LINEAR TRANSFORMATIONS IN HILBERT SPACE

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AN INTRODUCTION TO LINEAR TRANSFORMATIONS IN HILBERT SPACE

BY

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PREFACE

The theory of operators in Hilbert space has its roots in the theory of orthogonal functions and integral equations. Its growth spans nearly half a century and includes investigations by Fredholm, Hilbert, Weyl, Hellinger, Toeplitz, Riesz, Frechet, von Neumann and Stone. While this subject appeals to the imagination, it is also satisfying because due to its present abstract methods, questions of necessity and sufficiency are satisfactorily handled. One can therefore be confident that its developement is far from complete and eagerly await its further growth.

These notes present a set of results which we may call the group germ of this theory. We concern ourselves with the structure of a single normal operator and at the end present the reader with a reading guide which, we believe, will give him a clear and reasonably complete picture of the theory.

Fundamentally the treatment given here is based on the two papers of Professor J. von Neumann referred to at the end of Chapter I. An attempt however has been made to unify this treatment and also recast it in certain respects. (Cf. the introductory paragraphs of Chapter IX). The elementary portions of the subject were given as geometrical a form as possible and the integral representations of unitary, self-adjoint and normal operators were linked with the canonical resolution.

In presenting the course from which these notes were taken, the author had two purposes in mind. The first was to present the most elementary course possible on this subject. This seemed desirable since only in this way could one hope to reach the students of physics and of statistics to whom the subject can offer so much. The second purpose was to emphasize those notions which seem to be proper to linear spaces and in particular to Hilbert space and omitting other notions as far as possible. The importance of the combination of various notions cannot be over-emphasized but there is a considerable gain in clarity in first treating them separately. These purposes are not antagonistic. We may point out that the theoretical portions of this work, except §4 of Chapter III, can be read without a knowledge of Lebesgue integration.

On the other hand, for these very reasons, the present work cannot claim to have supplanted the well-known treatise of M. H. Stone or the lecture notes of J. von Neumann. It is simply hoped that the student will find it advantageous to read the present treatment first and follow the reading guides given in Chapters XI and XII in consulting Stone's treatise and the more recent literature.

To those familiar with the subject, it will hardly be necessary to point out that the influence of Professor von Neumann is effective throughout the present work. Professor Bochner of Princeton University has also taken a kind interest in this work and made a number of valuable suggestions. I am also deeply grateful to my brother, Mr. John E. Murray, whose valuable assistance in typing these lecture notes, was essential to their preparation.

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CHAPTER I

The expressions:

$$T_1 f = \int_0^1 k(x,y)f(y)dy$$

or

$$T_2 f = p(x)i\frac{d}{dx}f(x)+q(x)f(x)$$

or in the case of a function of two variables,

$$T_3 f = \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}$$

are linear operators. Thus the first two, when applicable, take a function defined on the unit interval into another function on the same interval.

Now if we confine our attention to functions $f(x)$ continuous on the closed unit interval and with a continuous derivative, we know that such a function can be expressed in the form,

$$f(x) = \Sigma_{\alpha=-\infty}^{\infty} x_\alpha \exp(2\pi i\alpha x)$$

where $x_\alpha = \int_0^1 f(x)\exp(-2\pi i\alpha x)dx$. If $T_1 f$ is of the same sort,

$$T_1 f = \Sigma_{\alpha=-\infty}^{\infty} y_\alpha \exp(2\pi i\alpha x)$$

where $y_\alpha = \int_0^1 T_1 f \exp(-2\pi i\alpha x)dx =$

$$\Sigma_{\beta=-\infty}^{\infty} x_\beta \int_0^1 T_1(\exp(2\pi i\beta x))\exp(-2\pi i\alpha x)dx = \Sigma_{\beta=-\infty}^{\infty} x_\beta a_{\beta,\alpha}.$$

Now for T_3 a somewhat similar argument holds, although it is customary to use a double summation.

The important thing to notice is that the operator equation

$$Tf = g$$

can be, in these cases, replaced by an infinite system of linear equations in an infinite number of unknowns. We shall prove that this can be done in far more general circumstances.

One might attempt to solve such an infinite system of equations by substituting a finite system and then passing to the limit, for example one might take the first n equations and ignore all but the first n unknowns. But this process is in-effective in general and introduces certain particular difficulties of its own.

Other methods must be sought. The choice of the functions $\exp(2\pi i\alpha x)$ corresponds to a choice of a system of coordinate axes in the case of a finite number of unknowns. In the finite case for a symmetrical operator, the coordinate system can be chosen, so that,

$$a_{\alpha,\beta} = \lambda_\alpha \delta_{\alpha,\beta} \qquad \begin{matrix} \delta_{\alpha,\beta} = 0 \quad \text{if } \alpha \neq \beta, \\ \delta_{\alpha,\alpha} = 1. \end{matrix}$$

(Cf. Chap.VII, §1, Lemma 4) Correspondingly in the infinite case we would seek a complete set of functions ϕ_α such that

$$T\phi_\alpha(x) = \lambda_\alpha \phi_\alpha(x).$$

When this has been done, inverting the equation becomes a simple process. For example, consider, $T = i\frac{d}{dx}$, $\phi_\alpha(x) = \exp(-2\pi i\alpha x)$. While this is in general impossible, nevertheless an effective method of generalizing the result in the finite dimensional case exists and we shall discuss it in the present work.

We shall want to give our discussion its most general form and for that reason we consider not the set of functions whose square is summable, but rather an abstract space which has just those properties of this set which are needed in our developement. This space, $\mathfrak{H}$, is called Hilbert space and we shall show in Chapter II the existence of something equivalent to orthogonal sets of functions.

In Chapter III, we discuss L_2 and other realizations of abstract Hilbert space. In Chapters IV, V and VI, linear transformations are studied and certain preliminary properties established. Also a notion "weak convergence," which is of considerable interest in the theory of abstract spaces, is introduced to establish Theorem V of Chapter V.

In Chapter VII, we shall develope the needed generalization of the notion of an operator in diagonal form. In Chapter IX, we shall show that a self-adjoint operator even if it is discontinuous, is expressable in the diagonal form.

Symmetry is not sufficient in the discontinuous case as we shall see. The distinction between symmetric and self-adjoint transformations is brought out in Chapter X. In Chapter XI, a brief outline of further developements in the theory is given.

Except for Chapter XI, our discussion is based on the following:

(1) F. Riesz and E. R. Lorch. Trans. of the Amer. Math. Soc. Vol. 39, pp. 331-340 (1936).

(2) M. H. Stone. Colloquium Publications of Amer. Math. Soc. Vol. XV (1932).

(3) J. von Neumann. Math. Annalen Bd. 102 pp. 49-131 (1929).

(4) J. von Neumann. Annals of Mathematics, 2nd series, Vol. 33, pp. 294-310 (1932).

CHAPTER II

The axiomatic treatment of Hilbert space was first given by J. von Neumann in (4) pp. 64-69. He proposed the definition given below. We follow here the discussion given by Stone (2). (Numerals in parentheses refer to the references cited at the end of Chapter I.)

DEFINITION 1.1. A class $\mathfrak{H}$ of elements $f, g, \ldots$ is called a Hilbert space if it satisfies the following postulates:

POSTULATE A. $\mathfrak{H}$ is a linear space; that is,

(1) there exists a commutative and associative operation denoted by $+$, applicable to every pair f, g of elements of $\mathfrak{H}$, with the property that $f+g$ is also an element of $\mathfrak{H}$.

(2) there exists a distributive and associative operation, denoted by $\cdot$, applicable to every pair (a,f), where a is a complex number and f is an element of $\mathfrak{H}$;

(3) in $\mathfrak{H}$ there exists a null element denoted by θ with the properties

$$f+\theta = f, \qquad a\cdot\theta = \theta, \qquad 0\cdot f = \theta.$$

POSTULATE B. There exists a numerically-valued function (f,g) defined for every pair f, g of elements of $\mathfrak{H}$, with the properties:

(1) $(af,g) = a(f,g)$.

(2) $(f_1+f_2,g) = (f_1,g)+(f_2,g)$.

(3) $(g,f) = \overline{(f,g)}$.

(4) $(f,f) \geqq 0$.

(5) $(f,f) = 0$ if and only if $f = \theta$.

The not-negative real number $(f,f)^{1/2}$ will be denoted for convenience by $|f|$.

POSTULATE C. For every n, $n = 1, 2, 3, \ldots$, there exists a set of n linearly independent elements