CONTRIBUTIONS TO THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS

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FOREWORD

In October 1952 a three day conference on partial differential equations was held at Arden House, Harriman, New York. The conference was organized and sponsored by the National Academy of Sciences - National Research Council.

This volume contains those papers, read at the conference, which were submitted by the authors for publication. The editors regret the unavoidable delay in publication and hope that this volume will prove to be useful to mathematicians working in this field.

The editing and preparing of this study was carried out entirely by Anneli Lax. The editors gratefully acknowledge her valuable assistance.

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CONTENTS

I. GREEN'S FORMULA AND ANALYTIC CONTINUATION

S. Bochner

For anyalytic functions in more than one complex variable there is a theorem of Hartog's that if a function is given on the connected boundary of a bounded domain, then it can be continued analytically into all of the domain. The class of functions to which this theorem applies was considerably generalized in our paper [1]: "Analytic and meromorphic continuation by means of Green's formula," Annals of Mathematics 44 (1943), 652-673; and it was further expanded in our recent note [2]: "Partial differential equations and analytic continuation," Proceedings of the National Academy of Sciences 38 (1952), 227-30. Now, in §1 of the present paper the leading theorem of [2] will be given its final version known to us (see Theorem 5) and, furthermore, details of the proof will be modified and added.

The real and imaginary parts of analytic functions of complex variables are solutions of a system of Cauchy-Riemann equations in real variables. In the case of more than one complex variable, this system is quite complicated and, as it turns out, much too restrictive for our theorem. At first in [1], and then more systematically and generally in [2], we introduced instead a system consisting of only two equations, both with constant coefficients: an elliptic one in all variables and some other one in fewer than all variables; the second equation was the one by which the actual continuation was brought about. However, the second equation could only operate if the function was first represented by a certain Green's formula, and it was the sole task of the elliptic equation to secure just such a formula. Now, in further analyzing certain aspects of our theorem, we found it pertinent to try to give up the elliptic equation altogether and to hypothesize directly a Green's formula having the requisite properties. This will be done in the present paper.

In §2 we will be dealing in a similar fashion with another theorem in several complex variables which although closely related to the previous one is different from it nevertheless. Following up a suggestion of Severi's, this theorem was presented more systematically than had been done before in Chapter IV of the book by Bochner-Martin: Several Complex Variables, Princeton, 1948. It will now be given a rather more general version than previously.

§1. EUCLIDEAN SPACES

In Euclidean $E_n: (\xi^1, \ldots, \xi^n)$ we take a p-form, $1 \leq p \leq n-1$

$$(1) \qquad \frac{1}{p!} \sum_{(\alpha)} A_{\alpha_1 \ldots \alpha_p} d\xi^{\alpha_1} \ldots d\xi^{\alpha_p}$$

and we assume that each component of the skew tensor is a finite linear partial differential expression involving an unspecified function $f(\xi)$ with coefficients which are functions of the difference $\xi - x = (\xi^\beta - x^\beta)$, where $x = (x^\beta)$ is another variable point of the given space. Thus we have

$$(2) \qquad A_{\alpha_1 \ldots \alpha_p} = \sum_{(\nu)} G^{\nu_1 \ldots \nu_n}_{\alpha_1 \ldots \alpha_p} (\xi - x) \wedge_{\nu_1 \ldots \nu_n} f(\xi)$$

where

$$(3) \qquad \wedge_{\nu_1 \ldots \nu_n} f(\xi) = \frac{\partial^{\nu_1 + \ldots + \nu_n} f(\xi)}{(\partial \xi^1)^{\nu_1} \ldots (\partial \xi^n)^{\nu_n}}$$

with $\nu_1 \geq 0, \ldots, \nu_n \geq 0$, $\nu_1 + \ldots + \nu_n \leq N - 1$ for some N sufficiently large but finite. In this sense we denote the form (1) by

$$(4) \qquad G_p(\xi - x;\ f(\xi);\ d\xi)$$

and we stipulate that the individual functions $G^{(\nu)}_{(\alpha)}(t)$, $t = \xi - x$, which occur in (2) shall be defined and real analytic in a certain open set T of the Euclidean $E_n: (t^\beta)$.

We now introduce the requirement

$$(5) \qquad d_\xi G_p \equiv 0$$

that is

$$(6) \qquad \sum_{q=1}^{p+1} (-1)^q \frac{\partial}{\partial \xi^{\alpha_q}} A_{\alpha_1 \ldots \alpha_{q-1} \alpha_{q+1} \ldots \alpha_{p+1}} = 0$$

and this is a system of equations

$$(7) \qquad \sum_{(\nu)} H^{\nu_1 \ldots \nu_n}_{\alpha_1 \ldots \alpha_{p+1}} (\xi - x) \wedge_{\nu_1 \ldots \nu_n} f(\xi) = 0$$

$$0 \leq \nu_1 + \ldots + \nu_n \leq N$$

in which the coefficients $\overset{(\nu)}{\underset{(\alpha)}{H}}$ are linear combinations of the coefficients $\overset{(\nu)}{\underset{(\alpha)}{G}}$ and their first derivatives.

We now introduce fictitiously an "elliptic operator" $\Delta_\xi f(\xi)$ and we will say that a function $f(\xi)$ which is defined and analytic in a domain U of E_n satisfies there the equation

$$\Delta_\xi f(\xi) = 0 \tag{8}$$

if the relations (7) are fulfilled for ξ in U and $\xi - x$ in T. For every U the space of solutions of (8) is a vector space having the following closure property: Let $f_1(\xi)$, $f_2(\xi)$, ... be a sequence of solutions. If every point ξ^β of U has a complex neighborhood into which all the functions of the sequence can be continued analytically, and if these functions converge uniformly in these neighborhoods, then the limit function is again a solution.

Furthermore, if we put $\xi = t + x$ in (7), we obtain

$$\sum_{(\nu)} \overset{(\nu)}{\underset{\alpha}{H}}(t) \wedge^t_{\nu_1 \cdots \nu_n} f(t+x) = 0 \tag{9}$$

and therefore, locally, if $f(\xi)$ is a solution of (7) then so is also the translated function $f(\xi+h)$ for a sufficiently small constant displacement $h = (h^\beta)$. This property is crucial to our purpose, and we express it symbolically by saying that our (fictitious) operator Δ has "constant coefficients"; and if we combine all properties enumerated, we obtain the further property that if $f(\xi)$ is a solution, then so are also all first derivatives $\wedge_\beta f(\xi)$, and hence, also all mixed partial derivatives, and hence, also every finite linear combination with constant coefficients

$$\wedge f = \sum_{(\mu)} a_{\mu_1 \cdots \mu_n} \wedge_{\nu_1 \cdots \nu_n} f(\xi) \tag{10}$$

$$0 \le \mu_1 + \cdots + \mu_n \le M$$

this operator being formed literally, and not just only fictitiously or symbolically.

We now take in U a p-dimensional chain B_p and we form the integral

$$g(x) = \int_{B_p} G(\xi - x;\ f(\xi);\ d\xi) \tag{11}$$

for those points x for which it is definable, that is, for the open set X which is such that for x in X and ξ in B_p, $\xi - x$ is in T.

If we hold B_p fixed, then $g(x)$ is analytic in T; and if we vary $f(\xi)$ and then introduce the functional

$$g(x) = L(f(\xi);\ x) \tag{12}$$

then the latter is distributive, that is,

$$L(c_1f_1 + c_2f_2;\ x) = c_1L(f_1;\ x) + c_2L(f_2;\ x). \tag{13}$$

Next, if two chains B_p, B'_p are homologous relative to $U - (x)$, then by (5) we have

$$\int_{B_p} = \int_{B'_p} \tag{14}$$

and to this extent the integral (11) is "independent of the path" if $f(\xi)$ is a solution of (8). But the decisive property is yet to be stated and it is as follows.

THEOREM 1. If B_p is a cycle then the functional (12) is commutative with translations and partial differentiations. That is, (12) implies

$$g(x+h) = L(f(\xi + h);\ x) \tag{15}$$

for small h locally, and also

$$\frac{\partial g}{\partial x^\beta} = L\left(\frac{\partial f}{\partial \xi^\beta};\ x\right) \tag{16}$$

and more generally

$$\wedge_x\, g(x) = L(\wedge_\xi f(\xi);\ x) \tag{17}$$

for any operator (10).

PROOF. We have

$$g(x+h) = \int_{B_p} G(\xi-x-h;\ f(\xi);\ d\xi) \tag{18}$$

and if we replace ξ by $\xi + h$ (18) is equal to

(19) $$\int_{B'_p} G(\xi - x;\ f(\xi + h);\ d\xi)$$

where $B'_p = B_p - (h)$ results from B_p by a translation. If x is given then for sufficiently small h, B'_p is homologous to B_p, and by (14) the integral (19) is

$$\int_{B_p} G(\xi - x;\ f(\xi + h);\ d\xi) = L(f(\xi + h);\ x)$$

as claimed in (15).

Due to (13) the result just obtained implies that the integral (11) carries a difference quotient

$$\frac{1}{h}[f(\xi^1, \ldots, \xi^\beta + h, \ldots, \xi^n) - f(\xi^1, \ldots, \xi^\beta, \ldots, \xi^n)]$$

into the corresponding difference quotient for $g(x^\alpha)$. If now we let $h \longrightarrow 0$, then we obtain (16), as can be easily proved, and then also (17), as claimed.

Theorem 1 will suffice for our present purposes. However, for a certain type of conclusion that was attempted in [1] and [2] a partial generalization of Theorem 1 is required for the case in which B_p is not a cycle, and we will state the generalization without proof.

THEOREM 2. If we are given a symbolic equation (8) and an operator (10), then there exists a (p-1)-form

$$Y_{p-1}(\xi - x;\ f(\xi);\ d\xi)$$

in which the partial derivatives of $f(\xi)$ occurring are of order $\leq N - 1 + M - 1$, and which has the following property: If B_p is a p-chain and B_{p-1} is its boundary and if $f(\xi)$ satisfies (8) in a domain containing $B_p + B_{p-1}$, then we have

$$L(\wedge_\xi f(\xi);\ x) - \wedge g(x) = \int_{B_{p-1}} Y_{p-1}(\xi - x;\ f(\xi);\ d\xi)$$

and thus we again have (17) provided that the function $f(\xi)$ and its derivatives of order $\leq N - 1 + M - 1$ are zero on the boundary B_{p-1}.

From now on we will consider only the dimension $p = n - 1$, and we will assume that the domain T on which the coefficients $G^{(\nu)}_{(\alpha)}(t)$ are defined is the entire E_n except for the origin, so that $L(f(\xi); x)$ is defined for x in $E_n - B_{n-1}$.

THEOREM 3. If B_{n-1} bounds a small subdomain D^o of U then we have

$$\oint_{B_{n-1}} G(\xi - x;\ f(\xi);\ d\xi) = \sum_{(r)} c_{r_1 \ldots r_n} \wedge_{r_1 \ldots r_n} f(x) \tag{20}$$

$$r_1 \geq 0, \ \ldots, \ r_n \geq 0$$

where $c_{r_1 \ldots r_n}$ are constants independent of $f(\xi)$.

PROOF. It follows from (14) that for the proof of (20) we may assume that D^o is a coordinate sphere $|\xi| < 2\rho$, and that even the double sphere $|\xi| < 4\rho$ is still contained in U, and that the points x are restricted to the small sphere $|x| < \rho$. Now, on replacing ξ by $\xi + x$ we obtain

$$g(x) = \oint_{B_{n-1}-(x)} G(\xi;\ f(\xi + x);\ d\xi)$$

$$= \oint_{B_{n-1}} G(\xi;\ f(\xi + x);\ d\xi)$$

and for sufficiently small ρ we may now insert the Taylor expansion

$$f(\xi + x) = \sum_{(s)} \frac{(\xi^1)^{s_1} \ldots (\xi^n)^{s_n}}{s_1! \ \ldots \ s_n!} \wedge_{s_1 \ldots s_n} f(x)$$

and integrate term by term. Hence the conclusion.

The theorem is of no practical importance to us, but it puts into a proper perspective the assumption we are now going to make that the right side in (20) shall be identically $f(x)$, that is,

$$\oint_{B_{n-1}} G(\xi - x;\ f(\xi);\ d\xi) = f(x) \tag{21}$$

and we express this assumption by saying that our integral (11) is a <u>Green's formula</u>. The first consequence of this assumption is:

THEOREM 4. If B is a simplex (which may be part of any chain over which we integrate) then the function

(22) $$g(x) = \int_B G(\xi - x;\ f(\xi);\ d\xi)$$

can be continued across B from either side and the difference of the two values in the vicinity of B is $f(x)$,

(23) $$g_+(x) - g_-(x) = \pm f(x)$$

the algebraic sign being determined by the orientation of B.

PROOF. (Cf. [1], p. 656-57.) Denoting one side of B as positive, we deform B into a simplex B' on its negative side but keep the edges fixed. By (14) we have for x on the positive side

$$g_+(x) = \int_B = \int_{B'}$$

and thus $g_+(x)$ can be continued a certain distance into the negative side. Denoting the continuation still by $g_+(x)$, we now have for points x in the domain bounded by $B' - B$

$$g_+(x) - g_-(x) = \int_{B'} - \int_B = \int_{B'-B}$$

and by (21) this is $\pm f(x)$ as claimed.

THEOREM 5. If we are given a Green's formula and if the coefficients $G^{(\nu)}_{(\alpha)}(t)$ in (2) are of slow growth at infinity in the sense that for some $r_0 > 0$ we have

(24) $$\Lambda_{r_1 \cdots r_n} G^{\nu_1 \cdots \nu_n}_{\alpha_1 \cdots \alpha_{n-1}}(t) = O(|t|^{-n-1}), \qquad |t| \longrightarrow \infty$$

for

(25) $$r_1 \geq 0,\ \ldots,\ r_n \geq 0, \qquad r_1 + \cdots + r_n \geq r_0$$

then, if D is a bounded domain having a connected boundary B_{n-1} and if U is a neighborhood of B_{n-1} and if an analytic function $f(x)$ in U satisfies the associated "elliptic equation"

(26) $$\Delta_x f(x) = 0$$

in all its n variables and an additional equation

$$\wedge f \equiv \sum_{(q)} a_{q_1 \dots q_m} \wedge_{q_1 \dots q_m} f(x) = 0 \quad (m<n)$$

with constant coefficients not all zero in fewer than all variables, then the function $f(x)$ has an analytic continuation into all of $D + U$.

PROOF. By Theorem 4, if we restrict U sufficiently, we can put

$$f(x) = g_+(x) - g_-(x)$$

where $g_+(x)$ exists and is analytic in $D + U$, $g_-(x)$ is analytic in $(E^n - D) + U$, and the assertion of the theorem will follow if we show that $g_-(x)$ has an analytic continuation into all of E^n.

Let us denote $g_-(x)$ by $\phi(x)$. We know the following about this function: It is defined and analytic in the exterior of a sphere

(27) $$|x| > R$$

and there exists an $r' > 0$ such that

$$\wedge_{r_1 \dots r_n} \phi(x) = O(|x|^{-n-1}), \quad |x| \longrightarrow \infty$$

for $r_1 + \dots + r_n \geq r'$ and we have

$$\wedge \phi(x) \equiv 0$$

If we now take fixed values $r_1, \dots, r_n$, and fixed coordinates $x_o^{m+1}, \dots, x_o^n$ for which

(28) $$(x_o^{m+1})^2 + \dots + (x_o^n)^2 > R^2$$

and introduce the function

$$\Psi(x^1, \ldots, x^m) = \wedge_{r_1 \ldots r_n} \phi(x^1, \ldots, x^m, x_0^{m+1}, \ldots, x_0^n)$$

then the latter has the following properties: It is defined and analytic in E_m: $(x^1, \ldots, x^m)$, we have

$$\wedge_{q_1 \ldots q_m} \Psi(x) = O(|x|^{-m-1}), \quad |x| \longrightarrow \infty \tag{29}$$

for any $q_1 \geq 0, \ldots, q_m \geq 0$, and we have

$$\wedge \Psi(x) \equiv 0 \tag{30}$$

We claim that under these circumstances we have

$$\Psi(x) \equiv 0$$

In fact, because of (29) we may introduce the Fourier transform

$$\chi(\alpha_j) = \int_{E^m} e^{i(\alpha_1 x^1 + \ldots + \alpha_m x^m)} \Psi(x) dv_x$$

and it follows easily that the Fourier transform of $\wedge \Psi(x)$ is then

$$\chi(\alpha_j) \quad \sum_{(q)} a_{q_1 \ldots q_m} (i\alpha_1)^{q_1} \ldots (i\alpha_m)^{q_m} \tag{31}$$

Now, (30) implies the vanishing of (31), and hence the vanishing of $\chi(\alpha)$, and by uniqueness of Fourier transforms the vanishing of $\Psi(x)$. Therefore we have

$$\wedge_{r_1 \ldots r_n} \phi(x^1, \ldots, x^m, x^{m+1}, \ldots, x^n) = 0 \tag{32}$$

for $(x^1, \ldots, x^m)$ in E_m and $(x^{m+1}, \ldots, x^n)$ in (28), but since the left side of (32) is analytic in (27), the relation (32) also holds identically in (27). And since this is true for all combinations $r_1, \ldots, r_n$ with $r_1 + \ldots + r_n = r'$ it follows that $\phi(x)$ is a polynomial of total degree $\leq r' - 1$, and thus is certainly continuable into all E_n.

§2. COORDINATE SPACES

We now take an arbitrary analytic coordinate space X_n: $(\xi^1, \ldots, \xi^n)$ and in it, immediately for $p = n - 1$, a form

$$G(\xi;\ x;\ f(\xi);\ d\xi) = \frac{1}{(n-1)!}\sum_{(\alpha)} A_{\alpha_1\cdots\alpha_{n-1}}\, d\xi^{\alpha_1} \ldots d\xi^{\alpha_{n-1}}$$

where

$$A_{\alpha_1\cdots\alpha_{n-1}} = \sum_{(\nu)} G^{\nu_1\cdots\nu_n}_{\alpha_1\cdots\alpha_{n-1}}(\xi;\ x)\Lambda_{\nu_1\cdots\nu_n} f(\xi)$$

for $\xi \neq x$. The point x itself also ranges over X_n, and any two points ξ_o, x_o $(\xi_o \neq x_o)$ have non-overlapping coordinate neighborhoods for which the tensoroid components $G^{(\nu)}_{(\alpha)}(\xi;\ x)$ are defined and analytic in $(\xi;\ x)$.

We again introduce the requirement $d_\xi G = 0$, that is

$$\sum_{q=1}^{n} (-1)^q \frac{\partial}{\partial \xi^{\alpha_q}} A_{\alpha_1\cdots\alpha_{q-1}\alpha_{q+1}\cdots\alpha_n} = 0 \qquad .$$

and it amounts to a tensoroid system of equations

$$\sum_{(\nu)} H^{(\nu)}_{(\alpha)}(\xi;\ x)\Lambda_{\nu_1\cdots\nu_n} f(\xi) = 0$$

which we again abbreviate to

$$\Delta_\xi f(\xi) = 0 \tag{33}$$

and we again consider analytic functions in open sets U which satisfy this "elliptic equation" there. We wish to point out, however, that we are not retaining in any manner whatsoever the previous assumption that the variables ξ, x shall occur as the difference $\xi - x$ only, and thus the "constancy of coefficients" in the operator Δ is being entirely dispensed with.

We again introduce the integral

$$g(x) = \int_{B_{n-1}} G(\xi;\ x;\ f(\xi);\ d\xi) \tag{34}$$

for a chain in U. It is again analytic and distributive, and we again have (14); and if B_{n-1} is the boundary of a small domain D^o, then for points of D^o we have

$$g(x) = \sum_{(r)} c^{r_1\cdots r_n}(x)\Lambda_{r_1\cdots r_n} f(x)$$

where the coefficients $c^{r_1 \ldots r_n}(x)$ are now functions of x.

We again call the integral a Green's formula if we have, in particular,

(35) $$\oint_{B_{n-1}} G(\xi;\ x;\ f(\xi);\ d\xi) = g(x)$$

and theorem 4 is valid again.

We now take a second analytic coordinate space Y_m: $(y^1, \ldots, y^m)$, $(m \gtreqless n)$ which will be a space of "parameters" and we form the product space

(36) $$Z_{n+m} = X_n \times Y_m$$

in the usual manner. Also, for any point set θ in Z_{n+m}, we will employ the representation by its layers in the X_n-space, thus

(37) $$\theta = \left\{ y \in Y_m;\ x \in D(y) \right\}$$

where for each y, $D(y)$ is a point set in X_n, perhaps empty.

We now take a domain (37) in (36) and another such domain

(38) $$\tilde{\theta} = \left\{ y \in Y_m;\ x \in \tilde{D}(y) \right\}$$

and we introduce the following

DEFINITION. We call the domain $\tilde{\theta}$ an enlargement of the domain θ if

(i) $\tilde{\theta} \supset \theta$ that is $\tilde{D}(y) \supset D(y)$ for every y,

(ii) for each (x,y) in $\tilde{\theta}$ there is in $D(y)$ a cycle $B_{n-1} = B_{n-1}(x,y)$ which bounds an n-complex $K_n(x,y)$ in $\tilde{D}(y)$, $B_{n-1} = \mathrm{bd}(K_n)$, such that $x \in K_n(x,y)$.

(iii) for any (x',y') in $\tilde{\theta}$ there is a neighborhood $N = N(x',y')$ in $\tilde{\theta}$ such that for $(x,y) \in N$ the cycle $B_{n-1}(x',y') - B_{n-1}(x,y)$ bounds a chain $H_n(x,y)$ in $D(y)$, with $x \notin H_n(x,y)$, that is $B_{n-1}(x',y') \approx B_{n-1}(x,y)$ in $D(y) - x$; and finally

(iv) there exists a point (x_o,y_o) in θ and a neighborhood $N_o(x_o,y_o)$ in θ such that for any (x,y) in $N_o(x_o,y_o)$ the complex $K_n(x,y)$ of (ii) lies in $D(y)$ itself, and not only in the larger $\tilde{D}(y)$.

We note that condition (iv) is implied by the simpler though less general condition

(iv)' for (x,y) in $N_o(x_o,y_o)$ we have $\tilde{D}(y) = D(y)$.

Now, the leading statement is as follows:

THEOREM 6. If $f(x,y)$ is defined in a domain θ of Z_{n+m} and is analytic as a function of (x,y) there, and if for each y it is a solution of our elliptic equation

$$\Delta_x f(x, y) = 0 \tag{39}$$

then $f(x, y)$ has an analytic continuation into any given enlargement $\tilde{\theta}$ of θ.

REMARK. There are enlargements which in a certain sense are not enlargeable themselves. If, however, we take any two enlargements $\tilde{\theta}_1$, $\tilde{\theta}_2$ of the kind introduced and if the point sets $\tilde{\theta}_1 - \theta$, $\tilde{\theta}_2 - \theta$ have a non-vacuous intersection which has parts not connected with θ itself, then we do not claim that the two continuations will necessarily coincide there; it is not known to us what the actual situation is.

PROOF. For every (x,y) in $\tilde{\theta}$, we introduce the quantity

$$g(x, y) = \int_{B_{n-1}(x,\ y)} G(\xi;\ x;\ f(\xi,\ y);\ d\xi). \tag{40}$$

Now, in a neighborhood $N(x',\ y')$ [defined on p. 15, property (iii)] we can replace the variable cycle $B_{n-1}(x,\ y)$ by the fixed cycle $B_{n-1}(x',y')$ and it now follows that $g(x,\ y)$ is analytic in $N(x',\ y')$ and hence everywhere in θ. But by property (iv) it coincides with $f(x,\ y)$ itself in a certain neighborhood N_o of θ, and by analytic continuation it coincides with $f(x,\ y)$ everywhere in θ, as claimed.

We now take in Z_{n+m} a closed point set which is the locus of a (finite or infinite) simplicial chain R_{n-1+m} and a neighborhood V_{n+m} of the latter and we also introduce the open set

$$\theta_{n+m} = Z_{n+m} - R_{n-1+m} \tag{41}$$

We are also introducing the representatives

$$R_{n-1+m} = \left\{ y \in Y_n;\quad x \in B_{n-1}(y) \right\}$$

$$V_{n+m} = \left\{ y \in Y_m; \quad x \in U_n\ (y) \right\}$$
$$\theta_{n+m} = \left\{ y \in Y_m; \quad x \in D_n\ (y) \right\}$$

and we are making the following assumptions.

(v) each $B_{n-1}(y)$ is the locus of a ("regular" or) "singular" cycle which we denote by the same symbol, and

(vi) for each (x', y') in θ_{n+m} there exists a neighborhood $N(x', y')$ and a cycle $B_{n-1}(x', y')$ in $U_n(y')$ such that for (x,y) in $N(x', y')$ the cycle $B_{n-1}(y) - B_{n-1}(x', y')$ bounds an n-dimensional complex in $U_n(y)$ which does not contain the point x.

THEOREM 7. Under the assumptions just made, if $f(x, y)$ is defined and analytic in V_{n+m} and satisfies (39) there, then the integral

$$g(x, y) = \int_{B_{n-1}(y)} G(\xi; x; f(\xi; y); d\xi) \tag{42}$$

defines an analytic function in θ_{n+m} having the following properties:

If we denote the (connected) components of θ_{n+m} by $\theta^1, \theta^2, \ldots,$ and denote the value of (42) in θ^a by $g^a(x, y)$, and if a "regular" simplex R^o_{n-1+m} of the chain R_{n-1+m} separates two components θ^a, θ^b; then the function $g^a(x, y)$ can be continued a certain distance across R^o into θ^b, and $g^b(x, y)$ can be so continued into θ^a, and in the vicinity of R^o we have the saltus relation

$$g^a(x, y) - g^b(x, y) = \pm f(x, y)$$

For the moment we call a component θ^a <u>non-bounded</u> if there is in Y_m some (small) neighborhood N_m such that for y in N_m the cross section $D^a_n(y)$ of $\theta^a_{n+m}(y)$ is the entire X_n. For these values y, the cycle $B_{n-1}(y)$ can be taken as null and thus $g^a(x, y) \equiv 0$. Hence the following consequences.

THEOREM 8. If in Theorem 7 all components θ^a are non-bounded then $f(x, y) \equiv 0$. If V_{n+m} is connected then the conclusion also holds if there are two non-bounded components θ^a, θ^b

having on their boundaries a joint simplex R^{o}_{n-1+m} separating them.

If there are altogether only two components θ^1, θ^2, and θ^2 is non-bounded, then $f(x, y)$ can be continued from the intersection of V_{n+m} with θ^1 into all of θ^1, and this is the most frequently occurring situation underlying Theorem 6.

II. STRONGLY ELLIPTIC SYSTEMS OF DIFFERENTIAL EQUATIONS

F. E. Browder

§1. INTRODUCTION

In a number of recent papers, we have presented a general theory of boundary-value problems for linear elliptic equations of arbitrary order and, more generally, for linear elliptic systems of differential equations ([3], [4], [5], [6], [7], [8]). It is the object of this paper to present a simple self-contained proof of the most basic results which we have obtained for the case of linear "strongly" elliptic systems of differential equations. These form a general subclass of the elliptic systems which contains single elliptic equations as well as such important special cases as the Laplace equation for exterior differential forms on Riemannian manifolds ([10]).

For the single elliptic equation, results similar to those of [3] have been announced by L. Gårding in [18]. The definition of strongly elliptic systems was given by M. I. Visik in [28]. Visik's theory of strongly elliptic systems presented in [29] has many points of contact with our results. Some major differences must be noted, however. The most important of these is that Visik obtains only weak solutions for his boundary-value problem (i.e., distributions in the sense of [24] or [26]) and establishes no analogue of the regularity theorem which is proved below. In particular, he obtains no results on fundamental solutions, Green's functions, or compactness and convergence theorems. In addition, Visik's abstract method rests upon results of Sobolev ([25], [26]) and Kondrashov ([21]) which are more complicated in character than the techniques which we employ. Morrey in [22] has also discussed strongly elliptic systems of second-order equations in two independent variables.

Our basic regularity theorem asserting that a weak solution of our equations is essentially a strict solution in the classical sense was established for a single elliptic equation in [5] and for a general elliptic system in [8] using an extension of the method of F. John ([19]) for the construction of a sufficiently differentiable fundamental solution in the small. In this paper, however, for the sake of directness and simplicity, we prove this theorem for strongly elliptic systems using the ideas and techniques of the Friedrichs mollifier method ([12], [13], [14]). Though it yields weaker estimates and is definitely restricted to the strongly elliptic case, this method is more closely related to our abstract approach than is the fundamental solution method. Friedrichs has recently presented such a proof in [15]. In [20], John has announced the construction, for a single linear elliptic equation, of a proof of similar type based on the method of spherical means discussed in Chapter IV of [19].

§2 presents the detailed formulation of the theorems which are proved in this paper. §3 contains the proof of auxiliary lemmas to be used in the later sections. In §4 the semi-boundedness of the general strongly elliptic linear system of differential equations is established. §5 is devoted to the proof of our basic regularity theorem for weak solutions of a strongly elliptic system of equations. In §6, the basic theorems concerning the Dirichlet problem are established including the Fredholm alternative, the discreteness of eigenvalues and finite dimensionality of eigenspaces, as well as the completeness of the eigenfunctions of self-adjoint systems. (For the proof of the completeness of the eigenfunctions of elliptic equations and strongly elliptic systems which are not necessarily self-adjoint, cf. [7] and [8]). §7 concludes the discussion with the proof of the existence and regularity of the Green's function for domains on which the Dirichlet problem has a unique solution.

§2. FORMULATION OF THEOREMS

Let D be a bounded domain in Euclidean n-space E^n. (Some partial extensions of our results to unbounded domains are given in [4] and [8]). We shall consider several families of complex-valued functions on D. If j is a non-negative integer, $C^j(D) = \{f \mid f$ and all its partial derivatives of order $\leq j$ are defined and continuous on $D\}$, $C^j(\overline{D}) = \{f \mid f$ and all its derivatives of order $\leq j$ are uniformly continuous on $D\}$; $L_2(D)$ is the Hilbert space of complex-valued square-summable functions on D. $C_c^{\infty}(D) = \{\phi \mid \phi$ and all its partial derivatives are defined and continuous on D, ϕ vanishes outside a compact subset of $D\}$. If $\phi \in C_c^{\infty}(D)$, $S(\phi)$ is the closure of the set $\{x \mid \phi(x) \neq 0\}$.

We shall consider r-vector functions, i.e., vector functions with r components $u = (u_1, \ldots, u_r)$; the i-th component u_i of u is a complex-valued function defined on D. $C^{j,r}(D)$, $C^{j,r}(\overline{D})$, $L_{2,r}(D)$, $C_c^{\infty,r}(D)$ are defined as the families of r-vector functions u such that for each i, u_i belongs to $C^j(D)$, $C^j(\overline{D})$, $L_2(D)$, $C_c^{\infty}(D)$ respectively.

A system K of differential equations on D of order $2m$ and rank r has the form

$$K_i(u) = \sum_{j=1}^{r} \sum_{\substack{k_1,\dots,k_s=1 \\ s \leq 2m}}^{n} a_{k_1\dots k_s;i,j}(x) \frac{\partial^s u_j}{\partial x_{k_1}\dots\partial x_{k_s}} \tag{2.1}$$

$$= v_i \qquad (i = 1, \dots, r)$$

where the indices $k_1, \dots, k_s$ range independently from 1 to n while for each set of indices $a_{k_1\dots k_s;i,j} \in C^s(D)$. The system of differential operators K transforms $u \in C^{2m,r}(D)$ into $v \in C^{0,r}(D)$. For the sake of simplicity we assume the coefficients real. For each system K, we may define a r by r characteristic matrix $A(x, \xi)$, defined for x in D and every real n-vector $\xi = (\xi_1, \dots, \xi_n)$ and depending only upon the highest order terms of the system K:

$$A(x, \xi) = (a_{ij}(x, \xi))$$

$$= \left(\sum_{k_1\dots k_{2m}} a_{k_1\dots k_{2m};i,j}(x)\xi_{k_1}\dots\xi_{k_{2m}} \right) \tag{2.2}$$

DEFINITIONS:

$\mathcal{E}$) K is said to be elliptic at x if $A(x, \xi)$ is non-singular for every $\xi \neq 0$.

$S\mathcal{E}_1$) K is said to be strongly elliptic at x in D if $A(x, \xi) + A^t(x, \xi)$ is positive definite for every $\xi \neq 0$, where A^t is the transpose of A. An equivalent formulation is the following:

$S\mathcal{E}_2$) Given $x \in D$, there exists $\rho > 0$ such that for every real n-vector ξ and r-vector η,

$$\sum_{i,j=1}^{r} \sum_{k_1\dots k_{2m}=1}^{n} a_{k_1\dots k_{2m};i,j}(x)\xi_{k_1}\dots\xi_{k_{2m}}\eta_i\eta_j$$

$$\geq \rho \left(\sum_{i=1}^{n} \xi_i^{2m} \right) \left(\sum_{j=1}^{r} \eta_j^2 \right) \tag{2.3}$$

K will be said to be uniformly strongly elliptic on D if there exists $\rho > 0$ for which $S\mathcal{E}_2$ is satisfied for all x in D.

For a single equation $(r = 1)$, $S\mathcal{E}_2$ reduces to the classical criterion of ellipticity. As a consequence the theory of strongly elliptic systems includes the theory of the linear elliptic differential equation of arbitrary order. From the definitions, it follows by a formal argument that the strongly elliptic systems are a proper sub-class of the elliptic systems. It has been shown by an example in [30] that such results of the theory of strongly elliptic systems as the discreteness of eigenvalues in the Dirichlet problem are not true for all elliptic systems.

For $\phi \in C_c^{\infty,r}(D)$, we define

$$\|\phi\|_m^2 = \sum_{j=1}^{r} \sum_{k_1 \dots k_m=1}^{n} \int_D \left| \frac{\partial^m \phi_j}{\partial x_{k_1} \dots \partial x_{k_m}} \right|^2 dx$$

(Integration is taken with respect to Lebesgue n-measure.)

For $u, v \in L_{2,r}(D)$,

$$(u \cdot v) = \sum_{j=1}^{r} \int_D u_j v_j^* \, dx$$

(z^* = complex conjugate of z).

The basic property of strongly elliptic systems which is not shared by the general class of elliptic systems is the semi-boundedness property expressed in the following theorem:

THEOREM 1. Suppose D is a bounded domain in E^n, K a system of differential operators which is uniformly strongly elliptic on D. Suppose that each of the coefficients $a_{k_1 \dots k_s; i, j} \in C^{\gamma(s)}(\bar{D})$, where $\gamma(s) = \max\{0, s - m\}$. Then there exist $\rho_1 > 0$, $k_0 \geq 0$ such that for all $\phi \in C_c^{\infty,r}(D)$,

$$\operatorname{Re}\{(-1)^m K(\phi) \cdot \phi\} \geq \rho_1 \|\phi\|_m^2 - k_0(\phi \cdot \phi)$$

If K is strongly elliptic on D, the conditions of Theorem 1 will be satisfied on all subdomains which are contained in compact subsets of D.

Theorem 1 enables us to translate boundary value problems into abstract problems concerning the existence of solutions of linear functional