(vi) later, there exists $\hat{\theta}_{\rm w}^{\rm s} \in (\theta_{\rm w}^{\rm d}, \theta_{\rm w}^{\rm s}]$ such that $|D\varphi_2(P_0)| < c_2$ for all $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \hat{\theta}_{\rm w}^{\rm s})$, and then the regular reflection-diffraction configuration is subsonic for these wedge angles.

Theorem 2.6.7 (Global Solutions up to the Detachment Angle when $u_1 \le c_1$)**.** *Consider all (ρ_0, ρ_1, γ) such that $u_1 \le c_1$. Then there is $\alpha = \alpha(\rho_0, \rho_1, \gamma) \in (0, \frac{1}{2})$ so that, when $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, there exists a solution φ of* **Problem 2.6.1** *such that*

$$\Phi(t, \mathbf{x}) = t\,\varphi(\frac{\mathbf{x}}{t}) + \frac{|\mathbf{x}|^2}{2t} \qquad \textit{for } \frac{\mathbf{x}}{t} \in \Lambda,\, t > 0$$

with

$$\rho(t, \mathbf{x}) = \Big(\rho_0^{\gamma-1} - (\gamma-1)\big(\Phi_t + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2\big)\Big)^{\frac{1}{\gamma-1}}$$

is a global weak solution of **Problem 2.2.1** *in the sense of Definition* 2.3.3*, which satisfies the entropy condition* (*cf. Definition* 2.1.2)*, and the type of reflection configurations* (*supersonic or subsonic*) *is determined by* $\theta_{\rm w}$:

- *If $|D\varphi_2(P_0)| > c_2$, then φ has the* supersonic *reflection configuration and satisfies all the properties in Theorem* 2.6.3*, which is the case for any wedge angle $\theta_{\rm w} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$;*

- *If $|D\varphi_2(P_0)| \le c_2$, then φ has the* subsonic *reflection configuration and satisfies*

$$\varphi \in C^\infty(\Omega) \cap C^{2,\alpha}(\overline{\Omega} \setminus \{P_0, P_3\}) \cap C^{1,\alpha}(\overline{\Omega}),$$

$$\varphi = \begin{cases} \varphi_0 & \textit{for } \xi_1 > \xi_1^0 \textit{ and } \xi_2 > \xi_1 \tan\theta_{\rm w}, \\ \varphi_1 & \textit{for } \xi_1 < \xi_1^0 \textit{ and above curve } P_0P_2, \\ \varphi_2(P_0) & \textit{at } P_0, \end{cases} \tag{2.6.4}$$

$D\varphi(P_0) = D\varphi_2(P_0)$, and the reflected-diffracted shock $\Gamma_{\rm shock}$ is $C^{1,\alpha}$ up to its endpoints and C^∞ except P_0. Also, the relative interior of shock $\Gamma_{\rm shock}$

lies in $\{\xi_2 > \xi_1 \tan\theta_{\rm w}, \xi_2 > 0\}$, *i.e., a domain bounded by the wedge and the symmetry line* $\{\xi_2 = 0\}$.

Furthermore, φ *satisfies the following properties*:

(i) *Equation* (2.2.8) *is strictly elliptic*:

$$|D\varphi| < c(|D\varphi|^2, \varphi) \tag{2.6.5}$$

in $\overline{\Omega} \setminus \{P_0\}$ *if* $|D\varphi_2(P_0)| = c_2$ *and in* $\overline{\Omega}$ *if* $|D\varphi_2(P_0)| < c_2$;

(ii) *In* Ω,

$$\varphi_2 \le \varphi \le \varphi_1. \tag{2.6.6}$$

Note that the regular reflection-diffraction solution has a subsonic reflection configuration for any $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \hat\theta_{\rm w}^{\rm s})$, *where* $\hat\theta_{\rm w}^{\rm s}$ *is from* (iii) *in* §2.5.

Moreover, the optimal regularity theorem, Theorem 2.6.6, *applies to any global regular reflection solutions of supersonic reflection configuration.*

Remark 2.6.8. *Solution* φ *in Theorem* 2.6.7 *is also an admissible solution in the sense of Definition* 15.1.1 *in the supersonic case and of Definition* 15.1.2 *in the subsonic case, so that* φ *satisfies the further conditions listed in Definitions* 15.1.1–15.1.2, *respectively, in Chapter* 15.

Now we address Case $u_1 > c_1$. In this case, the results of Theorem 2.6.7 hold from the wedge angle $\frac{\pi}{2}$ until either $\theta_{\rm w}^{\rm d}$ or the critical angle $\theta_{\rm w}^{\rm c} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$ when the shock hits the wedge vertex P_3.

Theorem 2.6.9 (Global Solutions up to the Detachment Angle when $u_1 > c_1$)**.** *Consider all* (ρ_0, ρ_1, γ) *such that* $u_1 > c_1$. *Then there are* $\theta_{\rm w}^{\rm c} \in [\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$ *and* $\alpha \in (0, \frac{1}{2})$ *depending only on* (ρ_0, ρ_1, γ) *so that the results of Theorem* 2.6.7 *hold for each wedge angle* $\theta_{\rm w} \in (\theta_{\rm w}^{\rm c}, \frac{\pi}{2})$.

If $\theta_{\rm w}^{\rm c} > \theta_{\rm w}^{\rm d}$, *then, for the wedge angle* $\theta_{\rm w} = \theta_{\rm w}^{\rm c}$, *there exists an attached solution* φ *as shown in Fig.* 2.5–2.6, *i.e., a solution* φ *of* **Problem 2.6.1** *with the properties as in Theorem* 2.6.7 *except that* $P_2 = P_3$. *Moreover, the attached solution* φ *has the following two cases*:

- *If* $|D\varphi_2(P_0)| > c_2$ *for* $\theta_{\rm w} = \theta_{\rm w}^{\rm c}$ (*the supersonic case*), *the reflected-diffracted shock of the attached solution satisfies all of the properties listed in Theorem* 2.6.5;

- *If* $|D\varphi_2(P_0)| \le c_2$ *for* $\theta_{\rm w} = \theta_{\rm w}^{\rm c}$ (*the subsonic case*),

$$\varphi \in C^\infty(\Omega) \cap C^{2,\alpha}(\overline{\Omega} \setminus \{P_0, P_3\}) \cap C^{1,\alpha}(\overline{\Omega} \setminus \{P_3\}) \cap C^{0,1}(\overline{\Omega}).$$

  The reflected-diffracted shock is Lipschitz up to its endpoints, $C^{1,\alpha}$ *except point* P_2, *and* C^∞ *except its endpoints* P_0 *and* P_2.

Remark 2.6.10. *We emphasize that all the results in the main theorems – Theorem* 2.6.3*, Theorems* 2.6.5–2.6.7*, and Theorem* 2.6.9 *– hold when* $\gamma = 1$*, which can be handled similarly with appropriate changes in the formulas in their respective proofs.*

Remark 2.6.11. *In Chen-Feldman-Xiang* [60]*, the strict convexity of self-similar transonic shocks has also been proved in the regular shock reflection-diffraction configurations in Theorem* 2.6.3*, Theorems* 2.6.5–2.6.7*, and Theorem* 2.6.9*.*

In Part IV, we further develop analytical techniques to complete the proofs of these theorems and further results, which provides a solution to **von Neumann's Detachment Conjecture**. The main challenge is the analysis of the transition from the supersonic to subsonic reflection configurations, which requires uniform *a priori* estimates for the solutions at the corner between the reflected-diffracted shock and the wedge when the wedge angle $\theta_{\rm w}$ decreases across the sonic angle $\theta_{\rm w}^{\rm s}$ up to the detachment angle $\theta_{\rm w}^{\rm d}$, or the critical angle $\theta_{\rm w}^{\rm c}$ in the attached case.

Chapter Three

Main Steps and Related Analysis in the Proofs of the Main Theorems

In this chapter, we give an overview of the main steps and related analysis in the proofs of the main theorems for the existence of global regular reflection-diffraction solutions. We first discuss the proof of the existence and properties of supersonic regular reflection-diffraction configurations up to the sonic angle in Theorems 2.6.3 and 2.6.5. Then we discuss the proofs of the existence and properties of regular reflection-diffraction configurations beyond the sonic angle, up to the detachment angle, in Theorems 2.6.7 and 2.6.9. The detailed proofs and analysis developed for the main theorems will be given in Parts III–IV.

3.1 NORMAL REFLECTION

When the wedge angle $\theta_{\mathrm{w}} = \frac{\pi}{2}$, the incident shock reflects normally (see Fig. 3.1). The reflected shock is also a plane at $\xi_1 = \bar{\xi}_1 < 0$. Then the velocity of state (2) is zero, $\bar{u}_2 = \bar{v}_2 = 0$, state (1) is of form (2.2.17), and state (2) is of the form:

$$\varphi_2(\boldsymbol{\xi}) = -\frac{|\boldsymbol{\xi}|^2}{2} + u_1(\bar{\xi}_1 - \xi_1^0), \tag{3.1.1}$$

where $\xi_1^0 = \frac{\rho_1 u_1}{\rho_1 - \rho_0} > 0$, which is the position of the incident shock in the self-similar coordinates $\boldsymbol{\xi}$.

The position: $\xi_1 = \bar{\xi}_1 < 0$ of the reflected shock and density $\bar{\rho}_2$ of state (2) can be determined uniquely from the Rankine-Hugoniot condition (2.2.13) at the reflected shock and the Bernoulli law (2.2.7).

3.2 MAIN STEPS AND RELATED ANALYSIS IN THE PROOF OF THE SONIC CONJECTURE

In this section, we always discuss the global solutions of **Problem 2.6.1** in §2.6 for the wedge angles $\theta_{\mathrm{w}} \in (\theta_{\mathrm{w}}^{\mathrm{s}}, \frac{\pi}{2})$, where $\theta_{\mathrm{w}}^{\mathrm{s}}$ is the sonic angle. Then the expected solutions are of the supersonic reflection configuration as described in §2.4.2.

To solve the free boundary problem (**Problem 2.6.1**) for the wedge angles $\theta_{\mathrm{w}} \in (\theta_{\mathrm{w}}^{\mathrm{s}}, \frac{\pi}{2})$, we first define a class of admissible solutions that are of the

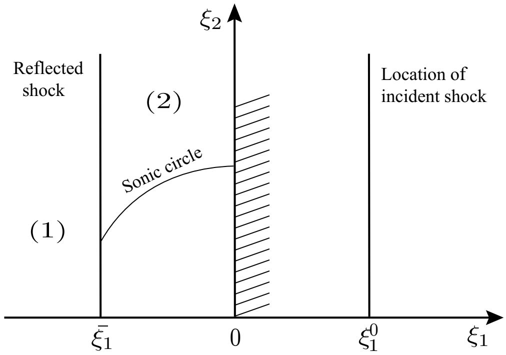


Figure 3.1: Normal reflection

structure of §2.4.2 and satisfy some additional properties as discussed below. Then we make *a priori* estimates of admissible solutions. Finally, based on the *a priori* estimates, we obtain the existence of admissible solutions as fixed points of the iteration procedure by employing the Leray-Schauder degree theory. We now discuss these steps in more detail.

3.2.1 Admissible solutions

To solve the free boundary problem (**Problem 2.6.1**), we first define a class of admissible solutions φ that are the solutions with supersonic reflection configuration as described in §2.4.2, which is the case when the wedge angle θ_{w} is between $\theta_{\mathrm{w}}^{\mathrm{s}}$ and $\frac{\pi}{2}$.

Let $\gamma > 1$ and $\rho_1 > \rho_0 > 0$ be given constants, and let $\xi_1^0 > 0$ and $u_1 > 0$ be defined by (2.2.18). Let the incident shock be defined by $\mathcal{S}_0 := \{\xi_1 = \xi_1^0\}$, and let state (0) and state (1) ahead of and behind $\mathcal{S}_0$ be given by (2.2.16)–(2.2.17), respectively, so that the Rankine-Hugoniot condition (2.2.13) holds on $\mathcal{S}_0$. As we will show in Theorem 7.1.1 (see also the discussion in §2.5), there exists $\theta_{\mathrm{w}}^{\mathrm{s}} \in (0, \frac{\pi}{2})$ such that, when the wedge angle $\theta_{\mathrm{w}} \in (\theta_{\mathrm{w}}^{\mathrm{s}}, \frac{\pi}{2})$, there is a unique weak state (2) of form (2.4.1) so that

(i) $u_2 > 0$. Then $v_2 = u_2 \tan\theta_{\mathrm{w}} > 0$, and $\mathcal{S}_1 := \{\varphi_1 = \varphi_2\}$ is a line. Lines $\mathcal{S}_0$ and $\mathcal{S}_1$ meet the wedge boundary $\{\xi_2 = \xi_1 \tan\theta_{\mathrm{w}}\}$ at point $P_0 \equiv P_0(\theta_{\mathrm{w}}) = (\xi_1^0, \xi_1^0 \tan\theta_{\mathrm{w}})$.

(ii) The entropy condition, $\rho_2 > \rho_1$, holds.

(iii) The Rankine-Hugoniot condition (2.2.13) holds for φ_1 and φ_2 along line $\mathcal{S}_1$.

(iv) φ_2 is supersonic at the reflection point P_0, *i.e.*, $|D\varphi_2(P_0)| > c_2$.

(v) u_2 and ρ_2 depend continuously on $\theta_{\mathrm{w}} \in (\theta_{\mathrm{w}}^{\mathrm{s}}, \frac{\pi}{2})$.

(vi) $\lim_{\theta_{\rm w}\to\frac{\pi}{2}-}(u_2(\theta_{\rm w}), \rho_2(\theta_{\rm w})) = (0, \bar{\rho}_2)$, where $\bar{\rho}_2$ is the unique density of state (2) for the normal reflection solution.

Using the properties of the uniform state solutions of (2.2.8), we show that, for each $\theta_{\rm w} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$, line $\mathcal{S}_1 = \{\varphi_1 = \varphi_2\}$ necessarily intersects with boundary $\partial B_{c_2}(u_2, v_2)$ of the sonic circle of state (2) in two points. Let P_1 be the nearest point of intersection of $\mathcal{S}_1$ with $\partial B_{c_2}(u_2, v_2)$ to $P_0 = (\xi_1^0, \xi_1^0 \tan\theta_{\rm w})$; see Fig. 2.3. Then P_1 necessarily lies within Λ, so does the whole segment P_0P_1.

With this, for any $\theta_{\rm w} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$, we define the points, segments, and curves shown in Fig. 2.3 as follows:

- Line $\mathcal{S}_1 := \{\varphi_1 = \varphi_2\}$.
- Point $P_0 := (\xi_1^0, \xi_1^0 \tan\theta_{\rm w})$.
- Point P_1 is the unique point of intersection of $\mathcal{S}_1$ with $\partial B_{c_2}(u_2, v_2)$ such that state (2) is supersonic at any $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathcal{S}_1$ satisfying $\xi_2 > \xi_{2P_1}$.
- Point $P_3 := \mathbf{0} := (0, 0)$.
- Point $P_4 := (q_2 + c_2)(\cos\theta_{\rm w}, \sin\theta_{\rm w})$, where $q_2 := \sqrt{u_2^2 + v_2^2}$; that is, P_4 is the upper point of intersection of the sonic circle of state (2) with the wedge boundary $\{\xi_2 = \xi_1 \tan\theta_{\rm w}\}$. From the definition,
$$\xi_{1P_1} < \xi_{1P_4}.$$
- Line segment $\Gamma_{\rm wedge} := P_3P_4 \subset \{\xi_2 = \xi_1 \tan\theta_{\rm w}\}$.
- $\Gamma_{\rm sonic}$ is the upper arc P_1P_4 of the sonic circle of state (2), that is,
$$\Gamma_{\rm sonic} := \{(\xi_1, v_2 + \sqrt{c_2^2 - (\xi_1 - u_2)^2}) \;:\; \xi_{1P_1} \le \xi_1 \le \xi_{1P_4}\}.$$

Now we define the admissible solutions of **Problem 2.6.1** for the wedge angles $\theta_{\rm w} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$. The admissible solutions are of the structure of supersonic reflection configuration described in §2.4.2. These conditions are listed in conditions (i)–(iii) of Definition 3.2.1. We also add conditions (iv)–(v) of Definition 3.2.1. This is motivated by the fact that, for the wedge angles sufficiently close to $\frac{\pi}{2}$, the solutions of **Problem 2.6.1** which satisfy conditions (i)–(iii) of Definition 3.2.1 also satisfy conditions (iv)–(v) of Definition 3.2.1, as we will prove in Appendix 8.3.

Definition 3.2.1. *Fix a wedge angle $\theta_{\rm w} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$. A function $\varphi \in C^{0,1}(\Lambda)$ is called an admissible solution of the regular shock reflection-diffraction problem if φ is a solution of* **Problem 2.6.1** *and satisfies the following*:

(i) *There exists a relatively open curve segment $\Gamma_{\rm shock}$ (without self-intersection) whose endpoints are $P_1 = (\xi_{1P_1}, \xi_{2P_1})$ and $P_2 = (\xi_{1P_2}, 0)$ with*
$$\xi_{1P_2} < \min\{0,\ u_1 - c_1\}, \qquad \xi_{1P_2} \le \xi_{1P_1},$$
such that $\Gamma_{\rm shock}$ satisfies that:

- *For the sonic circle $\partial B_{c_1}(u_1, 0)$ of state* (1),

$$\Gamma_{\text{shock}} \subset \big(\Lambda \setminus \overline{B_{c_1}(u_1,0)}\big) \cap \{\xi_{1P_2} \le \xi_1 \le \xi_{1P_1}\}; \tag{3.2.1}$$

- *Γ_{shock} is C^2 in its relative interior, and curve $\Gamma^{\text{ext}}_{\text{shock}} := \Gamma_{\text{shock}} \cup \Gamma^{-}_{\text{shock}} \cup \{P_2\}$ is C^1 at its relative interior (including P_2), where $\Gamma^{-}_{\text{shock}}$ is the reflection of Γ_{shock} with respect to the ξ_1–axis.*

Let $\Gamma_{\text{sym}} := P_2P_3$ be the line segment. Then Γ_{sonic}, Γ_{sym}, and Γ_{wedge} do not have common points except their common endpoints $\{P_3, P_4\}$. We require *that there be no common points between Γ_{shock} and curve $\overline{\Gamma_{\text{sym}}} \cup \overline{\Gamma_{\text{wedge}}} \cup \overline{\Gamma_{\text{sonic}}}$ except their common endpoints $\{P_1, P_2\}$. Thus, $\overline{\Gamma_{\text{shock}}} \cup \overline{\Gamma_{\text{sym}}} \cup \overline{\Gamma_{\text{wedge}}} \cup \overline{\Gamma_{\text{sonic}}}$ is a closed curve without self-intersection. Denote by Ω the open bounded domain restricted by this curve. Note that $\Omega \subset \Lambda$ and $\partial\Omega \cap \partial\Lambda = \overline{\Gamma_{\text{sym}}} \cup \overline{\Gamma_{\text{wedge}}}$.*

(ii) *φ satisfies*

$$\begin{aligned}
&\varphi \in C^{0,1}(\Lambda) \cap C^1(\overline{\Lambda} \setminus \overline{P_0P_1P_2}),\\
&\varphi \in C^3(\overline{\Omega} \setminus (\overline{\Gamma_{\text{sonic}}} \cup \{P_2, P_3\})) \cap C^1(\overline{\Omega}),\\
&\varphi = \begin{cases} \varphi_0 & \text{for } \xi_1 > \xi_1^0 \text{ and } \xi_2 > \xi_1 \tan\theta_{\text{w}},\\ \varphi_1 & \text{for } \xi_1 < \xi_1^0 \text{ and above curve } P_0P_1P_2,\\ \varphi_2 & \text{in } P_0P_1P_4. \end{cases}
\end{aligned} \tag{3.2.2}$$

(iii) *Equation* (2.2.8) *is strictly elliptic in $\overline{\Omega} \setminus \overline{\Gamma_{\text{sonic}}}$:*

$$|D\varphi| < c(|D\varphi|^2, \varphi) \qquad \text{in } \overline{\Omega} \setminus \overline{\Gamma_{\text{sonic}}}. \tag{3.2.3}$$

(iv) *In Ω,*

$$\varphi_2 \le \varphi \le \varphi_1. \tag{3.2.4}$$

(v) *Let $\mathbf{e}_{\mathcal{S}_1}$ be the unit vector parallel to $\mathcal{S}_1$ oriented so that $\mathbf{e}_{\mathcal{S}_1} \cdot D\varphi_2(P_0) > 0$; that is,*

$$\mathbf{e}_{\mathcal{S}_1} = \frac{P_1 - P_0}{|P_1 - P_0|} = -\frac{(v_2,\, u_1 - u_2)}{\sqrt{(u_1-u_2)^2 + v_2^2}}. \tag{3.2.5}$$

Then

$$\partial_{\mathbf{e}_{\mathcal{S}_1}}(\varphi_1 - \varphi) \le 0 \qquad \text{in } \Omega, \tag{3.2.6}$$

$$\partial_{\xi_2}(\varphi_1 - \varphi) \le 0 \qquad \text{in } \Omega. \tag{3.2.7}$$

Remark 3.2.2. *Condition* (3.2.4) *of Definition* 3.2.1 *implies that $\Omega \subset \{\varphi_2 < \varphi_1\}$, i.e., that Ω lies between line S_1 and the wedge boundary; see Fig.* 3.2.

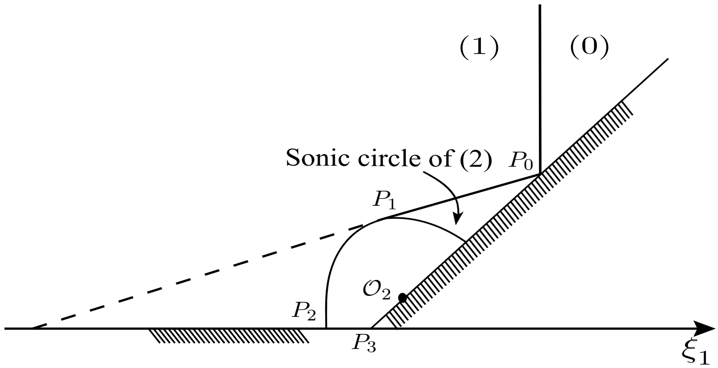


Figure 3.2: Location of domain Ω

Remark 3.2.3 (Cone of monotonicity directions)**.** *Conditions* (3.2.6) *and* (3.2.7) *imply that, if* φ *is an admissible solution of* **Problem 2.6.1** *in the sense of Definition* 3.2.1*, then*

$$\partial_{\mathbf{e}}(\varphi_1 - \varphi) \le 0 \qquad \text{in } \overline{\Omega}, \quad \text{for all } \mathbf{e} \in \mathrm{Cone}(\mathbf{e}_{\mathcal{S}_1}, \mathbf{e}_{\xi_2}),\ \mathbf{e} \neq 0, \tag{3.2.8}$$

where, for $\mathbf{e}, \mathbf{g} \in \mathbb{R}^2 \setminus \{0\}$ *with* $\mathbf{e}, \mathbf{g} \neq 0$ *and* $\mathbf{e} \neq c\mathbf{g}$*,*

$$\mathrm{Cone}(\mathbf{e}, \mathbf{g}) := \{a\mathbf{e} + b\mathbf{g} \ : \ a, b \ge 0\}. \tag{3.2.9}$$

We denote by $\mathrm{Cone}^0(\mathbf{e}, \mathbf{g})$ *the interior of* $\mathrm{Cone}(\mathbf{e}, \mathbf{g})$*.*

Remark 3.2.4 (Γ_{shock} does not intersect with Γ_{wedge} and the sonic circle of state (1))**.** *The property that* $\Gamma_{\text{shock}} \subset \Lambda \setminus \overline{B_{c_1}(u_1, 0)}$ *of* (3.2.1) *implies that* Γ_{shock} *intersects with neither* Γ_{wedge} *nor the sonic circle* $\partial B_{c_1}(u_1, 0)$ *of state* (1) *with*

$$B_{c_1}(u_1, 0) \cap \Lambda \subset \Omega. \tag{3.2.10}$$

Remark 3.2.5 (φ matches with φ_2 on Γ_{sonic})**.** *From Definition* 3.2.1(ii)*, it follows that*

$$\varphi = \varphi_2, \quad D\varphi = D\varphi_2 \qquad \text{on } \overline{\Gamma_{\text{sonic}}}.$$

Note that the Rankine-Hugoniot conditions (2.2.13)–(2.2.14) imply the following equalities on Γ_{shock}:

$$\rho(|D\varphi|^2, \varphi)\partial_{\boldsymbol{\nu}}\varphi = \rho_1 \partial_{\boldsymbol{\nu}}\varphi_1, \tag{3.2.11}$$

$$\partial_{\boldsymbol{\tau}}\varphi = \partial_{\boldsymbol{\tau}}\varphi_1, \tag{3.2.12}$$

$$\varphi = \varphi_1, \tag{3.2.13}$$

where, on the left-hand sides of (3.2.11)–(3.2.12), $D\varphi$ is evaluated on the Ω–side of Γ_{shock}.

Throughout the rest of this section, we always assume that φ is an admissible solution of **Problem 2.6.1**.

3.2.2 Strict monotonicity cone for $\varphi_1 - \varphi$ and its geometric consequences

First, we prove that, for any $\mathbf{e} \in \mathrm{Cone}^0(\mathbf{e}_{\mathcal{S}_1}, \mathbf{e}_{\xi_2})$,

$$\partial_{\mathbf{e}}(\varphi_1 - \varphi) < 0 \qquad \text{in } \overline{\Omega}, \tag{3.2.14}$$

where $\mathrm{Cone}^0(\mathbf{e}, \mathbf{g})$ is the interior of $\mathrm{Cone}(\mathbf{e}, \mathbf{g})$ defined by (3.2.9) for $\mathbf{e}, \mathbf{g} \in \mathbb{R}^2 \setminus \{0\}$. For the proof, we derive an equation for $w = \partial_{\mathbf{e}}(\varphi_1 - \varphi)$ in Ω, and employ the maximum principle and boundary conditions on $\partial\Omega$, including the conditions on Γ_{sonic} in Remark 3.2.5.

This implies that Γ_{shock} *is a graph in the directions of the cone.* That is, for $\mathbf{e} \in \mathrm{Cone}^0(\mathbf{e}_{\mathcal{S}_1}, \mathbf{e}_{\xi_2})$ with $\mathbf{e}^\perp$ being orthogonal to $\mathbf{e}$ and oriented so that $\mathbf{e}^\perp \cdot \mathbf{e}_{\mathcal{S}_1} < 0$ and $|\mathbf{e}| = |\mathbf{e}^\perp| = 1$, coordinates (S, T) with basis $\{\mathbf{e}, \mathbf{e}^\perp\}$, and $P_k = (S_{P_k}, T_{P_k})$, $k = 1, \dots, 4$, with $T_{P_2} < T_{P_1} < T_{P_4}$, there exists $f_{\mathbf{e},\text{sh}} \in C^1(\mathbb{R})$ such that

(i) $\Gamma_{\text{shock}} = \{S = f_{\mathbf{e},\text{sh}}(T) \,:\, T_{P_2} < T < T_{P_1}\}$ and
$\Omega \subset \{S < f_{\mathbf{e},\text{sh}}(T) \,:\, T \in \mathbb{R}\}$.

(ii) In the (S, T)–coordinates, $P_k = (f_{\mathbf{e},\text{sh}}(T_{P_k}), T_{P_k}), k = 1, 2$.

(iii) The tangent directions to Γ_{shock} are between the directions of line $\mathcal{S}_1$ and $\{t\mathbf{e}_{\xi_2} \,:\, t \in \mathbb{R}\}$, which are the tangent lines to Γ_{shock} at points P_1 and P_2, respectively. That is, for any $T \in (T_{P_2}, T_{P_1})$,

$$\begin{aligned} -\infty < \frac{\mathbf{e}_{\mathcal{S}_1} \cdot \mathbf{e}}{\mathbf{e}_{\mathcal{S}_1} \cdot \mathbf{e}^\perp} &= f'_{\mathbf{e},\text{sh}}(T_{P_1}) \\ &\le f'_{\mathbf{e},\text{sh}}(T) \le f'_{\mathbf{e},\text{sh}}(T_{P_2}) = \frac{\mathbf{e}_{\xi_2} \cdot \mathbf{e}}{\mathbf{e}_{\xi_2} \cdot \mathbf{e}^\perp} < \infty. \end{aligned}$$

Note that the last property gives an estimate of the Lipschitz constant of Γ_{shock} for an admissible solution in terms of the parameters of states (0), (1), and (2).

3.2.3 Monotonicity cone for $\varphi - \varphi_2$ and its consequences

We prove that, for any $\mathbf{e} \in \mathrm{Cone}^0(\mathbf{e}_{\mathcal{S}_1}, -\boldsymbol{\nu}_{\text{w}})$,

$$\partial_{\mathbf{e}}(\varphi - \varphi_2) \ge 0 \qquad \text{in } \overline{\Omega}, \tag{3.2.15}$$

where $\mathrm{Cone}^0(\mathbf{e}_{\mathcal{S}_1}, -\boldsymbol{\nu}_{\text{w}})$ is defined by (3.2.9), and $\boldsymbol{\nu}_{\text{w}}$ is the unit normal on Γ_{wedge}, interior to Ω.

As a consequence of this, we conclude that, in the local coordinates (x, y) with x the normal directional coordinate into Ω with respect to the sonic arc Γ_{sonic},

$$\partial_x(\varphi - \varphi_2) \ge 0$$

in a uniform neighborhood of Γ_{sonic}. This is important for the regularity estimates near Γ_{sonic}; see §3.2.5.2.

3.2.4 Uniform estimates for admissible solutions

We next discuss several uniform estimates for admissible solutions. Some of these estimates hold for any wedge angle $\theta_{\mathrm{w}} \in (\theta_{\mathrm{w}}^{\mathrm{s}}, \frac{\pi}{2})$. The universal constant C in these estimates depends only on the data: (ρ_0, ρ_1, γ).

In the other estimates, we have to restrict the range of wedge angles as follows: Fix any $\theta_{\mathrm{w}}^* \in (\theta_{\mathrm{w}}^{\mathrm{s}}, \frac{\pi}{2})$, and consider admissible solutions with $\theta_{\mathrm{w}} \in [\theta_{\mathrm{w}}^*, \frac{\pi}{2})$. In the case of Theorem 2.6.5, for some estimates, we need to restrict the wedge angles further by considering only $\theta_{\mathrm{w}}^* \in (\theta_{\mathrm{w}}^{\mathrm{c}}, \frac{\pi}{2})$, where $\theta_{\mathrm{w}}^{\mathrm{c}} \in [\theta_{\mathrm{w}}^{\mathrm{s}}, \frac{\pi}{2})$ is defined in §2.6 (also see §3.2.4.3). Then we obtain the uniform estimates for admissible solutions with the wedge angles $\theta_{\mathrm{w}} \in [\theta_{\mathrm{w}}^*, \frac{\pi}{2}]$. The universal constant C in these estimates depends only on the data and θ_{w}^*.

The proofs are achieved by employing the conditions of admissible solutions in Definition 3.2.1 and the monotonicity properties discussed in §3.2.2–§3.2.3. The arguments are based on the maximum principle via the strict ellipticity of the equation in Ω.

3.2.4.1 Uniform estimate of the size of Ω, the Lipschitz norm of the potential, and the density from above and below

In estimating $\mathrm{diam}(\Omega)$, a difficulty is that we cannot exclude the possibility that the ray:

$$\mathcal{S}_1^+ = \{P_0 + t(P_1 - P_0) \ : \ t > 0\}$$

does not intersect with the ξ_1–axis for $\theta_{\mathrm{w}} \in [\theta_{\mathrm{w}}^{\mathrm{s}}, \frac{\pi}{2})$. Then $\mathrm{diam}(\Omega)$ would not be estimated by the coordinates of the points of intersection of $\mathcal{S}_1^+$ with the ξ_1–axis.

By using the potential flow equation (2.2.8) and the conditions of admissible solutions in Definition 3.2.1, including the strict ellipticity in Ω, we show that there exists $C > 0$ such that, if φ is an admissible solution of **Problem 2.6.1** with $\theta_{\mathrm{w}} \in (\theta_{\mathrm{w}}^{\mathrm{s}}, \frac{\pi}{2})$, then

$$\Omega \subset B_C(\mathbf{0}), \tag{3.2.16}$$

$$\|\varphi\|_{0,1,\overline{\Omega}} \leq C, \tag{3.2.17}$$

$$a\rho_1 \leq \rho \leq C \qquad \text{in } \Omega \text{ with } a = \left(\tfrac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} > 0, \tag{3.2.18}$$

$$\rho_1 < \rho \leq C \qquad \text{on } \overline{\Gamma_{\mathrm{shock}}} \cup \{P_3\}. \tag{3.2.19}$$

These properties allow us to obtain the uniform $C^{2,\alpha}$–estimates of φ in Ω away from $\Gamma_{\mathrm{shock}} \cup \Gamma_{\mathrm{sonic}} \cup \{P_3\}$. With this, we obtain certain (preliminary) compactness properties of admissible solutions. In particular, we show that the admissible solutions tend to the normal reflection as the wedge angles tend to $\frac{\pi}{2}$, where the convergence is understood in the appropriate sense that implies the convergence of Γ_{shock} to the normal reflected shock $\Gamma_{\mathrm{shock}}^{\mathrm{norm}}$.

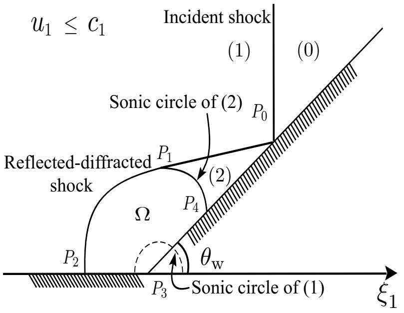


Figure 3.3: Supersonic regular reflection-diffraction configuration when $u_1 \leq c_1$

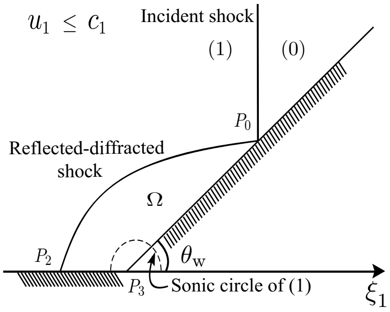


Figure 3.4: Subsonic regular reflection-diffraction configuration when $u_1 \leq c_1$

3.2.4.2 Separation of $\Gamma_{\rm shock}$ and $\Gamma_{\rm sym}$

There exists $\mu > 0$ depending only on the data such that, for any admissible solution φ of **Problem 2.6.1** with $\theta_{\rm w} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$,

$$f_{\rm sh}(\xi_1) \geq \min(\frac{c_1}{2}, \mu(\xi_1 - \xi_1^{P_2})) \qquad \text{for all } \xi_1 \in [\xi_1^{P_2}, \min\{\xi_1^{P_1}, 0\}],$$

where $\xi_2 = f_{\rm sh}(\xi_1)$ represents $\Gamma_{\rm shock}$ when $\xi_1 \in [\xi_1^{P_2}, \min\{\xi_1^{P_1}, 0\}]$.

3.2.4.3 Uniform positive lower bound for the distance between $\Gamma_{\rm shock}$ and $\Gamma_{\rm wedge}$

We now extend the set of admissible solutions by including the normal reflection as the unique admissible solution for $\theta_{\rm w} = \frac{\pi}{2}$. This is justified by the fact that all the admissible solutions converge to the normal reflection solution as the wedge angles tend to $\frac{\pi}{2}$; see §3.2.4.1. Fix $\theta_{\rm w}^* \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$.

If $u_1 \leq c_1$, which is determined by (ρ_0, ρ_1, γ), then there exists $C > 0$ such that

$$\operatorname{dist}(\Gamma_{\rm shock}, \Gamma_{\rm wedge}) > \frac{1}{C} \tag{3.2.20}$$

for any admissible solution of **Problem 2.6.1** with $\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2}]$. In this case, the reflected-diffracted shock does not hit the wedge vertex P_3, since point P_2 should be away from the sonic circle of state (1), as shown in Figs. 3.3–3.4.

Without assuming the condition that $u_1 \leq c_1$, we show the uniform lower bound of the distance between $\Gamma_{\rm shock}$ and $\Gamma_{\rm wedge}$ away from P_3 for any $\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2}]$: For any small $r > 0$, there exists $C_r > 0$ such that

$$\operatorname{dist}(\Gamma_{\rm shock}, \Gamma_{\rm wedge} \setminus B_r(P_3)) > \frac{1}{C_r} \tag{3.2.21}$$

for every admissible solution with $\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2}]$.

Recall that estimates (3.2.20)–(3.2.21) hold for the wedge angles $\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2}]$, and the constants in these estimates depend on $\theta_{\rm w}^*$. However, for the application in §3.2.4.4, we need an estimate of the distance between $\Gamma_{\rm shock}$ and $\Gamma_{\rm wedge}$ which holds for all the wedge angles $\theta_{\rm w} \in [\theta_{\rm w}^{\rm s}, \frac{\pi}{2}]$. We do not assume that $u_1 \le c_1$, which implies that our estimate has to be made away from P_3, as we discussed earlier. Moreover, for $\theta_{\rm w} = \theta_{\rm w}^{\rm s}$, $\Gamma_{\rm shock}$ and $\Gamma_{\rm wedge}$ meet at P_0, which implies that our estimate has to be made away from P_0. Then we obtain the following estimate: For every small $r > 0$, there exists $C_r > 0$ depending on $(\rho_0, \rho_1, \gamma, r)$ such that

$$\operatorname{dist}\left(\Gamma_{\rm shock},\ \Gamma_{\rm wedge} \setminus (B_r(P_0) \cup B_r(P_3))\right) \ge \frac{1}{C_r} \tag{3.2.22}$$

for any admissible solution of **Problem 2.6.1** with the wedge angle $\theta_{\rm w} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$.

If $u_1 > c_1$, the *critical angle* $\theta_{\rm w}^{\rm c}$ in Theorem 2.6.5 is defined as follows:

$$\theta_{\rm w}^{\rm c} = \inf \mathcal{A},$$

where

$$\mathcal{A} := \left\{ \theta_{\rm w}^* \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2}] : \begin{array}{l} \exists\, \varepsilon > 0 \text{ so that } \operatorname{dist}(\Gamma_{\rm shock}, \Gamma_{\rm wedge}) \ge \varepsilon \text{ for} \\ \text{any admissible solution with } \theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2}] \end{array} \right\}.$$

Since the normal reflection solution is the unique admissible solution for $\theta_{\rm w} = \frac{\pi}{2}$, the set of admissible solutions with angles $\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2}]$ is non-empty for any $\theta_{\rm w}^* \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2}]$. Moreover, since $\operatorname{dist}(\Gamma_{\rm shock}, \Gamma_{\rm wedge}) > 0$ for the normal reflection solution, we conclude that $\frac{\pi}{2} \in \mathcal{A}$, *i.e.*, $\mathcal{A} \ne \emptyset$. Furthermore, we show that $\theta_{\rm w}^{\rm c} < \frac{\pi}{2}$ by using that $\Gamma_{\rm shock} \to \Gamma_{\rm shock}^{\rm norm}$ as $\theta_{\rm w} \to \frac{\pi}{2}$; see §3.2.4.1. Then it follows directly from the definition of $\theta_{\rm w}^{\rm c}$ that, for each $\theta_{\rm w}^* \in (\theta_{\rm w}^{\rm c}, \frac{\pi}{2})$, there exists $C > 0$ such that

$$\operatorname{dist}(\Gamma_{\rm shock}, \Gamma_{\rm wedge}) \ge \frac{1}{C} \tag{3.2.23}$$

for any admissible solution with $\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2}]$.

We note that property (3.2.21) is employed in the proof of Theorem 2.6.5 to show the existence of the *attached* solution for $\theta_{\rm w} = \theta_{\rm w}^{\rm c}$ when $\theta_{\rm w}^{\rm c} > \theta_{\rm w}^{\rm s}$.

3.2.4.4 Uniform positive lower bound for the distance between $\Gamma_{\rm shock}$ and the sonic circle of state (1)

Employing the detail structure of the potential flow equation (2.2.8) for a solution φ that is close to a uniform state near its sonic circle, and the property of admissible solutions that $\varphi \le \varphi_1$ holds in Ω by (3.2.4), we use estimate (3.2.22) to prove that there exists $C > 0$ such that

$$\operatorname{dist}(\Gamma_{\rm shock}, B_{c_1}(\mathcal{O}_1)) > \frac{1}{C} \tag{3.2.24}$$

for any admissible solution φ of **Problem 2.6.1** with $\theta_{\rm w} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$, where $\mathcal{O}_1 = (u_1, 0)$ is the center of the sonic circle of state (1).

Estimate (3.2.24) is crucial, especially since it is employed for the ellipticity estimate in §3.2.4.5 below and the uniform estimate of the lower bound of the gradient jump across $\Gamma_{\rm shock}$ in the radial direction with respect to the sonic circle of state (1); see (3.2.29).

3.2.4.5 **Uniform estimate of the ellipticity of equation** (2.2.8) **in** Ω **up to** $\Gamma_{\rm shock}$

Set the Mach number

$$M^2 = \frac{|D\varphi|^2}{c^2} = \frac{|D\varphi|^2}{\rho_0^{\gamma-1} - (\gamma-1)(\varphi + \frac{1}{2}|D\varphi|^2)}, \tag{3.2.25}$$

where we have used (2.2.9) for the second equality. Note that, for an admissible solution of **Problem 2.6.1**, by (3.2.2),

$$M \in C(\overline{\Omega}) \cap C^2(\overline{\Omega} \setminus (\overline{\Gamma_{\rm sonic}} \cup \{P_2, P_3\})).$$

We conclude that there exists $\mu > 0$ depending only on (ρ_0, ρ_1, γ) such that, if φ is an admissible solution of **Problem 2.6.1** with $\theta_{\rm w} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$, then

$$M^2(\boldsymbol{\xi}) \le 1 - \mu\,\mathrm{dist}(\boldsymbol{\xi}, \Gamma_{\rm sonic}) \qquad \text{for all } \boldsymbol{\xi} \in \Omega(\varphi). \tag{3.2.26}$$

To achieve this, we show that a maximum of $M^2 + \mu d$, which is close to one, cannot be attained on $\overline{\Omega} \setminus \overline{\Gamma_{\rm sonic}}$, where $\mu > 0$ is a small constant and $d(\boldsymbol{\xi})$ is an appropriate function comparable with $\mathrm{dist}(\boldsymbol{\xi}, \Gamma_{\rm sonic})$.

First, the maximum of $M^2 + \mu d$ cannot be attained on $\Omega \cup \Gamma_{\rm wedge} \cup \Gamma_{\rm sym}$ if $1 - M^2 \ge 0$ is sufficiently small; see §5.2–§5.3 below. Also, we explicitly check that $M = 0$ at P_3 so that, by choosing μ small, we conclude that $M^2 + \mu d$ is small at P_3.

Thus, it remains to show that the maximum of $M^2 + \mu d$ cannot be attained on $\Gamma_{\rm shock} \cup \{P_2\}$. Crucially, the result of §3.2.4.4 on the positive lower bound on the distance between $\Gamma_{\rm shock}$ and the sonic circle of state (1) is employed, since it allows us to estimate that state (1) is *sufficiently hyperbolic* on the other side of $\Gamma_{\rm shock}$. Then, assuming that the maximum of $M^2 + \mu d$ is attained at $P \in \Gamma_{\rm shock}$, we use the first-order conditions at the maximum point, $\partial_{\boldsymbol{\tau}}(M^2 + \mu d)(P) = 0$ and $\partial_{\boldsymbol{\nu}}(M^2 + \mu d)(P) \le 0$ (where $\boldsymbol{\nu}$ is the interior normal to $\Gamma_{\rm shock}$), the fact that the equation holds at P, and the Rankine-Hugoniot condition:

$$\partial_{\boldsymbol{\tau}}\Big((\rho D\varphi - \rho_1 D\varphi_1) \cdot \frac{D\varphi - D\varphi_1}{|D\varphi - D\varphi_1|}\Big) = 0 \qquad \text{on } \Gamma_{\rm shock},$$

to obtain the four relations at P for the three components of $D^2\varphi$, which leads to a contradiction. Thus, the maximum of $M^2 + \mu d$ cannot be attained on $\Gamma_{\rm shock}$. The maximum at P_2 is handled similarly, since P_2 can be regarded as an interior point of $\Gamma_{\rm shock}$ after extending the solution by even reflection with respect to the ξ_1–axis. This completes the proof of (3.2.26).

Write equation (2.2.8) in the form:

$$\mathrm{div}\mathcal{A}(D\varphi, \varphi, \boldsymbol{\xi}) + \mathcal{B}(D\varphi, \varphi, \boldsymbol{\xi}) = 0 \tag{3.2.27}$$

with $\mathbf{p} = (p_1, p_2) \in \mathbb{R}^2$ and $z \in \mathbb{R}$, where

$$\mathcal{A}(\mathbf{p}, z, \boldsymbol{\xi}) \equiv \mathcal{A}(\mathbf{p}, z) := \rho(|\mathbf{p}|^2, z)\mathbf{p}, \quad \mathcal{B}(\mathbf{p}, z, \boldsymbol{\xi}) \equiv \mathcal{B}(\mathbf{p}, z) := 2\rho(|\mathbf{p}|^2, z)$$

with function $\rho(|\mathbf{p}|^2, z)$ defined by (2.2.9). We restrict $(\mathbf{p}, z)$ in a set such that (2.2.9) is defined, *i.e.*, satisfying $\rho_0^{\gamma-1} - (\gamma-1)(z + \frac{1}{2}|\mathbf{p}|^2) \geq 0$.

As a corollary of (3.2.26), we employ (3.2.18) to conclude that, for any $\theta_{\rm w}^* \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$, there exists $C > 0$ depending only on (ρ_0, ρ_1, γ) such that, if φ is an admissible solution of **Problem 2.6.1** with $\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2})$, equation (3.2.27) satisfies the strict ellipticity condition:

$$\frac{\mathrm{dist}(\boldsymbol{\xi}, \Gamma_{\rm sonic})}{C}|\boldsymbol{\kappa}|^2 \leq \sum_{i,j=1}^{2} \mathcal{A}_{p_j}^i(D\varphi(\boldsymbol{\xi}), \varphi(\boldsymbol{\xi}), \boldsymbol{\xi})\kappa_i\kappa_j \leq C|\boldsymbol{\kappa}|^2 \tag{3.2.28}$$

for any $\boldsymbol{\xi} \in \Omega$ and $\boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbb{R}^2$. Note that the ellipticity degenerates on $\Gamma_{\rm sonic}$.

3.2.5 Regularity and related uniform estimates

We consider $\theta_{\rm w}^* \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$ for Case $u_1 \leq c_1$ and $\theta_{\rm w}^* \in (\theta_{\rm w}^{\rm c}, \frac{\pi}{2})$ for Case $u_1 > c_1$. Then, from §3.2.4.3, we obtain the uniform estimate:

$$\mathrm{dist}(\Gamma_{\rm shock}, \Gamma_{\rm wedge}) \geq \frac{1}{C}$$

for any admissible solution with $\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2})$. This fixes the geometry of Ω for such a solution.

With the geometry of Ω and the strict ellipticity controlled, we can conclude the estimates in the properly scaled and weighted $C^{k,\alpha}$–spaces. We perform the estimates separately away from $\Gamma_{\rm sonic}$ where the equation is uniformly elliptic, and near $\Gamma_{\rm sonic}$ where the ellipticity degenerates.

3.2.5.1 Weighted $C^{k,\alpha}$–estimates away from $\Gamma_{\rm sonic}$

Away from the ε-neighborhood of $\Gamma_{\rm sonic}$, we use the uniform ellipticity to estimate admissible solutions with the bounds independent of the solution and the wedge angle $\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2})$. Also, in order to avoid the difficulty related to the corner at point P_2 of intersection of $\Gamma_{\rm shock}$ and $\Gamma_{\rm sym}$, we extend the elliptic domain Ω by reflection over the symmetry line and use the even extension of the solution. From the structure of the potential flow equation (2.2.8), it follows that (2.2.8) is satisfied in the extended domain, and the Rankine-Hugoniot conditions (2.2.13)–(2.2.14) are satisfied on the extended shock. Now the boundary part $\Gamma_{\rm sym}$ lies in the interior of the extended domain $\Omega^{\rm ext}$, and P_2 is the interior point of the extended shock $\Gamma^{\rm ext}_{\rm shock}$.

In the argument below, we consider the points of $\Omega^{\rm ext}$ which are on the distance, $d > 0$, from the original and reflected sonic arcs, and for which the constants in the estimates depend on d.

We use the estimates obtained in §3.2.4.2–§3.2.4.3 to control the geometry of domain Ω. Then, for any $\alpha \in (0,1)$, the $C^{2,\alpha}$–estimates in the interior of $\Omega^{\rm ext}$ and near $\Gamma_{\rm wedge}$ and the reflected $\Gamma^-_{\rm wedge}$ (away from corner P_3) follow from the standard elliptic theory, where we use the homogeneous Neumann boundary condition on $\Gamma_{\rm wedge}$, the uniform estimate of the distance between $\Gamma_{\rm shock}$ and $\Gamma_{\rm wedge}$, and the Lipschitz estimates of the solution.

For the estimates of the shock curve $\Gamma_{\rm shock}$ and the solution near $\Gamma_{\rm shock}$ (away from the ε–neighborhood of $\Gamma_{\rm sonic}$), we first show that the function:

$$\bar{\phi} := \varphi_1 - \varphi$$

is uniformly monotone in a uniform neighborhood of $\Gamma_{\rm shock}$ in the radial direction with respect to the center of the sonic circle $\mathcal{O}_1$ of state (1), *i.e.*, there exist $\delta, \sigma > 0$ such that

$$\partial_r \bar{\phi} \le -\delta \qquad \text{in } \ \mathcal{N}_\sigma(\Gamma^{\rm ext}_{\rm shock}) \cap \Omega^{\rm ext}. \tag{3.2.29}$$

Note that $\bar{\phi} = 0$ on $\Gamma_{\rm shock}$, by the Rankine-Hugoniot condition (2.2.14), and that $\bar{\phi} > 0$ in $\Omega^{\rm ext}$. Using that φ is an admissible solution of **Problem 2.6.1**, we show that the extended shock $\Gamma^{\rm ext}_{\rm shock}$ is a graph in the radial direction in the polar coordinates (r, θ) with center $\mathcal{O}_1$. With this, working in the (r,θ)–coordinates, we inductively make the $C^{k,\alpha}$–estimates of $\Gamma^{\rm ext}_{\rm shock}$ and φ near $\Gamma^{\rm ext}_{\rm shock}$, for $k = 1, 2, \dots$, as follows: φ satisfies the uniformly elliptic equation in $\Omega^{\rm ext}$ (away from the original and reflected sonic arcs) and an oblique boundary condition on $\Gamma^{\rm ext}_{\rm shock}$ from the Rankine-Hugoniot conditions. The nonlinear equation and boundary condition are given by smooth functions. Now we use the estimates due to Lieberman [192] (stated in §4.3 below) for two-dimensional elliptic equations with nonlinear boundary conditions, which show that the regularity of the solution is higher than that of the boundary. More precisely, from (3.2.17) and (3.2.29), we obtain the Lipschitz estimates of $\Gamma^{\rm ext}_{\rm shock}$ and of φ in $\Omega^{\rm ext}$. Then, from Theorem 4.3.2, we obtain the $C^{1,\alpha}$–estimates of φ near $\Gamma^{\rm ext}_{\rm shock}$ for some $\alpha \in (0,1)$. Moreover, by (3.2.29) and the fact that $\varphi = \varphi_1$ on $\Gamma^{\rm ext}_{\rm shock}$, we obtain the $C^{1,\alpha}$–estimates of $\Gamma^{\rm ext}_{\rm shock}$. Now, using Corollary 4.3.5, we obtain the $C^{2,\alpha}$–estimates of φ near $\Gamma^{\rm ext}_{\rm shock}$, which in turn implies the $C^{2,\alpha}$–estimates of $\Gamma^{\rm ext}_{\rm shock}$. We repeat this argument inductively for $k = 2, 3, \dots$.

Finally, we obtain the $C^{1,\alpha}$–estimates near corner P_3 for sufficiently small $\alpha > 0$ by using the results of Lieberman [189], stated in Theorem 4.3.13 below. For that, we work on the original domain Ω instead of the extended domain $\Omega^{\rm ext}$, and use the homogeneous Neumann conditions on $\Gamma_{\rm wedge} \cup \Gamma_{\rm sym}$. This is crucial, because the angle at the corner point P_3 of $\partial\Omega$ is less than π in this way, which allows us to obtain the $C^{1,\alpha}$–estimates.

Combining all the above estimates, we obtain

$$\varphi \in C^k(\overline{\Omega} \setminus (\mathcal{N}_d(\Gamma_{\text{sonic}}) \cup \{P_3\})) \cap C^{1,\alpha}(\overline{\Omega} \setminus \mathcal{N}_d(\Gamma_{\text{sonic}}))$$

and

$$\overline{\Gamma_{\text{shock}}} \setminus \mathcal{N}_d(\Gamma_{\text{sonic}}) \in C^k \qquad \text{for } k = 1, 2, \dots,$$

with uniform estimates.

3.2.5.2 Weighted and scaled $C^{2,\alpha}$–estimates near Γ_{sonic}

Near Γ_{sonic}, *i.e.*, in $\mathcal{N}_{\varepsilon_1}(\Gamma_{\text{sonic}}) \cap \Omega$ for sufficiently small $\varepsilon_1 > 0$, it is convenient to work in the coordinates flattening Γ_{sonic}. We consider the polar coordinates (r, θ) with respect to $\mathcal{O}_2 = (u_2, v_2)$, note that Γ_{sonic} is an arc of the circle with radius $r = c_2$ and center $\mathcal{O}_2$, and define

$$(x, y) = (c_2 - r, \theta).$$

Then there exists $\varepsilon_0 \in (0, \varepsilon_1)$ such that, for any $\varepsilon \in (0, \varepsilon_0]$,

$$\begin{aligned}
&\Omega_\varepsilon := \Omega \cap \mathcal{N}_{\varepsilon_1}(\Gamma_{\text{sonic}}) \cap \{x < \varepsilon\} = \{0 < x < \varepsilon,\ \theta_{\text{w}} < y < \hat{f}(x)\},\\
&\Gamma_{\text{sonic}} = \partial\Omega_\varepsilon \cap \{x = 0\},\\
&\Gamma_{\text{wedge}} \cap \partial\Omega_\varepsilon = \{0 < x < \varepsilon,\ y = \theta_{\text{w}}\},\\
&\Gamma_{\text{shock}} \cap \partial\Omega_\varepsilon = \{0 < x < \varepsilon,\ y = \hat{f}(x)\}
\end{aligned} \tag{3.2.30}$$

for some $\hat{f}(x)$ defined on $(0, \varepsilon_0)$. We perform the estimates in terms of the function:

$$\psi = \varphi - \varphi_2.$$

Note that $\psi(0, y) \equiv 0$, since $\varphi = \varphi_2$ on Γ_{sonic} by Definition 3.2.1(ii).

We write the potential flow equation (2.2.8) in terms of ψ in the (x, y)–coordinates. Then (3.2.28) implies that there exists $\delta > 0$ so that, for each admissible solution,

$$\psi_x \le \frac{2 - \delta}{1 + \gamma} x \qquad \text{in } \Omega_\varepsilon.$$

Combining this with the estimate that $\psi_x \ge 0$ shown in §3.2.3, we obtain

$$|\psi_x| \le Cx \qquad \text{in } \mathcal{N}_\varepsilon(\Gamma_{\text{sonic}}) \cap \Omega.$$

From this, using the monotonicity cone of ψ discussed in §3.2.3, we obtain

$$|\psi_y| \le Cx.$$

Now, since $|D\psi| \le Cx$, we can modify equation (2.2.8) to obtain that any admissible solution ψ satisfies an equation in $\mathcal{N}_\varepsilon(\Gamma_{\rm sonic})$:

$$\sum_{i,j=1}^{2} A_{ij}(D\psi,\psi,x)D_{ij}\psi + \sum_{i=1}^{2} A_i(D\psi,\psi,x)D_i\psi = 0 \tag{3.2.31}$$

with smooth $(A_{ij}, A_i)(\mathbf{p}, z, x)$ (independent of y), which is of the degenerate ellipticity structure:

$$\lambda|\boldsymbol{\xi}|^2 \le A_{11}(\mathbf{p},z,x)\frac{\xi_1^2}{x} + 2A_{12}(\mathbf{p},z,x)\frac{\xi_1\xi_2}{\sqrt{x}} + A_{22}(\mathbf{p},z,x)\xi_2^2 \le \frac{1}{\lambda}|\boldsymbol{\xi}|^2 \tag{3.2.32}$$

for $(\mathbf{p}, z) = (D\psi,\psi)(x,y)$ and for any $(x,y) \in \Omega_\varepsilon$, where we recall (3.2.30).

We use (3.2.31)–(3.2.32) for the estimates in the weighted and scaled $C^{2,\alpha}$–norms with the weights depending on x, which reflect the ellipticity structure. One way to define these norms is as follows: For any $(x_0, y_0) \in \Omega_\varepsilon$ and $\rho \in (0,1)$, let

$$\begin{aligned} \tilde{R}_\rho^{(x_0,y_0)} &:= \Big\{(s,t) \ : \ |s-x_0| < \frac{\rho}{4}x_0, |t-y_0| < \frac{\rho}{4}\sqrt{x_0}\Big\}, \\ R_\rho^{(x_0,y_0)} &:= \tilde{R}_\rho^{(x_0,y_0)} \cap \Omega. \end{aligned} \tag{3.2.33}$$

Rescale ψ from $R_\rho^{(x_0,y_0)}$ to the portion of the square with side-length 2ρ, *i.e.*, define the rescaled function:

$$\psi^{(x_0,y_0)}(S,T) = \frac{1}{x_0^2}\psi(x_0 + \frac{x_0}{4}S, y_0 + \frac{\sqrt{x_0}}{4}T) \qquad \text{in } Q_\rho^{(x_0,y_0)}, \tag{3.2.34}$$

where

$$Q_\rho^{(x_0,y_0)} := \Big\{(S,T) \in (-\rho,\rho)^2 \ : \ (x_0 + \frac{x_0}{4}S, \hat{y}_0 + \frac{\sqrt{x_0}}{4}T) \in \Omega\Big\}.$$

The *parabolic* norm of $\|\psi\|^{(\rm par)}_{2,\alpha,\Omega_\varepsilon}$ is the supremum over $(x_0,y_0) \in \Omega_\varepsilon$ of norms $\|\psi^{(x_0,y_0)}\|_{2,\alpha,\overline{Q_1^{(x_0,y_0)}}}$. Note that the estimate in norm $\|\cdot\|^{(\rm par)}_{2,\alpha,\Omega_\varepsilon}$ implies the $C^{1,1}$–estimate in Ω_ε.

In order to estimate $\|\psi\|^{(\rm par)}_{2,\alpha,\Omega_\varepsilon}$, we need to obtain the $C^{2,\alpha}$–estimates of the rescaled functions $\psi^{(x_0,y_0)}$. By the standard covering argument, it suffices to consider three cases:

(i) The interior rectangle: $R_{1/10}^{(x_0,y_0)} \subset \Omega$ for $(x_0,y_0) \in \Omega_\varepsilon$;

(ii) Rectangle $R_{1/2}^{(x_0,y_0)}$ centered at $(x_0,y_0) \in \Gamma_{\rm wedge} \cap \partial\Omega_\varepsilon$ (on the wedge);

(iii) Rectangle $R_{1/2}^{(x_0,y_0)}$ centered at $(x_0,y_0) \in \Gamma_{\rm shock} \cap \partial\Omega_\varepsilon$ (on the shock).

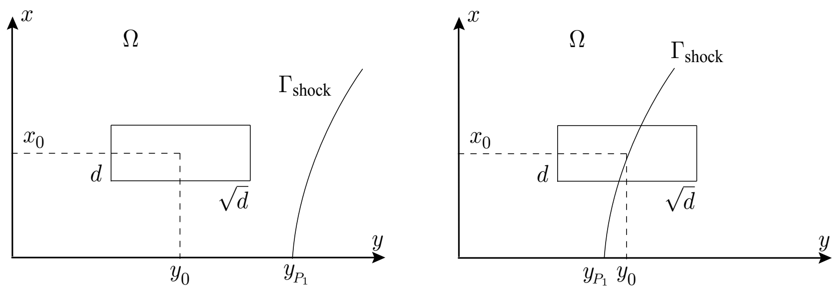


Figure 3.5: Rectangles in Cases (i) and (iii)

See Fig. 3.5 for $d = \frac{x_0}{4}$.

The gradient estimate $|D\psi| \le Cx$ and the property that $\psi(0, y) \equiv 0$ imply

$$|\psi| \le Cx^2,$$

so that

$$\|\psi^{(x_0,y_0)}\|_{L^\infty(\overline{Q_1^{(x_0,y_0)}})} \le C \qquad \text{for any } (x_0, y_0) \in \Omega_\varepsilon.$$

Also, writing equation (3.2.31) in terms of the rescaled function $\psi^{(x_0,y_0)}$ and using the ellipticity structure (3.2.32), we see that $\psi^{(x_0,y_0)}$ satisfies a uniformly elliptic homogeneous equation in $Q_1^{(x_0,y_0)}$, *with the ellipticity constants and certain Hölder norms of the coefficients independent of* (x_0, y_0). Then the $C^{2,\alpha}$–estimates of $\psi^{(x_0,y_0)}$ in the smaller square $Q_{1/20}^{(x_0,y_0)}$ in Case (i) follow from the interior elliptic estimates. In Case (ii), in addition to the equation, we use the boundary condition $\partial_{\boldsymbol{\nu}}\psi = 0$ on Γ_{wedge}, which holds under rescaling.

In Case (iii), we need to make the estimates up to Γ_{shock}, *i.e.*, the free boundary, for which only the Lipschitz estimates are *a priori* available. Thus, we rescale the region as in Cases (i)–(ii) to obtain the uniformly elliptic equation, and then follow the argument in §3.2.5.1 for the estimates near Γ_{shock}.

Owing to the non-isotropic rescaling (3.2.34), some difference from the estimates in §3.2.5.1 appears because:

(a) The Lipschitz estimate for ψ, combined with (3.2.34), does *not* imply the uniform Lipschitz estimate of $\psi^{(x_0,y_0)}$ with respect to $(x_0, y_0) \in \Gamma_{\text{shock}} \cap \partial\Omega_\varepsilon$. The estimate blows up as $d = \frac{x_0}{4} \to 0$, *i.e.*, for the rectangles close to Γ_{sonic}. Thus we have, *a priori*, only the L^∞ bound of $\psi^{(x_0,y_0)}$ uniform with respect to $(x_0, y_0) \in \Gamma_{\text{shock}} \cap \partial\Omega_\varepsilon$.

(b) The boundary condition for ψ on Γ_{shock} is uniformly oblique up to P_1 (*i.e.*, up to $x = 0$). However, the obliqueness of the rescaled condition for $\psi^{(x_0,y_0)}$ on Γ_{shock} degenerates as $d \to 0$. On the other hand, we can show that the rescaled boundary condition has an *almost tangential structure* with the constants uniform with respect to $(x_0, y_0) \in \Gamma_{\text{shock}} \cap \partial\Omega_\varepsilon$.

For these reasons, we cannot use the estimates of [192] (stated in §4.3 below) for the oblique derivative problem, with the bounds depending on the $C^{0,1}$–norm of the solution. Instead, we employ the estimates for the problem with *almost tangential structure*, when only the L^∞ bound of the solution is *a priori* known; see Theorems 4.2.4 and 4.2.8. These results give the gain-in-regularity similar to the estimates in [192], *i.e.*, we obtain the $C^{1,\alpha}$–estimate of the solution for the C^1–boundary with the Lipschitz estimate and the $C^{2,\alpha}$–estimate of the solution for the $C^{1,\alpha}$–boundary. This allows us to obtain the $C^{2,\alpha}$–estimates of Γ_{shock} and $\psi^{(x_0,y_0)}$ in Case (iii).

3.2.6 Existence of the supersonic regular reflection-diffraction configurations up to the sonic angle

Once the *a priori* estimates are established, we obtain a solution to **Problem 2.6.1** as a fixed point of an iteration map. The existence of a fixed point follows from the Leray-Schauder degree theory (*cf.* §3.4).

In order to apply the degree theory, the iteration set should be bounded and open in an appropriate function space (actually, in its product with the parameter space, *i.e.*, interval $[\theta_{\text{w}}^*, \frac{\pi}{2}]$ of the wedge angles), the iteration map should be defined and continuous on the closure of the iteration set, and any fixed point of the iteration map should not occur on the boundary of the iteration set. We choose this function space according to the norms and the other quantities in the *a priori* estimates. Then the *a priori* estimates allow us to conclude that the fixed point cannot occur on the boundary of the iteration set, if the bounds defining the iteration set are chosen appropriately large or small, depending on the context and the *a priori* estimates. This can be done for any $\theta_{\text{w}}^* \in (\theta_{\text{w}}^{\text{s}}, \frac{\pi}{2})$ if $u_1 \le c_1$ and for any $\theta_{\text{w}}^* \in (\theta_{\text{w}}^{\text{c}}, \frac{\pi}{2})$ if $u_1 > c_1$.

In our case, there is an extra issue of connecting the admissible solutions with the normal reflection solution in the setup convenient for the application of the degree theory. We use the strict monotonicity properties of the admissible solutions (proved as a part of the *a priori* estimates) in our definition of the iteration set. These strict monotonicity properties can be made uniform for any wedge angle θ_{w} away from $\frac{\pi}{2}$ and any point away from the appropriate parts of the boundary of the elliptic region by using the compactness of the set of admissible solutions, which is a corollary of the *a priori* estimates. However, the monotonicities become nonstrict when the wedge angle θ_{w} is $\frac{\pi}{2}$, *i.e.*, at the normal reflection solution. Then, for the wedge angles near $\frac{\pi}{2}$, we use the fact that *the admissible solutions converge to the normal reflection solution as* θ_{w} *tends to* $\frac{\pi}{2}$.

From this fact, we can derive the estimates similar to Chen-Feldman [54] for the admissible solutions and the *approximate* solutions for θ_{w} near $\frac{\pi}{2}$. Then, for the wedge angle θ_{w} near $\frac{\pi}{2}$, the iteration set $\mathcal{K}_{\theta_{\text{w}}}$ is a small neighborhood of the normal reflection solution, where the norms used and the size of neighborhood are related to the estimates of Chapters 9–11. For the wedge angle θ_{w} away from $\frac{\pi}{2}$, the iteration set $\mathcal{K}_{\theta_{\text{w}}}$ is defined by the bounds in the appropriate norms related to the *a priori* estimates and by the lower bounds of certain directional derivatives, corresponding to the strict monotonicity properties so that the actual solution cannot be on the boundary of the iteration set according to the *a priori* estimates. These two definitions are combined into one setup, with the bounds depending continuously on the wedge angle $\theta_{\rm w}$.

Also, since the elliptic domain Ω depends on the solution, we define the iteration set in terms of the functions on the unit square $Q^{\rm iter} = (0,1)^2 := (0,1)\times(0,1)$ and, for each such function and wedge angle $\theta_{\rm w}$, define the elliptic domain Ω of the *approximate solution* and a smooth invertible map from $Q^{\rm iter}$ to Ω, where Ω is of the same structure as the elliptic region of supersonic reflection configurations; see §2.4.2 and Fig. 2.3. This defines the iteration set:

$$\mathcal{K} = \cup_{\theta_{\rm w}\in[\theta_{\rm w}^*,\frac{\pi}{2}]}\mathcal{K}_{\theta_{\rm w}} \times \{\theta_{\rm w}\}$$

with $\mathcal{K}_{\theta_{\rm w}} \subset \mathcal{C}$, where $\mathcal{C}$ is a weighted and scaled $C^{2,\alpha}$–type space on $Q^{\rm iter}$ so that $\mathcal{C} \subset C^{2,\alpha}(Q^{\rm iter}) \cap C^{1,\alpha}(\overline{Q^{\rm iter}})$. For each $(u,\theta_{\rm w}) \in \mathcal{K}$, the map from $Q^{\rm iter}$ to $\Omega(u,\theta_{\rm w})$ can be extended to the smooth and smoothly invertible map $\overline{Q^{\rm iter}} \mapsto \overline{\Omega}$, where the sides of square $Q^{\rm iter}$ are mapped to the boundary parts $\Gamma_{\rm sonic}$, $\Gamma_{\rm shock}$, $\Gamma_{\rm sym}$, and $\Gamma_{\rm wedge}$ of Ω.

The iteration map $\mathcal{I}$ is defined as follows: Given $(u,\theta_{\rm w}) \in \overline{\mathcal{K}}$, define the corresponding *elliptic domain* $\Omega = \Omega(u,\theta_{\rm w})$ by both mapping from the unit square $Q^{\rm iter}$ to the *physical plane* and determining the iteration $\Gamma_{\rm shock}$ depending on $(u,\theta_{\rm w})$, and set up a boundary value problem in Ω for an elliptic equation that is degenerate near $\Gamma_{\rm sonic}$. Let $\hat{\varphi}$ be the solution of the boundary value problem in Ω. Then we define $\hat{u}$ on $Q^{\rm iter}$ by mapping $\hat{\varphi}$ back in such a way that the gain-in-regularity of the solution is preserved. This requires some care, since the original mapping between $Q^{\rm iter}$ and the *physical domain* is defined by u and hence has a lower regularity. Then the iteration map is defined by

$$\mathcal{I}(u,\theta_{\rm w}) = \hat{u}.$$

The boundary value problem in the definition of $\mathcal{I}$ is defined so that, at the fixed point $u = \hat{u}$, its solution satisfies the potential flow equation (2.2.8) with the ellipticity cutoff in a small neighborhood of $\Gamma_{\rm sonic}$, and both the Rankine-Hugoniot conditions (2.2.13)–(2.2.14) on $\Gamma_{\rm shock}$ and $D\hat{\varphi}\cdot\boldsymbol{\nu} = 0$ on $\Gamma_{\rm wedge}\cup\Gamma_{\rm sym}$. On the sonic arc $\Gamma_{\rm sonic}$ that is a fixed boundary, we can prescribe only one condition, the Dirichlet condition $\hat{\varphi} = \varphi_2$. However, it is not sufficient to have the potential flow equation (2.2.8) satisfied across $\Gamma_{\rm sonic}$. Indeed, the Rankine-Hugoniot conditions (2.2.13)–(2.2.14) need to be satisfied for $\hat{\varphi}$ and φ_2 on $\Gamma_{\rm sonic}$, and condition (2.2.13) implies that $D\varphi = D\varphi_2$ on $\Gamma_{\rm sonic}$, since φ_2 is sonic on $\Gamma_{\rm sonic}$. Thus, we need to prove that the last property holds for the solution of the iteration problem (at least for the fixed point). In this proof, we use the elliptic degeneracy of the iteration equation in Ω near $\Gamma_{\rm sonic}$ by obtaining the estimates of $\hat{\psi} = \hat{\varphi}-\varphi_2$ in the norms of $\|\cdot\|^{\rm (par)}_{2,\alpha,\mathcal{N}_\varepsilon(\Gamma_{\rm sonic})\cap\Omega}$ introduced in §3.2.5.2. These estimates imply that $D\hat{\psi} = 0$ on $\Gamma_{\rm sonic}$, *i.e.*, $D\varphi = D\varphi_2$ on $\Gamma_{\rm sonic}$.

Furthermore, the other conditions required in the definition of an admissible solution φ (including the inequalities, $\varphi_2 \le \varphi \le \varphi_1$, and the monotonicity properties) are satisfied for $\hat{\varphi}$ for any wedge angle $\theta_{\rm w}$ away from $\frac{\pi}{2}$ and for any point away from the appropriate parts of the boundary of the elliptic domain.

Then we prove the following facts:

(i) Any fixed point $u = \mathcal{I}(u, \theta_{\rm w})$, mapped to the *physical plane*, is an admissible solution φ. For that, we remove the ellipticity cutoff and prove the inequalities and monotonicity properties mentioned above for the regions and the wedge angles where they are not readily known from the definition of the iteration set. The fact that these estimates need to be proved only in the localized regions is crucial. This localization is achieved by using the uniform bounds and monotonicity properties which are a part of the *a priori* estimates.

(ii) The iteration set is open. We prove this by showing the existence of a solution for the iteration boundary value problem determined by any (v, θ) in a sufficiently small neighborhood of any $(u, \theta_{\rm w}) \in \mathcal{K}$.

(iii) The iteration map is compact. We prove this by using the gain-in-regularity of the solution of the iteration boundary value problem.

(iv) Any fixed point of the iteration map cannot occur on the boundary of the iteration set. This is shown by using the *a priori* estimates, which can be applied since the fixed point is, by (i) above, an admissible solution.

(v) The normal reflection solution $u^{(\rm norm)}$, expressed on the unit square, is in the iteration set, which shows that the iteration set is non-empty for $\theta_{\rm w} = \frac{\pi}{2}$.

Now the Leray-Schauder degree theory (see §3.4) guarantees that the fixed point index:

$$\mathbf{Ind}(\mathcal{I}^{(\theta_{\rm w})}, \overline{\mathcal{K}}(\theta_{\rm w}))$$

of the iteration map on the iteration set (for given $\theta_{\rm w}$) is independent of the wedge angle $\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2}]$.

It remains to show that $\mathbf{Ind}(\mathcal{I}^{(\theta_{\rm w})}, \overline{\mathcal{K}}(\theta_{\rm w}))$ is nonzero. In fact, at $\theta_{\rm w} = \frac{\pi}{2}$, we show that $\mathcal{I}_{\frac{\pi}{2}}(v) = u^{(\rm norm)}$ for any $v \in \mathcal{K}_{\frac{\pi}{2}}$. This implies that $\mathbf{Ind}(\mathcal{I}^{(\frac{\pi}{2})}, \overline{\mathcal{K}}(\frac{\pi}{2})) = 1$.

Then, for any $\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2}]$, $\mathbf{Ind}(\mathcal{I}^{(\theta_{\rm w})}, \overline{\mathcal{K}}(\theta_{\rm w})) = 1$, which implies that a fixed point exists. Moreover, the fixed point is an admissible solution of **Problem 2.6.1**.

Since $\theta_{\rm w}^*$ is arbitrary in interval $(\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$ if $u_1 \le c_1$ and in $(\theta_{\rm w}^{\rm c}, \frac{\pi}{2})$ if $u_1 > c_1$, we obtain the existence of admissible solutions in the intervals of the wedge angles $\theta_{\rm w}$ indicated in Theorems 2.6.3 and 2.6.5.

Moreover, for Case $u_1 > c_1$, if $\theta_{\rm w}^{\rm c} > \theta_{\rm w}^{\rm s}$, then, from the definition of $\theta_{\rm w}^{\rm c}$ in §3.2.4.3, there exists a sequence $\theta_{\rm w}^{(i)} \in [\theta_{\rm w}^{\rm c}, \frac{\pi}{2})$ with $\lim_{i\to\infty} \theta_{\rm w}^{(i)} = \theta_{\rm w}^{\rm c}$ and a corresponding admissible solution $\varphi^{(i)}$ with the wedge angle $\theta_{\rm w}^{(i)}$ such that

$$\lim_{i\to\infty} \mathrm{dist}(\Gamma_{\rm shock}^{(i)}, \Gamma_{\rm wedge}^{(i)}) = 0.$$

Taking the uniform limit in a subsequence of $\varphi^{(i)}$ and employing the geometric properties of the free boundary (shock) proved in §3.2.4.3, including (3.2.21), and the regularity of admissible solutions and involved shocks, we obtain an attached solution for the wedge angle $\theta_{\rm w} = \theta_{\rm w}^{\rm c}$ as asserted in Theorem 2.6.5.

In Part III, we give the detailed proofs of the steps described above for the main theorems for von Neumann's sonic conjecture, as well as related further finer estimates and analysis of the solutions.

3.3 MAIN STEPS AND RELATED ANALYSIS IN THE PROOF OF THE DETACHMENT CONJECTURE

In this section we discuss the solutions of **Problem 2.6.1** in §2.6 for the full range of wedge angles for which state (2) exists, *i.e.*, for any $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, where $\theta_{\rm w}^{\rm d}$ is the detachment angle. We make the whole iteration again, starting from the normal reflection, and prove the results for both the supersonic and subsonic reflection configurations. We follow the procedure discussed in §3.2 with the changes described below.

The difference with §3.2 is from the fact that, depending on $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, the expected solutions have the structure of either supersonic or subsonic reflection configurations described in §2.4.2 and §2.4.3, respectively; it is of supersonic (resp. subsonic) structure if state (2) is supersonic (resp. subsonic or sonic) at P_0, *i.e.*, $|D\varphi_2(P_0)| > c_2$ (resp. $|D\varphi_2(P_0)| \leq c_2$), where we recall that P_0 and (u_2, v_2, c_2) depend only on $\theta_{\rm w}$.

Then we will use the following terminology: $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$ is a *supersonic* (resp. *subsonic*, or *sonic*) wedge angle if $|D\varphi_2(P_0)| > c_2$ (resp. $|D\varphi_2(P_0)| < c_2$, or $|D\varphi_2(P_0)| = c_2$) for $\theta_{\rm w}$. Note that the sonic angle $\theta_{\rm w}^{\rm s}$, introduced above, is a sonic wedge angle according to this terminology; moreover, $\theta_{\rm w}^{\rm s}$ is the supremum of the set of sonic wedge angles (even though it is not clear if more sonic wedge angles other than $\theta_{\rm w}^{\rm s}$ exist).

Note that, if $\theta_{\rm w}^{(i)} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$ is a sequence of supersonic wedge angles, and $\theta_{\rm w}^{(i)} \to \theta_{\rm w}^{(\infty)}$ for a sonic wedge angle $\theta_{\rm w}^{(\infty)}$, then ${P_0}^{(i)} \to {P_0}^{(\infty)}$, ${P_1}^{(i)} \to {P_0}^{(\infty)}$, ${P_4}^{(i)} \to {P_0}^{(\infty)}$, and $\overline{\Gamma_{\rm sonic}}^{(i)}$ shrinks to point ${P_0}^{(\infty)}$. Thus, we define that, for the subsonic/sonic wedge angles, $P_1 = P_4 := P_0$ and $\overline{\Gamma_{\rm sonic}} := \{P_0\}$. That is, $P_0 = P_1 = P_4$ for the subsonic/sonic wedge angles.

Now we comment on the steps in §3.2 with the changes necessary in the present case.

3.3.1 Admissible solutions of Problem 2.6.1

The definition of admissible solutions of Problem 2.6.1 in §3.2.1 has included only the supersonic reflection solutions. Now we need to define admissible solutions of both supersonic and subsonic reflection configurations.

For the supersonic wedge angles $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, we define the admissible solutions by Definition 3.2.1.

For the subsonic/sonic wedge angles $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, we define the admissible solutions which correspond to the subsonic configuration described in §2.4.3 and shown on Fig. 2.4, which are elliptic in Ω as in Definition 3.2.1(iii), and satisfy conditions (iv)–(v) of Definition 3.2.1. Moreover, we require the property similar to that in Remark 3.2.5 to be held for the subsonic reflection configurations. Since $\overline{\Gamma_{\rm sonic}} = \{P_0\}$ in this case, Definition 3.2.1(ii) for the subsonic reflection solutions is changed into the following:

(ii) φ satisfies (2.6.4) and

$$\begin{aligned} &\varphi \in C^{0,1}(\Lambda) \cap C^1(\overline{\Lambda} \setminus \overline{\Gamma_{\rm shock}}), \\ &\varphi \in C^3(\overline{\Omega} \setminus \{P_0, P_2, P_3\}) \cap C^1(\overline{\Omega}), \end{aligned} \tag{3.3.1}$$

together with

$$\varphi(P_0) = \varphi_2(P_0), \quad D\varphi(P_0) = D\varphi_2(P_0). \tag{3.3.2}$$

3.3.2 Strict monotonicity cones for $\varphi_1 - \varphi$ and $\varphi - \varphi_2$

All of the results discussed in §3.2.2–§3.2.3 hold without change. In the proofs, the only difference is that, for subsonic reflection solutions, $\overline{\Gamma_{\rm shock}}$ is only one point, P_0. However, we use (3.3.2) instead of Remark 3.2.5 in this case, and then the argument works without change.

3.3.3 Uniform estimates for admissible solutions

We discuss the extensions of the estimates stated in §3.2.4 to the present case.

Some of the estimates hold for any $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, in which the universal constant C depends only on (ρ_0, ρ_1, γ).

In the other estimates, we have to restrict the range of angles by fixing any $\theta_{\rm w}^* \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$ and considering the admissible solutions with $\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2})$. The universal constant C in these estimates depends only on $(\rho_0, \rho_1, \gamma, \theta_{\rm w}^*)$. Note that both the supersonic and subsonic reflection configurations occur if $\theta_{\rm w}^* \in (\theta_{\rm w}^{\rm d}, \theta_{\rm w}^{\rm s}]$. We need to consider such $\theta_{\rm w}^*$, since we will prove the existence of solutions up to $\theta_{\rm w}^{\rm d}$.

3.3.3.1 Basic estimates of (φ, ρ, Ω), the distance between $\Gamma_{\rm shock}$ and the sonic circle of state (1), and separation of $\Gamma_{\rm shock}$ and $\Gamma_{\rm sym}$

The estimates in §3.2.4.1–§3.2.4.2 and §3.2.4.4 hold without change in the present case.

Specifically, the estimates of (φ, ρ, Ω) in §3.2.4.1 hold for admissible solutions (supersonic and subsonic) with $\theta_{\rm w}^* \in (\theta_{\rm w}^{\rm d}, \theta_{\rm w}^{\rm s}]$ for some $C > 0$. The proofs are the same as those in the previous case; indeed, the only difference is that, in the subsonic reflection case, we use (3.3.2) instead of Remark 3.2.5.

Then we obtain (3.2.24) with uniform C for any admissible solution for $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$.

The estimate in §3.2.4.2 is extended to all $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$ without change in its proof, since the supersonic and subsonic admissible solutions are of similar structures near $\Gamma_{\rm sym}$.

3.3.3.2 The distance between $\Gamma_{\rm shock}$ and $\Gamma_{\rm wedge}$

We note that estimates (3.2.20)–(3.2.21) of the distance between $\Gamma_{\rm shock}$ and $\Gamma_{\rm wedge}$ discussed in §3.2.4.3 cannot hold for the subsonic reflection configurations. Indeed, in this case, $\overline{\Gamma_{\rm shock}} \cap \overline{\Gamma_{\rm wedge}} = \{P_0\}$, *i.e.*, $\mathrm{dist}(\Gamma_{\rm shock}, \Gamma_{\rm wedge}) = 0$, even if $u_1 \le c_1$. Thus, we need to consider the distance between $\Gamma_{\rm shock}$ and $\Gamma_{\rm wedge}$ away from P_0, as we have done in estimate (3.2.22). Then the estimates in §3.2.4.3 in the present case have the following two forms:

If $u_1 \le c_1$, then, for every small $r > 0$, there exists $C_r > 0$ such that

$$\mathrm{dist}(\Gamma_{\rm shock}, \Gamma_{\rm wedge} \setminus B_r(P_0)) > \frac{1}{C_r} \tag{3.3.3}$$

for any admissible solution (supersonic and subsonic) of **Problem 2.6.1** with $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$. Note that, if $\theta_{\rm w}$ is supersonic, $P_0 \notin \overline{\Gamma_{\rm wedge}}$. Thus, choosing r sufficiently small, we see that $\Gamma_{\rm wedge} \setminus B_r(P_0) = \Gamma_{\rm wedge}$ so that, for such $\theta_{\rm w}$ and r, estimate (3.3.3) coincides with (3.2.20). Moreover, in this case, the reflected-diffracted shock does not hit the wedge vertex P_3 as shown in Figs. 3.3–3.4.

Without assuming the condition that $u_1 \le c_1$, we show the uniform lower bound of the distance between $\Gamma_{\rm shock}$ and $\Gamma_{\rm wedge}$ away from P_0 and P_3, *i.e.*, extending estimate (3.2.22) to the present case. That is, for any small $r > 0$, there exists $C_r > 0$ such that

$$\mathrm{dist}\,(\Gamma_{\rm shock},\ \Gamma_{\rm wedge} \setminus (B_r(P_0) \cup B_r(P_3))) \ge \frac{1}{C_r} \tag{3.3.4}$$

for every admissible solution with $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$.

If $u_1 > c_1$, the wedge angle $\theta_{\rm w}^{\rm c}$ in Theorem 2.6.9 is defined as follows: As in §3.2.4.3, we extend the set of admissible solutions by including the normal reflection as the unique admissible solution for $\theta_{\rm w} = \frac{\pi}{2}$. Let $r_1 := \inf_{\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})} |\Gamma_{\rm wedge}^{(\theta_{\rm w})}|$, which can be shown that $r_1 > 0$. Then we replace the definition of set $\mathcal{A}$ in §3.2.4.3 by

$$\mathcal{A} := \left\{ \theta_{\rm w}^* \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2}] \ :\ \begin{array}{l} \text{For each } r \in (0, r_1), \text{ there exists } \varepsilon > 0 \text{ such that} \\ \mathrm{dist}(\Gamma_{\rm shock}, \Gamma_{\rm wedge} \setminus B_r(P_0)) \ge \varepsilon \text{ for all admissible} \\ \text{solutions with the wedge angles } \theta_{\rm w} \in [\theta_{\rm w}^*, \dfrac{\pi}{2}] \end{array} \right\}.$$

Since the normal reflection solution is the unique admissible solution for $\theta_{\rm w} = \frac{\pi}{2}$, the set of admissible solutions with angles $\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2}]$ is non-empty for any $\theta_{\rm w}^* \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2}]$. Moreover, since $\mathrm{dist}(\Gamma_{\rm shock}, \Gamma_{\rm wedge}) > 0$ for the normal reflection solution, then $\frac{\pi}{2} \in \mathcal{A}$, *i.e.*, $\mathcal{A} \neq \emptyset$. Thus, we have

$$\theta_{\rm w}^{\rm c} = \inf \mathcal{A}.$$

Similarly to §3.2.4.3, we find that $\theta_{\rm w}^{\rm c} < \frac{\pi}{2}$. Therefore, for any $\theta_{\rm w}^* \in (\theta_{\rm w}^{\rm c}, \frac{\pi}{2})$ and $r \in (0, r_1)$, there exists $C_r > 0$ such that, for any admissible solution φ with $\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2})$,

$$\mathrm{dist}(\Gamma_{\rm shock}, \Gamma_{\rm wedge} \setminus B_r(P_0)) \ge \frac{1}{C_r}. \tag{3.3.5}$$

We note that, while the estimates of this section are weaker than the estimates in §3.2.4.3, since $\Gamma_{\rm wedge}$ is replaced by $\Gamma_{\rm wedge} \setminus B_r(P_0)$, the present estimates are used in the same way as the estimates in §3.2.4.3. Specifically, (3.2.20) and (3.2.23) are used in §3.2.5.1 to obtain the weighted $C^{k,\alpha}$–estimates away from $\Gamma_{\rm sonic}$. Clearly, (3.3.3) and (3.3.5) can be used for that purpose as well. Similarly, one can replace (3.2.21) by (3.3.4) in the proof that, for the *attached* solution for $\theta_{\rm w} = \theta_{\rm w}^{\rm c}$, the relative interior of $\Gamma_{\rm wedge}$ is disjoint from $\Gamma_{\rm shock}$.

3.3.3.3 Uniform estimate of the ellipticity of equation (2.2.8) in Ω up to $\Gamma_{\rm shock}$

We estimate the Mach number defined by (3.2.25).

First, we prove that (3.2.26) holds for all the supersonic admissible solutions with any supersonic wedge angle $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, with uniform $\mu > 0$.

For the subsonic admissible solutions, we obtain the following estimate of the Mach number:

$$M^2(\boldsymbol{\xi}) \le \max(1 - \hat{\zeta},\ \frac{|D\varphi_2(P_0)|^2}{c_2^2} - \hat{\mu}|\boldsymbol{\xi} - P_0|) \qquad \text{for all } \boldsymbol{\xi} \in \overline{\Omega(\varphi)},$$

where the positive constants $\hat{\zeta}$ and $\hat{\mu}$ depend only on (ρ_0, ρ_1, γ).

From these estimates, we obtain the following ellipticity properties of the potential flow equation (2.2.8), written in the form of (3.2.27): There exist $\hat{\zeta} > 0$ and $C > 0$ depending only on (ρ_0, ρ_1, γ) such that, if φ is an admissible solution of **Problem 2.6.1** with $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, then

(i) For any supersonic wedge angle $\theta_{\rm w}$, (3.2.28) holds;

(ii) For any subsonic/sonic wedge angle $\theta_{\rm w}$,

$$\begin{aligned} &\frac{1}{C}\min(c_2 - |D\varphi_2(P_0)| + |\boldsymbol{\xi} - P_0|,\ \hat{\zeta})|\boldsymbol{\kappa}|^2 \\ &\le \sum_{i,j=1}^{2} \mathcal{A}^i_{p_j}(D\varphi(\boldsymbol{\xi}), \varphi(\boldsymbol{\xi}))\kappa_i\kappa_j \le C|\boldsymbol{\kappa}|^2 \end{aligned} \tag{3.3.6}$$

for any $\boldsymbol{\xi} \in \overline{\Omega}$ and $\boldsymbol{\kappa} = (\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) \in \mathbb{R}^2$.

Note that, if $\theta_{\rm w}$ is a subsonic wedge angle, then $|D\varphi_2(P_0)| < c_2$ so that (3.3.6) shows the uniform ellipticity of (2.2.8) for φ in Ω. However, this ellipticity degenerates near P_0 as the subsonic wedge angles tend to a sonic angle. If $\theta_{\rm w}$ is a sonic angle, $|D\varphi_2(P_0)| = c_2$ and $\overline{\Gamma_{\rm sonic}} = \{P_0\}$ so that (3.3.6) coincides with (3.2.28) in this case.

3.3.4 Regularity and related uniform estimates

3.3.4.1 Weighted $C^{k,\alpha}$–estimates away from $\Gamma_{\rm sonic}$ or the reflection point

Now all the preliminary results used for the estimates in §3.2.5.1 are extended to all the admissible solutions (supersonic and subsonic) with $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, where there is some difference in the estimates of the distance between $\Gamma_{\rm shock}$ and $\Gamma_{\rm wedge}$. However, the estimates obtained there are sufficient, as discussed in §3.3.3.2. Then we obtain the weighted $C^{k,\alpha}$–estimates away from $\Gamma_{\rm sonic}$ or the reflection point for any admissible solutions (supersonic and subsonic) with $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$ by the same argument as that in §3.2.5.1.

3.3.4.2 Weighted and scaled $C^{k,\alpha}$–estimates near $\Gamma_{\rm sonic}$ or the reflection point

The main difference between the structure of supersonic and subsonic admissible solutions is near $\overline{\Gamma_{\rm sonic}}$, since $\overline{\Gamma_{\rm sonic}}$ is an arc for the supersonic wedge angles, and $\overline{\Gamma_{\rm sonic}} = \{P_0\}$ is one point for the subsonic and sonic wedge angles. Thus, the main difference from the argument in §3.2 is in the estimates near $\Gamma_{\rm sonic}$ or the reflection point, *i.e.*, near $\overline{\Gamma_{\rm sonic}}$.

Similarly to (3.2.30), we define and characterize Ω_ε, which now works for both supersonic and subsonic reflection solutions. We work in the (x,y)–coordinates introduced in §3.2.5.2, and note that $\overline{\Gamma_{\rm sonic}} \subset \{(x,y)\ :\ x = x_{P_1}\}$ for any wedge angle, where $x_{P_1} = 0$ for supersonic and sonic wedge angles, and $x_{P_1} > 0$ for subsonic wedge angles. Then, for appropriately small $\varepsilon_1 > \varepsilon_0 > 0$, we find that, for any $\varepsilon \in (0, \varepsilon_0]$,

$$
\begin{aligned}
&\Omega_\varepsilon := \Omega \cap \mathcal{N}_{\varepsilon_1}(\overline{\Gamma_{\rm sonic}}) \cap \{x < x_{P_1} + \varepsilon\}\\
&\qquad = \{x_{P_1} < x < x_{P_1} + \varepsilon,\ \theta_{\rm w} < y < \hat{f}(x)\},\\
&\Gamma_{\rm sonic} = \partial\Omega_\varepsilon \cap \{x = x_{P_1}\},\\
&\Gamma_{\rm wedge} \cap \partial\Omega_\varepsilon = \{x_{P_1} < x < x_{P_1} + \varepsilon,\ y = \theta_{\rm w}\},\\
&\Gamma_{\rm shock} \cap \partial\Omega_\varepsilon = \{x_{P_1} < x < x_{P_1} + \varepsilon,\ y = \hat{f}(x)\}
\end{aligned}
\tag{3.3.7}
$$

for some $\hat{f}(x)$ defined on $(x_{P_1},\ x_{P_1} + \varepsilon_0)$ and satisfying

$$
\begin{cases}
\hat{f}(x_{P_1}) = y_{P_1} > y_{P_4} = \theta_{\rm w} & \text{for supersonic reflection solutions,}\\
\hat{f}(x_{P_1}) = y_{P_0} = y_{P_1} = y_{P_4} = \theta_{\rm w} & \text{otherwise,}
\end{cases}
\tag{3.3.8}
$$

and

$$
0 < \omega \le \frac{d\hat{f}}{dx} < C \qquad \text{for any }\ x \in (x_{P_1},\ x_{P_1} + \varepsilon_0). \tag{3.3.9}
$$

To obtain the estimates near $\overline{\Gamma_{\rm sonic}}$, we consider four separate cases depending on the Mach number $\dfrac{|D\varphi_2|}{c_2}$ at P_0:

(a) Supersonic: $\frac{|D\varphi_2(P_0)|}{c_2} \geq 1+\delta$;

(b) Supersonic-almost-sonic: $1+\delta > \frac{|D\varphi_2(P_0)|}{c_2} > 1$;

(c) Subsonic-almost-sonic: $1 \geq \frac{|D\varphi_2(P_0)|}{c_2} \geq 1-\delta$;

(d) Subsonic: $\frac{|D\varphi_2(P_0)|}{c_2} \leq 1-\delta$.

We derive the uniform estimates in Ω_ε for any $\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2})$, where ε is independent of $\theta_{\rm w}$. Recall that $P_1 = P_0$ in the subsonic case. The choice of constants (ε, δ) will be described below with the following properties: δ is chosen small, depending on (ρ_0, ρ_1, γ), so that the estimates in Cases (b)–(c) work in Ω_ε for some $\varepsilon > 0$; then ε is further reduced so that all the estimates in Cases (a)–(d) work in Ω_ε.

We first present a general overview of the estimates. In Cases (a)–(b), equation (2.2.8) is degenerate elliptic in Ω near $\Gamma_{\rm sonic} = P_1P_4$; see Fig. 2.3. In Case (c), the equation is uniformly elliptic in $\overline{\Omega}$, but the ellipticity constant is small near P_0 in Fig. 2.4. Thus, in Cases (a)–(c), we use the local elliptic degeneracy, which allows us to find a comparison function in each case, to show the appropriately fast decay of $\varphi - \varphi_2$ near P_1P_4 in Cases (a)–(b) and near P_0 in Case (c). Similarly to the argument of §3.2.5.2, we perform the local non-isotropic rescaling (different in each of Cases (a)–(c)) near each point of Ω_ε so that the rescaled functions satisfy a uniformly elliptic equation and the uniform L^∞–estimates, which follow from the decay of $\varphi - \varphi_2$ obtained above. Then we obtain the *a priori* estimates in the weighted and scaled $C^{2,\alpha}$–norms, which are different in each of Cases (a)–(c), but they imply the standard $C^{1,1}$–estimates. This is an extension of the methods in §3.2.5.2. In the uniformly elliptic case, Case (iv), the solution is of subsonic reflection configuration (*cf.* Fig. 2.4) and the estimates are more technically challenging than those in Cases (a)–(c), owing to the fact that the lower *a priori* regularity (Lipschitz) of the free boundary presents a new difficulty in Case (d) and the uniform ellipticity does not allow a comparison function that shows the sufficiently fast decay of $\varphi - \varphi_2$ near P_0. Thus, we prove the C^α–estimates of $D(\varphi - \varphi_2)$ near P_0 by deriving the corresponding elliptic equations and oblique boundary conditions for appropriately chosen directional derivatives of $\varphi - \varphi_2$.

Now we discuss the estimates in Cases (a)–(c) in more detail.

The techniques described in §3.2.5.2, for $\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2})$ with $\theta_{\rm w}^* \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$, cannot be extended to all the supersonic reflection solutions. The reason for this is that, if the length of $\Gamma_{\rm sonic}$ is very small, rectangles $R_\rho^{(x_0,y_0)}$ specified in Cases (i)–(iii) in §3.2.5.2 do not *fit into* Ω in the following sense: The argument in §3.2.5.2 uses the property that the rectangles in Cases (i)–(ii) do not intersect with $\Gamma_{\rm shock}$ and the rectangles in Cases (i) and (iii) do not intersect with $\Gamma_{\rm wedge}$ (*cf.* Fig. 3.5) so that rectangles $R_\rho^{(x_0,y_0)}$ fit into Ω. From (3.3.7)–(3.3.9), rectangles $R_{1/2}^{(x_0,y_0)}$ in Cases (ii)–(iii) fit into Ω if $\sqrt{x_0} \lesssim y_{P_1} - y_{P_4}$, and do not fit into Ω in the opposite case; see Fig. 3.6. Thus, all the rectangles $R_{1/2}^{(x_0,y_0)}$ with

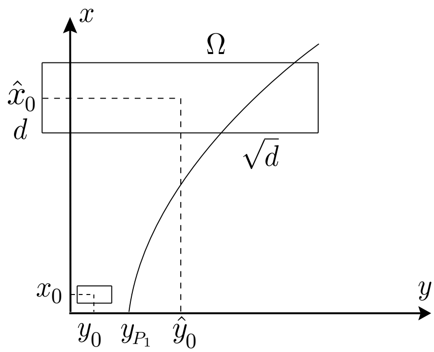


Figure 3.6: Rectangles when the sonic arc is short

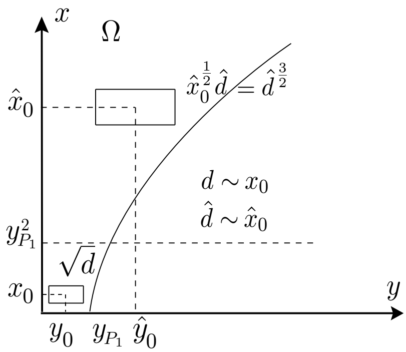


Figure 3.7: Estimates in the supersonic-almost-sonic case

$(x_0, y_0) \in \Gamma_{\rm wedge} \cup \Gamma_{\rm shock}$ fit into Ω only if $y_{P_1} - y_{P_4} \gtrsim \sqrt{\varepsilon}$, *i.e.*, when $\Gamma_{\rm sonic}$ is sufficiently long, depending only on ε. Note that making the rectangles smaller by choosing $\rho < \frac{1}{2}$ in (3.2.33) does not change the argument. The condition that $\frac{|D\varphi_2(P_0)|}{c_2} \geq 1 + \delta$ implies a positive lower bound $b > 0$ on the length of $\Gamma_{\rm sonic}$, depending on $\delta > 0$. We fix $\delta > 0$ below. Then the estimates in §3.2.5.2 apply to any $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$ satisfying $\frac{|D\varphi_2(P_0)|}{c_2} \geq 1+\delta$, and these estimates are obtained in Ω_ε with $\varepsilon \sim b^2$. This describes the estimates in Case (a) (supersonic).

In Case (b) (supersonic-almost-sonic), when $y_{P_1} - y_{P_4}$ is very small, we use (3.3.7)–(3.3.9) to note that there exists $k > 1$ so that the rectangles:

$$\hat{R}_{(x_0,y_0)} := \left\{|x - x_0| < \frac{x_0^{3/2}}{10k},\ |y - y_0| < \frac{x_0}{10k}\right\} \cap \Omega \tag{3.3.10}$$

for $(x_0, y_0) \in (\Gamma_{\rm wedge} \cup \Gamma_{\rm shock}) \cap \partial\Omega_\varepsilon$ fit into Ω in the sense described above. Note that the ratio of the lengths in the $x-$ and y–directions of $\hat{R}_{(x_0,y_0)}$ is $\sqrt{x_0}$, *i.e.*, the same as for the rectangles in (3.2.33). This implies that, rescaling $\hat{R}_{(x_0,y_0)}$ to the portion of square $(-1,1)^2 := (-1,1) \times (-1,1)$:

$$\hat{Q}_{(x_0,y_0)} := \big\{(S,T) \in (-1,1)^2 \,:\, (x_0 + x_0^{\frac{3}{2}} S, y_0 + \frac{x_0}{10k} T) \in \Omega\big\},$$

we obtain a uniformly elliptic equation for the function:

$$\psi^{(x_0,y_0)}(S,T) := \frac{1}{x_0^m} \psi(x_0 + x_0^{\frac{3}{2}} S, y_0 + \frac{x_0}{10k} T) \qquad \text{in } \hat{Q}_{(x_0,y_0)} \tag{3.3.11}$$

with any positive integer m. Thus, if the uniform L^∞ bound is obtained for functions $\psi^{(x_0,y_0)}$, we can follow the argument in §3.2.5.2 by using the rectangles in (3.3.10). However, if $m = 2$ is used, the resulting estimates, rescaled back into the (x,y)–variables, are weaker than the estimates obtained in §3.2.5.2, where we have used the rectangles in (3.2.33), and such estimates are not sufficient for the rest of the argument. In fact, we need to use $m = 4$. This requires the estimate: $\psi(x,y) \le Cx^4$, in order to obtain the uniform L^∞ bound of $\psi^{(x_0,y_0)}$. However, Theorem 2.6.6 implies that $\psi \in C^{2,\alpha}(\overline{\Omega_\varepsilon} \setminus \{P_1\})$ with $\psi_{xx} = \frac{1}{\gamma+1} > 0$ on Γ_{sonic} so that, recalling that $\psi = \psi_x = 0$ on Γ_{sonic}, we conclude that the estimate, $\psi(x,y) \le Cx^4$, does not hold near Γ_{sonic}. For this reason, we decompose Ω_ε into two subdomains; see Fig. 3.7. For $\theta_{\text{w}} \in (\theta_{\text{w}}^{\text{d}}, \frac{\pi}{2})$ satisfying the condition of Case (b), define

$$b_{\text{so}} := y_{P_1} - y_{P_4}.$$

As we have discussed above, in $\Omega_{b_{\text{so}}^2}$, we can use the argument in §3.2.5.2 to obtain the estimates described there. Furthermore, for each $m = 2, 3, \dots$, if δ is small in the condition of Case (b) depending only on $(\rho_0, \rho_1, \gamma, m)$, we obtain

$$0 \le \psi(x,y) \le Cx^m \qquad \text{in } \Omega_\varepsilon \cap \{x > \frac{b_{\text{so}}^2}{10}\}, \tag{3.3.12}$$

where $C > 0$ and $\varepsilon \in (0, \varepsilon_0]$ depend only on $(\rho_0, \rho_1, \gamma, m)$. The main point here is that $C > 0$ and ε are independent of b_{so}. Estimate (3.3.12) is proved by showing that

$$0 \le \psi(x,y) \le C(x + Mb_{\text{so}}^2)^m \qquad \text{in } \Omega_\varepsilon$$

with C, M, and ε depending only on $(\rho_0, \rho_1, \gamma, m)$. We use $m = 4$ in (3.3.12). This fixes δ for Cases (a)–(b). Then, as we have discussed above, we obtain the estimates in $\Omega_\varepsilon \cap \{x > \frac{b_{\text{so}}^2}{2}\}$ by using the rectangles in (3.3.10) and the rescaled functions (3.3.11) with $m = 4$. Combining this with the estimates in $\Omega_{b_{\text{so}}^2}$, we complete the uniform estimates in Ω_ε for Case (b).

If $\theta_{\text{w}} \in (\theta_{\text{w}}^{\text{d}}, \frac{\pi}{2})$ satisfies the condition of Case (c), we argue similar to Case (b), by changing the size of the rectangles (*i.e.*, the scaling) according to the geometry of the domain; see Fig. 3.8. Specifically, for each $m = 2, 3, \dots$, if δ is small depending only on $(\rho_0, \rho_1, \gamma, m)$ in the condition of Case (c), we obtain

$$0 \le \psi(x,y) \le C(x - x_{P_0})^m \qquad \text{in } \Omega_\varepsilon \tag{3.3.13}$$

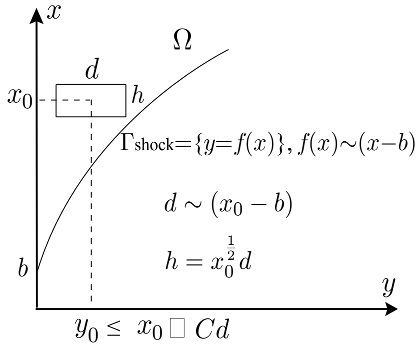


Figure 3.8: Estimates in the subsonic-almost-sonic case

with C, M, and ε depending only on $(\rho_0, \rho_1, \gamma, m)$. Recall that $x_{P_0} = x_{P_1} > 0$ in the subsonic case. Also, for sufficiently large $k > 1$, the rectangles:

$$\hat{R}_{(x_0,y_0)} := \Big\{|x - x_0| < \frac{\sqrt{x_0}}{10k}(x - x_{P_0}),\ |y - y_0| < \frac{1}{10k}(x - x_{P_0})\Big\} \cap \Omega$$

for $(x_0, y_0) \in (\Gamma_{\text{wedge}} \cup \Gamma_{\text{shock}}) \cap \partial\Omega_\varepsilon$ fit into Ω in the sense described above. The ratio of the side lengths in the x– and y–directions of $\hat{R}_{(x_0,y_0)}$ is $\sqrt{x_0}$, as in the previous cases. Thus, rescaling $\hat{R}_{(x_0,y_0)}$ to the portion of square $(-1,1)^2$:

$$\hat{Q}_{(x_0,y_0)} := \big\{(S,T) \in (-1,1)^2 : (x_0 + \frac{\sqrt{x_0}}{10k}(x - x_{P_0})S, y_0 + \frac{1}{10k}(x - x_{P_0})T) \in \Omega\big\},$$

we obtain a uniformly elliptic equation in $\hat{Q}_{(x_0,y_0)}$ for the function:

$$\psi^{(z_0)}(S,T) := \frac{1}{(x - x_{P_0})^m}\psi(x_0 + \frac{\sqrt{x_0}}{10k}(x - x_{P_0})S, y_0 + \frac{1}{10k}(x - x_{P_0})T). \quad (3.3.14)$$

We use $m = 5$ in (3.3.13), which fixes δ for Cases (c)–(d). Then, repeating the argument of the previous cases for the rescaled functions (3.3.14) with $m = 5$, we obtain the uniform estimates of ψ in $C^{2,\alpha}(\overline{\Omega_\varepsilon})$ with $\varepsilon(\rho_0, \rho_1, \gamma)$.

Next we consider Case (d). If $\theta_{\text{w}}^* \in (\theta_{\text{w}}^{\text{d}}, \frac{\pi}{2})$ is fixed, and $\theta_{\text{w}} \in (\theta_{\text{w}}^{\text{d}}, \frac{\pi}{2})$ satisfies the condition of Case (d) with δ fixed above, we use the uniform ellipticity (independent of θ_{w}) in the estimates. The main steps of these estimates are described in §16.6.1. We note the following points of this argument:

- We use the fact that φ_2 in (3.3.2) is the *weak* state (2);
- We use the monotonicity cone of $\varphi_1 - \varphi$ (*cf.* §3.3.2), and the convexity of the shock polar;
- We obtain the estimates in $C^{1,\alpha}$ up to P_0, which is a weaker regularity than that in Cases (a)–(c);

- The constants in the estimates depend on $\theta_{\rm w}^*$, in addition to (ρ_0, ρ_1, γ), and blow up as $\theta_{\rm w}^* \to \theta_{\rm w}^{\rm d}+$.

3.3.5 Existence of the regular reflection-diffraction configuration up to the detachment angle

Let $\theta_{\rm w}^* \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$. We show that there exists an admissible solution for any wedge angle $\theta_{\rm w} \in (\theta_{\rm w}^*, \frac{\pi}{2}]$.

We follow the argument described in §3.2.6 with the changes necessary to handle both cases of supersonic and subsonic reflection solutions in the argument. This includes the following three steps:

1. As in §3.2.6, the iteration set $\mathcal{K}$ consists of pairs $(u, \theta_{\rm w})$, for a function u on the unit square $Q^{\rm iter}$ and $\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2}]$:

$$\mathcal{K} = \cup_{\theta_{\rm w} \in [\theta_{\rm w}^*, \frac{\pi}{2}]} \mathcal{K}_{\theta_{\rm w}} \times \{\theta_{\rm w}\}$$

with $\mathcal{K}_{\theta_{\rm w}} \subset \mathcal{C}$, where $\mathcal{C}$ is a weighted and scaled $C^{2,\alpha}$–type space on $Q^{\rm iter}$ for some $\alpha = \alpha(\rho_0, \rho_1, \gamma) \in (0, 1)$, which satisfies

$$\mathcal{C} \subset C^{2,\alpha}(Q^{\rm iter}) \cap C^{1,\alpha}(\overline{Q^{\rm iter}}).$$

For each $(u, \theta_{\rm w}) \in \mathcal{K}$, the elliptic domain Ω of the *approximate solution* and a smooth invertible map $\mathcal{G}_{u,\theta_{\rm w}} : Q^{\rm iter} \mapsto \Omega$ are defined. As in §3.2.6, for any supersonic wedge angle $\theta_{\rm w} \in (\theta_{\rm w}^*, \frac{\pi}{2}]$, region Ω is of the same structure as an elliptic region of supersonic reflection solutions; see §2.4.2 and Fig. 2.3. Map $\mathcal{G}_{u,\theta_{\rm w}} : Q^{\rm iter} \mapsto \Omega(u, \theta_{\rm w})$ can be extended to the smooth and smoothly invertible map $\overline{Q^{\rm iter}} \mapsto \overline{\Omega}$, where the sides of square $Q^{\rm iter}$ are mapped to the boundary parts $\Gamma_{\rm sonic}, \Gamma_{\rm shock}, \Gamma_{\rm sym}$, and $\Gamma_{\rm wedge}$ of Ω. However, for any subsonic/sonic wedge angle $\theta_{\rm w}$, $\Omega(u, \theta_{\rm w})$ is of the structure described in §2.4.3 and Fig. 2.4, *i.e.*, has a triangular shape $P_0P_2P_3$. Thus, map $\mathcal{G}_{u,\theta_{\rm w}} : \overline{Q^{\rm iter}} \mapsto \overline{\Omega}$ is smooth but not invertible; one of the sides of $Q^{\rm iter}$ is now mapped into point $\overline{\Gamma_{\rm sonic}} = \{P_0\}$.

2. The singularity of mapping $\mathcal{G}_{u,\theta_{\rm w}} : \overline{Q^{\rm iter}} \mapsto \overline{\Omega}$, described above, affects the choice of the function space $\mathcal{C}$ introduced above. The norm in $\mathcal{C}$ is a weighted and scaled $C^{2,\alpha}$–type norm on $Q^{\rm iter}$ such that

- If $(u, \theta_{\rm w}) \in \mathcal{K}$ and $v \in \mathcal{C}$, then, expressing v as a function w on $\Omega(u, \theta_{\rm w})$ by $w = v \circ \mathcal{G}_{u,\theta_{\rm w}}^{-1}$, we obtain that $w \in C^{1,\alpha}(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$ and some more detailed properties.

- If φ is an admissible solution for the wedge angle $\theta_{\rm w}$, there exists $u \in \mathcal{K}_{\theta_{\rm w}}$, which is related to φ through map $\mathcal{G}_{u,\theta_{\rm w}}$. The *a priori* estimates of the admissible solutions for all the cases described in §3.3.4.1–§3.3.4.2 imply the estimates for u in a norm which is stronger than the norm of $\mathcal{C}$. This allows us to define an iteration map which is compact in the norm of $\mathcal{C}$ and to show that there is no fixed point of the iteration map on the boundary of the iteration set.

3. The properties of the potential flow equation (2.2.8) for admissible solutions, near Γ_{sonic} or the reflection point, are different for $\theta_{\rm w}$ belonging to the different cases (a)–(d) in §3.3.4.2. This affects the definition of the equation in the boundary value problem used in the definition of the iteration map for the corresponding angle $\theta_{\rm w}$. Also, in solving this problem and deriving the estimates of its solutions, we employ techniques similar to the estimates of admissible solutions in §3.3.4.1–§3.3.4.2 for Cases (a)–(d). This allows us to define the iteration map and obtain its compactness.

3.4 APPENDIX: THE METHOD OF CONTINUITY AND FIXED POINT THEOREMS

For completeness, we now present several fundamental theorems regarding the method of continuity and fixed point theorems that are used in this book.

Theorem 3.4.1 (Method of Continuity)**.** *Let $\mathcal{B}$ be a Banach space and $\mathcal{V}$ a normed linear space, and let $\mathbf{L}_0$ and $\mathbf{L}_1$ be bounded linear operators from $\mathcal{B}$ into $\mathcal{V}$. Suppose that there is a constant C such that, for any $\tau \in [0,1]$,*

$$\|\mathbf{x}\|_{\mathcal{B}} \leq C\big\|\big((1-\tau)\mathbf{L}_0 + \tau\mathbf{L}_1\big)\mathbf{x}\big\|_{\mathcal{V}} \qquad \text{for any } \mathbf{x} \in \mathcal{B}. \tag{3.4.1}$$

Then $\mathbf{L}_1$ maps $\mathcal{B}$ onto $\mathcal{V}$ if and only if $\mathbf{L}_0$ maps $\mathcal{B}$ onto $\mathcal{V}$.

Definition 3.4.2. *Let X and Y be metric spaces. A map $h : X \mapsto Y$ is compact provided that*

(i) *h is continuous*;

(ii) *$f(A)$ is compact whenever $A \subset X$ is bounded.*

Theorem 3.4.3 (Leray-Schauder Fixed Point Theorem)**.** *Let $\mathbf{T}$ be a compact mapping of a Banach space $\mathcal{B}$ into itself. Suppose that there exists a constant M such that, for all $\mathbf{x} \in \mathcal{B}$ and $\tau \in [0,1]$ satisfying $\mathbf{x} = \tau\mathbf{T}\mathbf{x}$,*

$$\|\mathbf{x}\|_{\mathcal{B}} \leq M. \tag{3.4.2}$$

Then $\mathbf{T}$ has a fixed point.

Theorems 3.4.1 to 3.4.3 can be found as Theorem 5.2 and 11.3 in [131].

Theorem 3.4.4 (Schauder Fixed Point Theorem)**.** *Let $\mathcal{K}$ be a closed and convex subset of a Banach space, and let $\mathbf{J} : \mathcal{K} \mapsto \mathcal{K}$ be a continuous map such that $\mathbf{J}(\mathcal{K})$ is precompact. Then $\mathbf{J}$ has a fixed point.*

More details can be found in [131], including Corollary 11.2.

Next we present some further basic definitions and facts in the Leray-Schauder degree theory.

Definition 3.4.5. *Let G be an open bounded set in a Banach space X. Denote by $V(G,X)$ the set of all the maps $\mathbf{f} : \bar{G} \mapsto X$ satisfying the following*:

(i) $\mathbf{f}$ *is compact*;

(ii) $\mathbf{f}$ *has no fixed points on boundary ∂G.*

Definition 3.4.6. *Two maps $\mathbf{f}, \mathbf{g} \in V(G,X)$ are called compactly homotopic on ∂G if there exists a map $\mathbf{H}$ with the following properties*:

(i) $\mathbf{H} : \bar{G} \times [0,1] \mapsto X$ *is compact*;

(ii) $\mathbf{H}(\mathbf{x},\tau) \neq \mathbf{x}$ *for any* $(\mathbf{x},\tau) \in \partial G \times [0,1]$;

(iii) $\mathbf{H}(\mathbf{x},0) = \mathbf{f}(\mathbf{x})$ *and* $\mathbf{H}(\mathbf{x},1) = \mathbf{g}(\mathbf{x})$ *on* $\bar{G}$.

We write $\partial G : \mathbf{f} \tilde{=} \mathbf{g}$. This map $\mathbf{H}$ is called a compact homotopy.

Then we have the following Leray-Schauder degree theory.

Theorem 3.4.7. *Let G be an open bounded set in a Banach space X. Then, to each map $\mathbf{f} \in V(G,X)$, an integer number $\mathbf{Ind}(\mathbf{f},G)$ can be uniquely assigned such that*

(i) *If* $\mathbf{f}(\mathbf{x}) \equiv \mathbf{x}_0$ *for any* $\mathbf{x} \in \bar{G}$ *and some fixed* $\mathbf{x}_0 \in G$*, then* $\mathbf{Ind}(\mathbf{f},G) = 1$;

(ii) *If* $\mathbf{Ind}(\mathbf{f},G) \neq 0$*, there exists* $\mathbf{x} \in G$ *such that* $\mathbf{f}(\mathbf{x}) = \mathbf{x}$;

(iii) $\mathbf{Ind}(\mathbf{f},G) = \sum_{j=1}^{n} \mathbf{Ind}(\mathbf{f},G_j)$*, whenever* $\mathbf{f} \in V(G,X) \cap \left(\cup_{j=1}^{n} V(G_j,X) \right)$*, where $\{G_j\}$ is a regular partition of G, i.e., G_j are pairwise disjoint and* $\bar{G} = \cup_{j=1}^{n} \bar{G}_j$;

(iv) *If* $\partial G : \mathbf{f} \tilde{=} \mathbf{g}$*, then* $\mathbf{Ind}(\mathbf{f},G) = \mathbf{Ind}(\mathbf{g},G)$.

The integer number $\mathbf{Ind}(\mathbf{f},G)$ is called the fixed point index of f on G.

We also need to consider the case in which set G varies with the homotopy parameter t; see §13.6($A4^*$) in [283].

Theorem 3.4.8 (Generalized Homotopy Invariance of the Fixed Point Index)**.** *Let X be a Banach space, $t_2 > t_1$. Let $U \subset X \times [t_1,t_2]$, and let $U_t := \{\mathbf{x} : (\mathbf{x},t) \in U\}$. Then*

$$\mathbf{Ind}(\mathbf{h}(\cdot,t),U_t) = \textit{const.} \qquad \textit{for all } t \in [t_1,t_2],$$

provided that U is bounded and open in $X \times [t_1,t_2]$, and operator $\mathbf{h} : \overline{U} \mapsto X$ is compact with $\mathbf{h}(\mathbf{x},t) \neq \mathbf{x}$ on ∂U.

Note that set U is open with respect to the subspace topology on $X \times [t_1,t_2]$. That is, U is an intersection of an open set in $X \times \mathbb{R}$ with $X \times [t_1,t_2]$.

More details about the degree theory can be found in Chapters 12–13 in [283].

Part II

Elliptic Theory and Related Analysis for Shock Reflection-Diffraction