COSMOLOGY IN $(2+1)$-DIMENSIONS, CYCLIC MODELS, AND DEFORMATIONS OF $M_{2,1}$

BY

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FOREWORD

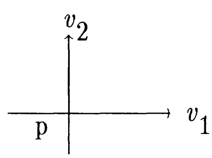
In this paper a "cyclic model" will mean a compact Lorentz manifold with the property that all its null-geodesics are periodic. Such a model is cyclic in the sense that every space-time event gets replicated infinitely often; it has an infinite number of antecedents with identical "pasts" and "futures". We should warn the non-expert that this is not what relativists usually mean by cyclicity. This term is almost always used to describe periodic solutions of Einstein's equations. In (2+1)-dimensions this implies that the metric involved is conformally flat; and, as we will see in §11, this is practically incompatible with cyclicity in our sense.

We will call a Lorentz metric all of whose null-geodesics are periodic a Zollfrei metric. (For the etymology of this term, see §1.) Notice that the property of being Zollfrei is conformally invariant. This is because two Lorentz metrics have the same null-geodesics if they differ by a conformality factor. (Another way of stating this fact is that the trajectories of light rays are independent of the metric structure of space-time but only depend on its causal structure: i.e., the specification of the future of every space-time event.)

The Zollfrei problem is interesting even in dimension 2; in fact, as a warm-up for the problem in dimension 3 we will briefly describe what happens in dimension 2:

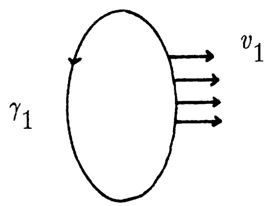
<u>Theorem</u>. Let g_{can} be the standard Zollfrei metric on $S^1 \times S^1$; i.e., the metric, $d\theta_1\, d\theta_2$, where θ_1 and θ_2 are the standard angle variables on the first and second factors. Let (X, g) be any oriented Zollfrei two-fold. Then there exists a covering map $\pi: X \to S^1 \times S^1$ such that $\pi^* g_{\text{can}}$ and g are conformally equivalent

<u>Proof</u>: First of all notice that every oriented compact Lorentzian two-fold has to be diffeomorphic to $S^1 \times S^1$ since its Euler characteristic is zero. Now suppose that X is a compact Lorentzian two-fold all of whose null-geodesics are periodic. The null-cone at $p \in X$ consists of two lines in T_p (See figure.)



so the conformal geometry of X is completely described by a pair of tranverse line element fields. Let v_1 and v_2 be vector fields defining these line element fields. By assumption the integral curves of v_1 and v_2 are all closed.

Choose an oriented curve, γ_1, which intersects each integral curve of v_1 transversally. (This is always possible. See [41], page 9.) Let p be the intersection number of γ_1 with the trajectory of v_1 through x. It is clear that this number is independent of x. Thus γ_1 has to intersect each trajectory in *exactly* p points since the orientation numbers at the points of intersect have to be all of the same sign. (See figure.)



Suppose in particular that x is on γ_1. Let f(x) be the next point at which the trajectory through x intersects γ_1. The map $f: \gamma_1 \to \gamma_1$ which sends x to f(x) is a diffeomorphism of γ_1, and the points, x, f(x), $f^2(x),\ldots,f^{p-1}(x)$, are the distinct points where the trajectory through x intersects γ_1. Thus f defines a *free* action of the finite cyclic group, Z_p, on γ_1. In particular there exists a covering map

$$\phi_1 : \gamma_1 \longrightarrow S^1$$

whose fibers are the Z_p orbits. Now extend ϕ_1 to all of X by associating to the point $x \in X$ the Z_p orbit in which the trajectory through x intersects γ_1. Notice that the level curves of ϕ_1 are identical with the integral curves of v_1.

Next choose a cycle, γ_2, in X which intersects each integral curves of v_2 transversally and repeat this argument. Let ϕ_2: $X \to S^1$ be the analogue of the mapping, ϕ_1, for the v_2 trajectories, and let

$$\phi\colon X \to S^1 \times S^1$$

be the product of ϕ_1 and ϕ_2. We leave it for the reader to convince himself that ϕ is a covering map and that $\phi^* g_{\mathrm{can}}$ is conformally equivalent to g. Q.E.D.

An easy corollary of this theorem is that every oriented Zollfrei two-fold is of the form $\mathbb{R}^2/L$, L being a rational lattice subgroup of $\mathbb{R}^2$ and the null-geodesics being the projections of the lines parallel to the x and y axes.

Lets next turn to the Zollfrei problem in dimension three. We pointed out above that for a compact oriented two-manifold to be a Lorentz manifold it has to be diffeomorphic to T^2. Unfortunately the fact that a compact 3-manifold, M, is a Lorentz manifold is no constraint at all on the topology of M. (The only topological obstruction to the existence of a Lorentz structure on M is the vanishing of its Euler characteristic, which is automatic in dimension three.) We suspect, however, that for M to be Zollfrei it must have a very simple topological structure. To be more specific, Thurston's classification of three-dimensional geometries suggests four obvious possibilities for the diffeotype of M and we suspect these are the only possibilities. We recall that a geometry for Thurston is a simply connected homogeneous space of the form G/H where G is a connected Lie group and H a compact subgroup. Thurston calls a compact manifold geometrizable if it is universal cover is such a space. He has conjectured that all three-manifolds can be obtained from the geometrizable ones by simple topological operations like "connected sum." The three-dimensional geometries are easy to classify and turn out to be eight in number; so every geometrizable three-manifold belongs to one of eight distinct categories. Our conjecture is tha the Zollfrei examples are all geometrizable and belong to the simplest of these eight categories, namely $S^2 \times \mathbb{R}$, with structure group, $G = SO(3) \times \mathbb{R}$. The compact manifolds with $S^2 \times \mathbb{R}$ as universal cover are just four in number: $S^2 \times S^1$ and the three spaces with $S^2 \times S^1$ as double cover corresponding to the three involutions of $S^2 \times S^1$:

i) $(x,y,z,t) \longrightarrow (-x,-y,-z,t)$

ii) $(x,y,z,t) \longrightarrow (-x,-y,-z,t+\pi)$

iii) $(x,y,z,t) \longrightarrow (-x,-y,z,t+\pi)$.

(See [34], page 458). All of these spaces have Zollfrei metrics which are covered by the standard Einstein metric, $(dx)^2 + (dy)^2 + (dz)^2 - (dt)^2$, on $S^2 \times \mathbb{R}$. We will henceforth call metrics of this type standard Zollfrei metrics; so our conjecture can be reformulated in the form:

Conjecture. Every Zollfrei manifold in dimension three has the same diffeotype as one of the standard examples.

A somewhat safer conjecture is that this conclusion is true with the additional hypothesis that the universal cover of M satisfies the causality condition (i.e., has no closed space-like or time-like curves. See [29], page 407. Incidentally, for the Floquet theory, which we will describe below, this property is highly desirable.)

If the above conjecture were true, the natural place to look for Zollfrei metrics in dimension three would be in the vicinity of the standard models. In fact an obvious question to ask is: Do the standard models admit non-trivial "Zollfrei deformations"?

This question will occupy us for the next 150 pages. We will, for the most part, concentrate on the simplest and most symmetric standard model, the conformal compactification of Minkowski three space, which has the toplogy of the second space on the list above. We will henceforth denote this model by $M_{2,1}$. (See §2.) We will show in §12 that it has lots of non-trivial Zollfrei deformations.

It would be interesting to develop a deformation theory along the lines of this monograph for some of the other standard models; and, in fact, we hope to do so sometime in the future. Particularly intriguing is the third space on the list, above. This space is diffeomorphic to the connected sum of $\mathbb{RP}^3$ with itself and is the only geometrizable three-manifold which is a connected sum ([34], page 457. It is quite a challenge, by the way, to describe the Zollfrei metric of $\mathbb{RP}^3 \# \mathbb{RP}^3$ as a "connected sum" of metrics on the individual $\mathbb{RP}^3$'s.)

There are interesting cyclicity phenomena in dimension three which we unfortunately won't have time to pursue in this article. We will, however, briefly describe the most bizarre of these: One of the eight geometries of Thurston is the universal cover of $SL(2,\mathbb{R})$. (This geometry plays an important role in the study of Seifert fiber spaces. See [24].) From the Killing form on the Lie algebra of $SL(2,\mathbb{R})$ one gets a bi-invariant Lorentz metric on $SL(2,\mathbb{R})$ with the property that all time-like geodesics are periodic. The null-geodesics on the other hand are not periodic. Compact examples of this phenomenon can be obtained by quotienting $SL(2,\mathbb{R})$ by a discrete co-compact subgroup. (See [24] for details.)

In this article we won't, for the most part, discuss Zollfrei metrics in dimension greater than three. This is not because this problem is uninteresting but because it seems to be much harder to find examples. (Some examples do exist: compactified Minkowski n-space, $S^n \times S^1$, $\mathbb{C}P^n \times S^1$ etc. More generally if M is any of the n-dimensional $SO(n)$-invariant Zoll manifolds constructed by Weinstein in [2], $M \times S^1$ is an example.) It is not unlikely that other methods than ours (for instance twistorial methods) will yield a larger supply of examples.

Having given some indication of the contents of this monograph we will say a few words about our motives for writing it. One of our main motives is the flickering (and probably unwarranted) hope that there are interesting solutions of Einstein's equations in (3+1) dimensions associated with the cyclic models described above. More explicitly the standard $M_{2,1}$ is what is left of the anti-deSitler universe after it undergoes gravitational collapse. We suspect that there may be interesting solutions of Einstein's equations in (3+1) dimensions which are related in the same way to cyclic deformations of $M_{2,1}$. The evidence for this is unfortunately still rather skimpy: First of all, as we will see in §12, the solutions of the linearized Einstein equations on $M_{3,1}$ (aka "free gravitons") are in one-one correspondence with the infinitesmal cyclic deformations of $M_{2,1}$. Secondly, there are methods, developed independently by Lebrun and by Fefferman and Graham, for putting (3+1)-dimensional Einstein "collars" on (2+1)-dimensional conformal spaces, which are particularly well adapted to (2+1)-dimensional cyclic models in our sense. (See the comments at the end of §15.)

Our second motive for writing this monograph is more defensible, at least on mathematical ground. Namely, as we will see below, Zollfrei manifolds turn out to have lots of interesting non-local conformal invariants. To construct these invariants we make use of some ideas of Paneitz and Segal of which we will give a short description here. (More details will be provided in §18.) Let M be a compact manifold, $\Box$ a differential operator on M and $\tilde{M}$ the universal cover of M. Corresponding to $\Box$ is a differential operator on $\tilde{M}$ which we will denote by $\tilde{\Box}$. It is clear that the action of the fundamental group of M on $\tilde{M}$ leaves $\tilde{\Box}$ fixed; hence there is a canonical representation of the fundamental group of M on the space of solutions of the equation $\tilde{\Box} = 0$. A classical example of this situation is Hill's equation

$$\frac{d}{dt^2} + q(t) = \Box = 0$$

on the circle (i.e. $q(t) = q(t + 2\pi)$.) The deck transformation, $\sigma\colon t \to t+2\pi$, acts on the two dimensional space of solutions of $\tilde{\Box} = 0$ and the two-by-two matrix, $A(\sigma)$, representing

σ is the classical Floquet matrix. Paneitz and Segal point out that if M is a compact space–time and $\square$ is a conformally invariant differential operator, the space of solutions of $\tilde{\square}$ will very often be a fairly manageable space, (e.g. Hilbertizable). They also show that the Floquet representation on this space describes what is traditionally referred to as the "scattering phenomena" associated with $\tilde{\square}$. (See [30], [31], [21] and [37].)

We will review the theory of these Floquet operators in part five and will show that if M is Zollfrei they have the form

$$e^{i\sigma}I + K.$$

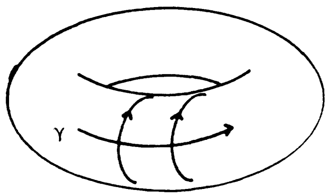
Here σ is an integral multiple of $\pi/4$ and K is a compact operator. (Incidentally we will also show that the converse of this assertion is true. If the Floquet operators are of this simple form M has to be Zollfrei.) In particular for Zollfrei manifolds these operators have discrete spectrum; so these manifolds have a large number of discrete conformal invariants.

It would be nice to relate these invariants to other conformal invariants of M, for instance the Chern-Simon invariant [5], or the invariants studied by Fefferman-Graham [8] and Branson and Oersted [3]. At first glance, however, they seem to be a good deal more complicated. (We will describe some of our efforts to compute these invariants in §19.)

We will conclude this introduction by warning the reader about some of the technical complications involved in dealing with closed geodesics on Lorentzian manifolds. As was pointed out to us by John Beem a closed null-geodesic does not need to be complete. The problem is that the following situation can occur. (In fact it can occur generically. See [1].) Let H be the Hamiltonian function on T^*M defining geodesic flow and (p,ξ) a point on the null-energy surface $H = 0$. A trajectory of geodesic flow whose initial point is (p,ξ) can return after a finite period of time, T, not to the point (p,ξ) itself but to the point $(p,\lambda\xi)$ with $\lambda > 1$. The projection of this trajectory onto M will look like a perfectly respectable closed null-geodesic. Notice, however, that the next circuit which this trajectory makes will go from $(p,\lambda\xi)$ to $(p,\lambda^2\xi)$ in time T/λ and the next circuit after that from $(p,\lambda^2\xi)$ to (p,λ^3) in time T/λ^2. The ultimate destiny of this trajectory is clear: It will cease to exist after a finite period of time.

To avoid this kind of behavior we will categorically decree from now on that Zollfrei $\Rightarrow$ the trajectories of geodesic flow are periodic on the null-energy surface $H = 0$.

There is another type of pathology which is not quite as serious as this but which we will rule out to make life simpler for ourselves. Namely, it is possible for all trajectories of geodesic flow on $H = 0$ to be periodic, but with some trajectories much shorter than the rest. This situation is illustrated in the figure below:



(γ is the short geodesic and all the others spiral around it giving rise to a Seifert fibration of the solid torus).

To avoid this type of behavior we will decree that Zollfrei $\Rightarrow$ geodesic flow is a fibration (in the usual sense) of the energy surface, $H = 0$, by S^1's.

Before we get down to business we would like to express our appreciation to the many persons who have helped us with the preparation of this manuscript. The material on the infinitesmal deformations of compactified Minkowski space could not have been written without the help of David Vogan. (Our original version of this material, using spherical harmonics, was three times as long as the Harish-Chandra module approach described in §9-12.) Similarly the sections on the microlocal properties of the x-ray transform,, §14–17, are much better in the final manuscript than they were in their original version

thanks to Richard Melrose's unstinting aid. Other persons with whom we discussed the contents of this manuscript and who provided us with valuable suggestions for improving it are John Beem, Luis Casian, Michael Eastwood, John Morgan, Bent Oersted, Michele Vergne,, Gunther Uhlmann and Alejandro Uribe. Last but not least, the stimulus for writing this paper was Irving Segal's monograph, [35] (which convinced us that the Einstein static universe still has to be taken seriously as a cosmological model.)

PART I

A RELATIVISTIC APPROACH TO ZOLL PHENOMENA

§1. A Riemannian metric on a compact manifold is called a Zoll metric if all of its geodesics are simply periodic of period 2π. For instance the standard metric, $(dx)^2$, on S^2 has this property. Seventy–five years ago, Funk wrote a seminal paper on Zoll two–folds [9] in which he posed the following problem: find all Zoll metrics on S^2 which are C^2 close to the standard one. In particular he proposed an algorithm for constructing such metrics: Given a function f on S^2 define the Funk transform, $\hat{f}$ of f to be the function

$$\hat{f}(p) = \int_{\gamma_p} f ds ,$$

where γ_p is the hemispherical circle on S^2 obtained by situating p at the north pole. It is easy to see that $\tilde{f} \equiv 0$ if and only if f is odd, i.e. $f(-p) = -f(p)$ for all $p \in S^2$. Starting with an odd function, f_0, Funk shows how to construct a sequence of functions, $f_1, f_2, \ldots$, by solving recursively integral equations of the form

$$\hat{f}_i = F_i(f_1,\ldots,f_{i-1}),$$

and conjectures, first of all, that the series

$$f_t = \sum f_i(x)t^i \tag{*}$$

converges for all t in a sufficiently small interval about $t = 0$, and, secondly, that $(1+f_t)(dx)^2$ is a Zoll metric for all t on this interval.

Given the convergence of (*) the second assertion is quite plausible; but it is not known to this day whether (*) converges except for very special choices of f_0. (In fact it seems unlikely that it does.) There is, however, an updated version of the Funk algorithm involving Nash–Moser techniques, for which (*) does converge, and that gives essentially the same result as that which Funk had hoped to get from the scheme above. (For a survey of what is known about Zoll metrics and the Funk problem, see [2].)

Now suppose we are given a Riemann metric, $(d\gamma)^2$, on S^2. Let t be the standard angle variable on S^1 and consider the pseudometric of signature $(2+1)$:

$$(d\gamma)^2 - (dt)^2 , \tag{1.1}$$