THE REAL ANALYSIS LIFESAVER

The Real Analysis Lifesaver

All the tools you need to understand proofs

RAFFI GRINBERG

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Although I recommend that you read all of the chapters in order to cover a typical real analysis curriculum (and because the material builds on itself), it is possible to take a "fastest path" through this book using only the chapters marked with a *.

PRELIMINARIES

CHAPTER 1

Introduction

Slow down there, hotshot. I know you're smart—you might have always been good with numbers, you might have aced calculus—but I want you to *slow down*. Real analysis is an entirely different animal from calculus or even linear algebra. Besides the fact that it's just plain harder, the way you learn real analysis is not by memorizing formulas or algorithms and plugging things in. Rather, you need to read and reread definitions and proofs until you understand the larger concepts at work, so you can apply those concepts in your own proofs. The best way to get good at this is to take your time; read slowly, write slowly, and think carefully.

What follows is a short introduction about why I wrote this book and how you should go about reading it.

Why I Wrote This Book

Real analysis is hard. This topic is probably your introduction to proof-based mathematics, which makes it even harder. But I very much believe that anyone can learn anything, as long as it is explained clearly enough.

I struggled with my first real analysis course. I constantly felt like I was my own teacher and wished there was someone who could explain things to me in a clear, linear fashion. The fact that I struggled—and eventually pulled through—makes me an excellent candidate to be your guide. I easily recall what it was like to see this stuff for the first time. I remember what confused me, what was never really clear, and what stumped me. In this book, I hope I can preempt most of your questions by giving you the explanations I would have most liked to have seen.

My course used the textbook *Principles of Mathematical Analysis*, 3rd edition, by Walter Rudin (also known as Baby Rudin, or That Grueling Little Blue Book). It is usually considered the classic, standard real analysis text. I appreciate Rudin now—his book is well organized and concise. But I can tell you that when I used it to learn the material for the first time, it was a *slog*. It never explains anything! Rudin lists definitions without giving examples and writes polished proofs without telling you how he came up with them.

Don't get me wrong: having to figure things out for yourself can be of tremendous value. Being challenged to understand why things work—without linear steps handed to you on a silver platter—makes you a better thinker and a better learner. But I believe that as a pedagogical technique, “throwing you in the deep end” without teaching you how to swim is only good in moderation. After all, your teachers want you to learn, not drown. I think Rudin can provide all the throwing, and this book can be a lifesaver when you need it.

I wrote this book because if you are an intelligent-but-not-a-genius student (like I was), who genuinely wants to learn real analysis... you need it.

What Is Real Analysis?

Real analysis is what mathematicians would call the *rigorous* version of calculus. Being “rigorous” means that every step we take and every formula we use must be proved. If we start from a set of basic assumptions, called *axioms* or *postulates*, we can always get to where we are now by taking one justified step after another.

In calculus, you might have proved some important results, but you also took many things for granted. What exactly *are* limits, and how do you really know when an infinite sum “converges” to one number? In an introductory real analysis course, you are reintroduced to concepts you’ve seen before—continuity, differentiability, and so on—but this time, their foundations will be clearly laid. And when you are done, you will have basically proven that calculus *works*.

Real analysis is typically the first course in a pure math curriculum, because it introduces you to the important ideas and methodologies of pure math in the context of material you are already familiar with.

Once you are able to be rigorous with familiar ideas, you can apply that way of thinking to unfamiliar territory. At the core of real analysis is the question: “how do we expand our intuition for certain concepts—such as sums—to work in the infinite cases?” Puzzles such as infinite sums cannot be properly understood without being rigorous. Thus, you must build your hard-core proving skills to apply them to these new (not-from-high-school-calculus), more interesting problems.

How to Read This Book

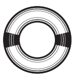
This book is not intended to be concise. Take a look at Chapter 7 as an example; I spend several pages covering what Rudin does in just two. The definitions are followed by examples in an attempt to make them less abstract. The proofs here are intended to show you not just *why* the theorem is true but also *how* you could go about proving it yourself. I try to state every fact being used in an argument, instead of omitting the more basic ones (as advanced mathematical literature would do).

If you are using Rudin, you’ll find that I’ve purposely tried to cover all the definitions and theorems that he covers, mostly in the same order. There isn’t a one-to-one mapping between this book and Rudin’s (Chapter 7 math joke!); for example, the next chapter explains the basic theory of sets, whereas Rudin holds off on that until after covering real numbers. I also include a few extra pieces of information for your enrichment. But by following his structure and notations as closely as possible, you should be able to go back and forth between this book and his with ease.

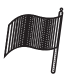
Unlike some other math books—which are meant to be glanced at, skimmed, or just referenced—you should read this one linearly. The chapters here are deliberately short and should contain the equivalent of an easily digestible one-hour lecture. Start at the beginning of a chapter and don’t jump around until you make it to the end.

Now for some advice: read actively. Fill in the blanks where I tell you to. (I purposely didn't include the answers to these; the temptation to peek would just be too great.) Make notes even where I don't tell you to. Copy definitions into your notebook if you learn by repetition; draw lots of figures if you learn visually. Write any questions you may have in the margins. If, after reading a chapter twice, you still have unanswered questions, ask your study group, ask your TA, ask your professor (or ask all three; the more times you hear something, the better you'll learn it). Within each chapter, try to summarize its main ideas or methods; you'll find that almost every topic has one or two tricks that are used to do most of the proofs.

If your time is limited or you are reviewing material you've already learned, you can use the following icons to guide your skimming:



- Here begins an example or a proof that is figured out step by step.



- This is an important clarification or thing to keep in mind.

- Try this fill-in-the-blank exercise!

- This is a more complicated topic that is only mentioned briefly.

Extra resources never hurt. In fact, the more textbooks you read, the better your chances of success in learning advanced mathematics. The best strategy is to have one or two primary textbooks (for example, this one with Rudin) whose material you are committed to learning. Complement those with a library of other books from which to get extra practice and to look up an explanation if your primaries are not satisfactory. If you choose to disregard this and try to learn *all* the material in *all* the real analysis books out there... good luck to you!

This book covers most of a typical first-semester real analysis course, though it's possible your school covers more material. If this book ends before your course does, don't panic! Everything builds on what comes before it, so the most important factor for success is an understanding of the fundamentals. We will cover those fundamentals in detail, to make sure you have a solid foundation with which to swim onward (while avoiding mixed metaphors, such as this one).

For a list of some recommended books, along with my comments and criticisms, see the Bibliography.

Once you turn the page, we'll begin learning by going over some basic mathematical and logical concepts; they are critical background material for a rigorous study of real analysis. (How many times have I used the word *rigorous* so far? This many: $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} + 7$.)

CHAPTER 2

Basic Math and Logic

If you've seen some of this stuff before, great! If not, don't worry—we'll take it nice and slow.

Some Notation

What follows are some notational conventions which you should become comfortable with.

The symbol $\forall$ stands for "for all" or "for every" and can also be read "as long as." For example, the definition of even numbers tells us: *n is divisible by 2* $\forall$ *n even*. Read: "n is divisible by 2 for all even numbers n," or "n is divisible by 2 as long as n is even."

The symbol $\exists$ stands for "there exists" or "there is some." For example, one definition of the number e tells us: $\exists a$ *such that* $\frac{d}{dx}a^x = a^x$. That statement is true, since such a number a does exist; it is $e = 2.71828\ldots$

Note that the following two statements have completely different meanings:

$$\forall x, \exists y \text{ such that } y > x$$

$$\exists y \text{ such that } y > x, \forall x$$

The first means that given any x, there is some y greater than it. The second means that there exists some y which is greater than *every* possible x. If x and y are real numbers, then the first statement is true, since for any x we can set $y = x + 1$. The second is false, since no matter how big a y we choose, there will always be another number bigger than it.

A *sequence* is a list of numbers, indexed in order by integers. For example, $2, 4, 6, \ldots$ is a sequence, and the... symbol "..." indicates that it extends infinitely in a similar pattern. In the $2, 4, 6, \ldots$ example, the 10th element of the sequence is 20. By definition, a sequence continues on forever (so just the numbers 2, 4, 6 is not a sequence).

Sequences can also be made up of variables, such as $x_1, x_2, x_3, \ldots$. We say that x_i is the ith element of the sequence, as long as i is a positive integer (so using the notation from above, x_i is the ith element of the sequence, $\forall i \geq 1$). The integer subscript of a particular x is the *index* of that element of the sequence.

The sum of elements in a pattern can be concisely expressed in *summation notation* using the Greek letter $\sum$ (capital sigma). For example, the sum of the first n integers can be written as $\sum_{i=1}^{n} i$, which is read: "the sum from $i = 1$ until $i = n$ of i." As you might have noticed, by convention, the index of the sum always takes on integer values, starting at the subscript of the sigma and ending at the superscript of the sigma. Another example is $\sum_{i=1}^{n} 1$, which is read: "The sum from $i = 1$ until $i = n$ of 1," which is just the sum of $1 + 1 + 1 + \dots$, n times, which equals n.

Summations can also be written over a sequence (which, remember, is always infinite), and these are called *infinite series*, or just *series*. For example, the sum of all the elements in the aforementioned sequence $2, 4, 6, \dots$, can be written as the series $\sum_{i=1}^{\infty} 2i = 2 + 4 + 6 + \dots$. Another example is $\sum_{i=0}^{\infty} \frac{1}{i!}$, which actually equals the number e.

There are certain groups of numbers that have their own symbols:

- $\mathbb{N}$ is the set of all natural numbers. These are the positive integers, not including 0.
- $\mathbb{Z}$ is the set of all integers, including 0 and negative integers.
- $\mathbb{Q}$ is the set of all rational numbers These are defined as numbers of the form $\frac{m}{n}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$.
- $\mathbb{R}$ is the set of all real numbers. We will define what *real numbers* actually are later.

You can remember these symbols by the following mnemonics: N is for Natural numbers, R is for Real numbers, Q is for Quotients, and Z is for integerZ.

Through Chapter 4, we assume all the usual facts that let us perform algebra on numbers in $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$. In Chapter 5, we'll take a closer look at these properties.

Formal Logic

What follows are some concepts from the study of logic, which we will use over and over again in proofs.

A logical statement is *equivalent* to another statement whenever it is only possible for them to be either both true or both false. For example, "I have been alive for 5 years" is equivalent to "I am 5 years old"—since if one is true, then so is the other; if one is false, then so is the other.

The symbol $\implies$ stands for "implies." For example, the following four statements are equivalent to each other:

Statement 1. If $n = 5$, then n is in $\mathbb{N}$.
Statement 2. n is in $\mathbb{N}$ if $n = 5$.
Statement 3. $n = 5$ only if n is in $\mathbb{N}$.
Statement 4. $n = 5 \implies n$ is in $\mathbb{N}$.

Note that these are *not* equivalent to "$n = 5$ if n is in $\mathbb{N}$." (Also, that statement is clearly not true, since there exist natural numbers that are not equal to 5—my personal favorite being 246,734.)

The symbol $\iff$ stands for "if and only if" (abbreviated *iff*), and it is used to state that both directions of an implication are true. For example, "n is even $\iff n$ is divisible by 2." The left statement implies the right statement, and vice versa. This particular *iff* statement is true, since it is the definition of even numbers.

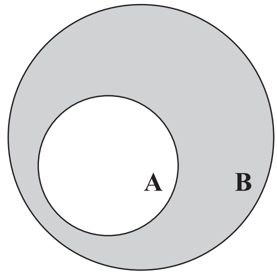


Figure 2.1. The fact **A** is completely contained in **B**. If x is in **A**, then x is also in **B**.

There is a slightly confusing mathematical convention for writing definitions. Theoretically, all definitions should be written with "if and only if." For example, "a number is called even if and only if it is divisible by 2." The "if" goes both ways, since "even" is just a name we assign to certain numbers. However, mathematicians are lazy; to save time, they usually write definitions with just "if" instead of "if and only if." Don't be confused! If you see the following:

Definition *(Even)*
A number is called ***even*** *if it is divisible by 2.*

You should read:

Definition *(Even)*
A number is called ***even*** *if and only if it is divisible by 2.*

An arbitrary statement we might want to prove can be expressed as $\mathbf{A} \implies \mathbf{B}$, where **A** and **B** are any facts.

The statement's *converse* is $\mathbf{B} \implies \mathbf{A}$. Just because a statement $\mathbf{A} \implies \mathbf{B}$ is true, does not mean its converse $\mathbf{B} \implies \mathbf{A}$ is true. For example, we saw that $n = 5 \implies n$ *is in* $\mathbb{N}$, but n *is in* $\mathbb{N}$ does not imply $n = 5$.

The statement's *inverse* is $\neg\mathbf{A} \implies \neg\mathbf{B}$ (here, the symbol $\neg$ means "not"). Again, just because a statement $\mathbf{A} \implies \mathbf{B}$ is true, does not mean its inverse $\neg\mathbf{A} \implies \neg\mathbf{B}$ is true. For example, we saw that $n = 5 \implies n$ *is in* $\mathbb{N}$, but $n \neq 5$ does not imply n *is not in* $\mathbb{N}$ (because n could, for instance, be the number 246,734).

If $\mathbf{A} \implies \mathbf{B}$ is a statement, then $\neg\mathbf{B} \implies \neg\mathbf{A}$ is its *contrapositive*. The statement $\mathbf{A} \implies \mathbf{B}$ is actually always equivalent to the statement $\neg\mathbf{B} \implies \neg\mathbf{A}$. If one of those statements is true, so is the other; if one of them is false, so is the other.

Why is every statement equivalent to its contrapositive? It helps to think of $\mathbf{A} \implies \mathbf{B}$ as saying "if x is in **A**, then x is in **B**." In that reading, we can represent **A** as a set that is completely contained in **B**.

Figure 2.1 helps us visualize: if x is not in **B**, then it certainly cannot be in **A**.

Last, note that if A is some property the number x can have, then the following two statements are equivalent:

Statement 1. $\neg\ (\forall x, x$ has property $A)$.
Statement 2. $\exists x$ such that $\neg\ (x$ has property $A)$.

The first statement says "it is not true that every x has property A," and the second statement says "there is some x such that x does not have property A." Read these two out loud, and it should be obvious why they are the same.

Similarly, the following two statements are also equivalent to each other:

Statement 3. $\neg$ ($\exists x$ such that x has property A).
Statement 4. $\forall x$, $\neg$ (x has property A).

Try to read the statements out loud, translating all the symbols into English.

Proof Techniques

There are many different ways to prove a theorem; sometimes, more than one method will work. There are five main techniques used throughout this book:

1. Proof by counterexample.
2. Proof by contrapositive.
3. Proof by contradiction.
4. Proof by induction.
5. Direct proof, in two steps.

Proof by Counterexample. In some cases, a proof may just be one counterexample. How would you prove the fact that not every integer is even? If I say "every integer is even," you just need to find one example of an integer that is not even, for instance, the number 3, to prove me wrong. Proofs by counterexample work for any statement of the form "$\exists x$ such that x has property A," or "$\neg(\forall x$, x has property A)." For the first, we just need to find one x that has property A; for the second, we just need to find one x that does not have property A.

Example 2.1. (Proof by Counterexample)
Let's try to prove the following statement: "Not every continuous function is differentiable." To do so, we just need one counterexample—any function that is continuous but not differentiable will do—for instance, $f(x) = |x|$.

Now we need to prove rigorously that $|x|$ is continuous and that it is not differentiable. You'll learn how to do so later on in your study of real analysis.

This example shows us that thinking up a counterexample is only half the work; the hard part is to prove rigorously that it indeed meets all the necessary conditions.

Proof by Contrapositive. As we understood earlier, $\neg\mathbf{B} \implies \neg\mathbf{A}$ is equivalent to $\mathbf{A} \implies \mathbf{B}$. So in order to prove $\mathbf{A} \implies \mathbf{B}$, we could alternatively assume $\mathbf{B}$ is false and show that $\mathbf{A}$ is also false.

Example 2.2. (Proof by Contrapositive)
Let's try to prove the following statement: *For any two numbers x and y, $x = y$ if and only if* $\forall \epsilon > 0$, $|x - y| < \epsilon$. This asserts that two numbers are equal if they are arbitrarily close (meaning we can choose an arbitrary distance ϵ, and they will be closer to each other than that distance). Since the statement has *iff,* the implication is bidirectional, and we must prove both directions.

1. $x = y \implies \forall \epsilon > 0, |x - y| < \epsilon$.
   Proving this direction is simple. Assume $x = y$. Then $x - y = 0$, so $|x - y| = 0$. Since any ϵ we choose must be greater than 0, we have $|x - y| = 0 < \epsilon$. Thus $\forall \epsilon > 0$, $|x - y| < \epsilon$.

2. $\forall \epsilon > 0, |x - y| < \epsilon \implies x = y$.
   This statement should make intuitive sense. It's saying that if the distance between x and y is less than every positive number, the distance between them must equal 0.
   To prove this direction, we'll use the contrapositive. In this case, the statement $\neg\mathbf{B} \implies \neg\mathbf{A}$ is

$$x \neq y \implies \neg(\forall \epsilon > 0,\ |x - y| < \epsilon).$$

   Remember from our discussion of logic that we can simplify the right-hand side to

$$x \neq y \implies \exists \epsilon > 0 \text{ such that } \neg(|x - y| < \epsilon).$$

   Well, if $x \neq y$, then x must equal y plus some number z, where $z \neq 0$. So $|x - y| = |z|$. The absolute value of any non-zero number is always positive, so if we let $\epsilon = |z|$, then $\epsilon > 0$ and $|x - y| = \epsilon$. We have shown that $\exists \epsilon > 0$ such that $\neg(|x - y| < \epsilon)$, so we're done!

Proof by Contradiction. Not to be confused with a proof by contrapositive, a proof by contradiction is something entirely different. Let's say we are trying to show that $\mathbf{A} \implies \mathbf{B}$. If we assume that **A** is true but **B** is false, then something should go horribly wrong; we should end up with a *contradiction*, something that violates a fundamental mathematical axiom or definition, such as "$0 = 1$" or "5.3 is an integer." When this happens, we have shown that if **A** is true, it is impossible for **B** *not* to be true—otherwise the definitions of math would break down.

Example 2.3. (Proof by Contradiction)
Let's try to prove the theorem "$\sqrt{2}$ is not a rational number." In this case, if we put the theorem into the form $\mathbf{A} \implies \mathbf{B}$, the statement **B** is "$\sqrt{2}$ is not a rational number." Notice that there really isn't any statement **A**—since the theorem is claiming that it is not necessary for anything besides the usual mathematical axioms to be true, in order for **B** to be true. Thus proof by contrapositive won't work. How about a proof by contradiction? Assume that $\sqrt{2}$ *is* in $\mathbb{Q}$, and show that something goes horribly wrong.

Before getting started on the proof, here are some general facts about numbers that will come in handy.

Fact 1. Any rational number $\frac{m}{n}$ can be simplified so that m or n (or both) are not even. (If m and n are both even, we can just divide the top and bottom by 2 and obtain a more simplified version of the same rational number.)
Fact 2. If $a = 2b$ for some integers a and b, then a must be even, since it is divisible by 2.
Fact 3. If a number a is odd, then a^2 is also odd, since a^2 is an odd number added to itself an odd number of times.

Now we can start. If $\sqrt{2}$ is rational, then by Fact 1 it can be expressed as a simplified fraction, so there exist integers m and n (not both even) such that $\left(\frac{m}{n}\right)^2 = 2$. Then $m^2 = 2n^2$, so by Fact 2, m^2 must be even. By Fact 3, if m were odd, m^2 would also be odd, so m must be even.

We can express m as $2b$ for some number b, so $m^2 = (2b)^2 = 4b^2$, which implies that m^2 is divisible by 4. Then $2n^2$ is also divisible by 4, so n^2 is even. By Fact 3 again, n must also be even.

Wait! Fact 1 told us that if $\sqrt{2}$ is rational, we can express it as $\frac{m}{n}$, where m and n are not both even. But we just showed that they both *are* even! We have contradicted a basic axiom about fractions, so the only possible logical conclusion is that our main assumption—that $\sqrt{2}$ is rational—must be false.

By the way, the same methodology of this argument also works for proving that the square root of *any* prime number is not rational.

A proof by contradiction is generally considered to be a last-resort method. In many cases, if you prove something by contradiction, you can apply the same key steps to easily prove the theorem directly. In the $\sqrt{2}$ example, that is not the case, but just be aware: proof by contradiction is a good way to start thinking about a problem, but always check to see if you can go further and prove it directly (for bonus mathematical etiquette points).

Proof by Induction. Mathematical induction works the same way as dominoes: if we set them all up, and then knock over just the first one, they will all fall down. Induction works for any proof in which we need to prove an infinite number of cases (actually, it must be a *countably* infinite number of cases—you'll understand what this means in Chapter 8).

Let's say we're able to set up the dominoes by proving the following: *if we assume the theorem is true for case* 1*, then it is also true for case* 2*; if we assume the theorem is true for case* 2*, then it is also true for case* 3; and so on. This can by summarized by proving that *if the theorem is true for* $n - 1$*, then it is also true for* n. Now all we need to do is knock down the first domino by proving that the theorem is true for case 1. They all fall down, since our setup tells us that once case 1 is true, so is case 2; and now that case 2 is true, so is case 3; and so on.

Knocking the first domino down is easier, so we usually do it first (this step is called the *base case*). Then we assume the theorem is true for the $n - 1$ case (this assumption is called the *inductive hypothesis*) and show that it is also true for the n case (this step is called the *inductive step*).

Example 2.4. (Proof by Induction)
Let's try to find a formula for the sum of the first n natural numbers, $1 + 2 + 3 + \ldots + n$. Using our notation from the previous section, this sum is equivalent to $\sum_{i=1}^{n} i$. If you play with this long enough, you might stumble on the answer:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Try plugging in a few values to convince yourself that the formula works. To prove it, we'll need to be more rigorous (a few examples isn't a proof, since they don't exclude the possibility that a counterexample exists). We need to show that this formula works for every possible choice of n, which can be any positive integer. Therefore, induction is probably the best technique.

1. *Base Case.* We just need to show that the formula holds for the case $n = 1$. Well, $\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$, so the first step is done. Yay! (Base cases are usually a breeze.)

2. *Inductive Step.* The inductive hypothesis lets us assume that the formula is true for $n-1$, so we can assume that $\sum_{i=1}^{n-1} i = \frac{(n-1)(n)}{2}$. Using this assumption, we want to show that the formula holds for n, meaning $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. By making a substitution and simplifying, we can write

$$\sum_{i=1}^{n} i = \left(\sum_{i=1}^{n-1} i\right) + n = \frac{(n-1)(n)}{2} + n = \frac{n^2 - n}{2} + \frac{2n}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2},$$

and that's it!

Although it seems almost too simple, remember, there's no magic involved. We didn't "bootstrap" the proof or use circular logic. We just used the inductive step to say, "if it works for 1, then it works for 2; if it works for 2, then it works for 3; and so on," and because the base case says, "it works for 1," we have thus proved it for every possible positive integer choice of n (in other words, for every n in $\mathbb{N}$).

Direct Proof in Two Steps. None of the tricks we have covered show how we can prove $\mathbf{A} \implies \mathbf{B}$ directly, by assuming **A** is true then taking logical steps to end up with **B**.

Coming up with a direct proof requires you to play around for a while, until you figure out what the crux of the problem is and how to solve it. In many cases, the crux will involve finding some magic function or variable that makes everything fall into place. Unless you are writing a textbook, the reader of your proof does not care *how* you solved the crux, he or she just wants to see why the theorem is true. Once you have a good idea of what key steps you'll use to prove the theorem, the next step is to write it cleanly in a linear fashion.

Example 2.5. (Direct Proof in Two Steps)
Remember from calculus that we define the *limit* of a function as $\lim_{x \to p} f(x) = q$ if and only if for every $\epsilon > 0$, there is some $\delta > 0$ such that

$$|x - p| < \delta \implies |f(x) - q| < \epsilon.$$

You'll understand what this means in more detail when you study the topic of continuity. For now, though, let's just look at the following statement: "For $f(x) = 3x + 1$, $\lim_{x \to 2} f(x) = 7$." You know this should be true, since all polynomials are continuous—and there are probably multiple ways to prove this particular statement—but let's try to do a direct proof, using only the definition we just saw.

First we do the scratchwork to figure out the key steps. For every possible choice of $\epsilon > 0$, we need to find the right $\delta > 0$ so that

$$|x - 2| < \delta \implies |f(x) - 7| < \epsilon,$$

or equivalently

$$-\delta + 2 < x < \delta + 2 \implies -\epsilon < 3x - 6 < \epsilon.$$

Thus our δ needs to make

$$\frac{-\epsilon + 6}{3} < x < \frac{\epsilon + 6}{3},$$

so we just need

$$\delta + 2 = \frac{\epsilon + 6}{3} \implies \delta = \frac{\epsilon}{3}.$$

Now that we have found our magic δ, we can write up the proof concisely: For any $\epsilon > 0$, let $\delta = \frac{\epsilon}{3}$. Then $\delta > 0$, and

$$\begin{aligned} |x - p| < \delta &\implies |x - 2| < \frac{\epsilon}{3} \\ &\implies 2 - \frac{\epsilon}{3} < x < 2 + \frac{\epsilon}{3} \\ &\implies -\epsilon < 3x - 6 < \epsilon \\ &\implies |(3x + 1) - 7| < \epsilon \\ &\implies |f(x) - q| < \epsilon. \end{aligned}$$

Thus we have $\lim_{x \to 2} (3x + 1) = 7$.

One more hint about writing proofs: if you get stuck, look at the facts you haven't used yet. In the previous example, the only real "fact" available was the definition of a limit I gave you; but in later topics, you'll have a host of definitions and theorems to call on. Chances are, applying one you have forgotten will pull you out of the morass.

In the future, we will place the symbol $\square$, which signifies Q.E.D., at the end of every proof. It stands for the Latin *quod erat demonstrandum*, which basically means (and here I paraphrase liberally), "we have proved what we said we would prove."

And we're off! Now you know everything you'll need to start learning real analysis. As promised, we'll spend the next chapter learning about sets before we look at the real numbers.

CHAPTER 3

Set Theory

Before we dive into real analysis, a basic knowledge of sets (and how to manipulate them) will be useful. What are sets? Well, not all numbers are real numbers. In fact, not all "things" we wish to consider are numbers at all. *Sets* are a useful abstraction. They contain *elements*, which can be real numbers, imaginary numbers, dollars, people, beluga whales, and so on.

In this chapter, we'll go over the basic notation and theorems used to describe abstract sets. When you think of operations on numbers, addition, subtraction, multiplication, and division usually come to mind. For sets, however, the basic operations we will learn about are *union, intersection,* and *complement.*

Definition 3.1. *(Set)*
A ***set*** *is a collection of* ***elements****. A set with an infinite number of elements is called an* ***infinite set****.*

Example 3.2. (Sets)
Here are some examples of sets and their notation:

- $\{1, 2, 3\}$
  The set containing the numbers 1, 2, and 3. We write $1 \in \{1, 2, 3\}$ to mean that 1 is an element of the set.
- A
  The set named A.
- $A = \{1, 2, 3\}$
  The set named A which contains the numbers 1, 2, and 3.
- $\{a, b, c\}$
  The set containing the elements named a, b, and c. (These elements are not necessarily numbers.)
- $\{A, B, C\}$
  The set containing the elements named A, B, and C. In general, uppercase letters are used to denote sets, so this set might contain three other sets.
- $\mathbb{R}$
  The set containing all the real numbers. For example, $\pi \in \mathbb{R}$. This is an infinite set.