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Lectures on the H-Cobordism Theorem

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LECTURES ON THE h-COBORDISM THEOREM

BY

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NOTES BY

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§0. Introduction

These are notes for lectures of John Milnor that were given as a seminar on differential topology in October and November, 1963 at Princeton University.

Let W be a compact smooth manifold having two boundary components V and V' such that V and V' are both deformation retracts of W. Then W is said to be a h-cobordism between V and V'. The h-cobordism theorem states that if in addition V and (hence) V' are simply connected and of dimension greater than 4, then W is diffeomorphic to $V \times [0, 1]$ and (consequently) V is diffeomorphic to V'. The proof is due to Stephen Smale [6]. This theorem has numerous important applications — including the proof of the generalized Poincaré conjecture in dimensions > 4 — and several of these appear in §9. Our main task, however, is to describe in some detail a proof of the theorem.

Here is a very rough outline of the proof. We begin by constructing a Morse function for W (§2.1), i.e. a smooth function $f : W \longrightarrow [0, 1]$ with $V = f^{-1}(0)$, $V' = f^{-1}(1)$ such that f has finitely many critical points, all nondegenerate and in the interior of W. The proof is inspired by the observation (§3.4) that W is diffeomorphic to $V \times [0, 1]$ if (and only if) W admits a Morse function as above with no critical points. Thus in §§4-8 we show that under the hypothesis of the theorem it is possible to simplify a given Morse function

f until finally all critical points are eliminated. In §4, f is adjusted so that the level $f(p)$ of a critical point p is an increasing function of its index. In §5, geometrical conditions are given under which a pair of critical points p, q of index λ and $\lambda + 1$ can be eliminated or 'cancelled'. In §6, the geometrical conditions of §5 are replaced by more algebraic conditions —— given a hypothesis of simple connectivity. In §8, the result of §5 allows us to eliminate all critical points of index 0 or n, and then to replace the critical points of index 1 and $n - 1$ by equal numbers of critical points of index 3 and $n - 3$, respectively. In §7 it is shown that the critical points of the same index λ can be rearranged among themselves for $2 \leq \lambda \leq n - 2$ (§7.6) in such a way that all critical points can then be cancelled in pairs by repeated application of the result of §6. This completes the proof.

Two acknowledgements are in order. In §5 our argument is inspired by recent ideas of M. Morse [11][32] which involve alteration of a gradient-like vector field for f, rather than by the original proof of Smale which involves his 'handlebodies'. We in fact never explicitly mention handles or handlebodies in these notes. In §6 we have incorporated an improvement appearing in the thesis of Dennis Barden [33], namely the argument on our pages 72-73 for Theorem 6.4 in the case $\lambda = 2$, and the statement of Theorem 6.6 in the case $r = 2$.

The h-cobordism theorem can be generalized in several directions. No one has succeeded in removing the restriction that V and V' have dimension > 4. (See page 113.) If we omit the restriction that V and (hence) V' be simply connected, the theorem becomes false. (See Milnor [34].) But it will remain true if we at the same time assume that the inclusion of V (or V') into W is a simple homotopy equivalence in the sense of J. H. C. Whitehead. This generalization, called the s-cobordism theorem, is due to Mazur [35], Barden [33] and Stallings. For this and further generalizations see especially Wall [36]. Lastly, we remark that analogous h- and s-cobordism theorems hold for piecewise linear manifolds.

Contents

Section 1. The Cobordism Category

First some familiar definitions. Euclidean space will be denoted by $R^n = \{(x_1,\ldots,x_n) | x_i \in R, \ i = 1,\ldots,n\}$ where R = the real numbers, and Euclidean half-space by $R^n_+ = \{(x_1,\ldots,x_n) \in R^n | x_n \geq 0\}$.

Definition 1.1. If V is any subset of R^n, a map $f: V \longrightarrow R^m$ is smooth or differentiable of class C^∞ if f can be extended to a map $g: U \longrightarrow R^m$, where $U \supset V$ is open in R^n, such that the partial derivatives of g of all orders exist and are continuous.

Definition 1.2. A smooth n-manifold is a topological manifold W with a countable basis together with a smoothness structure $\mathcal{S}$ on M. $\mathcal{S}$ is a collection of pairs (U,h) satisfying four conditions:

(1) Each $(U,h) \in \mathcal{S}$ consists of an open set $U \subset W$ (called a coordinate neighborhood) together with a homeomorphism h which maps U onto an open subset of either R^n or R^n_+.

(2) The coordinate neighborhoods in $\mathcal{S}$ cover W.

(3) If (U_1,h_1) and (U_2,h_2) belong to $\mathcal{S}$, then

$$h_1 h_2^{-1}: h_2(U_1 \cap U_2) \longrightarrow R^n \text{ or } R^n_+$$

is smooth.

(4) The collection $\mathcal{S}$ is maximal with respect to property (3); i.e. if any pair (U, h) not in $\mathcal{S}$ is adjoined to $\mathcal{S}$, then property (3) fails.

The <u>boundary of W</u>, denoted $\mathrm{Bd}\, W$, is the set of all points in W which do not have neighborhoods homeomorphic to R^n (see Munkres [5, p.8]).

<u>Definition 1.3</u>. $(W; V_0, V_1)$ is a <u>smooth manifold triad</u> if W is a compact smooth n-manifold and $\mathrm{Bd}\, W$ is the disjoint union of two open and closed submanifolds V_0 and V_1.

If $(W; V_0, V_1)$, $(W'; V_1', V_2')$ are two smooth manifold triads and $h: V_1 \longrightarrow V_1'$ is a diffeomorphism (i.e. a homeomorphism such that h and h^{-1} are smooth), then we can form a third triad $(W \cup_h W'; V_0, V_2')$ where $W \cup_h W$ is the space formed from W and W' by identifying points of V_1 and V_1' under h, according to the following theorem.

<u>Theorem 1.4. There exists a smoothness structure $\mathcal{S}$ for $W \cup_h W'$ compatible with the given structures (i.e. so that each inclusion map $W \longrightarrow W \cup_h W'$, $W' \longrightarrow W \cup_h W'$ is a diffeomorphism onto its image.)</u>

<u>$\mathcal{S}$ is unique up to a diffeomorphism leaving V_0, $h(V_1) = V_1'$, and V_2' fixed.</u>

The proof will be given in § 3 .

<u>Definition 1.5</u>. Given two closed smooth n-manifolds M_0 and M_1 (i.e. M_0 , M_1 compact, $\mathrm{Bd}\, M_0 = \mathrm{Bd}\, M_1 = \emptyset$), a <u>cobordism</u> from M_0 to M_1 is a 5-tuple, $(W; V_0, V_1; h_0, h_1)$, where $(W; V_0, V_1)$ is a smooth manifold triad and $h_i: V_i \longrightarrow M_i$ is a diffeomorphism, $i = 0, 1$. Two cobordisms $(W; V_0, V_1; h_0, h_1)$ and $(W'; V_0', V_1'; h_0', h_1')$ from M_0 to M_1 are <u>equivalent</u> if there exists a diffeomorphism $g: W \longrightarrow W'$ carrying V_0 to V_0' and V_1 to

V_i' such that for $i = 0,1$ the following triangle commutes:

$$\begin{tikzcd} V_i \arrow[rr, "g|V_i"] \arrow[dr, "h_i"'] & & V_i' \arrow[dl, "h_i'"] \\ & M_i & \end{tikzcd}$$

Then we have a category (see Eilenberg and Steenrod, [2,p.108]) whose objects are closed manifolds and whose morphisms are equivalence classes c of cobordisms. This means that cobordisms satisfy the following two conditions. They follow easily from 1.4 and 3.5, respectively.

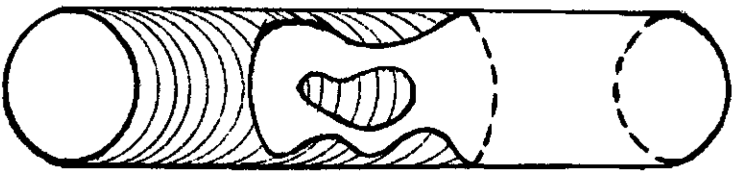
(1) Given cobordism equivalence classes c from M_0 to M_1 and c' from M_1 to M_2, there is a well-defined class cc' from M_0 to M_2. This composition operation is associative.

(2) For every closed manifold M there is the identity cobordism class ι_M = the equivalence class of $(M \times I; M \times 0, M \times 1; p_0, p_1)$, $p_i(x,i) = x$, $x \in M$, $i = 0,1$. That is, if c is a cobordism class from M_1 to M_2, then

$$\iota_{M_1} c = c = c\,\iota_{M_2} .$$

Notice that it is possible that $cc' = \iota_M$, but c is not ι_M. For example

FIGURE 1



c is shaded. c' is unshaded.

Here c has a right inverse c', but no left inverse. Note that the manifolds in a cobordism are not assumed connected.

Consider cobordism classes from M to itself, M fixed. These form a monoid H_M, i.e. a set with an associative composition with an identity. The invertible cobordisms in H_M form a group G_M. We can construct some elements of G_M by taking $M = M'$ below.

Given a diffeomorphism $h: M \longrightarrow M'$, define c_h as the class of $(M \times I; M \times 0, M \times 1; j, h_1)$ where $j(x,0) = x$ and $h_1(x,1) = h(x)$, $x \in M$.

Theorem 1.6. $c_h c_{h'} = c_{h'h}$ for any two diffeomorphisms $h: M \longrightarrow M'$ and $h': M' \longrightarrow M''$.

Proof: Let $W = M \times I \cup_h M' \times I$ and let $j_h: M \times I \longrightarrow W$, $j_{h'}: M' \times I \longrightarrow W$ be the inclusion maps in the definition of $c_h c_{h'}$. Define $g: M \times I \longrightarrow W$ as follows:

$$g(x,t) = j_h(x,2t) \qquad 0 \leq t \leq \tfrac{1}{2}$$

$$g(x,t) = j_{h'}(h(x),2t-1) \qquad \tfrac{1}{2} \leq t \leq 1 .$$

Then g is well-defined and is the required equivalence.

<u>Definition 1.7</u>. Two diffeomorphisms $h_0, h_1: M \longrightarrow M'$ are (smoothly) <u>isotopic</u> if there exists a map $f: M \times I \longrightarrow M'$ such that

(1) f is smooth,

(2) each f_t, defined by $f_t(x) = f(x,t)$, is a diffeomorphism,

(3) $f_0 = h_0$, $f_1 = h_1$.

Two diffeomorphisms $h_0, h_1: M \longrightarrow M'$ are <u>pseudo-isotopic</u>* if there is a diffeomorphism $g: M \times I \longrightarrow M' \times I$ such that $g(x,0) = (h_0(x),0)$, $g(x,1) = (h_1(x),1)$.

<u>Lemma 1.8. Isotopy and pseudo-isotopy are equivalence relations.</u>

<u>Proof:</u> Symmetry and reflexivity are clear. To show transitivity, let $h_0, h_1, h_2: M \longrightarrow M'$ be diffeomorphisms and assume we are given isotopies $f, g: M \times I \longrightarrow M'$ between h_0 and h_1 and between h_1 and h_2 respectively. Let $m: I \longrightarrow I$ be a smooth monotonic function such that $m(t) = 0$ for $0 \leq t \leq 1/3$, and $m(t) = 1$ for $2/3 \leq t \leq 1$. The required isotopy $k: M \times I \longrightarrow M'$ between h_0 and h_1 is now defined by $k(x,t) = f(x,m(2t))$ for $0 \leq t \leq 1/2$, and $k(x,t) = g(x,m(2t-1))$ for $1/2 \leq t \leq 1$. The proof of transitivity for pseudo-isotopies is more difficult and follows from Lemma 6.1 of Munkres [5,p.59].

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* In Munkres' terminology h_0 is "I-cobordant" to h_1. (See [5,p.62].) In Hirsch's terminology h_0 is "concordant" to h_1.

It is clear that if h_0 and h_1 are isotopic then they are pseudo-isotopic, for if $f: M \times \dot{I} \longrightarrow M'$ is the isotopy, then $\hat{f}: M \times I \longrightarrow M' \times I$, defined by $\hat{f}(x,t) = (f_t(x),t)$, is a diffeomorphism, as follows from the inverse function theorem, and hence is a pseudo-isotopy between h_0 and h_1. (The converse for $M = S^n$, $n \geq 8$ is proved by J. Cerf [39].) It follows from this remark and from 1.9 below that if h_0 and h_1 are isotopic, then $c_{h_0} = c_{h_1}$.

<u>Theorem 1.9</u>. $c_{h_0} = c_{h_1} \Longleftrightarrow h_0$ <u>is pseudo-isotopic to</u> h_1

<u>Proof</u>: Let $g: M \times I \longrightarrow M' \times I$ be a pseudo-isotopy between h_0 and h_1. Define $h_0^{-1} \times I: M' \times I \longrightarrow M \times I$ by $(h_0^{-1} \times 1)(x,t) = (h_0^{-1}(x),t)$. Then $(h_0^{-1} \times 1) \circ g$ is an equivalence between c_{h_1} and c_{h_0}.

The converse is similar.