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FOREWORD

This book is based on the notes from lectures given by the second author at Princeton University in the year 1970-71, which were subsequently expanded and revised by the first author. Our aim has been to provide a thorough and coherent account of the solution of the $\bar{\partial}$-Neumann problem and certain of its applications and ramifications, and we have taken the opportunity afforded by the monograph format to employ a somewhat more leisurely style than is common in the original journal articles. It is our hope that this book may thereby be accessible to a fairly wide audience and that it may also provide a sort of working introduction to some of the recent techniques in partial differential equations.

In keeping with this philosophy, we have tried to make the book as self-contained as possible. On the geometrical side, we assume the reader is familiar with differentiable manifolds and their native flora and fauna: vector fields, differential forms, partitions of unity, etc. On the analytical side, we assume only an elementary knowledge of functional analysis and Fourier analysis. In the body of the text we also assume an acquaintance with Sobolev spaces and pseudodifferential operators, but we have included an appendix which develops these theories as far as they are needed.

G. B. F.
J. J. K.

JUNE, 1972

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CHAPTER I

FORMULATION OF THE PROBLEM

1. *Introduction*

In the nineteenth century two approaches to the theory of functions of a complex variable were initiated by Weierstrass and Riemann, respectively. The first was to study power series, canonical products, and such, staying within the analytic category; the second was to work in the $\mathcal{C}^\infty$ category, using the differential equations and associated variational problems arising from the situation. The first approach was generalized to functions of several variables by K. Oka, H. Cartan, and others, and it is along these lines that the modern theory of several complex variables has largely developed. The second approach has been used with great success in the case of compact complex manifolds (the Hodge theory, cf. Weil [46]), and more recently these methods have been extended to open manifolds. This extension, however, poses rather delicate analytical problems. In particular, it leads to a non-coercive boundary value problem for the complex Laplacian, the *$\bar\partial$-Neumann problem.* It is our purpose here to present a detailed solution of this problem for domains with smooth boundary satisfying certain pseudoconvexity conditions and to indicate its applications to complex function theory.

By way of introduction, let us consider functions (and, more generally, differential forms) on a bounded domain M in C^n with smooth boundary bM. The Cauchy-Riemann operator $\bar\partial$ defined on functions by $\bar\partial f = \Sigma_1^n \frac{\partial f}{\partial \bar z_i} d\bar z_i$ extends naturally to yield the *Dolbeault complex*

$$0 \longrightarrow \Lambda^{p,0}(M) \xrightarrow{\bar\partial} \Lambda^{p,1}(M) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Lambda^{p,n}(M) \longrightarrow 0$$

where $\Lambda^{p,q}(M)$ is the space of smooth forms of type (p,q) on M. The holomorphic functions are precisely the solutions of the homogeneous equation $\bar{\partial}f = 0$.

The inhomogeneous equation $\bar{\partial}f = \phi$ is also of interest. Consider the following version of the *Levi problem*: given $p \in bM$, is there a holomorphic function on M that blows up at p? In general, the answer is no: for example, if M is the region between two concentric spheres, Hartogs' theorem [18] says that any holomorphic function on M extends holomorphically to the interior of the inner sphere. However, if M is strongly convex at p (meaning that there is a neighborhood U of p such that for any $q \in (M \cup bM) \cap U$, the line segment between p and q lies in M), a classical construction of E. E. Levi guarantees the existence of a neighborhood V of p and a holomorphic function w on $M \cap V$ which blows up only at p. Now suppose the equation $\bar{\partial}f = \phi$ (where ϕ satisfies the compatibility condition $\bar{\partial}\phi = 0$) is always solvable in M in such a way that f is smooth up to bM (i.e., can be extended smoothly across bM) whenever ϕ is. Then we can solve the Levi problem. Indeed, let ψ be a smooth function with support in V which is identically one near p. Then ψw is defined on all of M and is smooth away from p; since $\bar{\partial}(\psi w) = 0$ near p, $\bar{\partial}(\psi w)$ is smooth up to the boundary. Therefore there is a function f, smooth up to the boundary, which satisfies $\bar{\partial}f = \bar{\partial}(\psi w)$. Finally, $f - \psi w$ is holomorphic in M and blows up at p. (We shall discuss this construction in greater detail in §4.2.)

Let us consider the equation $\bar{\partial}f = \phi$ where f and ϕ are supposed square-integrable. If a solution f exists, it is determined only modulo the space $\mathcal{H} = \{g \in L^2(M): \bar{\partial}g = 0\}$. By general Hilbert space theory, $\mathcal{H}^{\perp}$ is the closure of the range of the Hilbert space adjoint $\bar{\partial}^*$ of $\bar{\partial}$. Thus we are led to study the equation

$$\bar{\partial}\bar{\partial}^*\theta = \phi \qquad (\bar{\partial}\phi = 0)\,. \tag{1.1.1}$$

For general ϕ, in analogy with the deRham-Hodge construction for the exterior derivative, the proper equation is

$$(1.1.2) \qquad (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\theta = \phi .$$

(Note that (1.1.2) reduces to (1.1.1) when $\bar{\partial}\phi = 0$, for then

$$\bar{\partial}\bar{\partial}^*\bar{\partial}\theta = 0 \implies (\bar{\partial}\bar{\partial}^*\bar{\partial}\theta, \bar{\partial}\theta) = 0 \implies (\bar{\partial}^*\bar{\partial}\theta, \bar{\partial}^*\bar{\partial}\theta) = 0 \implies \bar{\partial}^*\bar{\partial}\theta = 0 .)$$

The equation (1.1.2) is a boundary value problem in disguise, for $\bar{\partial}^*$ is a differential operator obtained from the equation $(\bar{\partial}^*\phi, \psi) = (\phi, \bar{\partial}\psi)$ by formal integration by parts, and the forms in the domain of $\bar{\partial}^*$ must therefore satisfy conditions which guarantee that the boundary terms in the integration by parts always vanish. (1.1.2) may also be considered as a variational problem, cf. Morrey [34].

Thus, with much hand-waving and little precision, we have set up the *$\bar{\partial}$-Neumann problem*: prove existence and regularity for solutions of (1.1.2). The $\bar{\partial}$-Neumann problem was proposed by D. C. Spencer in the early 1950's as a means of extending Hodge theory to open manifolds and obtaining existence theorems for holomorphic functions; he also pioneered the generalization of this approach in the study of more general overdetermined systems. Related boundary value problems were studied by Garabedian and Spencer [12], Kohn and Spencer [29], and Conner [8] using integral operators, but these methods were not powerful enough to solve the $\bar{\partial}$-Neumann problem. Later Morrey [33] introduced the "basic estimate," and the problem was solved by Kohn [22] by establishing regularity. However, the regularity proof in [22] has since been supplanted by a better proof using the technique of elliptic regularization developed by Kohn and Nirenberg [27]. It is the latter method which we shall employ here; another version of this proof may be found in the book of Morrey [34]. A different approach has been developed by Hormander [16], [18]; we shall discuss his work briefly in §6.1.

We shall now retrace our steps with more care and in greater generality. The natural setting for the $\bar{\partial}$-Neumann problem is the class of compact complex manifolds with boundary. However, we shall go one step further and work with integrable almost-complex manifolds so that we will be in a position to prove the Newlander-Nirenberg theorem to the effect that every integrable almost-complex manifold is in fact complex. This presents no additional complications, and the reader who wishes to envision all our manifolds as complex is free to do so.

2. *Almost-complex manifolds and differential operators*

Let M be a real $\mathcal{C}^\infty$ manifold of dimension m. An *almost-complex structure* on M is a splitting of the complexified tangent bundle $\mathbb{C}TM = TM \otimes_{\mathbf{R}} \mathbf{C}$ by projections $\Pi_{1,0}$ and $\Pi_{0,1}$ such that $\Pi_{1,0} + \Pi_{0,1} = 1$, $\Pi_{1,0}\Pi_{0,1} = \Pi_{0,1}\Pi_{1,0} = 0$, and $\Pi_{0,1} = \overline{\Pi_{1,0}}$. (The last equation means that for $\xi \in \mathbb{C}TM$, $\Pi_{0,1}\xi = \overline{(\Pi_{1,0}\bar\xi)}$ where $\overline{}$ denotes complex conjugation.) We write $T_{1,0}M = \text{Range}(\Pi_{1,0})$ and $T_{0,1}M = \text{Range}(\Pi_{0,1})$; note that $\dim_{\mathbf{C}} T_{1,0}M = \dim_{\mathbf{C}} T_{0,1}M = \frac{m}{2}$, so m must be even; we write $m = 2n$. One can easily verify that an almost-complex structure induces a preferred orientation on M, by restricting the coordinate transformations to those which preserve $\Pi_{1,0}$ and $\Pi_{0,1}$.

The projections $\Pi_{1,0}$ and $\Pi_{0,1}$ naturally induce a splitting of the exterior powers of the complexified cotangent bundle, $\Lambda^k\mathbb{C}T^*M = \bigoplus_{\substack{p+q=k\\ 0\le p,q\le n}} \Lambda^{p,q}\mathbb{C}T^*M$, and we denote the projection $\Lambda^k\mathbb{C}T^*M \to \Lambda^{p,q}\mathbb{C}T^*M$ by $\Pi_{p,q}$. The space of $\mathcal{C}^\infty$ sections of $\Lambda^{p,q}\mathbb{C}T^*M$, i.e., the *forms of type* (p,q) on M, will be denoted by $\Lambda^{p,q}(M)$. We define the operators $\partial\colon \Lambda^{p,q}(M) \to \Lambda^{p+1,q}(M)$ and $\bar\partial\colon \Lambda^{p,q}(M) \to \Lambda^{p,q+1}(M)$ by $\partial\phi = \Pi_{p+1,q}d\phi$, $\bar\partial\phi = \Pi_{p,q+1}d\phi$. Since $d = \bar d$ and $\overline{\Pi_{p,q}} = \Pi_{q,p}$, we have $\bar\partial\phi = \overline{(\partial\bar\phi)}$. It is clear from the corresponding properties of d that ∂ and $\bar\partial$ act locally and satisfy the derivation law:

$$\bar\partial(\phi \wedge \psi) = \bar\partial\phi \wedge \psi + (-1)^{p+q}\phi \wedge \bar\partial\psi \quad \text{for} \quad \phi \in \Lambda^{p,q}(M)\,.$$

The *torsion tensor* of the almost-complex structure is the bilinear map T from complex vector fields to complex vector fields defined by

$$T(X,Y) = \Pi_{1,0}[\Pi_{0,1}X, \Pi_{0,1}Y] + \Pi_{0,1}[\Pi_{1,0}X, \Pi_{1,0}Y]\,.$$

(1.2.1) PROPOSITION. *The following properties are equivalent*:

(1) $\quad$ (1) $T = 0$; $\quad$ (2) $\bar{\partial}^2 = 0$; $\quad$ (3) $\partial^2 = 0$; $\quad$ (4) $d = \partial + \bar{\partial}$.

Proof: That (2) $\Longleftrightarrow$ (3) is obvious. That (4) $\Longrightarrow$ (2) and (3) follows immediately from the equation $d^2 = 0$ and the fact that forms of different type are linearly independent. To show that (1) $\Longleftrightarrow$ (4), we use the identity

$$2d\phi(X, Y) = X\phi(Y) - Y\phi(X) - \phi([X, Y])$$

for one-forms ϕ. If X, Y are sections of $T_{1,0}M$, (1) implies that $[X, Y]$ is also, so $d\phi(X, Y) = 0$ for $\phi \in \Lambda^{0,1}(M)$. Thus $d(\Lambda^{0,1}(M)) \subset \Lambda^{1,1}(M) + \Lambda^{0,2}(M)$, so $d = \partial + \bar{\partial}$ on $\Lambda^{0,1}(M)$. Similarly $d = \partial + \bar{\partial}$ on $\Lambda^{1,0}(M)$. But $d = \partial + \bar{\partial}$ trivially on functions, so by the derivation law and the fact that all forms are locally products of functions and one-forms, $d = \partial + \bar{\partial}$ everywhere. The implication (4) $\Longrightarrow$ (1) follows by reversing this argument. Finally, if (2) holds, from the definition of $\bar{\partial}$ we have for any sections X, Y of $T_{0,1}M$ and any function f,

$$\begin{aligned} 0 &= 2\bar{\partial}^2 f(X, Y) = 2d\bar{\partial} f(X, Y) \\ &= X\bar{\partial} f(Y) - Y\bar{\partial} f(X) - \bar{\partial} f([X, Y]) \\ &= XYf - YXf - (\Pi_{0,1}[X, Y])f, \end{aligned}$$

and hence $[X, Y] = \Pi_{0,1}[X, Y]$. Likewise, by (3), $\Pi_{1,0}[X, Y] = [X, Y]$ for all sections X, Y of $T_{1,0}M$. Thus (1) holds. Q.E.D.

An almost-complex structure satisfying the conditions of Proposition (1.2.1) is called *integrable*. Condition (1) says that the sections of $T_{1,0}M$ and $T_{0,1}M$ form Lie algebras, i.e., they are integrable distributions in the sense of Frobenius. Condition (2) says that the sequence

$$(1.2.2) \qquad 0 \longrightarrow \Lambda^{p,0}(M) \xrightarrow{\bar{\partial}} \Lambda^{p,1}(M) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Lambda^{p,n}(M) \longrightarrow 0$$

is a *complex*. This is the property which will be crucial for our purposes.

If M is actually a complex manifold, that is, it possesses a covering by charts with complex coordinates $\{z_j = x_j + iy_j\}_1^n$ which are