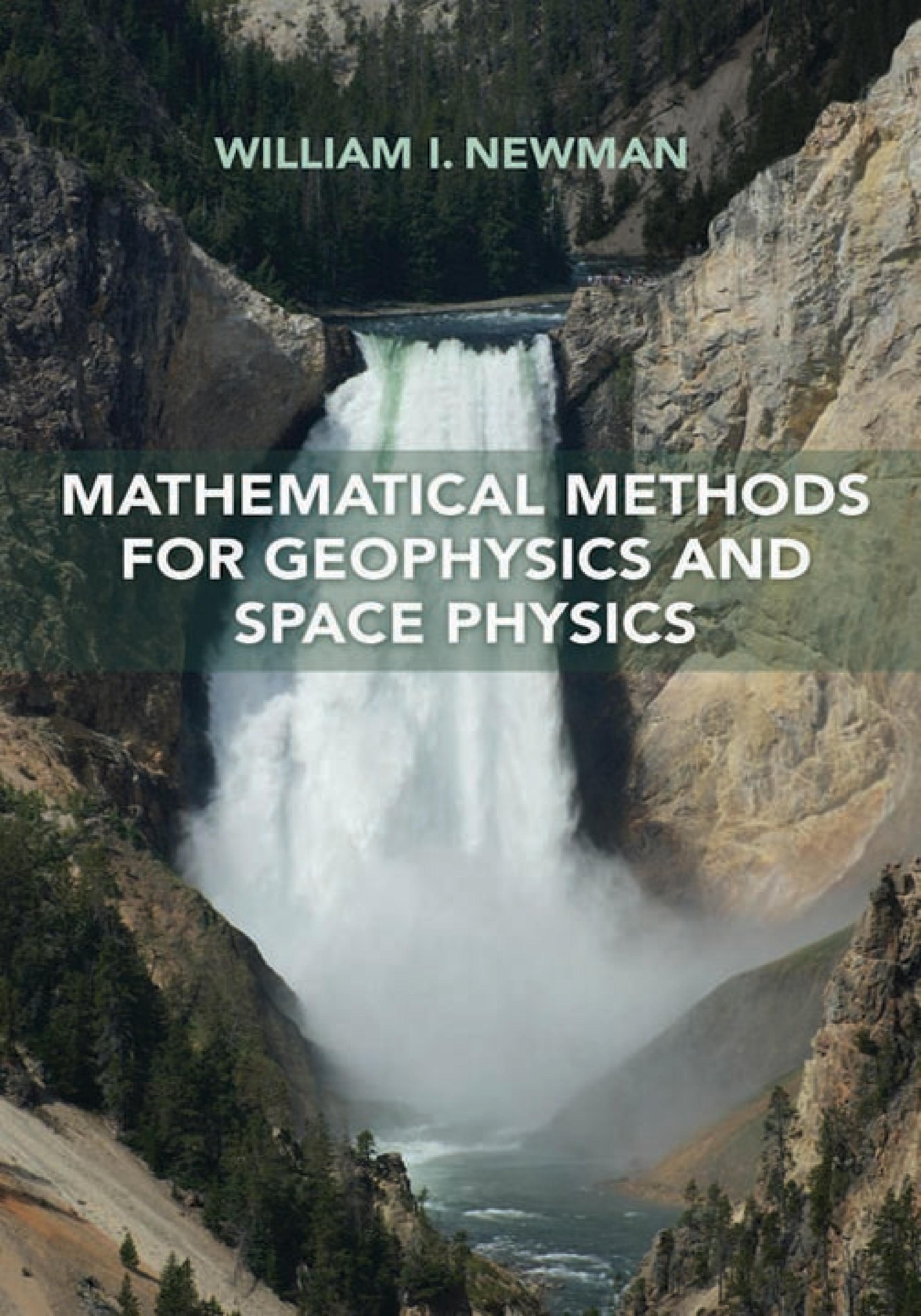
WILLIAM I. NEWMAN

MATHEMATICAL METHODS FOR GEOPHYSICS AND SPACE PHYSICS



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Preface

Graduate students in the earth sciences, particularly those in geophysics and atmospheric, oceanographic, planetary, and space physics, as well as astronomy, require a substantial degree of mathematical preparation—for the sake of brevity, we will simply refer to these application areas as being in geophysics. While there is significant overlap between their needs and those of graduate students in physics or in applied mathematics, there are important differences in the preparation needed and, notably, the sequence of presentation required as well as the overall quantity of material that is necessary. Most textbooks that address mathematical methods for physics and engineering begin from the standpoint that the student already knows the underlying equations, generally partial differential equations, but needs to learn how to solve them. Since the background of most entering or second-year graduate students in geophysics is highly variable, I felt it necessary to provide derivations in a number of circumstances for those equations to help students appreciate better where they arise and how their solution must be addressed. Moreover, most mathematical methods textbooks were published before the renaissance in thinking, especially about geophysical problems, that introduced the concepts of chaos and complexity, as well as the significance of probability and statistics and of numerical methods. Significant attention is given to the ordinary and partial differential equations that have played a pivotal role in the evolution of geophysics. In addition, in order to round out our treatment of mathematical methods, a succinct survey of statistical and computational issues is introduced. A brief but comprehensive summary of solution methods is presented, including many exercises. In so doing, it is my hope that this book will address that need during the course of one academic semester or quarter. In essence, we treat some central problem areas in depth, while providing a measure of literacy in others.

Students also can find helpful materials in the following works. The text that is closest to our presentation is that due to

Mathews and Walker (1970) which is out of print. A relatively contemporary text on the topic is that of Arfken and Weber (2005), but its newest edition (Arfken et al., 2013) has become a "comprehensive guide"; a helpful lead-in to the latter, designed more for advanced undergraduates, is Weber and Arfken (2004). The graduate textbook by Stone and Goldbart (2009) is also helpful, although the examples selected are drawn from physics and have a more formal flavor. Finally, the classic two-volume definitive texts on the subject are those by Morse and Feshbach (1999) and by Courant and Hilbert (1962). While the former is now back in print, the latter remains out of print. Regarding specific applications to geophysics and planetary, atmospheric, oceanographic, and space physics, chapters in existing graduate-level textbooks in those specialties contain appropriate derivations. As we encounter each new topic, additional citations to reference materials will be provided. We shall attempt to integrate some of the most important of these into this book.

Given the time available in a single academic quarter or semester, we are fundamentally limited in the quantity of material that can be presented. Basically, we provide an overarching survey of the relevant issues, a brief treatment of how to treat these problems, and an indication for each of these topics where the student can find a more thorough and rigorous treatment. Our objective is to give each student sufficient instruction to solve elementary problems and then advance to more exhaustive treatments of the individual topics, whether they originated in geophysics and its associated disciplines, physics, astronomy, or engineering.

The first chapter reviews many mathematical preliminaries that students should have studied previously, but also serves as a review. Vectors, indicial or "Einstein" notation, vector operators, cylindrical and spherical geometry, and the theorems of Gauss, Green, and Stokes are presented here. Since the focus of this chapter is on geometry, we introduce matrices in the context of the rotation of vectors. Then, we present tensors, which are matrices whose physical properties remain unchanged under a rotation and preserve other physical (e.g., variational) principles, including a very brief description of the eigenvalue problem. Here we introduce the concept of generalized functions through the Dirac δ function, and some of its relatives, inasmuch as they will form the basis later for our treatment of Green's functions. We present a number of assignment problems.

In the second chapter, we review features of ordinary differential equations. The Laplacian operator in partial differential equations permits the use of the method of separation of variables, which yields a set of second-order ordinary differential equations in the different geometric variables. We introduce the concept of Green's functions. Accordingly, we treat the separation of variables issue from the standpoint of ordinary differential equations and we introduce the derivation underlying Bessel functions and spherical harmonics, including the Legendre polynomials. (We complete the discussion of Poisson's, Laplace's, and Helmholtz's equation in chapter 4 because of their utility in solving partial differential equations of elliptic type.) We introduce problems describable by coupled ordinary differential equations, which, ultimately, provide the basis for chaos theory and are largely overlooked in classical mathematical methods of physics textbooks. Geophysical examples provide a wonderful testbed for ordinary differential equation approaches. For example, efforts to model the geodynamo using the interaction of mechanical and electrical components yielded strictly cyclical behavior with no field reversals. Efforts to resolve this problem demonstrated an epiphanic paradigm shift in moving to systems with three equations, such as the Lorenz model for convection and turbulence. This chapter also provides hands-on experience in performing perturbation theory analysis. Since chaotic behavior often yields fractal geometry, as in the Lorenz model trajectory, we provide a brief survey of fractal concepts and applications, as well as mappings as an adjunct to understanding transition to chaos.

In the third chapter, we introduce the evaluation of integrals, including a brief overview of complex analysis and elementary contour integration, saddle point methods, and some special problems in geophysics that yield elliptic integrals. We continue to address integral transforms following a brief introduction to Fourier series and transforms. We prove the sampling theorem and describe the phenomenon of aliasing. While these latter topics are overlooked in most textbooks, they play an important role in geophysics, particularly in the context of data collection and analysis. We introduce the fast Fourier transform and some approximation methods for spectral analysis. We conclude this chapter by briefly touching upon Laplace transforms and the Bromwich integral, and we introduce some integral equations, including the Abel and Radon transforms, as well as the Herglotz–Wiechert problem of seismology.

In chapter 4, we introduce the fundamental partial differential equations of mathematical physics, in general, and geophysics, in particular. We whet the student's appetite by introducing the three fundamental types of partial differential equations that are pervasive in geophysics: the wave equation, the potential equation, and the diffusion equation. This chapter embeds practical examples of real-world problems with the theory. Classic mathematical methods of physics books rarely provide examples, especially those that are appropriate to the earth sciences. Remarkably, some of the most beautiful yet practical examples of these types of equations appear in geophysics. We introduce, for linear problems, integral transform methods, and introduce eigenfunctions, eigenvalues, and Green's functions in those time-dependent contexts. We exploit these methods to solve both the diffusion equation and the wave equation in three dimensions. We employ spherical harmonics, introduced in the second chapter, to solve the gravitational potential equation relating a planet's mass distribution to its potential in three dimensions. Further, we exploit Fourier methods in order to identify dispersion relations for linear problems, including the role of diffusion and dispersion. At this stage, we associate with dispersion relations for partial differential equations the role of instability. Perturbation theory in this context is presented via a simple example, the propagation of sound in a fluid. However, since partial differential equations incorporate an infinite number of modes—associated with spherical harmonics, for example—the chaotic nature of a fundamentally infinite degree of freedom system underscores what is called *complexity*. We consider collective, nonlinear modes of behavior as exemplified by solitary waves and, especially, solitons. As illustrations, we derive the solution for solitary waves exemplified by Burgers's equation and for solitons via the Korteweg–de Vries equation. Scaling arguments underlying the emergence of turbulence are presented, as well as a simple derivation for the Kolmogorov spectrum.

The remaining chapter surveys two topics that are central to modern geophysics yet have been orphaned from essentially all elementary treatments. We briefly survey topics in probability and statistics, including the binomial, Poisson, and Gaussian (normal) distributions as well as the central limit theorem. A sketch is provided for methods of random number generation, central to Monte Carlo simulation. We also identify some of the themes associated with regression and the fitting of experimental data. Finally, we survey some questions emergent from numerical methods. Here, we briefly address the nature of computational and round-off errors. As an example, we survey the determination of the roots of polynomials, which play a fundamental role in the dispersion relations of modern geophysics. We provide a brief overview of numerical methods of solving ordinary and partial differential equations, with a focus on finite difference methods, but mention spectral approaches.

As is evident, this textbook provides a whirlwind survey of many topics and helps bring together many different concepts yet provide a brief practical introduction to problem solving in geophysics. This book was developed in consultation with my colleagues and is the outcome of several offerings at UCLA of this survey course to entering and second-year graduate students in geophysics and planetary and space physics, but was also designed to be helpful to students in allied disciplines, including atmospheric and ocean sciences, and in physics and astronomy. We very much hope that this volume will help stimulate thinking about these problem areas and further investigation and study of the different topics reviewed.

While completing this volume, my editor asked me to provide a cover image for this book and recommended that a photograph be adopted instead of a geometrical design or blank cover as is so often employed in technical books. This presented a special challenge inasmuch as how could a photograph convey what underscores the mathematics implicit to the earth, planetary, and space sciences? What kind of image would capture the outcome of a combination of many different geologic events? Yellowstone National Park is a truly special place, and the Grand Canyon of the Yellowstone is a focal point for much of its varied geologic history. This area was shaped by a caldera eruption 600,000 years ago and a series of lava flows. The area was also faulted by the caldera dome before the eruption. The site of this canyon was possibly established by this faulting, which magnified the rate of erosion. Glaciation also took place, although glacial deposits are largely absent. This photograph features the Lower Falls, 308 feet in height, as viewed from Lookout Point. The rich colors of the rock in this photograph are likely an outcome of the hydrothermal alteration of the rhyolite containing different iron compounds and their subsequent "cooking." Exposure to the elements and oxidation added to this effect, and are not due to sulfur. The falling water provides a quick reminder of the power of the flow. Thinking about all of the various physical and chemical effects present in creating this scene, it is clear how this image captures so many different influences and that challenge of providing a quantitative description of them. I took this photograph on August 24, 2009, with a Sony A350 DSLR at F8 with a 1/320-second exposure time using a 160-mm zoom lens.

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Mathematical Methods for Geophysics and Space Physics

CHAPTER ONE

Mathematical Preliminaries

The underlying theory for geophysics, planetary physics, and space physics requires a solid understanding of many of the methods of mathematical physics as well as a set of specialized topics that are integral to the diverse array of real-world problems that we seek to understand. This chapter will review some essential mathematical concepts and notations that are commonly employed and will be exploited throughout this book. We will begin with a review of vector analysis focusing on indicial notation, including the Kronecker δ and Levi-Civita ϵ permutation symbol, and vector operators. Cylindrical and spherical geometry are ubiquitous in geophysics and space physics, as are the theorems of Gauss, Green, and Stokes. Accordingly, we will derive some of the essential vector analysis results in Cartesian geometry in these curvilinear coordinate systems. We will proceed to explore how vectors transform in space and the role of rotation and matrix representations, and then go on to introduce tensors, eigenvalues, and eigenvectors. The solution of the (linear) partial differential equations of mathematical physics is commonly used in geophysics, and we will present some materials here that we will exploit later in the development of Green's functions. In particular, we will close this chapter by introducing the ramp, Heaviside, and Dirac δ functions. As in all of our remaining chapters, we will provide a set of problems and cite references that present more detailed investigations of these topics.

1.1 Vectors, Indicial Notation, and Vector Operators

This book primarily will pursue the kinds of geophysical problems that emerge from scalar and vector quantities. While mention will be made of tensor operations, our primary focus will be upon vector problems in three dimensions that form the basis of geophysics. Scalars and vectors may be regarded as tensors of a specific rank. *Scalar* quantities, such as density and temperatures, are *zero-rank* or *zero-order* tensors. *Vector* quantities such as velocities have an associated direction as well as a magnitude. Vectors are *first-rank* tensors and are usually designated by boldface lower-case letters. *Second-rank tensors*, or simply *tensors*, such as the stress tensor are a special case of square matrices. Matrices are generally denoted by boldface, uppercase letters, while tensors are generally denoted by boldface, uppercase, sans-serif letters (Goldstein et al., 2002). For example, $\boldsymbol{M}$ would designate a matrix while $\boldsymbol{\mathsf{T}}$ would designate a tensor. [There are other notations, e.g., Kusse and Westwig (2006), that employ overbars for vectors and double overbars for tensors.] Substantial simplification of notational issues emerges upon adopting *indicial* notation.

In lieu of x, y, and z in describing the Cartesian components for position, we will employ x_1, x_2, and x_3. Similarly, we will denote by $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, and $\hat{\boldsymbol{e}}_3$ the mutually orthogonal *unit vectors* that are in the direction of the x_1, x_2, and x_3 axes. (Historically, the use of $\boldsymbol{e}$ emerged in Germany where the letter "e" stood for the word *Einheit*, which translates as "unit.") The indicial notation implies that any repeated index is summed, generally from 1 through 3. This is the *Einstein summation convention*.

It is sufficient to denote a vector $\boldsymbol{v}$, such as the velocity, by its three components (v_1, v_2, v_3). We note that $\boldsymbol{v}$ can be represented vectorially by its component terms, namely,

$$\boldsymbol{v} = \sum_{i=1}^{3} v_i \hat{\boldsymbol{e}}_i = v_i \hat{\boldsymbol{e}}_i. \tag{1.1}$$

Suppose $\boldsymbol{\mathsf{T}}$ is a tensor with components T_{ij}. Then,

$$\boldsymbol{\mathsf{T}} = \sum_{i=1,j=1}^{3} T_{ij} \hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j = T_{ij} \hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j. \tag{1.2}$$

We now introduce the *inner product*, also known as a scalar product or dot product, according to the convention

$$\boldsymbol{u} \cdot \boldsymbol{v} \equiv u_i v_i. \tag{1.3}$$

Moreover, we define u and v to be the lengths of $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively, according to

$$u \equiv \sqrt{u_i u_i} = |\boldsymbol{u}|; \qquad v \equiv \sqrt{v_i v_i} = |\boldsymbol{v}|; \tag{1.4}$$

we can identify an angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$ that we define according to

$$\boldsymbol{u} \cdot \boldsymbol{v} \equiv uv \cos\theta, \tag{1.5}$$

which corresponds directly to our geometric intuition.

We now introduce the Kronecker δ according to

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{1.6}$$

The Kronecker δ is the indicial realization of the identity matrix. It follows, then, that

$$\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = \delta_{ij}, \tag{1.7}$$

and that

$$\delta_{ii} = 3. \tag{1.8}$$

This is equivalent to saying that the *trace*, that is, the sum of the diagonal elements, of the identity matrix is 3. An important consequence of Eq. (1.7) is that

$$\delta_{ij}\hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}_i. \tag{1.9}$$

A special example of these results is that we can now derive the general scalar product relation (1.3), namely,

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i\hat{\boldsymbol{e}}_i \cdot v_j\hat{\boldsymbol{e}}_j = u_i v_j \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = u_i v_j \delta_{ij} = u_i v_i, \tag{1.10}$$

by applying Eq. (1.7).

We introduce the Levi-Civita or permutation symbol ϵ_{ijk} in order to address the *vector product* or *cross product*. In particular, we define it according to

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i\,j\,k \text{ are an even permutation of } 1\,2\,3, \\ -1 & \text{if } i\,j\,k \text{ are an odd permutation of } 1\,2\,3, \\ 0 & \text{if any two of } i, j, k \text{ are the same.} \end{cases} \tag{1.11}$$

We note that ϵ_{ijk} changes sign if any two of its indices are interchanged. For example, if the 1 and 3 are interchanged, then the sequence 1 2 3 becomes 3 2 1. Accordingly, we define the cross product $\boldsymbol{u} \times \boldsymbol{v}$ according to its ith component, namely,

$$(\boldsymbol{u} \times \boldsymbol{v})_i \equiv \epsilon_{ijk} u_j v_k, \tag{1.12}$$

or, equivalently,

$$\boldsymbol{u} \times \boldsymbol{v} = (\boldsymbol{u} \times \boldsymbol{v})_i \hat{\boldsymbol{e}}_i = \epsilon_{ijk} \hat{\boldsymbol{e}}_i u_j v_k = -(\boldsymbol{v} \times \boldsymbol{u}). \tag{1.13}$$

It is observed that this structure is closely connected to the definition of the determinant of a 3×3 matrix, which emerges from expressing the scalar triple product

$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \epsilon_{ijk} u_i v_j w_k, \tag{1.14}$$

and, by virtue of the cyclic permutivity of the Levi-Civita symbol, demonstrates that

$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{u}) = \boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v}). \tag{1.15}$$

The right-hand side of Eq. (1.14) is the determinant of a matrix whose rows correspond to $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

Indicial notation facilitates the calculation of quantities such as the vector triple cross product

$$\begin{aligned} \boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) &= \boldsymbol{u} \times \epsilon_{ijk} \hat{\boldsymbol{e}}_i v_j w_k = \epsilon_{lmi} \hat{\boldsymbol{e}}_l u_m \epsilon_{ijk} v_j w_k \\ &= (\epsilon_{ilm} \epsilon_{ijk}) \hat{\boldsymbol{e}}_l u_m v_j w_k. \end{aligned} \tag{1.16}$$

It is necessary to deal first with the $\epsilon_{ilm}\epsilon_{ijk}$ term. Observe, as we sum over the i index, that contributions can emerge only if $l \neq m$ and $j \neq k$. If these conditions both hold, then we get a contribution of 1 if $l = j$ and $m = k$, and a contribution of -1 if $l = k$ and $m = j$. Hence, it follows that

$$\epsilon_{ilm} \epsilon_{ijk} = \delta_{\ell j} \delta_{mk} - \delta_{lk} \delta_{mj}. \tag{1.17}$$

Returning to (1.16), we obtain

$$\begin{aligned} \boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) &= (\delta_{lj} \delta_{mk} - \delta_{lk} \delta_{mj}) \hat{\boldsymbol{e}}_l u_m v_j w_k \\ &= \hat{\boldsymbol{e}}_l v_l u_m w_m - \hat{\boldsymbol{e}}_l w_l u_m v_m \\ &= \boldsymbol{v}(\boldsymbol{u} \cdot \boldsymbol{w}) - \boldsymbol{w}(\boldsymbol{u} \cdot \boldsymbol{v}), \end{aligned} \tag{1.18}$$

thereby reproducing a familiar, albeit otherwise cumbersome to derive, algebraic identity. Finally, if we replace the role of $\boldsymbol{u}$ in the triple scalar product (1.18) by $\boldsymbol{v} \times \boldsymbol{w}$, it immediately follows that

$$\begin{aligned} (\boldsymbol{v} \times \boldsymbol{w}) \cdot (\boldsymbol{v} \times \boldsymbol{w}) &= |\boldsymbol{v} \times \boldsymbol{w}|^2 = \epsilon_{ijk} v_j w_k \epsilon_{ilm} v_l w_m \\ &= (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) v_j w_k v_l w_m \\ &= v^2 w^2 - (\boldsymbol{v} \cdot \boldsymbol{w})^2 = v^2 w^2 \sin^2 \theta, \end{aligned} \tag{1.19}$$

where we have made use of the definition for the angle θ given in (1.5).

The Kronecker δ and Levi-Civita ϵ permutation symbols simplify the calculation of many other vector identities, including those with respect to *derivative* operators. We define ∂_i according to

$$\partial_i \equiv \frac{\partial}{\partial x_i}, \tag{1.20}$$

and employ it to define the gradient operator $\boldsymbol{\nabla}$, which is itself a vector:

$$\boldsymbol{\nabla} = \partial_i \hat{\boldsymbol{e}}_i. \tag{1.21}$$

Another notational shortcut is to employ a subscript of "$,i$" to denote a derivative with respect to x_i; importantly, a comma "," is employed together with the subscript to designate differentiation. Hence, if f is a scalar function of $\boldsymbol{x}$, we write

$$\frac{\partial f}{\partial x_i} = \partial_i f = f_{,i}; \tag{1.22}$$

but if $\boldsymbol{g}$ is a vector function of $\boldsymbol{x}$, then we write

$$\frac{\partial g_i}{\partial x_j} = \partial_j g_i = g_{i,j}. \tag{1.23}$$

Higher derivatives may be expressed using this shorthand as well, for example,

$$\frac{\partial^2 g_i}{\partial x_j \partial x_k} = g_{i,jk}. \tag{1.24}$$

Then, the usual divergence and curl operators become

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \partial_i u_i = u_{i,i} \tag{1.25}$$

and

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \epsilon_{ijk} \hat{\boldsymbol{e}}_i \partial_j u_k = \epsilon_{ijk} \hat{\boldsymbol{e}}_i u_{k,j}. \tag{1.26}$$

Our derivations will employ Cartesian coordinates, primarily, since curvilinear coordinates, such as cylindrical and spherical coordinates, introduce a complication insofar as the unit vectors defining the associated directions change. However, once we have obtained the fundamental equations, curvilinear coordinates can be especially helpful in solving problems since they help capture the essential geometry of the Earth.

1.2 Cylindrical and Spherical Geometry

Two other coordinate systems are widely employed in geophysics, namely, cylindrical coordinates and spherical coordinates. As we indicated earlier, our starting point will always be the fundamental equations that we derived using Cartesian coordinates and then we will convert to coordinates that are more "natural" for solving the problem at hand. Let us begin in two dimensions with polar coordinates (r, θ) and review some fundamental results.

As usual, we relate our polar and Cartesian coordinates according to

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta, \end{aligned} \tag{1.27}$$

which can be inverted according to

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan\left(y/x\right). \end{aligned} \tag{1.28}$$

Unit vectors in the new coordinates can be expressed

$$\begin{aligned} \hat{\boldsymbol{r}} &= \quad \cos\theta\,\hat{\boldsymbol{x}} + \sin\theta\,\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{\theta}} &= -\sin\theta\,\hat{\boldsymbol{x}} + \cos\theta\,\hat{\boldsymbol{y}}. \end{aligned} \tag{1.29}$$

We recall how to obtain the various differential operations, such as the gradient, divergence, and curl, by using the chain rule of multivariable calculus. Suppose that f is a scalar function of x and y, and we wish to transform its Cartesian derivatives into derivatives with respect to polar coordinates. From the chain rule, it follows that

$$\frac{\partial f}{\partial x} = \left.\frac{\partial r}{\partial x}\right|_y \left.\frac{\partial f}{\partial r}\right|_\theta + \left.\frac{\partial \theta}{\partial x}\right|_y \left.\frac{\partial f}{\partial \theta}\right|_r = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}, \tag{1.30}$$

where the vertical bar followed by a subscript designates the variable or variables that are held fixed. In like fashion, we can derive

$$\frac{\partial f}{\partial y} = \sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}. \tag{1.31}$$

Finally, we can obtain the *Laplacian* of a scalar quantity in two dimensions, ∇^2, defined according to

$$\nabla^2 f \equiv \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}. \tag{1.32}$$

Integrals in two dimensions require the transformation of differential area elements from $\mathrm{d}x\,\mathrm{d}y$ to $r\,\mathrm{d}\theta\,\mathrm{d}r$. Therefore, the integral of f over some area A can be expressed equivalently as

$$\int_A f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_A f(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \int_A f(\boldsymbol{x})\,\mathrm{d}A$$
$$= \int_A f(\boldsymbol{x})\,\mathrm{d}^2x = \int_A f(r,\theta)r\,\mathrm{d}r\,\mathrm{d}\theta, \quad (1.33)$$

where the areas of integration are kept the same and integration over two variables is implicit.

We now move on to review three-dimensional geometry wherein polar coordinates become either cylindrical or spherical polar coordinates. We begin with cylindrical coordinates, which now introduce the third or z dimension. Accordingly, we observe that the Laplacian becomes

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}, \quad (1.34)$$

where we assume that r is measured in the x-y plane, that is, it is not the radial distance from the origin to the point in question. Suppose, as before, that $\boldsymbol{g}$ is a vector function and we wish to obtain its divergence and curl. We will designate its components in the cylindrical coordinate system (r,θ,z) by (g_r,g_θ,g_z). These can be calculated directly by taking projections of $(f_1,f_2,f_3) \equiv (f_x,f_y,f_z)$ onto the (r,θ,z) directions. The z direction requires no elaboration. However, we note that

$$\begin{aligned} g_r &= \cos\theta g_x + \sin\theta g_y \\ g_\theta &= -\sin\theta g_x + \cos\theta g_y \end{aligned} \quad (1.35)$$

and

$$\begin{aligned} g_x &= \cos\theta g_r - \sin\theta g_\theta \\ g_y &= \sin\theta g_r + \cos\theta g_\theta. \end{aligned} \quad (1.36)$$

With these results in hand, we can show that the divergence of $\boldsymbol{g}$ becomes

$$\boldsymbol{\nabla}\cdot\boldsymbol{g} = \frac{1}{r}\frac{\partial}{\partial r}(rg_r) + \frac{1}{r}\frac{\partial g_\theta}{\partial \theta} + \frac{\partial g_z}{\partial z}, \quad (1.37)$$

where $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{z}}$ are unit vectors in the associated directions. Similarly, we write the curl of $\boldsymbol{g}$ as

$$\nabla \times \boldsymbol{g} = \frac{1}{r}\left\{\left[\frac{\partial g_z}{\partial \theta} - \frac{\partial (r g_\theta)}{\partial z}\right]\hat{\boldsymbol{r}} + \left[\frac{\partial g_r}{\partial z} - \frac{\partial g_z}{\partial r}\right] r\hat{\boldsymbol{\theta}} + \left[\frac{\partial (r g_\theta)}{\partial r} - \frac{\partial g_r}{\partial \theta}\right]\hat{\boldsymbol{z}}\right\}. \tag{1.38}$$

Finally, the integral over some volume V of a scalar function f can be written equivalently as

$$\begin{aligned}\int_V f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_V f(x_1,x_2,x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \\ &= \int_V f(\boldsymbol{x})\,\mathrm{d}V = \int_V f(\boldsymbol{x})\,\mathrm{d}^3x \\ &= \int_V f(r,\theta,z) r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z. \end{aligned} \tag{1.39}$$

This concludes our summary of cylindrical coordinates.

We now adopt spherical coordinates (Figure 1.1) according to

$$\begin{aligned} x &= r \sin\theta \cos\varphi \\ y &= r \sin\theta \sin\varphi \\ z &= r\cos\theta, \end{aligned} \tag{1.40}$$

which can be inverted according to

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \arccos(z/r) = \arccos\left(z/\sqrt{x^2+y^2+z^2}\right) \\ \varphi &= \arctan(y/x). \end{aligned} \tag{1.41}$$

Without elaboration, we list here some essential results.

1. Unit vector relationships, from which g_r, g_θ, and g_φ can also be extracted:

$$\begin{aligned} \hat{\boldsymbol{r}} &= \sin\theta\cos\varphi\hat{\boldsymbol{x}} + \sin\theta\sin\varphi\hat{\boldsymbol{y}} + \cos\theta\hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\varphi\hat{\boldsymbol{x}} + \cos\theta\sin\varphi\hat{\boldsymbol{y}} - \sin\theta\hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\varphi}} &= -\sin\varphi\hat{\boldsymbol{x}} + \cos\varphi\hat{\boldsymbol{y}}. \end{aligned} \tag{1.42}$$

   We note, as a check, that all three of these unit vectors are of unit length and are mutually orthogonal.

2. Gradient of a scalar f:

$$\nabla f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}. \tag{1.43}$$

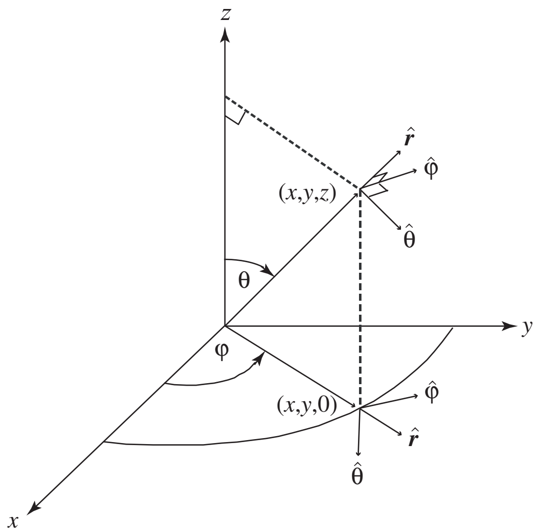

Figure 1.1. Spherical coordinates.

3. Laplacian of a scalar f:

$$\nabla^2 f = \frac{1}{r^2 \sin\theta}\left\{ \sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 f}{\partial \varphi^2}\right\}. \quad (1.44)$$

Note that the Laplacian of vector quantities will differ from the above due to the dependence of the projected components on the coordinates.

4. Divergence of a vector $\boldsymbol{g}$:

$$\boldsymbol{\nabla}\cdot\boldsymbol{g} = \frac{1}{r^2 \sin\theta}\left[\sin\theta \frac{\partial (r^2 g_r)}{\partial r} + r\frac{\partial(\sin\theta g_\theta)}{\partial\theta} + r\frac{\partial g_\varphi}{\partial\varphi}\right]. \quad (1.45)$$

5. Curl of a vector $\boldsymbol{g}$:

$$\boldsymbol{\nabla}\times\boldsymbol{g} = \frac{1}{r^2 \sin\theta}\left\{\left[\frac{\partial(r\sin\theta g_\varphi)}{\partial\theta} - \frac{\partial(r g_\theta)}{\partial\varphi}\right]\hat{\boldsymbol{r}} + \left[\frac{\partial g_r}{\partial\varphi} - \frac{\partial(r\sin\theta g_\varphi)}{\partial r}\right] r\hat{\boldsymbol{\theta}} + \left[\frac{\partial(r g_\theta)}{\partial r} - \frac{\partial g_r}{\partial\theta}\right] r\sin\theta\hat{\boldsymbol{\varphi}}\right\}. \quad (1.46)$$

6. Volume integral of a scalar f:

$$\int_V f(\boldsymbol{x})\,\mathrm{d}^3x = \int_V f(r,\theta,\varphi) r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi. \quad (1.47)$$

We will now review some of the integral relations involving vector quantities.

1.3 Theorems of Gauss, Green, and Stokes

We wish to present some familiar results from integral calculus. We will not provide proofs but will present a brief sketch as to how they can be obtained. In Figure 1.2, we depict the relevant geometry.

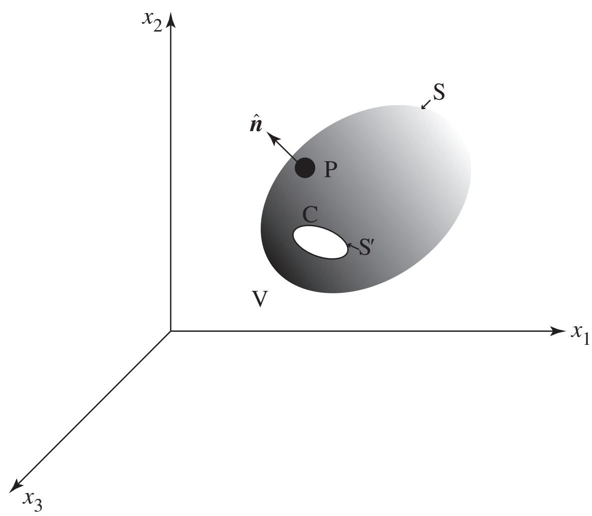

Figure 1.2. Geometry of volume and surface.

We denote by V the volume under consideration, and S denotes the surface of that volume. We identify a point P on the surface of that volume, and show by an arrow the unit vector $\hat{\boldsymbol{n}}$ emerging out from that surface. Finally, we draw a closed curve C on that surface that contains a surface area S′. We denote by $\boldsymbol{g}$ a vector function and by f and h two different scalar functions. We assume that f, $\boldsymbol{g}$, and h all go to zero as our distance from the origin goes to infinity. As before, we denote surface and volume elements by d^2x and d^3x, respectively.

Gauss's theorem can be expressed by

$$\int_{\mathrm{V}} \boldsymbol{\nabla} \cdot \boldsymbol{g}\, \mathrm{d}^3x = \int_{\mathrm{S}} \boldsymbol{g} \cdot \hat{\boldsymbol{n}}\, \mathrm{d}^2x. \tag{1.48}$$

This result can be proved by subdividing the volume V into a set of cubes, going to the limit that the sides of the cubes become