CONTRIBUTIONS TO THE THEORY OF NONLINEAR OSCILLATIONS

VOLUME IV

H. ANTOSIEWICZ
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PREFACE

The present volume of the Contributions, fourth in the series, covers, like its predecessors, a great variety of topics in non-linear differential equations. The first paper by Kakutani and Markus deals with a differential-difference equation arising in the theory of growth phenomena. Beyond general considerations of the functional properties of the solutions, the authors have obtained very detailed information concerning the oscillations and asymptotic behavior of the solutions. The second paper by Lefschetz, a complement to Barocio's Mexican thesis, contains a rather detailed description of singularities of a pair of analytic differential equations in the plane. The third by Bushaw is a noteworthy contribution in the study of discontinuous forcing terms. The particular point amply covered is the rapidity with which the origin is reached by any solution — an important question in control problems. The paper by de Vogelaere deals with the periodic solutions of Störmer's problem arising in electro-magnetic theory. Slotnick's paper is concerned with the instabilities of Hamiltonian systems. This work continues an investigation initiated by J. Moser. Kyner's contribution relates the theory of periodic surfaces along the line developed by S. Diliberto which was amply described in Contributions III. The paper by Seifert deals with the qualitative behavior of planar differential systems by the method of rotating vector fields. Antosiewicz, in his contribution, gives a survey of the second method of Lyapounov. As is well known, this method has been extensively treated in the Soviet Union but is also acquiring great importance in other areas in view of its elasticity and general power.

The contributions of Mendelson and Bass are concerned with the qualitative behavior of the solutions of non-linear differential systems, with many degrees of freedom, near a critical point. Mendelson investigates the phase portrait near an isolated critical point where one characteristic root is zero and the others have real parts of the same sign. Bass studies the instability near an equilibrium from which repulsive forces act.

A number of these papers have been contributed by various Governmental organizations. These are indicated in connection with each paper.

Solomon Lefschetz

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VOL. IV

I. ON THE NON-LINEAR DIFFERENCE-DIFFERENTIAL EQUATION $y'(t) = [A - By(t - \tau)]y(t)$

S. Kakutani and L. Markus

1. INTRODUCTION

If one assumes that the net birth rate $y'(t)/y(t)$ of a population $y(t)$ is the constant coefficient A, then elementary considerations yield an exponential growth (or decay) of population, $y(t) = y(0)e^{At}$. A more feasible mathematical model, for certain discussions, could be obtained by assuming that the birth rate coefficient is diminished by a quantity proportional to the population of the preceding generation. These assumptions lead to the functional equation appearing in the title,

$$y'(t) = [A - By(t - \tau)]y(t), \tag{1}$$

where $\tau > 0$, A and B are real numbers. This delay-differential equation also occurs in the theory of certain servo-mechanisms [3].

If $B = 0$, the resulting differential equation is elementary and we shall henceforth assume $B \neq 0$. We simplify equation (1) by writing $z(t) = B\tau y(\tau t)$. Then

$$z'(t) = [a - z(t - 1)]z(t) \quad , \tag{2}$$

where $a = A\tau$. We shall investigate the functional equation (2) and the results can easily be reinterpreted for (1).

DEFINITION. A solution $z(t)$ of (2) is a real continuous function defined on $0 \leq t < 1 + \epsilon$, $\epsilon > 0$, where $z(t) \in C^{(1)}$ on $1 < t < 1 + \epsilon$, and there satisfies the functional equation (2).

Clearly the constants $z = 0$ and $z = a$ are solutions. It is interesting to note that these are the only solutions of period one since

$z'(t) = [a - z(t)]z(t)$ has no other periodic solutions.

Quite general existence theorems are available [1, 2] for difference-differential equations. However we obtain precise knowledge of the behavior of the solutions of (2) and in our detailed investigations the general theory is not directly applicable.

2. GENERAL PROPERTIES OF THE SOLUTIONS

THEOREM 1. Let $\varphi(t)$, $0 \leq t \leq 1$, be a continuous, real-valued function prescribed as an initial condition. Then there exists a solution $z(t)$ of (2), defined on $0 \leq t < \infty$, for which $z(t) = \varphi(t)$ on $0 \leq t \leq 1$. Moreover $z(t)$ is unique in that each solution of (2), agreeing with $\varphi(t)$ on $0 \leq t \leq 1$, also agrees with $z(t)$ on their common domain of definition.

PROOF. Let $z(t) = \varphi(t)$ on $0 \leq t \leq 1$. Define

$$z(t) = \varphi(1) \exp\left[a(t - 1) - \int_0^{t-1} \varphi(s)ds \right]$$

on $1 \leq t \leq 2$. If $z(t)$ is well-defined on $0 \leq t \leq n$, $n = 2, 3, 4, \ldots$, then define

$$z(t) = z(n) \exp\left[a(t - n) - \int_{n-1}^{t-1} z(s)ds \right]$$

on $n \leq t \leq n + 1$. Clearly $z(t)$ is continuous on $0 \leq t < \infty$ and moreover $z(t) \in C^{(1)}$ on $1 < t < \infty$, except possibly at $t = n$.

At $t = n$,

$$z'(n + 0) = z(n)[a - z(n - 1)]$$

and

$$z'(n - 0) = z(n - 1)[a - z(n - 1)]\exp\left[a - \int_{n-2}^{n-1} z(s)ds \right]$$

$$= z(n)[a - z(n - 1)].$$

Thus $z(t) \in C^{(1)}$ for $1 < t < \infty$ and there satisfies

$$z'(t) = [a - z(t-1)]z(t).$$

If $w(t)$ is another solution, corresponding to the initial function $\varphi(t)$, then let $t_0 \geq 1$ be the l.u.b. $\{t \mid w(t) = z(t)\}$. But on $t_0 \leq t \leq t_0 + 1$,

$$z(t) = z(t_0) \exp\left[a(t - t_0) - \int_{t_0 - 1}^{t-1} z(s)ds \right]$$

and

$$w(t) = w(t_0) \exp\left[a(t - t_0) - \int_{t_0 - 1}^{t-1} w(s)ds \right] .$$

Since $w(t_0) = z(t_0)$ and furthermore $w(s) = z(s)$ on $t_0 - 1 \leq s \leq t_0$, we have $w(t) = z(t)$ on $t_0 \leq t \leq t_0 + 1$ which contradicts the existence of the finite number t_0. Thus $w(t) = z(t)$ on their common domain of definition. Q.E.D.

Hereafter, by a solution of (2), we shall mean a solution defined on $0 \leq t < \infty$.

COROLLARY. The solution $z(t) \in C^{(1)}$ on $0 \leq t < \infty$ if and only if $\varphi(t) \in C^{(1)}$ on $0 \leq t \leq 1$ and also $\varphi'(1) = [a - \varphi(0)]\varphi(1)$.

PROOF. If $z(t) \in C^{(1)}$ on $0 \leq t < \infty$ then $\varphi(t) \in C^{(1)}$ on $0 \leq t \leq 1$. Furthermore $z(t)$ satisfies the functional equation (2) at $t = 1 + \epsilon$, $\epsilon > 0$, and thus at $t = 1$. But at $t = 1$, $z(1) = \varphi(1)$, $z'(1) = \varphi'(1)$ and $\varphi'(1) = [a - \varphi(0)]\varphi(1)$.

Conversely if $\varphi(t) \in C^{(1)}$ on $0 \leq t \leq 1$, then $z(t) \in C^{(1)}$ on $0 \leq t < \infty$, except possibly at $t = 1$. But both $z'(1 - 0) = \varphi'(1)$ and $z'(1 + 0) = [a - \varphi(0)]\varphi(1)$ exist and are equal. Thus $z(t) \in C^{(1)}$ on $0 \leq t < \infty$. Q.E.D.

Clearly, if we require $\varphi(t) \in C^{(\infty)}$ on $0 \leq t \leq 1$ and also that the derivatives of $\varphi(t)$ at $t = 1$ be related to those at $t = 0$ by the functional equation (2) and by those equations obtained from (2) by differentiation, then the solution $z(t) \in C^{(\infty)}$ on $0 \leq t < \infty$. It is not apparent whether or not there are analytic solutions of (2). The next theorem shows that there are no (entire) analytic solutions of (2) (other than $z = a$, $z = 0$) which can be expressed in terms of elementary functions.

THEOREM 2. Let $z(t)$ be an entire function of the complex variable t and let $z(t)$ satisfy $z'(t) = [a - z(t-1)]z(t)$. Then, unless $z \equiv 0$ or $z \equiv a$, for each integer k,

$$\max_{|t|=r} |z(t)| = M(r) > \exp \exp \ldots \exp r$$

(k repetitions) for all sufficiently large r.

PROOF. If $z(t_o) = 0$, then $z'(t_o) = 0$ and clearly $z^{(n)}(t_o) = 0$ so that $z \equiv 0$. Thus assume that $z(t)$ has no zeros. If $z(t)$ is of finite order, then by Hadamard's theorem, $z(t) = \exp P(t)$ where $P(t)$ is a polynomial. But then $P'(t) = [a - \exp P(t-1)]$ which is impossible unless $P(t)$ is a constant. But the only constant solutions are $z \equiv 0$ and $z \equiv a$.

Now allow $z(t)$ to be of infinite order. Let $\operatorname{Log} z(t)$ be an entire function such that $\exp \operatorname{Log} z(t) = z(t)$. For any entire function $f(t)$ one knows[1]

$$\max_{|t|=r} |f'(t)| \leq \frac{CR}{(R-r)^2} \{A(R) + f(0)\} \quad ,$$

where

$$A(R) = \max_{|t|=R} \operatorname{Re}\{f(t)\}$$

provided $A(R) \geq 0$, C is a positive constant, and $0 < r < R$. Set $R = 2r$, $f(t) = \operatorname{Log} z(t)$ and we obtain

$$\max_{|t|=r} \left| \frac{d}{dt} \operatorname{Log} z(t) \right| \leq \frac{C_1}{r} \left\{ \max_{|t|=2r} \operatorname{Log} |z(t)| + C_2 \right\} \quad ,$$

where C_1, C_2 are positive constants and the above result applies since

$$\max_{|t|=2r} \operatorname{Log} |z(t)| = \log \left\{ \max_{|t|=2r} |z(t)| \right\} > 0 \quad .$$

Set

$$M(u) = \max_{|t|=u} |z(t)| \quad .$$

[1] Dr. J. Wermer suggested this argument.

Then

$$\max_{|t|=r} \left| \frac{z'(t)}{z(t)} \right| \leq \frac{C_3}{r} \log M(2r)$$

for a positive constant C_3. But

$$|z(t)| \leq \left| \frac{z'(t+1)}{z(t+1)} \right| + |a|$$

and so

$$M(r) \leq \frac{C_4}{(r+1)} \log M(2r + 2)$$

for a positive constant C_4.

Now, proceeding by induction, suppose there are no non-constant entire solutions $z(t)$ with $M(r) \leq \exp \exp \ldots \exp r$ (k repetitions), for all large r. If $z(t)$ is an entire solution with

$$M(r) \leq \exp \exp \ldots \exp r$$

($k + 1$ repetitions), then

$$M(r) \leq \frac{C_4}{r+1} \exp \exp \ldots \exp(2r + 2)$$

(k repetitions). Using the same reduction again we have

$$M(r) \leq \frac{C_4}{r+1} \left[\log C_4 - \log(r + 1) + \exp \exp \ldots \exp(4r + 6)\right]$$

($k - 1$ repetitions). But this states that

$$M(r) \leq \exp \exp \ldots \exp r$$

(k repetitions) for all large r. This contradicts the induction hypothesis. Q.E.D.

E. M. Wright [5] has shown that there are real-valued entire solutions for $a > 0$ and, at least for small a, these solutions can be positive on $t \geq 0$.

Returning to the consideration of real solutions, we note that a solution $z(t)$ is nowhere zero for $1 \leq t < \infty$ in case $\varphi(1) \neq 0$.

THEOREM 3. If $\varphi(1) > 0$, then the corresponding solution $z(t) > 0$ for $1 \leq t < \infty$. If $\varphi(1) < 0$, then $z(t) < 0$ for $1 \leq t < \infty$. If $\varphi(1) = 0$, then $z(t) \equiv 0$ for $1 \leq t < \infty$.

PROOF. Consider only the case $\varphi(1) > 0$. Here

$$z(t) = \varphi(1) \exp \left[a(t - 1) - \int_0^{t-1} \varphi(s)ds \right] > 0$$

on $1 \leq t \leq 2$. Similarly, if $z(t) > 0$ on $1 \leq t \leq n$, then $z(n) > 0$ and

$$z(t) = z(n) \exp \left[a(t - n) - \int_{n-1}^{t-1} z(s)ds \right] > 0$$

on $n \leq t \leq n + 1$. Therefore $z(t) > 0$ on $1 \leq t < \infty$. Similar proofs hold in the other cases $\varphi(1) < 0$ and $\varphi(1) = 0$. Q.E.D.

COROLLARY. Let two solutions $z_1(t)$ and $z_2(t)$ agree on a unit interval $0 \leq t_1 \leq t \leq t_1 + 1$. Then either $z_1(t) = z_2(t)$ on $0 \leq t < \infty$ or else $z_1(t) = z_2(t) = 0$ on $1 \leq t < \infty$.

PROOF. By the argument for uniqueness used in Theorem 1, $z_1(t) = z_2(t)$ on $t_1 \leq t < \infty$. Now $z_i(t)$ $(i = 1$ or $2)$ vanishes at some point on $t_1 \leq t < \infty$ if and only if $z_i(1) = 0$ and then $z_i(t) = 0$ on $1 \leq t < \infty$. Thus either $z_1(t) = z_2(t) = 0$ on $1 \leq t < \infty$ or else neither solution vanishes anywhere on $1 \leq t < \infty$. But then the equation $z(t - 1) = a - z'(t)/z(t)$ determines that $z_1(t) = z_2(t)$ on $0 \leq t < \infty$. Q.E.D.

In the following analysis we shall be primarily interested in the case where the intersects of a solution curve $z = z(t)$ with the line $z = a$ form a discrete set.

THEOREM 4. The intersections of the solution curve $z = z(t)$ with the line $z = a$ are discrete on $0 \leq t < \infty$ if and only if there are a finite number of zeros of $\varphi(t) - a$ on $0 \leq t \leq 1$.

PROOF. If the zeros of $z(t) - a$ are discrete on $0 \leq t < \infty$, then, a fortiori, the zeros of $\varphi(t) - a$ are discrete on $0 \leq t \leq 1$.

Conversely, suppose the zeros of $\varphi(t) - a$ are discrete on $0 \leq t \leq 1$. Then, since we may take $\varphi(1) \neq 0$, the zeros of $z'(t)$ on $1 \leq t \leq 2$ are discrete. Thus there are only a finite number of zeros of $z(t) - a$ on $1 \leq t \leq 2$. By an induction argument one shows that there are only a finite number of zeros of $z(t) - a$ on each unit interval $n \leq t \leq n + 1$, $n = 1, 2, \ldots$. Q.E.D.

3. THE PRINCIPAL CASE, $a > 0$

THEOREM 5. Let $a > 0$, $\varphi(1) > 0$ and let $z(t)$ be the solution of (2) corresponding to the initial function $\varphi(t)$. Then $0 < m \leq z(t) \leq M < \infty$ on $1 \leq t < \infty$ where

$$M = \max\left\{ \max_{1\leq t\leq 3} z(t),\ ae^{a} \right\}$$

and

$$m = \min\left\{ \min_{1\leq t\leq 3} z(t),\ ae^{a-M} \right\}.$$

PROOF. Since $\varphi(1) > 0$, $z(t) > 0$ and $a - z(t) < a$ on $1 \leq t < \infty$. Suppose there exists a $t_1 \geq 3$ with $z(t_1) > M \geq ae^{a}$. Then $z'(t) = [a - z(t - 1)]z(t) < a\, z(t)$ on $t_1 - 1 \leq t \leq t_1$ and $z(t_1 - 1) > a$. Thus $z'(t_1) < 0$ and one sees easily that $z(t)$ is monotone decreasing on $3 \leq t \leq t_1$. But then $z(3) > M$ which contradicts the definition of M. Therefore $z(t) \leq M$ on $1 \leq t < \infty$.

Now $a - z(t) \geq a - M$ on $1 \leq t < \infty$. Suppose $z(t_2) < m \leq ae^{a-M}$ for some $t_2 \geq 3$. Then $z(t_2 - 1) < a$, since $z'(t) \geq (a - M)\, z(t)$ on $t_2 - 1 \leq t \leq t_2$. Thus $z'(t_2) > 0$ and so $z(t)$ is monotonely increasing on $3 \leq t \leq t_2$. Therefore $z(3) < m$ which is a contradiction. Q.E.D.

THEOREM 6. Let $a > 0$ and $\varphi(1) > 0$ and let $z(t)$ be the solution of (2) corresponding to the initial function $\varphi(t)$. Then either

(i) $z(t)$ is asymptotic to $z = a$; that is, $z(t)$ and $z'(t)$ are monotone for large t and

$$\lim_{t=\infty} z(t) = a, \qquad \lim_{t=\infty} z'(t) = 0 \quad ,$$

or

(ii) $z(t)$ oscillates about $z = a$; that is, $z(t) - a$ assumes both positive and negative values for arbitrarily large t.

PROOF. If $z(t) \geq a$ for all large t, then $a - z(t - 1) \leq 0$ and $z'(t) \leq 0$. Thus $z(t)$ decreases monotonely to a limit which is easily seen to be a. Furthermore

$$z''(t) = \{[a - z(t - 1)]^2 - z'(t - 1)\}\, z(t) \geq 0$$

so that $z'(t)$ increases monotonely to a limit which is easily seen to be zero.

If $z(t) \leq a$ for all large t, then $a - z(t - 1) \geq 0$ and $z'(t) \geq 0$. Thus $z(t)$ increases to the limit a. Also $z'(t) \geq [a - z(t)]\, z(t) \geq [a - z(t)]^2$ for large t so that $z''(t) \leq 0$. Thus $z'(t)$ decreases monotonely to the limit zero.

If neither $z(t) \geq a$ nor $z(t) \leq a$ for all large t, then $z(t)$ oscillates about the line $z = a$. Q.E.D.

As we shall later see, each of these alternatives is possible for certain values of the parameter a. However, before determining which values of the parameter a produce which behaviors, we shall investigate the qualitative form of the oscillatory solutions.

THEOREM 7. Let $a > 0$, $\varphi(1) > 0$ and $z(t)$, the solution of (2) corresponding to $\varphi(t)$, oscillate about $z = a$. Assume the zeros of $z(t) - a$ are a discrete set on $0 \leq t < \infty$. Then, for sufficiently large t, each zero of $z(t) - a$ is simple and there is exactly one zero of $z'(t)$ between consecutive zeros of $z(t) - a$.

PROOF. Let the number of zeros of $z(t) - a$ on $n \leq t \leq n + 1$, $n = 1, 2, 3, \ldots$, be $k(n)$. Then define the number $\bar{k}(n)$ of potential zeros to be $\bar{k}(n) = k(n)$ if the last zero $t_\ell = n + 1$ or if

$$\frac{d}{dt}\,|z(t) - a| > 0$$

on $\max(n, t_\ell) < t < n + 1$ and define $\bar{k}(n) = k(n) + 1$ in all other cases. We show that $\bar{k}(n)$ is a non-increasing function of n.

Suppose $\bar{k}(n) = k(n)$ and $t_\ell = n + 1$. Then there are at most $k(n) - 1$ bend points of $z(t) - a$ in $n + 1 \leq t < n + 2$. Thus $k(n + 1) \leq k(n)$ and for equality one must have a situation in which $\bar{k}(n + 1) = k(n + 1)$. Therefore in this case $\bar{k}(n + 1) \leq \bar{k}(n)$.

Next suppose $\bar{k}(n) = k(n)$ and

$$\frac{d}{dt}\,|z(t) - a| > 0$$

on $\max\,(n, t_\ell) < t < n + 1$. Then on $n + 1 \leq t \leq n + 2$ there are at most $k(n)$ bend points, at least one of which must occur before the first zero of $z(t) - a$. Thus $k(n + 1) \leq k(n)$. If $k(n + 1) = k(n)$, then no zero of $z'(t)$ occurs following the last zero of $z(t) - a$ on $n + 1 \leq t \leq n + 2$, and hence $\bar{k}(n+1) = k(n+1)$. Therefore $\bar{k}(n+1) \leq \bar{k}(n)$ in this case.

Finally suppose $\bar{k}(n) = k(n) + 1$. There are at most $k(n)$ bend points of $z(t) - a$ on $n + 1 \leq t \leq n + 2$ and thus $k(n+1) \leq k(n) + 1$. But if $k(n+1) = k(n) + 1$ the situation is such that $\bar{k}(n+1) = k(n+1)$. Therefore $\bar{k}(n+1) \leq \bar{k}(n)$ in all cases.

Now let

$$\lim_{n=\infty} \bar{k}(n) = \bar{k}$$

and say, for $n > N$, $\bar{k}(n) = \bar{k}$. Suppose on $N + r \leq t \leq N + r + 1$, $r = 1, 2, 3, \ldots,$ there are at least two zeros of $z'(t)$ on the open interval between two successive zeros of $z(t) - a$. Then there are at least $\bar{k}(N+r)$ zeros of $z'(t)$ on $N + r \leq t \leq N + r + 1$. But then there are at least $\bar{k}(N+r)$ zeros of $z(t) - a$ on $N + r - 1 \leq t \leq N + r$, that is, $k(N + r - 1) \geq \bar{k}(N+r)$. Furthermore equality holds only when $\bar{k}(N + r - 1) = k(N + r - 1) + 1$. Thus $\bar{k}(N + r - 1) > \bar{k}(N + r)$ which contradicts the property that $\bar{k}(n) = \bar{k}$ for $n \geq N$.

A similar argument shows that a double root of $z(t) - a$ also results in a decrease in $\bar{k}(n)$ and so can not occur for $n \geq N$. Q.E.D.

COROLLARY. Let $a > 0$, $\varphi(1) > 0$ and let the solution $z(t)$ oscillate about $z = a$. Let $\varphi(t) - a \in C^{(1)}$ on $0 \leq t \leq 1$ have at most one potential zero (in particular if $\varphi(t) - a \neq 0$), then the interval from each zero of $z(t) - a$, $t \geq 1$, to the following extremum is of length one. The interval from an extremum of $z(t) - a$, of amplitude $\epsilon > 0$, to the following zero is of length

$$d_\epsilon > \frac{1}{\epsilon} \log (1 + \epsilon/a) \quad .$$

If $a > 1$ and if ϵ is sufficiently small, then

$$\frac{1}{a}\left[1 - \epsilon/2a\right] < d_\epsilon < 2 \quad .$$

PROOF. If $\varphi(t) - a$ has no zeros on $0 \leq t \leq 1$, then $z'(t) \neq 0$ on $1 \leq t \leq 2$ and $\bar{k}(1) = 1$. If $\varphi(t) - a$ has one zero followed by a zero of $\varphi'(t)$ on $0 \leq t \leq 1$, then again $z'(t)$ has one zero on $1 \leq t \leq 2$ and $\bar{k}(1) = 1$.

Now consider an extremum where $z'(t_0) = 0$ on $n \leq t_0 \leq n + 1$, $n = 1, 2, \ldots$. Then on $n - 1 \leq t \leq n$ there is one, and thus only one, zero of $z(t) - a$, since $\bar{k}(n-1) = k(n-1) = 1$. If there were another extremum on $t_0 - 1 \leq t \leq t_0$, then $\bar{k}(n) \geq 2$ which is impossible. Therefore

$z(t) - a$ is strictly monotone on $t_o - 1 \leq t \leq t_o$.

Let t_1 be an extremum where $z(t_1) = a + \epsilon$. Then

$$z(t) > (a + \epsilon)e^{-\epsilon(t-t_1)}$$

for $t_1 < t < t_1 + d_\epsilon$. But for

$$t - t_1 = \frac{1}{\epsilon} \log (1 + \epsilon/a) ,$$

$$(a + \epsilon)e^{-\epsilon(t-t_1)} = (a + \epsilon) \left(\frac{a}{a+\epsilon}\right) = a \quad .$$

Thus $d_\epsilon > \frac{1}{\epsilon} \log (1 + \epsilon/a)$.

For a minimum where $z(t_2) = a - \epsilon$,

$$z(t) < (a - \epsilon)e^{\epsilon(t-t_2)} < a$$

when

$$(t - t_2) \leq \frac{1}{\epsilon} \log \left(\frac{a}{a-\epsilon}\right) \quad .$$

Thus

$$d_\epsilon > \min \left\{\frac{1}{\epsilon} \log (1 + \epsilon/a), \quad \frac{1}{\epsilon} \log \frac{a}{a-\epsilon}\right\} = \frac{1}{\epsilon} \log (1 + \epsilon/a) \quad .$$

If ϵ is small, say $\epsilon < a$, then

$$d_\epsilon > \frac{1}{\epsilon} \left[\epsilon/a - \frac{1}{2} (\epsilon/a)^2 + \frac{1}{3} (\epsilon/a)^3 - \dots\right] > \frac{1}{\epsilon} \left[\epsilon/a - \frac{1}{2} (\epsilon/a)^2\right] \quad .$$

Thus

$$d_\epsilon > \frac{1}{a} \left[1 - \frac{1}{2}\frac{\epsilon}{a}\right] \quad .$$

Now take $a > 1$. Suppose that following a maximum $z(t_1) = a + \epsilon$, $z(t) \geq a$ on $t_1 \leq t \leq t_1 + 2$. But then on $t_1 + 1 \leq t \leq t_1 + 2$, $- z'(t) > \epsilon_1 z(t)$ and

$$z(t) < (a + \epsilon_1)e^{-\epsilon_1(t-t_1-1)}$$

where $\epsilon_1 = z(t_1 + 1) - a$. Then $z(t_1 + 2) < (a + \epsilon_1)(1 - \epsilon_1 + \epsilon_1^2/2) < a$ which is a contradiction. In this case $d_\epsilon < 2$.

Similarly, after a minimum where $z(t_2) = a - \epsilon$, suppose $z(t) \leq a$ on $t_2 \leq t \leq t_2 + 2$. Then on $t_2 + 1 \leq t \leq t_2 + 2$, $z'(t) > \epsilon_2 z(t)$ and

$$z(t) > (a + \epsilon_2)e^{\epsilon_2(t-t_2-1)}$$

where $\epsilon_2 = a - z(t_2 + 1)$. Then $z(t_2 + 2) > (a - \epsilon_2)(1 + \epsilon_2) > a$. Thus here also $d_\epsilon < 2$. Q.E.D.

The most interesting oscillations are those which are strictly monotone from each zero of $z(t) - a$, for a unit length, until the following extremum. Such an oscillation is concave up on $t_1 + 1 < t < t_2 + 1$, where t_1 and t_2 are successive maxima and minima respectively. Thus on

$$t_2 - \frac{1}{\epsilon} \log (1 + \epsilon/a) \leq t \leq t_2 + 1$$

or on

$$t_2 - \frac{1}{2a} \leq t \leq t_2 + 1 ,$$

where $\epsilon < a$ is the amplitude of the maximum, the solution $z(t)$ is concave upwards.

THEOREM 8. Let $0 < a \leq 1$ and let $z(t) - a$ oscillate with discrete zeros. Then the oscillations are damped and

$$\lim_{t=\infty} z(t) = a \quad .$$

PROOF. Let $\epsilon_1, \epsilon_2, \ldots$ and $\delta_1, \delta_2, \ldots$ be the amplitudes of successive maxima and minima respectively, with δ_i occurring before ϵ_i and we consider only large t so that the oscillations have the form described in Theorem 7.

Suppose ϵ_{1+k} is the maximum which follows by one unit the first zero of $z(t) - a$ after δ_1. Then clearly ϵ_{n+k} is the maximum following by one unit the first zero of $z(t) - a$ after δ_n, $n = 1, 2, \ldots$. Also δ_{n+k+1} is the minimum following by one unit the first zero of $z(t) - a$ after ϵ_n.

A trivial estimate yields

$$a + \epsilon_{n+k} < ae^{\delta_n}$$

and

$$a - \delta_{n+k+1} > ae^{-\epsilon_n} \quad .$$

But then

$$\epsilon_n < a \left\{\exp a \left[1 - e^{-\epsilon_{n-2k-1}}\right] - 1\right\} \quad .$$

We show that $\epsilon_n < \epsilon_{n-2k-1}$.

Consider the function

$$f(\epsilon) = a\left\{\exp a\left[1 - e^{-\epsilon}\right] - 1\right\} .$$

Now $f(0) = 0$ and

$$f'(\epsilon) = a^2 \exp a\left[1 - e^{-\epsilon} - \epsilon/a\right] .$$

Thus, for $\epsilon > 0$,

$$f'(\epsilon) < a^2 \exp a\left[1 - \epsilon - e^{-\epsilon}\right] < a^2 \leq 1 .$$

Thus $\epsilon \longrightarrow f(\epsilon)$ is a strict contraction of the half-line $\epsilon > 0$ onto itself. Since $\epsilon_n < f(\epsilon_{n-2k-1}) < \epsilon_{n-2k-1}$ we have

$$\lim_{n=\infty} \epsilon_{n_i+(2k+1)n} = 0$$

where $n_i = 1, 2, \ldots, 2k$. But then

$$\lim_{n=\infty} \epsilon_n = 0 .$$

Since

$$\delta_{n+k+1} < a\left[1 - e^{-\epsilon_n}\right], \quad \lim_{n=\infty} \delta_n = 0 .$$

Therefore

$$\lim_{t=\infty} z(t) = a .$$

Q.E.D.

Thus for $0 < a \leq 1$, $\varphi(1) > 0$, every solution $z(t)$ is either asymptotic to $z = a$ or is a damped oscillation and thus

$$\lim_{t=\infty} z(t) = a .$$

It seems likely that for $a > 1$, some of the oscillations are not damped (there are only oscillations, see Theorem 9) and do not approach a limit value.

THEOREM 9. Let $a > 1/e$ and $\varphi(1) > 0$. Then no solution $z(t)$ is asymptotic to $z = a$ (except $z(t) \equiv a$).

PROOF. If $z'(t) = [a - z(t - 1)]z(t)$, let $y(t) = \frac{1}{a} z(t)$ and then $y(t)$ is a solution of $y'(t) = a[1 - y(t - 1)]y(t)$ which is asymptotic to $y = 1$ if and only if $z(t)$ is asymptotic to $z = a$. Let $\zeta(t) = 1 - y(t)$ and then $\zeta(t) \searrow 0$ with $\zeta'(t) \nearrow 0$ in case $z(t) \nearrow a$ or $y(t) \nearrow 1$ as $t \longrightarrow \infty$.

Suppose $z(t)$ is asymptotic to $z = a$ from below. Now