V.V. NEMYTSKII
V.V. STEPANOV

Qualitative Theory of Differential Equations

PRINCETON LEGACY LIBRARY

QUALITATIVE THEORY
OF DIFFERENTIAL EQUATIONS

PRINCETON MATHEMATICAL SERIES

Editors: Marston Morse and A. W. Tucker

1. The Classical Groups, Their Invariants and Representatives. By Hermann Weyl.
2. Topological Groups. By L. Pontrjagin. Translated by Emma Lehmer.
3. An Introduction to Differential Geometry with Use of the Tensor Calculus. By Luther Pfahler Eisenhart.
4. Dimension Theory. By Witold Hurewicz and Henry Wallman.
5. The Analytic Foundations of Celestial Mechanics. By Aurel Wintner.
6. The Laplace Transform. By David Vernon Widder.
7. Integration. By Edward James McShane.
8. Theory of Lie Groups: I. By Claude Chevalley.
9. Mathematical Methods of Statistics. By Harald Cramér.
10. Several Complex Variables. By Salomon Bochner and William Ted Martin.
11. Introduction to Topology. By Solomon Lefschetz.
12. Algebraic Geometry and Topology. Edited by R. H. Fox, D. C. Spencer, and A. W. Tucker.
13. Algebraic Curves. By Robert J. Walker.
14. The Topology of Fibre Bundles. By Norman Steenrod.
15. Foundations of Algebraic Topology. By Samuel Eilenberg and Norman Steenrod.
16. Functionals of Finite Riemann Surfaces. By Menahem Schiffer and Donald C. Spencer.
17. Introduction to Mathematical Logic, Vol. I. By Alonzo Church.
18. Algebraic Geometry. By Solomon Lefschetz.
19. Homological Algebra. By Henri Cartan and Samuel Eilenberg.
20. The Convolution Transform. By I. I. Hirschman and D. V. Widder.
21. Geometric Integration Theory. By Hassler Whitney.
22. Qualitative Theory of Differential Equations. By V. V. Nemickii and V. V. Stepanov. Translated by Solomon Lefschetz.
23. Topological Analysis. By Gordon T. Whyburn.
24. Analytic Functions. By L. Ahlfors, H. Behnke and H. Grauert, L. Bers, et al.
25. Continuous Geometry. By John von Neumann. Edited by Israel Halperin.
26. Riemann Surfaces. By Lars V. Ahlfors and Leo Sario.

QUALITATIVE THEORY OF DIFFERENTIAL EQUATIONS

BY

V. V. NEMYTSKII

AND

V. V. STEPANOV

PRINCETON, NEW JERSEY

PRINCETON UNIVERSITY PRESS

1960

PUBLISHED, 1960, BY PRINCETON UNIVERSITY PRESS

L. C. CARD 60-12240

The translation and editing of this text was supported by the United States Air Force, Office of Scientific Research, of the Air Research and Development Command, under Contract AF 33 (038)–9993. Reproduction, translation, publication, use and disposal, in whole or in part, by or for The United States Government is permitted.

PRINTED IN THE UNITED STATES OF AMERICA

Preface to the English Language Edition

The English language edition of the Nemickii-Stepanov treatise has gone through many vicissitudes. Several years ago when only the first edition was available in this country a complete translation was made by Dr. Thomas Doyle, at the time a member of the faculty of Dartmouth College. This translation was edited by Donald Bushaw and John McCarthy, at the time graduate students at Princeton University. Hardly was this done when there appeared a much enlarged second edition of the book. Dr. Arnold Ross of the University of Notre Dame undertook to prepare an English translation of the first four chapters which he actually had to rewrite for the most part. Undoubtedly American mathematicians are greatly in debt to Dr. Ross for the enormous amount of work which he has done in this connection. The last two chapters, which did not differ too much in the two editions, were finally put in proper shape by Dr. Robert Bass, who utilized in the process translations of the few new sections by Dr. McCarthy and by Dr. Lawrence Markus. It seems fair to say that this edition contains all the material of the second Russian edition of the book.

A couple of years ago there appeared a brief summary written by Nemickii giving a resumé of the recent work done under his guidance by the very active Moscow school. The English language version of this resumé, prepared by Dr. McCarthy, is included at the end of Part One.

The book falls naturally into two parts: Part One on classical differential equations, and Part Two on topological dynamics and ergodic theory. The first part has its own bibliography and index, and the last two chapters, making up the second part, as well as the Appendix, have individual bibliographies.

Readers may be interested in the supplement to Chapter 5, written by Nemickii, which was published by the American Mathematical Society as Translation No. 103 (1954).

In conclusion we wish to say that the work was done under the auspices of the Air Research and Development Command under Contract AF 18(600)-332.

January 1, 1956

Princeton, N. J. S. LEFSCHETZ

TABLE OF CONTENTS

PART TWO

PART ONE

CHAPTER I

Existence and Continuity Theorems

1. Existence Theorems

In the qualitative theory of differential equations one considers systems of differential equations of the form

$$(1.01) \qquad \frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n), \qquad i = 1, 2, \ldots, n,$$

or

$$(1.02) \qquad \frac{dx_i}{dt} = f_i(x_1, \ldots, x_n, t), \qquad i = 1, 2, \ldots, n,$$

where the f_i are assumed to be continuous functions of their arguments in a certain domain G of the Euclidean space $R^n = \{(x_1, \ldots, x_n)\}$ the *phase space,* and in an interval $a < t < b$.

1.11. THEOREM. (*Existence of solutions* [45], [52], [54] [1]). *Consider a system of differential equations* (1.01) *where the functions $f_i(x_1, \ldots, x_n)$ are assumed to be continuous in a certain closed and bounded domain $\bar{G}$. Let $A_0(x_{10}, x_{20}, \ldots, x_{n0})$ be an arbitrary interior point of $\bar{G}$. Then there exists a solution of the system* (1.01), *which passes through A_0 at the time t_0 and which is defined in the interval*

$$t_0 - \frac{D}{M\sqrt{n}} \leqq t \leqq t_0 + \frac{D}{M\sqrt{n}},$$

where D is the distance of A_0 from the boundary of the domain $\bar{G}$ and M is an upper bound of $|f_i(x_1, \ldots, x_n)|$ in the domain $\bar{G}$.

The proof of this theorem follows.

1.12. ε-solutions. We call a system of n functions $\bar{x}_1(t), \ldots, \bar{x}_n(t)$ defined on $a \leqq t \leqq b$ a solution of the system (1.01) up to the error ε or simply an *ε-solution* of (1.01) if each of these functions is

[1]The numbers in square brackets refer to the bibliography at the end of each part of the book.

continuous, sectionally smooth[2], and satisfies the following system of integral equations

$$(1.121)\qquad \bar{x}_i(t) = \bar{x}_{i0} + \int_{t_0}^{t} f_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)\, dt + \int_{t_0}^{t} \theta_i(t)\, dt,$$

where $\theta_i(t)$ are piecewise continuous functions on $[a, b]$, less than ε in absolute value.

1.13. Euler polygons. Consider a point $A_0(x_{10}, \ldots, x_{n0})$ of $\bar{G}$ at distance $D > 0$ from the boundary. Let M be an upper bound of $|f_i(x_1, \ldots, x_n)|$ in the domain $\bar{G}$. In view of the uniform continuity of the functions $f_i(x_1, \ldots, x_n)$ in the domain $\bar{G}$, for every $\varepsilon > 0$ there exists a $\delta \geqq 0$ such that the inequality $|x_i' - x_i''| \leqq \delta$ implies that for all i

$$|f_i(x_1', x_2', \ldots, x_n') - f_i(x_1'', x_2'', \ldots, x_n'')| < \varepsilon \quad (i = 1, 2, \ldots, n).$$

We subdivide our domain $\bar{G}$ into cubes with sides of length δ.

Proceeding in the direction of increasing t we draw a segment along the straight line

$$x_i = x_{i0} + f_i(x_{10}, x_{20}, \ldots, x_{n0})(t - t_0)$$

from A_0 to the intersection $A_1(x_{11}, x_{21}, \ldots, x_{n1}) \neq A_0$, say at time t_1, of this line with one of the faces, say l, of a cube containing A_0. We write

$$x_{i1} = x_{i0} + f_i(x_{10}, x_{20}, \ldots, x_{n0})(t_1 - t_0), \quad t_0 < t_1.$$

Through the point A_1 we draw the line

$$x_i = x_{i1} + f_i(x_{11}, x_{21}, \ldots, x_{n1})(t - t_1),$$

and proceed in the direction of increasing t until we reach the point $A_2(x_{12}, x_{22}, \ldots, x_{n2}) \neq A_1$ of intersection of our line and a face different from l of a cube containing A_1.

This construction yields a polygon (an Euler polygon)

$$x_i = \bar{x}_i(t), \qquad t \geqq t_0$$

where, if $\bar{x}_i'$ is the right derivative of $\bar{x}_i$,

$$(1.131)\qquad \bar{x}_i'(t) = f_i(x_{1j}, x_{2j}, \ldots, x_{nj}) \quad \text{for} \quad t_j \leqq t < t_{j+1}.$$

A similar construction carried out in the direction of decreasing

[2] A function defined in an interval $[a, b]$ is sectionally smooth if it is continuous in this interval and is differentiable at every point of the interval except for at most a finite number of points where it has right and left derivatives. Moreover, we assume that the right and the left derivatives are bounded in the whole interval $[a, b]$.

t yields a polygon

$$x_i = \bar{x}_i(t), \qquad t \leqq t_0$$

with successive vertices $A_1(t_1'),\ A_2(t_2'), \ldots, t_0 > t_1' > t_2' \ldots$. In a manner similar to the above

$$\bar{x}_i'(t) = f_i(x_{1j}, x_{2j}, \ldots, x_{nj}) \quad \text{for} \quad t_j' > t \geqq t_{j+1}'. \tag{1.132}$$

Let us determine how far we may continue the above construction in either direction without leaving the domain $\bar{G}$. Our polygon remains in $\bar{G}$ as long as

$$\left| \int_{t_0}^{t} \sqrt{\Sigma_i(\bar{x}_i'(t))^2}\,dt \right| \leqq D.$$

But by (1.131) and (1.132), we have

$$\left| \int_{t_0}^{t} \sqrt{\Sigma(\bar{x}_i'(t))^2}\,dt \right| \leqq |t - t_0|\, M\sqrt{n}.$$

Thus our construction may be continued as long as

$$|t - t_0| M\sqrt{n} \leqq D,$$

that is as long as

$$t_0 - \frac{D}{M\sqrt{n}} \leqq t \leqq t_0 + \frac{D}{M\sqrt{n}}. \tag{1.133}$$

1.14. Let us show that our polygon is an "ε-solution". By construction each of the functions $\bar{x}_i(t)$ is continuous and sectionally smooth. It remains to verify that these functions satisfy equations (1.121).

The system of integral equations which the functions $\bar{x}_i(t)$ must satisfy is equivalent to the system

$$\bar{x}_i'(t) = f_i(\bar{x}_1(t),\ \bar{x}_2(t), \ldots, \bar{x}_n(t)) + \theta_i(t),$$

where $\bar{x}_i'(t)$ designates, say, the right-hand derivative of $\bar{x}_i(t)$.

Consider a fixed value of t and the corresponding point $B(t) = (\bar{x}_1(t), \ldots, \bar{x}_n(t))$ of our polygon. Then

$$\bar{x}_i(t) = \tilde{x}_i + f_i(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)(t - \tilde{t}),$$

where $\tilde{x}_1, \ldots, \tilde{x}_n$ are the coordinates of the vertex immediately preceding B. Let us denote this vertex by $C(t)$. Thus, for the given value of t, $\bar{x}_i'(t) = f_i(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$.

If we define step functions $\tilde{x}_i(t)$ by the equalities $\tilde{x}_i(t) = \bar{x}_i(\tilde{t}) = \tilde{x}_i$, then

$$\bar{x}_i'(t) = f_i(\tilde{x}_1(t), \tilde{x}_2(t), \ldots, \tilde{x}_n(t)).$$

If we let

$$\theta_i(t) = f_i(\tilde{x}_1(t), \ldots, \tilde{x}_n(t)) - f_i(\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_n(t)),$$

then

$$\bar{x}_i'(t) = f_i(\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_n(t)) + \theta_i(t).$$

Since the points $C(t)$ and $B(t)$ lie in the same cube of our partition, we have

$$|\theta_i(t)| = |f_i(\tilde{x}_1(t), \ldots, \tilde{x}_n(t)) - f_i(\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_n(t))| < \varepsilon.$$

Moreover, since

$$f_i(\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_n(t))$$

are continuous and

$$f_i(\tilde{x}_1(t), \tilde{x}_2(t), \ldots, \tilde{x}_n(t))$$

assume only a finite number of values, the functions $\theta_i(t)$ are piecewise continuous. This completes the proof of our assertion.

The following observation will be useful in the sequel.

1.15. In constructing ε-solutions we may replace Euler polygons by what we shall call "universal polygons". Let the domain $\bar{G}$ be partitioned into cubes with sides of length $\delta/2$. We take a point in each one of these cubes, say the center, and determine the value of the functions $f_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$, at each of these points. Beginning at a point A_0 we construct a polygon by a method similar to that used in the construction of Euler polygons. Here, however, the direction of each of the sides of our polygon is determined by the value of $f_i(x_1, x_2, \ldots, x_n)$ at the previously selected point of the corresponding cube.

1.16. Let us now take a sequence of positive numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \ldots$ tending to zero and proceeding as in 1.13, let us construct consecutively an ε_1-solution, through a point A interior to G, an ε_2-solution through A, and so on. Since the interval (1.131) for which our approximate solutions are defined does not depend on ε, all of these solutions can be constructed for one and the same interval, say,

$$t_0 - \frac{D}{M\sqrt{n}} \leqq t \leqq t_0 + \frac{D}{M\sqrt{n}}.$$

We denote an ε_k-solution by $\{x_i^{\varepsilon_k}(t)\}$. We shall prove that the family of solutions

$$\{x_i^{\varepsilon_k}(t)\}, \qquad k = 1, 2, \ldots,$$

forms an equicontinuous and uniformly bounded family of functions.

Since

$$x_i^{\varepsilon_k}(t) = x_{i0} + \int_{t_0}^{t} f_i(x_1^{\varepsilon_k}, x_2^{\varepsilon_k}, \ldots, x_n^{\varepsilon_k})dt + \int_{t_0}^{t} \theta_i^{\varepsilon_k}(t)\, dt,$$

we have

$$|x_i^{\varepsilon_k}(t)| \leqq L + M\frac{D}{M\sqrt{n}} + \varepsilon_k \frac{D}{M\sqrt{n}},$$

where L is an upper bound of the absolute values of the coordinates of points in $\overline{G}$. Furthermore,

$$x_i^{\varepsilon_k}(t+h) - x_i^{\varepsilon_k}(t) = \int_{t}^{t+h} f_i(x_1^{\varepsilon_k}, x_2^{\varepsilon_k}, \ldots, x_n^{\varepsilon_k})dt + \int_{t}^{t+h} \theta_i^{\varepsilon_k}(t)\, dt,$$

and therefore

$$|x_i^{\varepsilon_k}(t+h) - x_i^{\varepsilon_k}(t)| \leqq hM + h\varepsilon_k.$$

The last two inequalities establish our assertion.

1.17. In view of Arzela's theorem [3] there exists a sequence of indices $n_1, n_2, \ldots, n_k, \ldots$ such that the n sequences $x_i^{\varepsilon_{n_k}}(t)$, $i = 1, 2, \ldots, n$, converge in the interval

$$t_0 - \frac{D}{M\sqrt{n}} \leqq t \leqq t_0 + \frac{D}{M\sqrt{n}}$$

to continuous functions

$$x_1(t), \ldots, x_n(t).$$

Passing to the limit in the equalities

$$x_i^{\varepsilon_{n_k}}(t) = x_{i0} + \int_{t_0}^{t} f_i(x_1^{\varepsilon_{n_k}}, x_2^{\varepsilon_{n_k}}, \ldots, x_n^{\varepsilon_{n_k}})\, dt + \int_{t_0}^{t} \theta_i^{\varepsilon_{n_k}}(t)\, dt,$$

and observing that the $f_i(x_1, x_2, \ldots, x_n)$ are uniformly continuous in $\overline{G}$ and that

$$|\theta_i^{\varepsilon_{n_k}}(t)| < \varepsilon_{n_k}, \quad \text{for every } \varepsilon_{n_k},$$

[3]This theorem states that every infinite family of functions uniformly bounded and equicontinuous on a closed interval [a, b] contains a uniformly convergent sequence of functions. Cf. *Memorie Acad. Bologna* (5) vs. 5 (1895) and 8 (1899).

we obtain

$$x_i(t) = x_{i0} + \int_{t_0}^{t} f_i(x_1(t), \ldots, x_n(t))\, dt,$$

or

$$\frac{dx_i(t)}{dt} = f_i(x_1(t), \ldots, x_n(t)).$$

This completes the proof of Theorem 1.11.

1.2. We shall extend our existence theorem to systems of type (1.02).

Let f_i be defined and continuous for points $A(x_1, x_2, \ldots, x_n)$ of a closed and bounded domain $\overline{G}$ and for values of t in an interval $[t_0 - b, t_0 + b]$. We introduce a new independent variable τ such that $dt/d\tau = 1$. Then the given system (1.02) may be written in the form

$$\text{(1.201)} \qquad \begin{cases} \dfrac{dx_i}{d\tau} = f_i(x_1, x_2, \ldots, x_n, t), \\ \dfrac{dt}{d\tau} = 1. \end{cases}$$

Applying our existence theorem to the closed and bounded domain of the $(n+1)$-dimensional space $(x_1, \ldots, x_n, t)$ determined by $\overline{G}$ and the interval $t_0 - b \leqq t \leqq t_0 + b$, we assure the existence of a solution of system (1.02) in the interval $(t_0 - h, t_0 + h)$, where

$$h = \frac{\min(D, b)}{1 + M\sqrt{n}}.$$

We shall speak of (1.201) as the *parametric system* corresponding to the system (1.02).

1.21 As a simple corollary of our existence theorem, we obtain the following result which is very important for the theory of dynamical systems.

1.21. THEOREM. *If as time increases, a given trajectory (an integral curve) remains in a closed bounded region Γ imbedded in an open domain G for which the conditions of our existence theorem are fulfilled, then the motion (the solution) may be continued for the whole infinite interval $[t_0, +\infty]$.*

Let $2D$ be the distance of the boundary of G from the boundary of Γ. Then successive applications of our existence theorem always lead to points whose distance from the boundary of G is not less than D. Consequently, at each step we can continue our solution for another interval of at least the length $D/M\sqrt{n}$.

1.3. Theorem 1.21 does not allow us to decide from the form of a given system of equations whether or not its solutions can be continued for the infinite interval $-\infty < t < +\infty$. We indicate several sufficient conditions for such continuation. [58], [59].

1.31. THEOREM. *If the functions*

$$f_1(x_1, x_2, \ldots, x_n), \ldots, f_n(x_1, x_2, \ldots, x_n)$$

are continuous for $-\infty < x_i < +\infty$, *and, moreover, if*

$$f_i(x_1, x_2, \ldots, x_n) = O(|x_1| + |x_2| + \ldots + |x_n|)$$

for $|x_1| + \ldots + |x_n| \to +\infty$, *then the solutions of the system*

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n)$$

are defined on the whole axis $-\infty < t < +\infty$.

1.32. It follows from the hypotheses of our theorem that

$$(1.321) \qquad |f_i(x_1, x_2, \ldots, x_n)| < A \max\,(|x_1|, \ldots, |x_n|, 1),$$

where A is some positive constant. For, if $|x_1| + \ldots + |x_n| > D > 0$, where D is some sufficiently large number, then the ratios

$$\frac{|f_i(x_1, x_2, \ldots, x_n)|}{\Sigma\,|x_j|}$$

remain bounded, whereas the functions $f_i(x_1, \ldots, x_n)$ themselves are bounded in the region $|x_1| + \ldots + |x_n| \leqq D$.

1.33. Let us consider first the cube $|x_i - x_{i0}| \leqq b$ $(i = 1, 2, \ldots, n)$, and let M be an upper bound of $|f_i(x_1, x_2, \ldots, x_n)|$ in this cube. According to the existence theorem, the solution passing through A_0 is defined in the whole interval $[t_0, t_0 + (b/M\sqrt{n})]$.

Set x_{i0}, t_0, and b equal to c_i, 0, and 1 respectively. Then it follows from the inequality (1.321) and the condition $|x_i(t) - c_i| \leqq 1$ that we may take $M = A(c+1) = A \max\,[c+1, 1]$ with $c = \max |c_i|$ $(i = 1, 2, \ldots, n)$. Write

$$t_1 = \frac{b}{M\sqrt{n}} = \frac{1}{M\sqrt{n}} = \frac{1}{A(c+1)\sqrt{n}}.$$

Then our solution is defined for $0 \leqq t \leqq t_1$, and in this interval

$$|x_i(t)| \leqq c + 1.$$

Next, let us take x_{i0}, t_0, b equal to $x_1(t_1)$, t_1, 1 respectively. Then we may take $M = A \max(c+2, 1) = A(c+2)$. We write

$$t_2 = \frac{b}{M\sqrt{n}} = \frac{1}{M\sqrt{n}} = \frac{1}{A(c+2)\sqrt{n}},$$

and observe that the solution is defined in $t_1 \leqq t \leqq t_1 + t_2 = \tau_2$.

Combining both of the above results we see that our solution is defined in the interval $[0, \tau_2]$. The inequality $|x_i(t) - x_i(t_1)| \leqq 1$ for $t_1 \leqq t \leqq \tau_2 = t_1 + t_2$, implies that $|x_i(\tau_2)| \leqq c + 2$. Continuing this process for m steps we obtain a number $t_m = 1/(c+m)A\sqrt{n}$ such that our solution is defined in the interval $[0, \tau_m]$ where $\tau_m = t_1 + t_2 + \ldots + t_m$ and $|x_i(\tau_m)| \leqq c + m$. The series

$$\frac{1}{A\sqrt{n}} \sum_{m=0}^{\infty} \frac{1}{c+m+1}$$

diverges. Therefore by means of a sufficiently large number of steps we can continue our solution for an interval of arbitrarily large length.[4]

1.34. Corollary. *If*

$$f_i(x_1, x_2, \ldots, x_n, t) = O(|x_1| + |x_2| + \ldots + |x_n|)$$

uniformly in t, then solutions of the system $dx_i/dt = f_i$ may be continued to the whole t-axis.

Indeed, let us consider the corresponding parametric system

$$\frac{dx_i}{d\tau} = f_i(x_1, x_2, \ldots, x_n, t),$$

$$\frac{dt}{d\tau} = 1.$$

[4]From the estimates given in the proof it follows that

$$|x_i(t)| = O(e^{ct})$$

where the constant c may be chosen independently of the initial conditions. In fact, after the mth step in the process of continuation we have

$$|x_i(t)| \leqq c + m,$$

for

$$t = t_1 + t_2 + \ldots + t_m = \frac{1}{A\sqrt{n}} \sum_{j=0}^{m-1} \frac{1}{c+j+1}.$$

Thus t is asymptotically equal to $A^{-1}n^{-1/2} \log m$ or m is asymptotically equal to $e^{An^{\frac{1}{2}}t}$. This proves our assertion since A is chosen independently of the initial conditions.

Since

$$|f_i(x_1, x_2, \ldots, x_n, t)| < A \max(|x_1|, \ldots, |x_n|, 1)$$

where A is independent of t, then obviously

$$|f_i(x_1, x_2, \ldots, x_n, t)| < A \max(|x_1|, \ldots, |x_n|, |t|, 1).$$

Consequently the conditions of Theorem 1.31 are fulfilled by the parametric system.

1.35. The last result may be somewhat generalized.

If functions $f_1, f_2, \ldots, f_n$ are continuous in an $(n+1)$-dimensional domain $0 < t < +\infty$ and $-\infty < x_i < +\infty$, if there exists a function $L(r)$ continuous for $0 < r < +\infty$ and such that $\int_0^\infty (1/L(r))\,dr = \infty$, and if $|f_i(x_1, \ldots, x_n, t)| < L(r)$, where $r^2 = x_1^2 + \ldots + x_n^2$, then all the solutions of the system $dx_i/dt = f_i$ may be continued over the entire t-axis.

We omit the proof of this theorem even though it is quite simple and refer the reader to the original work of Wintner [58].

2. Certain Uniqueness and Continuity Theorems

In what follows we shall consider systems of equations (1.01) in which the functions $f_i(x_1, \ldots, x_n)$ satisfy Lipschitz conditions in a bounded closed domain $\overline{G}$ called the Lipschitz domain. That is

$$|f_i(x_1', x_2', \ldots, x_n') - f_i(x_1'', x_2'', \ldots, x_n'')| < L \sum_{i=1}^{n} |x_i' - x_i''|.$$

The number L is called a Lipschitz constant. To indicate explicitly the connection between the domain $\overline{G}$ and the constant L we shall write $\overline{G}_L$ instead of $\overline{G}$.

We establish first the following simple lemma [5] which is quite essential for what follows.

2.11 Lemma. *If a function $y(t)$ satisfies the inequality*

$$|y(t)| < M\left(1 + k\int_{t_0}^{t} |y(t)|\,|f(t)|\,dt\right) \tag{2.111}$$

where $f(t)$ is continuous, then we have the inequality

$$|y(t)| < Me^{kM\int_{t_0}^{t}|f(t)|dt} \quad (t > t_0). \tag{2.112}$$

Multiplying (2.111) by $|f(t)|$, we get

$$|y(t)|\,|f(t)| < M\,|f(t)|\left(1 + k\int_{t_0}^{t} |y(t)|\,|f(t)|\,dt\right). \tag{2.113}$$

Let $v(t) = \int_{t_0}^{t} |y(t)f(t)|\, dt$. Then the inequality (2.113) may be written in the form

$$v'(t) < M\,|f(t)|(1 + kv),$$

or

$$\frac{v'(t)}{1 + kv} < M\,|f(t)|.$$

Thus

$$\log\,(1 + kv(t)) < kM \int_{t_0}^{t} |f(t)|\, dt,$$

and hence

$$1 + k\int_{t_0}^{t} |f(t)y(t)|\, dt < e^{kM\int_{t_0}^{t}|f(t)|dt}.$$

By hypothesis

$$\frac{|y(t)|}{M} < 1 + k\int_{t_0}^{t} |f(t)y(t)|\, dt,$$

whence

$$|y(t)| < Me^{kM\int_{t_0}^{t}|f(t)|dt}.$$

2.12. We shall use our lemma to establish a fundamental inequality.

Consider two ε-solutions

$$\{x_i^{(1)}(t)\}\,, \qquad \{x_i^{(2)}(t)\}\,.$$

In view of (1.121),

$$\begin{aligned} x_i^{(1)}(t) &- x_i^{(2)}(t) \\ &= (x_{i0}^{(1)} - x_{i0}^{(2)}) + \int_{t_0}^{t} [f_i(x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}) - f_i(x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)})]dt \\ &\quad + \int_{t_0}^{t} [\theta_i^{(1)}(t) - \theta_i^{(2)}(t)]\, dt. \end{aligned}$$

Making use of Lipschitz inequalities, we get

$$\begin{aligned} |x_i^{(1)}(t) - x_i^{(2)}(t)| &< |x_{i0}^{(1)} - x_{i0}^{(2)}| \\ &\quad + \int_{t_0}^{t} L \cdot \sum_{j=1}^{n} |x_j^{(1)} - x_j^{(2)}|\, dt + \int_{t_0}^{t} |\theta_i^{(1)}(t) - \theta_i^{(2)}(t)|\, dt \end{aligned}$$

for $i = 1, 2, \ldots, n$, and $t > t_0$. Adding these inequalities and writing

$$\delta = \max\, |x_{i0}^{(1)} - x_{i0}^{(2)}| \qquad (i = 1, 2, \ldots, n),$$

and remembering that

$$|\theta_i^{(1)}(t)| \leqq \varepsilon \quad \text{and} \quad |\theta_i^{(2)}(t)| \leqq \varepsilon,$$

we obtain the inequality

$$\sum_{i=1}^{n} |x_i^{(1)}(t) - x_i^{(2)}(t)| < n\delta + n\int_{t_0}^{t} L \sum_{i=1}^{n} |x_i^{(1)}(t) - x_i^{(2)}(t)|\, dt$$
$$+ 2n\varepsilon(t - t_0) \leqq (2n\varepsilon(T - t_0) + n\delta)\left[1 + \frac{n}{2n\varepsilon(T - t_0) + n\delta}\right.$$
$$\left.\cdot \int_{t_0}^{t} L \sum_{i=1}^{n} |x_i^{(1)}(t) - x_i^{(2)}(t)|\, dt\right], \quad t_0 < t \leqq T.$$

Applying Lemma 2.11, we obtain

$$\sum_{i=1}^{n} |x_i^{(1)}(t) - x_i^{(2)}(t)| < [2n\varepsilon(T - t_0) + n\delta]\, e^{n\int_{t_0}^{t} L\,dt}$$

for $t_0 < t \leqq T$. If $t_0 > t$ we assume that $t_0 > t \geqq T$ and invert the order of integration throughout. Setting $t = T$ and simplifying, we obtain, in either case,

$$(2.124) \quad \sum_{i=1}^{n} |x_i^{(1)}(t) - x_i^{(2)}(t)| < 2n|t - t_0|\,\varepsilon e^{nL|t-t_0|} + n\delta e^{nL|t-t_0|}.$$

In what follows we shall refer to this estimate as *the fundamental inequality*.

2.2. We shall discuss next a number of immediate consequences of the fundamental inequality, all of which are of basic importance in the theory of differential equations.

2.21. Theorem (Uniqueness). *If the right-hand members of system* (1.01) *satisfy Lipschitz conditions, then there exists a unique solution satisfying given initial conditions.*

Let $\{x_i^{(1)}(t)\}$, $\{x_i^{(2)}(t)\}$ be two solutions defined on a segment $[t_0, t_1]$ and satisfying the same initial conditions at t_0 (or at t_1). We may consider these solutions as ε-solutions for an arbitrarily small ε. Applying the fundamental inequality and observing that $\delta = 0$, we obtain

$$\sum_{i=1}^{n} |x_i^{(1)}(t) - x_i^{(2)}(t)| < 2n(t_1 - t_0)\,\varepsilon e^{nL|t_1-t_0|}.$$

Since ε is arbitrarily small, we have

$$\sum_{i=1}^{n} |x_i^{(1)}(t) - x_i^{(2)}(t)| = 0 \quad \text{for} \quad t_0 \leqq t \leqq t_1,$$

which proves our assertion.

2.22. THEOREM (*continuity in the initial conditions*). *Let the right-hand members of* (1.01) *satisfy Lipschitz conditions in a domain* G_L. *If a solution* $\{x_i\} = \{x_i(t, t_0, x_{i0}, \ldots, x_{n0})\} = x(t)$ *is defined for* $t_0 \leqq t \leqq T$, *then for every* $\eta > 0$ *there is a* $\delta > 0$ *such that for* $|\bar{x}_{i0} - x_{i0}| < \delta$ $(i = 1, 2, \ldots, n)$ *the solution* $x_i = x_i(t, t_0, \bar{x}_{i0}, \ldots, \bar{x}_{n0}) = \bar{x}_i(t)$ *is also defined for* $t_0 \leqq t \leqq T$ *and for all values of* t *in this interval* $|\bar{x}_i(t) - x_i(t)| < \eta$.

The fundamental inequality (2.124) with $\varepsilon = 0$ yields

$$\sum_{i=1}^{n} |\bar{x}_i(t) - x_i(t)| < n\delta e^{nL(t-t_0)} \tag{2.221}$$

for every value of t in $t_0 \leqq t \leqq T$ for which $\bar{x}_i(t)$ is defined. For some $d > 0$, the d-neighborhood of the segment $C : x_i(t)$, $t_0 \leqq t \leqq T$, lies in the interior of G_L. If $\eta < d$, we let

$$\delta \leqq \frac{\eta}{n e^{nL(T-t_0)}}. \tag{2.222}$$

If we take the (closed) d-neighborhood of C as the domain $\bar{G}$ of Theorem 1.11, then $D \geqq d - \eta > 0$, we see at once that the solution $\bar{x}(t)$ through $(\bar{x}_{i0}, \ldots, \bar{x}_{n0})$ can be extended at least as far as T in view of the inequality (2.221) and the choice of δ. Also, throughout the interval $t_0 \leqq t \leqq T$, we have

$$|\bar{x}_i(t) - x_i(t)| < \eta.$$

2.3. One should note that the choice of δ depends not only upon the degree of the desired approximation, that is upon η, but also upon the length $(T - t_0)$ of our time interval. In many problems of mechanics, it is essential to seek solutions in which δ can be chosen independently of the length of the time interval. Such motions possess a certain degree of stability with respect to the change in the initial conditions. Detailed study of such motions and of the methods of their characterization was carried out by the inspired Russian scientist Liapounoff. We shall meet these ideas and methods in the subsequent chapters.

2.4. Stability of solutions with respect to changes in the right-hand members of our system. Let a system (1.01) be replaced by a system

$$\frac{d\bar{x}_i}{dt} = f_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) + \theta_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n), \tag{2.411}$$

and let $|\theta_i| \leqq \varepsilon$ for all values of $\bar{x}_i$ in a closed domain $\overline{G}_L$. Then every solution $\bar{x}_i(t)$ of systems (2.411) is obviously an ε-solution of system (1.01). If $x_i(t)$ is a solution of system (1.01) satisfying the same initial condition as a solution $\bar{x}_i(t)$ of system (2.411), then, in view of the fundamental inequality, we obtain

$$|\bar{x}_i(t) - x_i(t)| \leqq 2n\,|t - t_0|\,\varepsilon e^{nL|t-t_0|}. \tag{2.412}$$

It follows from this estimate that for a fixed interval of time we may make the difference of the above solutions arbitrarily small by choosing ε sufficiently small.

2.42. Frequent use is made of the process of linearization, i.e., of a replacement of a given nonlinear system by a linear system. In particular, such a method is considered permissible if the nonlinear terms have small parameters. The above inequality (2.412) makes it possible to obtain a numerical estimate of the error resulting from linearization.

2.5. A method of approximate integration [35]. In deriving the fundamental inequality we required that the functions $\theta_i(t)$ should be piecewise continuous.

Observing this, one may develop the following method of approximate integration of (1.01).

2.51. For a given $\varepsilon > 0$ we partition the domain G_L into cubes of side δ, where δ is so small that the inequalities $|x_i' - x_i''| \leqq \delta (i = 1, \ldots, n)$ imply

$$|f_i(x_1', x_2', \ldots, x_n') - f_i(x_1'', x_2'', \ldots, x_n'')| \leqq \frac{\varepsilon}{2}.$$

We construct new functions $\bar{f}_i(x_1, x_2, \ldots, x_n)$ which assume throughout each cube the values of the corresponding functions $f_i(x_1, \ldots, x_n)$ at the center of the cube. Obviously,

$$|f_i(x_1, x_2, \ldots, x_n) - \bar{f}_i(x_1, x_2, \ldots, x_n)| \leqq \varepsilon.$$

On the boundaries of the cubes we allow each $\bar{f}_i$ to be manyvalued.

Let us consider next the system of equations

$$\frac{d\bar{x}_i}{dt} = \bar{f}_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n). \tag{2.511}$$

Within each cube the solutions of (2.511) form a family of parallel straight line segments whose direction is determined by the values of $f_i(x_1, \ldots, x_n)$ at the center of this cube.

By a solution $\bar{x}_i(t)$ of the system (2.511) we shall mean a polygon constructed as follows: Given a point A_0 we choose one (there may be more than one) of the above segments A_0A_1, say, $A_{-1}A_0A_1$ passing through A_0. If A_1 is the initial point of a segment solution A_1A_2 of (2.511), we choose A_1A_2 as the second link, and so on until we exhaust that interval of time for which we seek an approximation to a solution of (1.01).

2.52. A solution of (2.511) is an ε-solution of the system (1.01). For, if $\bar{x}_j = \bar{x}_j(t)$ $(j = 1, \ldots, n)$ is a solution of (2.511), this system may be written

$$\frac{d\bar{x}_i}{dt} = f_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) + [\bar{f}_i(\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_n(t)) - f_i(\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_n(t))]$$

where the difference in the brackets is numerically smaller than ε in the domain $\overline{G}_L$ and is piecewise continuous in t. For $\bar{x}_j(t)$, as well as $f_i(x_1, x_2, \ldots, x_n)$ are continuous, and $\bar{f}_i(x_1, x_2, \ldots, x_n)$ assumes only a finite number of values.

2.53. We observe that in constructing an approximate solution we need not start our polygon at the given initial point of the desired solution.

Let us construct polygon solutions of (2.511) starting at the center of each cube of our partition. Let $A_1, \ldots, A_s$ be the family of all such solutions. Then, for every solution $\{x_i(t)\}$ of (1.01) defined for a time interval T, there exists a polygon $A_j = \{\bar{x}_i^{(j)}(t)\}$ such that

$$|x_i(t) - \bar{x}_i^{(j)}(t)| \leqq 2n\varepsilon Te^{nLT} + n\delta e^{nLT}. \tag{2.531}$$

Since we may assume that $\delta \leqq \varepsilon$, the inequality (2.531) yields

$$|x_i(t) - \bar{x}_i^{(j)}(t)| \leqq 2n\varepsilon(T + \tfrac{1}{2})e^{nLT}.$$

Thus, for a fixed T, the error may be made arbitrarily small by choosing ε sufficiently small.

2.6. Toroidal and cylindrical phase spaces. We shall conclude this section with a few remarks regarding the generality of the theorems considered above.

In all of our proofs we considered an n-tuple $(x_1, x_2, \ldots, x_n)$ as a point in an n-dimensional Euclidean space. This assumption was not necessary. We may assume that our solution space is a manifold every point of which has a neighborhood homeomorphic to an n-dimensional sphere of an n-dimensional Euclidean space R^n. In particular it can be an arbitrary domain in an n-dimensional Euclidean space. In case the space is only locally Euclidean, then the estimates of the interval of existence of solutions must obviously be changed.

A special role is played by systems of differential equations (1.01) in which the right-hand members are defined for all values of the variables $x_1, x_2, \ldots, x_n$ but in which certain of these variables are cyclic, i.e., they take values only in a finite interval of length γ_i. The domains of definition of these variables may be extended to the whole infinite line. Here we shall identify points whose ith coordinates differ by γ_i.

Consider for example a system of two equations

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y).$$

If (x, y) are plane coordinates then the solution space is a plane. If x varies from $-\infty$ to $+\infty$ but y is a cyclic coordinate, then the solution space is a cylinder. If both coordinates are cyclic, then the space is a torus. The theorem on unlimited continuation of solutions applies to the cylindrical as well as to the toroidal solution space.

3. Dynamical systems defined by a system of differential equations.

We shall give here only a few basic definitions and elementary results pertaining to dynamical systems.

3.1. First, we study an important property of systems of differential equations satisfying the uniqueness conditions of (2.21).

3.11. To indicate the dependence of solutions upon the initial conditions explicitly, we write

$$x_i = x_i(t, t_0, x_1^{(0)}, \ldots, x_n^{(0)}) \tag{3.111}$$

for that solution of (1.01) which passes through the point $x_i^{(0)}$ when $t = t_0$. If $t_0 = 0$, then we abbreviate (3.111) by writing

$$x_i = x_i(t, x_1^{(0)}, \ldots, x_n^{(0)}). \tag{3.112}$$

Next, we consider

$$x_i = x_i(t - t_0, x_1^{(0)}, \ldots, x_n^{(0)}). \tag{3.113}$$

Since the right-hand members of (1.01) do not contain t explicitly, (3.113) is a solution of (1.01). Moreover, we observe that it is the solution which passes through $x_i^{(0)}$ for $t = t_0$.

In particular, we have the important relation

$$\begin{aligned} x_i(t_2, x_1(t_1, x_1^{(0)}, \ldots, x_n^{(0)}), \ldots, x_n(t_1, x_1^{(0)}, \ldots, x_n^{(0)})) \\ = x_i(t_1 + t_2, x_1^{(0)}, \ldots, x_n^{(0)}). \end{aligned} \tag{3.114}$$

For both the right-hand member and the left-hand member of (3.114), considered as functions of t_2, represent solutions passing through the same point $x_i(t_1, x_1^{(0)}, \ldots, x_n^{(0)})$ for $t_2 = 0$.

3.12. Let us denote the solution (3.112) passing through the point $p(x_1^{(0)}, \ldots, x_n^{(0)})$ by the symbol $f(p, t)$. Thus for every t, $f(p, t) = q$ is a definite point on the trajectory through p and in particular $f(p, 0) = p$. Moreover, if for every p in $\overline{G}_L$ the function $f(p, t)$ is defined for $t \in T = (-\infty, +\infty)$ then

$$\begin{cases} f(p, t) \text{ is continuous in both of its arguments in} \\ \overline{G}_L \times T, \end{cases} \tag{3.121}$$

and in view of (3.114),

$$f(p, t_1 + t) = f(f(p, t_1), t). \tag{3.122}$$

Thus $f(p, t)$ defines a one-parameter group of transformations of the solution space $\overline{G}_L$ into itself. It is customary to speak of the set of all the transformations of this group as a *dynamical system* and of the totality of all the points $f(p, t)$ for a fixed p and $-\infty < t < +\infty$ as a *trajectory* of this dynamical system.

3.2. In general, even if the $f_i(x_1, \ldots, x_n)$ satisfy Lipschitz conditions or other conditions assuring uniqueness of solutions in a domain G of an n-dimensional Euclidean space or of a locally Euclidean manifold, the corresponding system (1.01) does not necessarily define a dynamical system, since it may have solutions which cannot be continued for all values of t. Some sufficient conditions for unlimited continuation were given in Sections **1.2** and **1.3.**

We shall show, however, that by merely changing the independent variable, i.e., by changing the parametrization of the integral curves of the given system, we can arrive at a system whose solutions do determine a dynamical system.

In other words, if we are interested only in the geometrical or, more precisely, topological properties of individual integral curves or of the whole family of integral curves, then we may limit ourselves to the study of differential equations which define dynamical systems.

3.21. DEFINITION. *Two systems* (1.01) *are called equivalent if their solutions (including the singular points) coincide geometrically. A system* (1.01) *will be called a D-system if its solutions define a dynamical system.*

A point $p(x_{10}, \ldots, x_{n0})$ is called a *singular point* of (1.01) if $f_i(x_{10}, \ldots, x_{n0}) = 0$ simultaneously for all the right-hand members of f_i of a system (1.01).

3.22. THEOREM (R. E. Vinograd) [55]. *Consider a system* (1.01) *satisfying Lipschitz conditions in an open domain $G_L \subset R^n$. There exists a D-system defined over the whole R^n and equivalent to* (1.01) *in G_L.*

3.23. Let us prove first that every system (1.01) may be replaced by an equivalent system with bounded right-hand members.

We define $\varphi_i(x)$ so that [5]

$$\varphi_i(x) = 1 \qquad \text{if} \quad |f_i(x)| \leqq 1,$$

$$\varphi_i(x) = \frac{1}{f_i(x)} \quad \text{if} \quad f_i(x) > 1,$$

$$\varphi_i(x) = \frac{-1}{f_i(x)} \quad \text{if} \quad f_i(x) < -1,$$

and we write $\varphi(x) = \Pi_{i=1}^n \varphi_i(x)$. Obviously $0 < \varphi_i(x) \leqq 1$, $|f_i(x)\varphi_i(x)| \leqq 1$, and $\varphi_i(x)$ are continuous. Therefore $0 < \varphi(x) \leqq 1$, $|f_i(x)\varphi(x)| \leqq 1$ and $\varphi(x)$ is continuous.

The system

$$\frac{dx_i}{dt} = f_i(x)\varphi(x)$$

is equivalent to the given system in G_L and its right-hand members are bounded.

3.24. We may assume therefore that system (1.01) has bounded right-hand members.

[5]Here x is an abbreviation for $(x_1, x_2, \ldots, x_n)$.

We observe that in this case we have

$$\int_0^{t'} v(x)\,dt \leqq |t'|c,$$

where

$$v(x) = \sqrt{\sum_{i=1}^{n} f_i^2(x)}, \qquad c = M\sqrt{n}.$$

Thus,

3.241. *For a finite t' the length of the trajectory*

$$x(t, x^{(0)}), \qquad 0 \leqq t \leqq t',$$

is finite.

3.242. Now let $x(t, x^{(0)})$ be a solution of (1.01) which cannot be continued beyond $t = t_1$. The trajectory defined by this solution must have a limit point $x^{(1)}$ on the boundary B of G_L for otherwise, in view of Theorem 1.21, the solution could have been continued indefinitely.

By 3.241, the above limit point $x^{(1)}$ on the boundary is unique and it is approached along the trajectory as $t \to t_1$.

3.243. Write $F = R^n - G_L$ and let

$$\psi(x) = \frac{\varrho(x, F)}{\varrho(x, F) + \varrho(x, x^0) + 1}$$

where $\varrho(x, y)$ is the distance between x and y, $\varrho(x, F) = \min_{y \in F} \varrho(x, y)$, and $x^{(0)}$ is a fixed point of G_L.

The function $\psi(x)$ is continuous everywhere in R^n, $0 \leqq \psi(x) < 1$, and $\psi(x) = 0$ in F and nowhere else.

Let us consider the system

$$\frac{dx_i}{dt} = f_i(x)\psi(x). \tag{3.243}$$

It is equivalent to the given system and has bounded right-hand members in G_L. To prove that (3.243) determines a dynamical system, it suffices, in view of (3.242), to show that we can extend indefinitely solutions corresponding to half-trajectories of finite length s_0 terminating at a point $x^{(1)} \in B \subset F$.

We observe that

$$t = \int_0^s \frac{ds}{v(x)\psi(x)},$$

where we may for definiteness assume $s > 0$. Next, for every point $x = x(s)$ on our trajectory

$$\psi(x) = \frac{\varrho(x, F)}{\varrho(x, F) + \varrho(x, x^{(0)}) + 1} < \varrho(x, F) = \min_{y \in F} \varrho(x, y)$$
$$\leqq \varrho(x, x^{(1)}) \leqq s_0 - s.$$

Since $0 < v(x) \leqq c$, then

$$t \geqq \frac{1}{c} \int_0^s \frac{ds}{s_0 - s} = -\frac{1}{c} \log \frac{s_0 - s}{s_0}.$$

Thus $t \to \infty$ as $s \to s_0$.

3.244. We extend the domain of definition of the right-hand members of (3.243) by setting them equal to zero in F. Since $f_i(x)$ are bounded in G_L and $\psi(x)$ is continuous and vanishes on the boundary B, the extended system has continuous right-hand members. The new system is a D-system for which all points of F are singular points. This completes the proof of Theorem 3.22. The above reasoning also yields the following:

3.245. THEOREM. *Given a system* (1.01) *and a closed set $\Phi \subset G$, there exists a D-system which is equivalent to the given system on $G - \Phi$ and has all the points of Φ as equilibrium points.*

3.25. We shall return now to the study of the properties of dynamical systems.

3.251. DEFINITION. A point q is called an ω-limit point of a trajectory $f(p, t)$ if there exists a sequence $t_1, t_2, \ldots, t_n \to +\infty$ such that $\lim \varrho(f(p, t_n), q) = 0$. A point q is called an α-limit point of a trajectory $f(p, t)$ if there exists a sequence $t_1, t_2, \ldots, t_n, \ldots, \to -\infty$ such that $\lim \varrho(f(p, t_n), q) = 0$. The set of all ω-limit points of a given trajectory we shall call its ω-limit set, and we shall denote this set by Ω_p. Similarly, the α-limit set A_p of a given trajectory is the set of all its α-limit points. Both Ω_p and A_p are closed sets.

3.252. THEOREM. *If q is either an ω- or an α-limit point of a trajectory $f(p, t)$, then all other points of the trajectory $f(q, t)$ are also ω- or α-limit points respectively of the given trajectory $f(p, t)$.*

Let $r = f(q, \bar{t})$ be a point on the trajectory $f(q, t)$. Since q is an ω-limit point, there exists a sequence $\{t_n\}$ with $t_n \to +\infty$ and such that $f(p, t_n) \to q$. Then by Theorem 2.22, $f(p, t_n + \bar{t}) \to f(q, \bar{t})$ and since $t_n + \bar{t} \to +\infty$, the point r is an ω-limit point.

This theorem may also be stated as follows:

3.253. *Both ω- and α-limit sets of a trajectory consist of whole trajectories.*

3.254. We now classify trajectories according to the properties of their α- and ω-sets.

3.2541. We say that a solution (or a trajectory) recedes in the positive direction if it has no ω-limit points.

3.2542. A solution (or a trajectory) $f(p, t)$ is called asymptotic in the positive direction if there exist ω-limit points, but they do not belong to this solution.

3.2543. A solution (or a trajectory) $f(p, t)$ is called stable in the positive direction in the sense of Poisson if it has ω-limit points which belong to this solution.

We introduce similar definitions describing behavior of solutions as $t \to -\infty$.

3.255. We now consider two important classes of solutions of (1.01) stable in the sense of Poisson. These are the singular points and the periodic solutions.

It is clear that if a point $p(x_{10}, \ldots, x_{n0})$ is a singular point, then the set of functions $x_i(t) = x_{i0}$ $(i = 1, \ldots, n)$ is a solution of (1.01). Thus a singular point $p(x_{10}, \ldots, x_{n0})$ is a trajectory and $f(p, t) = p$ for all t. Therefore every singular point is its own α- as well as ω-limit point, and hence is a trajectory stable in the sense of Poisson.

The set of all singular points is a closed set, and by 3.245, it can be an arbitrary closed set.

If a trajectory $f(p, t)$ has a unique limit point either for $t \to +\infty$ or for $t \to -\infty$ then, in view of 3.253, this limit point is a singular point.

3.256. THEOREM. *If every neighborhood, however small, of a point p contains a trajectory traversed over an arbitrarily long time span, then p is a singular point.*

If p is not a singular point, there exists a t_1 such that $p_1 = f(p, t_1) \neq p$. Then $p_{-1} = f(p, -t_1) \neq p$ as well. Let $d = \min [\varrho(p, p_1), \varrho(p, p_{-1})]$. By the continuity in the initial conditions we can find a $\delta > 0$ such that $\varrho(p, x) < \delta$ implies that $\varrho(f(p, t), f(x, t)) < d/3$ for $-t_1 \leqq t \leqq t_1$. We may assume that $\delta < d/3$. Then the trajectory $f(x, t)$ through any point x in the δ-neighborhood of p does not remain in this neighborhood for $|t| = |t_1|$.

3.257. Consider next the periodic solutions of (1.01), i.e., the solutions $x_i(t)$ in which all the functions $x_i(t)$ are periodic with a common period T. The trajectory of a periodic solution $f(p, t)$ is a closed curve in the phase space and $f(p, t+T) = f(p, t)$.

Thus every point of a periodic solution is an α- as well as an ω-limit point and therefore such a solution is stable in the sense of Poisson.

3.258. It is easy to find examples of systems of differential equations whose solutions are receding, periodic, or are singular points. The problem of constructing asymptotic solutions and solutions which are non-periodic and stable in the sense of Poisson is somewhat more difficult.

We note that a solution which is not a singular point and which has a single α- or ω-limit point is asymptotic. For, as was shown in 3.255, this limit point is a singular point and the solution cannot reach a singular point (which itself is a solution) in finite time in view of the uniqueness condition. It is clear therefore that any system whose singularities include a saddle point or a nodal point will have asymptotic solutions. Consider for example the system

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = y, \quad x = C_1 e^t, \quad y = C_2 e^t.$$

The point $x = 0$, $y = 0$ is a singular point. All other solutions are asymptotic for $t \to -\infty$ and recede for $t \to +\infty$.

We shall consider next the more complicated examples of asymptotic solutions whose ω-limit sets contain more than one point.

3.26. Limit cycles. Consider a system

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y). \tag{3.2601}$$

A periodic solution of (3.2601) is called a *limit cycle* if it is either the α- or the ω-limit set of another solution of this system. Let C be the closed trajectory of a limit cycle. If C is the ω-limit set for solutions contained in its interior, as well as for solutions lying in its exterior, then the limit cycle is called *stable.* If C is the α-limit set for trajectories in the interior and for those in the exterior of C, then the limit cycle is called *unstable.* If, however, C is the α-limit set for the trajectories in the interior (exterior), but is the ω-limit set of the trajectories in the exterior (interior) of C, then the limit cycle is called *semi-stable.*

3.261. *Example.* Given the system

$$\frac{dx}{dt} = -y + \frac{x}{\sqrt{x^2 + y^2}} (1 - (x^2 + y^2)),$$

$$\frac{dy}{dt} = x + \frac{y}{\sqrt{x^2 + y^2}} (1 - (x^2 + y^2)).$$

Passing to polar coordinates, we let $x = r \cos \theta$ and $y = r \sin \theta$. Then

$$\frac{dx}{dt} = -y + \frac{x}{r} (1 - r^2); \quad \frac{dy}{dt} = x + \frac{y}{r} (1 - r^2).$$

Multiplying the first of these equations by x and the second by y and adding, we obtain

$$\frac{dr}{dt} = 1 - r^2 \qquad (r > 0).$$

Next, multiplying the first equation by y, the second one by x, subtracting, and making use of the identity

$$x \frac{dy}{dt} - y \frac{dx}{dt} = r^2 \frac{d\theta}{dt},$$

we get

$$\frac{d\theta}{dt} = 1.$$

Integrating the equation

$$\frac{dr}{1 - r^2} = dt$$

we get

$$\log \left| \frac{1 + r}{1 - r} \right| = 2t + \log A, \quad \text{where} \quad A = \left| \frac{1 + r_0}{1 - r_0} \right|.$$

Thus

$$r = \frac{Ae^{2t} - 1}{Ae^{2t} + 1} \quad \text{for} \quad 0 < r < 1, \quad \text{and} \quad r = \frac{Ae^{2t} + 1}{Ae^{2t} - 1} \quad \text{for} \quad r > 1.$$

We observe that in both cases $r \to 1$ as $t \to +\infty$. Consequently all the solutions outside the circle $r = 1$, as well as all those inside, are spirals approaching this circle. Therefore the periodic solution $x = \cos (\theta_0 + t)$, $y = \sin (\theta_0 + t)$ is a stable limit cycle.

3.262. *Example.* Given the system

$$\frac{dx}{dt} = -y + x(x^2 + y^2 - 1), \quad \frac{dy}{dt} = x + y(x^2 + y^2 - 1),$$

which in polar coordinates has the form

$$\frac{dr}{dt} = r(r^2 - 1), \quad \frac{d\theta}{dt} = 1 \quad (r \geqq 0).$$

Integrating these equations we get $\theta = \theta_0 + t$ and

$$r = 0, \quad r = \frac{1}{\sqrt{1 + Ae^{2t}}} \quad \text{for} \quad 0 < r_0 < 1, \quad A = (1 - r_0^2)/r_0^2;$$

$$r = 1, \quad r = \frac{1}{\sqrt{1 - Ae^{2t}}} \quad \text{for} \quad r_0 > 1, \quad A = (r_0^2 - 1)/r_0^2.$$

The parameter A is always positive and we see at once that for the solutions outside the circle $r = 1$, as well as for the solutions inside this circle, we have $r \to 1$ as $t \to -\infty$. Thus the circle $r = 1$ is the α-limit set of the solutions originating outside the circle as well as for those originating inside. We note that these latter spiral toward the origin $r = 0$ as $t \to +\infty$. Thus the solution $x = \cos(\theta_0 + t)$, $y = \sin(\theta_0 + t)$ is an unstable limit cycle and the solution $x = y = 0$ is a position of stable equilibrium.

3.263. *Example.* Consider the system

$$\frac{dx}{dt} = x(x^2 + y^2 - 1)^2 - y,$$

$$\frac{dy}{dt} = y(x^2 + y^2 - 1)^2 + x.$$

In polar coordinates this becomes

$$\text{(3.2631)} \qquad \frac{dr}{dt} = r(r^2 - 1)^2, \qquad \frac{d\theta}{dt} = 1.$$

If we let $r^2 = u$, we get

$$\frac{du}{dt} = 2u(u - 1)^2,$$

or

$$\frac{du}{u(u - 1)^2} = 2dt.$$

In view of the identity

$$\frac{1}{u(u-1)^2} = \frac{1}{u} - \frac{1}{u-1} + \frac{1}{(u-1)^2},$$

we get

$$\log\left|\frac{u}{u-1}\right| - \frac{1}{u-1} = \log C + 2t,$$

or

$$\left|\frac{u}{u-1}\right| e^{-\frac{1}{u-1}} = Ce^{2t}.$$

Finally, setting $u - 1 = v$, we get

$$\left(\frac{1}{v} + 1\right) e^{-1/v} = Ce^{2t} \quad \text{for} \quad v > 0 \ (r > 1), \tag{3.2632}$$

and

$$\left(-1 - \frac{1}{v}\right) e^{-1/v} = Ce^{2t} \quad \text{for} \quad v < 0 \ (r < 1). \tag{3.2633}$$

Let us consider the behavior of solutions $v = v(C, t)$ of (3.2632) and (3.2633) in the neighborhood of $v = 0$. For t positive and sufficiently large (3.2633) has a unique solution $v(C, t) < 0$. Moreover, as $t \to \infty$, $v(C, t) \to 0$ and hence $r = \sqrt{v+1} \to 1$. For t negative and sufficiently large numerically (3.2632) has a unique solution $v(C, t) > 0$. This solution $v(C, t) \to 0$ as $t \to -\infty$, whence $r \to 1$ in this case as well. Thus in this example the solution $x = \cos(\theta_0 + t)$, $y = \sin(\theta_0 + t)$ is a semi-stable limit cycle.

3.264. *Example.* Let us consider the system

$$\begin{cases} \dfrac{dx}{dt} = -y + (x^2 + y^2 - 1)x \sin \dfrac{1}{x^2 + y^2 - 1} \\ \dfrac{dy}{dt} = x + (x^2 + y^2 - 1)y \sin \dfrac{1}{x^2 + y^2 - 1} \\ \text{for } x^2 + y^2 \neq 1, \text{ and} \\ \dfrac{dx}{dt} = -y, \quad \dfrac{dy}{dt} = x \quad \text{for} \quad x^2 + y^2 = 1. \end{cases} \tag{3.2641}$$

In polar coordinates this system takes the form

$$(3.2642)\qquad \begin{aligned} \frac{dr}{dt} &= r(r^2-1)\sin\frac{1}{r^2-1} && \text{for} \quad r \neq 1, \\ \frac{dr}{dt} &= 0 && \text{for} \quad r = 1, \end{aligned}$$

and in both cases $d\theta/dt = 1$.

Thus in every neighborhood of the periodic solution

$$(3.2643)\qquad x = \cos(\theta_0 + t), \quad y = \sin(\theta_0 + t)$$

of (3.2641) there are infinitely many periodic solutions

$$(3.2644)\qquad x = r_k \cos(\theta_0 + t), \quad y = r_k \sin(\theta_0 + t)$$

where $r_k = \sqrt{1 + (1/k\pi)}$ satisfies the condition $\sin(r_k^2 - 1)^{-1} = 0$. In each ring-shaped region between two consecutive circles (3.2644), the trajectories are spirals approaching these two circles. Thus every solution (3.2644) is a limit cycle.

3.265. We should now give some examples of nonperiodic solutions stable in the sense of Poisson.

In the next chapter we shall see that there exist no such solutions either in the plane or on the surface of a two-dimensional cylinder. However, there do exist such solutions on a torus.

Let us introduce real Cartesian coordinates (φ, ϑ) in the plane and let us identify any two points (φ, ϑ) and $(\varphi + n, \vartheta + m)$ whose coordinates differ by integers n and m respectively.

On the resulting torus consider the system

$$(3.265)\qquad \frac{d\varphi}{dt} = 1, \qquad \frac{d\vartheta}{dt} = \alpha.$$

Whenever we are interested only in the geometrical arrangement of integral curves, we may consider the one equation $d\vartheta/d\varphi = \alpha$. There are two essentially different cases: one in which $\alpha = p/q$ is a rational number and the other in which α is irrational.

3.266. *Example.* Consider the integral curves of the equation

$$(3.2661)\qquad \frac{d\vartheta}{d\varphi} = \frac{p}{q}$$

where q is a natural number, p is an integer, and the fraction p/q is irreducible. The solution corresponding to the initial conditions

$\varphi = 0$, $\vartheta = \vartheta_0$ has the form

$$\vartheta = \vartheta_0 + \frac{p}{q}\varphi. \tag{3.2662}$$

As φ takes on the value q, the coordinate ϑ in (3.2662) takes the value $\vartheta_0 + p$, the resulting point of our integral curve on the torus coincides with the initial point $(0, \vartheta_0)$, and the curve is closed. Thus the torus is covered by closed integral curves of (3.2661).

3.267. *Example.* We consider next the equation

$$\frac{d\vartheta}{d\varphi} = \alpha \tag{3.2671}$$

where α is an irrational number.

In this case there are no closed curves among the integral curves

$$\vartheta = \vartheta_0 + \alpha\varphi \tag{3.2672}$$

of (3.2671). For, suppose that a point (φ_1, ϑ_1) on the integral curve (3.2672) coincides with the initial point $(0, \vartheta_0)$. Then

$$\vartheta_1 = \vartheta_0 + \alpha\varphi_1 = \vartheta_0 + n\alpha = \vartheta_0 + m$$

(m, n integers), whence $n\alpha = m$, and $\alpha = m/n$ is a rational number.

Since all the trajectories can be obtained from the trajectory $\vartheta = \alpha\varphi$ by a translation along the ϑ axis, we need to consider only this trajectory in detail. Its intersections with the meridian $\varphi = 0$ are $\varphi = 0$, $\vartheta_n = n\alpha$, $n = 0, \pm 1, \pm 2, \ldots,$. *These points are everywhere dense in this meridian.*

Write $(\alpha) = \alpha - [\alpha]$, where $[\alpha]$ is the greatest integer in α. To prove our assertion we need only to show that the set $(n\alpha)$, $n = 0, 1, 2, \ldots$, is everywhere dense in the interval $[0, 1]$. Since α is irrational, the $p + 1$ numbers

$$0,\ (\alpha), \ldots, (p\alpha) \tag{3.2673}$$

are all distinct and since they are all distributed among the p intervals

$$I_h : \frac{h}{p} \leqq \vartheta < \frac{h+1}{p} \qquad (h = 0, 1, \ldots, p-1), \tag{3.2674}$$

one of these intervals must contain at least two of the numbers (3.2673). Let $(k_1\alpha)$ and $(k_2\alpha)$ be two such numbers. They differ by less than $1/p$ since each of the intervals I_h is of length $1/p$.

If $k_2 > k_1$, we write $k = k_2 - k_1$. Then either

$$(k\alpha) \in I_0 \text{ or } (k\alpha) \in I_{p-1}.$$

In either case, the sequence

$$(k\alpha),\ (2k\alpha),\ (3k\alpha), \ldots,$$

continued as long as may be necessary, will partition the interval $[0, 1]$ into segments of length less than $1/p$.

To show that every ε-neighborhood of a point in $[0, 1]$, contains a point of the set $(n\alpha)$, it suffices to take $p > 1/\varepsilon$ in the above discussion.

Thus the set $(n\alpha)$ is everywhere dense in $[0, 1]$, and therefore every point of the meridian $\varphi = 0$ is a limit point for the set of points $\varphi = n$, $\vartheta = n\alpha$ of our trajectory. Similarly, every point $\varphi = \varphi_0$, $\vartheta = \vartheta_0$, is a limit point for the set of points

$$\varphi = n + \varphi_0, \quad \vartheta = \alpha(n + \varphi_0)$$

of the same trajectory.

It follows that the trajectory $\vartheta = \alpha\varphi$ and hence every trajectory of (3.2671) is everywhere dense on the torus. *In particular, every trajectory, even though it is not closed, contains some of its ω-limit points.*

3.268. *Example.* Consider the system

$$\frac{d\varphi}{dt} = (\varphi^2 + \vartheta^2), \qquad \frac{d\vartheta}{dt} = \alpha(\varphi^2 + \vartheta^2). \tag{3.268}$$

Trajectories of this system lie on the trajectories of the system (3.265). However, system (3.268) has a singular point at $\varphi = 0$, $\vartheta = 0$. This singular point splits the trajectory of (3.265) through the origin (stable in the sense of Poisson) into three trajectories of (3.268), viz., the singular point $(0, 0)$, and two other trajectories each of which is asymptotic in one direction and stable according to Poisson in the other.

3.27. The qualitative theory of differential equations whose right-hand members do not contain time explicitly, concerns itself with the solution of the following two problems.

3.271. The classification of solutions and the study of relationships between different classes of solutions. This problem is essentially solved and the results of such investigations will be presented in the following chapters.

3.272. The search for methods of determining the types of solutions admitted by a given system of differential equations on the basis of information supplied by the analytic properties of the right-hand members of this system. This problem is far from being completely solved. The reader will find the basic known results in the subsequent chapters of our book.

4.1. Families of Integral Curves

We consider now a family S of integral curves filling either a region G or a closed region $\bar{G}$ in R^n.

4.11. Definition. *A family S of trajectories filling a domain G* (not necessarily open) *in R^n, is called a regular family* (a notion due to Hassler Whitney [58]) *if there exists a homeomorphism (one to one and bicontinuous mapping) of the domain G onto a set $E \subset R^n$ or R^{n+1}, which maps trajectories into parallel straight lines so that the images of different integral curves lie on different straight lines.*

It is clear that a regular family of trajectories cannot contain trajectories which are either stable in the sense of Poisson or are asymptotic. On the other hand, there exist dynamical systems whose integral curves recede in both directions but whose families of trajectories are, nevertheless, not regular. Consider, for example, the system

$$\frac{dx}{dt} = \sin y, \quad \frac{dy}{dt} = \cos^2 y.$$

The integral curves of this system are the curves $x + c = (\cos y)^{-1}$ and the straight lines $y = k\pi + \pi/2$, $k = 0, \pm 1, \ldots$. We consider only the strip

$$\bar{G} : -\frac{\pi}{2} \leqq y \leqq \frac{\pi}{2}.$$

Although all the integral curves situated within this strip recede in both directions (cf. Fig. 1), the family of integral curves filling this strip is not regular.

To prove this, draw a segment PQ with the endpoints P and Q on the lines $y = -\pi/2$ and $+\pi/2$ respectively, and consider a sequence of points P_n on this segment, converging to P. Write L_n for the trajectory passing through P_n, and L and L' respectively for the lower and the upper boundaries of the strip. Assume that our family of trajectories is regular, and let f be a homeomorphism of Definition 4.11. Then the sequence $f(P_n) \in f(L_n)$ converges to the point $f(P) \in f(L)$. Moreover, since $f(L_n)$ and $f(L)$ are parallel straight lines, any convergent sequence of points $y_n \in f(L_n)$ has its limit point of $f(L)$. To obtain a contradiction we observe that if $\{Q_n\}$ is a sequence such that $Q_n \in L_n$ and $Q_n \to Q \in L'$, then $y_n = f(Q_n) \in f(L_n)$ and $f(Q_n) \to f(Q) \in f(L')$. Thus $f(Q)$ must lie on $f(L)$ as well as on $f(L')$. This is a contradiction.

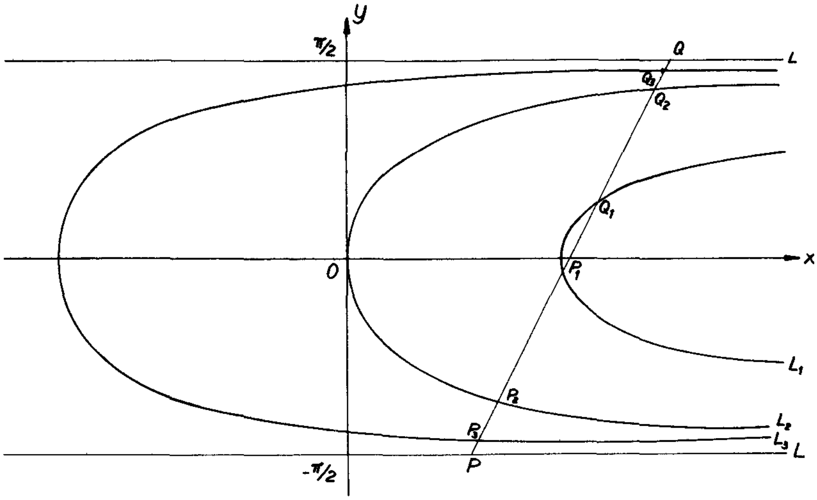

Fig. 1

The following theorem elucidates the part played by regular families in the theory of differential equations.

4.12. Theorem. *Let G be a domain in which the system* (1.01) *satisfies both the uniqueness and the existence conditions* (*cf. Sections* 1 *and* 2) *and let q be a non-singular point of G. Then there exists a neighborhood of q such that the family of integral curves filling this neighborhood forms a regular family.*

Since q is not a singular point, then at q the integral curve L passing through q has a well-defined tangent and hence a well-defined normal hyperplane N as well.

By the continuity of the right-hand members of (1.01), there exists a closed spherical neighborhood $S_0(q, R) \subset G$ with center at q and of radius R, such that the directions of tangent vectors to integral curves at any point inside or on the boundary of the sphere S_0 deviate from the direction of the tangent vector at q by less than $\pi/4$.

Consider the closed $(n - 1)$-sphere $N_1(q, R) = N \cap S_0$. Through every point $p \in N_1$ there passes a solution $f(p, t)$ defined for $|t| \leqq h_p$. Since N_1 is closed, then for sufficiently small R, $0 < h_0 = g.l.b\, h_p$ in view of Theorem 1.11. Also, the periods of the periodic solutions passing through N_1 (if there are such) have a lower bound t_0, provided R is chosen small enough. Write

$$h = \min\left(h_0, \frac{t_0}{2}\right).$$

Then through every point $p \in N_1$ there is an integral arc $f(p, t)$ defined for $-h \leqq t \leqq h$. The totality of these integral arcs forms a *tube* τ_{2h} of length of time $2h$.

A closed set which has one and only one point in common with every trajectory arc of the tube, is called a *section* of the tube. The set N_1 in our construction is a section of the tube τ_{2h}.

Write $T = [-h, +h]$ and consider the circular cylinder $N_1 \times T$. The correspondence between the points $f(p, t)$, $p \in N_1$, $-h \leqq t \leqq h$ of the tube τ_{2h} and the points (p, t) of $N_1 \times T$ is one-to-one. Since, moreover, in one direction this correspondence is a continuous mapping of a compactum, this correspondence is a homeomorphism.

The image of q is an inner point of the cylinder and therefore q is an inner point of the tube τ_{2h}. Any neighborhood of q completely contained in τ_{2h} will serve as the desired neighborhood.

4.2 We shall discuss now conditions under which a system of differential equations will define a regular dynamical system (i.e. with a regular family of trajectories).

4.21. THEOREM. (E. A. Barbashin [3]). *Consider a system of differential equations* (1.01) *which defines a dynamical system in a domain G. If there exists a single-valued function $u(x_1, \ldots, x_n)$ satisfying the condition*

$$\sum_{i=1}^{n} \frac{\partial u}{\partial x_i} f_i = 1 \tag{4.211}$$

in G, then our dynamical system is regular.

Let $u(p) = u(x_1, \ldots, x_n)$ be a single-valued function defined in a domain G, having in G derivatives of the first order, and satisfying the condition (4.211). If $f(p, t) = (x_1(t), \ldots, x_n(t))$ is a trajectory of a dynamical system defined by (1.01), then

$$\sum_{i=1}^{n} \frac{\partial u(x_1(t), \ldots, x_n(t))}{\partial x_i} f_i(x_1(t), \ldots, x_n(t)) = 1,$$

or

$$1 = \sum_{i=1}^{n} \frac{\partial u(x_1(t), \ldots, x_n(t))}{\partial x_i} \cdot \frac{dx_i}{dt} = \frac{du(t)}{dt}$$

where

$$u(t) = u(f(p, t)) = u(x_1(t), \ldots, x_n(t)).$$

Integrating, we get $u(t) = u(0) + t$, or

$$u(f(p, t)) = u(p) + t. \tag{4.212}$$

Let F be the set of all points q for which $u(q) = 0$. It follows from (4.212) that every trajectory has one and only one point in common with F.

Consider the topological product Z of the set F and the real axis T.[6]

If $p \in G$, then by (4.212), $f(p, -u(p)) = q \in F$. The mapping ψ defined by $\psi(p) = (q, t_p)$, where $t_p = -u(p)$ is a one-to-one mapping (since $u(p)$ is single-valued) of G onto Z, and maps trajectories into parallel straight lines in $Z \subset R^{n+1}$. We shall show next that both ψ and ψ^{-1} are continuous.

Let a sequence of points $p_1, p_2, \ldots, p_k, \ldots$ converge to a point p_0. From the continuity of $u(p)$ it follows that $u(p_k)$ tends to $u(p_0)$ and from the continuity of $f(p, t)$ it follows that

$$f(p_k, -u(p_k)) = q_k \to f(p_0, -u(p_0)) = q_0.$$

Thus,

$$\psi(p_k) \to \psi(p_0).$$

Conversely, if a sequence $(q_k, t^{(k)})$ converges, say, to $(q, \bar{t})$, then $q_k \to q$ and $t^{(k)} \to \bar{t}$. Thus the initial points q_k of the integral arcs

$$f(q_k, t), \qquad 0 \leqq t \leqq t^{(k)} \tag{4.213}$$

tend to the initial point q of the integral arc

$$f(q, t), \qquad 0 \leqq t \leqq \bar{t}, \tag{4.214}$$

and the time intervals $t^{(k)}$ tend to $\bar{t}$. By the continuity of $f(p, t)$,

[6]The set Z consists of the points of an $(n + 1)$-dimensional space, situated on parallel straight lines passing through the points of the set F.

the endpoints

$$p_k = \psi^{-1}((q_k, t^{(k)})) = f(q_k, t^{(k)}),$$

of the arcs (4.213) tend to the endpoint

$$p = \psi^{-1}(q, \bar{t}) = f(q, \bar{t})$$

of the arc (4.214).

Theorem 4.21 yields important corollaries.

4.22 *If solutions of* (1.02) *are defined for* $-\infty < t < +\infty$, *then the associated parametric system* (1.201) *is regular.*

The conditions (4.211) corresponding to the parametric system (1.201) will read

$$\sum_{i=1}^{n} \frac{\partial u}{\partial x_i} f_i + \frac{\partial u}{\partial t} = 1.$$

It will be satisfied by the single-valued and continuous function $u = t$.

4.23. *The conclusions of Theorem* 4.21 *still hold if we replace condition* (4.211) *by*

$$N = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} f_i \geqq K^2 > 0. \tag{4.231}$$

Let u satisfy the condition (4.231). Make the substitution

$$t' = \int_0^t N\, dt.$$

If $|t| \to \infty$, then $|t'| \to \infty$. The new system

$$\frac{dx_i}{dt'} = \frac{1}{N} f_i \tag{4.232}$$

is equivalent to the original system and also defines a dynamical system. The condition (4.211) for (4.232) becomes

$$\sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{1}{N} f_i = 1. \tag{4.233}$$

It is clear that u satisfies (4.233).

4.24. A system of the form

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial x_i}. \tag{4.241}$$

is said to possess a velocity potential $F(x_1, \ldots, x_n)$. Corollary 4.23 yields

4.241. *If*

$$\sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i}\right)^2 \geqq K^2 > 0,$$

then the system (4.241) *is regular*

5.1. Fields of Linear Elements

Consider again a system of type (1.01). Such a system assigns a vector $(f_1, f_2, \ldots, f_n)$ to every point $p(x_1, \ldots, x_n)$ at which all the functions $f_i(x_1, \ldots, x_n)$ are defined and at which they do not all vanish. In a domain G in which f_i are all continuous, system (1.01) defines a vector field continuous except at the singular points. It may sometimes happen that we may augment the definition of our vector field so that it will become continuous everywhere. More precisely, we may sometimes find a function $\psi(x_1, \ldots, x_n)$ continuous everywhere except possibly at the singular points and such that the functions $f_i\psi$ are continuous everywhere and do not vanish simultaneously.

5.11. The theory of differential equations also studies systems in the so-called *symmetric* form:

$$\frac{dx_1}{X_1(x_1, \ldots, x_n)} = \frac{dx_2}{X_2(x_1, \ldots, x_n)} = \ldots = \frac{dx_n}{X_n(x_1, \ldots, x_n)}. \tag{5.11}$$

We note that system (1.01) assigns to each point p a *vector* $(f_1, \ldots, f_n)$ whereas system (5.11) assigns to each point a *linear element* (line position) $dx_1 : dx_2 : \ldots : dx_n = X_1 : X_2 : \ldots : X_n$. This linear element is associated with two vectors, the vector $(X_1, \ldots, X_n)$ and the vector $(-X_1, \ldots, -X_n)$.

5.12. We now ask if there exists a system of type (1.01) whose trajectories are the integral curves of our initial system (5.11) and which has no singular points other than those of (5.11).

To state the problem geometrically, we ask if it is possible to choose a positive sense on each linear element and a suitable value for vector length so that the resulting vector field should be continuous everywhere except at the singular points.

Analytically, the problem consists in finding a function $\psi(x_1, ..., x_n)$

such that the products $X_1\psi, X_2\psi, \ldots, X_n\psi$ are continuous in D_1 and do not vanish simultaneously anywhere in D_1, if D_1 is a domain where (5.11) has no singular points.

5.13. We note that the problem of *orientation* of a field of linear elements defined in a domain D_1 is equivalent to the problem of establishing a positive direction along each integral curve of a system of type (5.11) in such a way that every two integral curves which are near each other must agree in direction.

5.14. *It is not always possible to orient a field of elements in the plane.* This can be seen from the following example.

Consider the field of linear elements in the plane defined by the differential equation

$$\frac{dy}{dx} = \cot\frac{\varphi}{2} \tag{5.141}$$

where φ is the polar angle. As is usual, in a neighborhood of a point near which the absolute value of the right-hand member is not bounded, we consider the equation

$$\frac{dx}{dy} = \tan\frac{\varphi}{2}.$$

It is clear that the field of linear elements is defined and is continuous everywhere except at the point (0, 0). Introducing polar coordinates, we obtain

$$\cos\frac{3\varphi}{2}\,dr = r\sin\frac{3\varphi}{2}\,d\varphi.$$

Solving this equation, we obtain three integral half-lines

$$\varphi = \frac{\pi}{3}, \quad \varphi = \pi, \quad \varphi = \frac{5\pi}{3}, \qquad r > 0,$$

and three families of similar curves (Fig. 2)

$$r = \frac{a}{\left(\cos\frac{3\varphi}{2}\right)^{2/3}}$$

with the parameter a and

$$\text{(I)}\quad -\frac{\pi}{3} < \varphi < \frac{\pi}{3}, \qquad \text{(II)}\quad \frac{\pi}{3} < \varphi < \pi, \qquad \text{(III)}\quad \pi < \varphi < \frac{5\pi}{3}.$$

The above field cannot be oriented. For, let us choose the direction away from the origin on the half-line $\varphi = \pi/3$, $r > 0$ as positive. Take a point p on this half-line and draw a circle through p with the center at $(0, 0)$. As we move along the circumference, say in the counterclockwise direction, considerations of continuity will

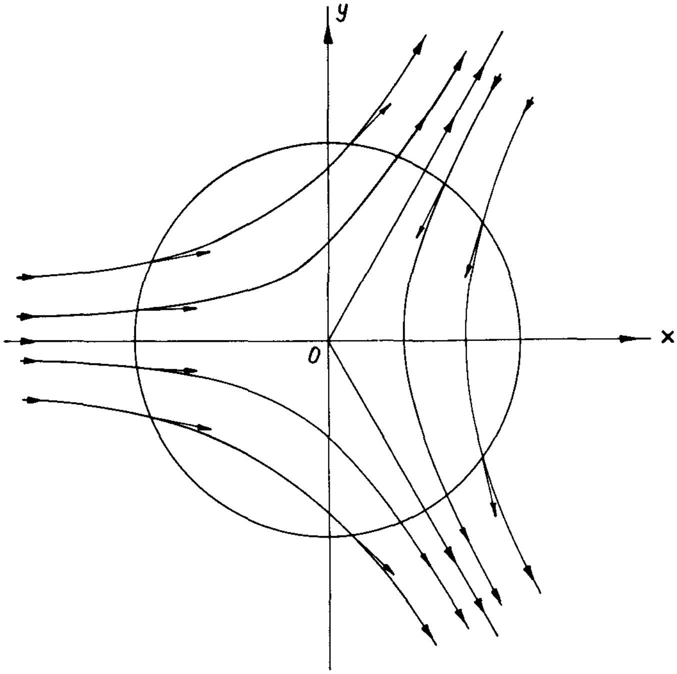

Fig. 2

assign as positive the direction toward the origin on the halfline $\varphi = \pi$, $r > 0$ and the direction away from the origin on the halfline $\varphi = 5\pi/3$, $r > 0$. We find that in view of the orientation induced in the family (III) by considerations of continuity as we move along the circumference, we shall arrive at the initial point with orientation opposite to that chosen originally.

5.141. The selected positive direction for the linear element through a point q on the circle in Fig. 2 may be indicated by a tangent unit vector $v(q)$. The angle ($2n\pi$ or $2n\pi + \pi$) through which v rotates as q spans the circle once in the positive sense, is a property of our field of directions. The rotation angle for the circle in our example is $-\pi$. In this discussion the circle could be replaced by any other simply-closed curve containing the origin.

5.142. We observe that if we write system (5.141) in the form

$$\frac{dy}{dx} = \frac{\sin\varphi}{1 - \cos\varphi} = \frac{r\sin\varphi}{r(1 - \cos\varphi)} = \frac{y}{\sqrt{x^2 + y^2} - x}$$

and then replace it by the system

$$\frac{dx}{dt} = \sqrt{x^2 + y^2} - x \qquad \frac{dy}{dt} = y$$

of type (1.01), then we introduce new singular points (points of equilibrium) filling the positive half of the x-axis.

5.143. In the above example the *domain* D (cf. 5.13) *is not simply-connected* (it does not contain the point (0, 0)). As will be seen from the next theorem, this is not accidental.

5.15. THEOREM. *A continuous field of linear elements in a simply-connected domain D of the plane can be oriented.*

The domain D can be approximated from within by a bounded domain D_1 composed of squares, and therefore it suffices to prove our theorem for such domains D_1. Let us choose the sides of our squares so small that within each one of them the oscillation of the direction of linear elements is less than $\pi/4$. If we choose a positive direction on one of the elements, then considerations of continuity will lead to a unique definition of positive direction for all other elements of the same square. To extend the definition to the element through a point p outside the square we again use considerations of continuity and proceed stepwise along a chain of adjacent squares until we reach p.

The positive direction for the linear element through p is defined uniquely, for if we proceed toward p along two different paths, which yield different orientations at p, then the closed path defines a reversal of the direction of the vector field. Indeed in this case our vector field turns through an angle of $\pi + 2k\pi$ (k is an integer) around some closed curve consisting of the edges of the squares; however the algebraic sum of the rotation angle around each of the squares in the interior of our path must equal the rotation around the outside path, and this is zero modulo 2π. This is a contradiction and the theorem is proved.

CHAPTER II

Integral Curves of a System of Two Differential Equations

1. General Properties of Integral Curves in the Plane.

1.1 Consider the system

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y). \tag{1.01}$$

Let the functions $P(x, y)$ and $Q(x, y)$ satisfy Lipschitz conditions in some domain G_L (a Lipschitz domain) of the plane. We shall study the behavior of the integral curves of this system. In view of I, 3.2 we may assume without loss of generality that the system (1.01) defines a dynamical system in G_L.

The basic results in this case were obtained by Bendixson [7] and Poincaré [47].

We make use of the continuity of the vector field $[P(x, y), Q(x, y)]$ defined by the right-hand members of (1.01), through the following basic lemmas.

1.11. LEMMA. *If a point P_0 of G_L is not a singular point, then there exists an $\varepsilon > 0$ such that the circle $S(P_0, \varepsilon)$ with center at P_0 and of radius ε does not contain singular points either on its boundary or in its interior, and such that the angle between the vector of the field $[P(x, y), Q(x, y)]$ at P_0 and the vector of the field at an arbitrary point of the circle, $S(P_0, \varepsilon)$ is less than $\pi/4$.*

In what follows we shall speak of such a circle as a *small neighborhood of P_0*.

We write $f^+(Q_0)$ for the half-trajectory (semi-trajectory) $f(Q_0, t)$, $0 \leqq t < +\infty$ and $f^-(Q_0)$ for the half-trajectory $f(Q_0, t)$, $-\infty < t \leqq 0$.

1.12. LEMMA. *Let $S(P_0, \varepsilon)$ be a small neighborhood of a point P_0. Let N and N' be the points of intersection of the circumference of $S(P_0, \varepsilon)$ with the normal at P_0 to the trajectory through P_0. Then* *there exists a positive* $\delta < \varepsilon$ *such that for every point* Q_0 *of* $S(P_0, \delta)$, *either the half-trajectory* $f^+(Q_0)$ *or the half-trajectory* $f^-(Q_0)$ *cuts across the segment* NN' *of the normal, before leaving* $S(P_0, \varepsilon)$.

This lemma is an immediate consequence of the theorem on local regularity of the family of integral curves in a neighborhood of a nonsingular point (cf. I.4.12).

Lemma 1.12 can be proved directly as well. Let $Q_0 \in S(P_0, \varepsilon)$. Then the trajectory through Q_0 lies within two vertical right angles whose common vertex is Q_0 and whose bisector is parallel to the

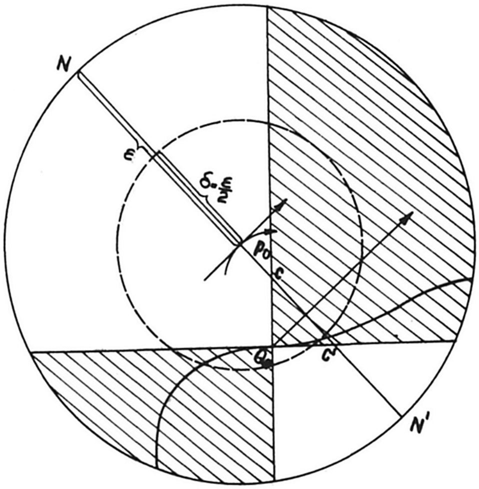

Fig. 3

tangent at P_0 of the trajectory through P_0 (cf. Fig. 3). Let $\delta = \varepsilon/2$ and let $Q_0 \in S(P_0, \delta)$. Then the points C, C' of the intersection of the sides of one of the above vertical angles and the normal, lie on the segment NN' of this normal. The half-trajectory which lies within this angle cuts across the segment NN' before leaving $S(P_0, \varepsilon)$.

We note that both half-trajectories $f^+(Q_0)$ and $f^-(Q_0)$ do leave the small neighborhood $S(P_0, \varepsilon)$. For suppose that $f^+(Q_0)$, say, remains in $S(P_0, \varepsilon)$. Then its ω-limit set contains exactly one point which must therefore be a singular point, which contradicts our hypothesis regarding $S(P_0, \varepsilon)$. The same conclusion may be arrived at by observing that since the velocity vector $V(p)$ defined by the right-hand members of (1.01), does not vanish in $\bar{S}(P_0, \varepsilon)$, then $||V(p)|| \geqq \mu > 0$ for $p \in S(P_0, \varepsilon)$. Moreover, the component $V_\tau(p)$

of $V(p)$ along the tangent at P_0, does not change its direction and $|V_\tau| \geqq \mu/2\sqrt{2}$. Therefore for $|t| > 2\varepsilon\sqrt{2}/\mu$, $f(P_0, t)$ lies outside of $S(P_0, \varepsilon)$.

1.13. The normal NP_0N' divides the small neighborhood $S(P_0, \varepsilon)$ of P_0 into two parts, D_1 and D_2. Suppose that the trajectory through P_0 cuts across the segment NN' from D_1 to D_2 with increasing t. We shall speak of D_1 and D_2 as the negative and the positive sides of the segment NN' respectively. *With increasing t all the trajectories cutting across NN' pass from the negative to the positive side of NN'.*

1.2. THEOREM. *Every trajectory of* (1.01) *possessing at least one-sided stability in the sense of Poisson, is either a singular point or a periodic solution.*

Let $f(A, t)$ be a trajectory, not a singular point, stable in the sense of Poisson for, say, $t \geqq 0$, and let P_0 be an ω-limit point of $f(A, t)$ lying on this trajectory. Since P_0 is not a singular point, it has a small (cf. 1.11) neighborhood $S(P_0, \varepsilon)$. Choose δ as in Lemma 1.12. Since P_0 is an ω-limit point of $f(A, t)$, every half-trajectory $f(P_0, t)$, $t \geqq t_0 > 0$ enters $S(P_0, \delta)$ (reenters—if t_0 is sufficiently large) and therefore, by the choice of δ, it cuts the segment NN'. Let P_1 be the first such intersection following P_0 on $f(P_0, t)$.

If $P_1 = P_0$ then the solution $f(A, t)$ is periodic.

Suppose $P_1 \neq P_0$. We shall show that this contradicts the hypothesis that P_0 is an ω-limit point.

If $P_1 \neq P_0$, only the two arrangements indicated in Fig. 4 are possible. Denote by $\overline{G}_1$ the closed domain bounded by the arc P_0P_1 of our trajectory and by the segment P_1P_0 of the normal. The arrangement in Fig. 4 (a) implies that the trajectory $f(P_1, t)$ remains in $\overline{G}_1$ for all $t > 0$. Moreover, if we take a small neighborhood $S(P_0, \varepsilon_1)$ not containing P_1, then $f^+(P_1)$ cannot enter the corresponding δ_1-neighborhood of the point P_0. For if $f^+(P_1)$ should enter the δ_1-neighborhood of P_0, then by Lemma 1.12 it would have to cut across the normal segment P_0P_1 from the positive to the negative side of NN' (cf. 1.13), which is impossible. We dispose of the alternative in Fig. 4(b) in a similar manner.

1.3. Assume that a Lipschitz domain $\overline{G}_L$ is bounded, closed, and *contains no singular points*. Then, in view of the uniform continuity of $P(x, y)$ and $Q(x, y)$, there exists $\varrho_0 > 0$ such that

$\varrho(P_1, P_2) \leqq \varrho_0$ for $P_1 P_2 \in \overline{G}_L$, implies that the angle formed by the vectors of the field at P_1 and P_2, is less than $\pi/4$.

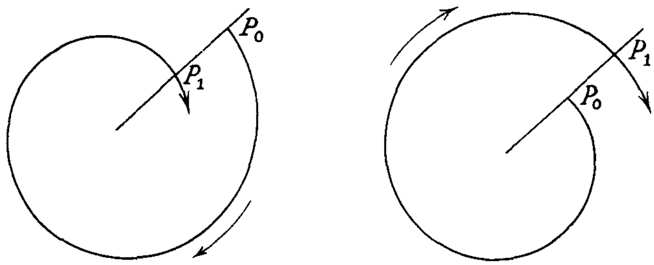

Fig. 4

Let L_{ab} be the arc $a \leqq t \leqq b$ of a (nonsingular) trajectory $f(P_0, t)$, and let $(L_{ab})_\varepsilon$ be an ε-neighborhood of L_{ab}. Since the set of singular points is closed, ε may be chosen so small that $(L_{ab})_\varepsilon$ contains no singular points *either* in its interior or on its boundary. Taking $(L_{ab})_\varepsilon$ as the domain $\overline{G}_L$ we choose $\varepsilon_1 \leqq \varrho_0$. If NP_1N' is a normal to L_{ab} at $P_1 = f(P_0, t_1)$, $a \leqq t_1 \leqq b$, then all the trajectories cutting across NP_1N' within $(L_{ab})_{\varepsilon_1}$ *agree in direction* (cf. 1.13) with L_{ab} at P_1.

We shall speak of such an ε_1-neighborhood of L_{ab} as a small neighborhood of L_{ab} (cf. 1.11).

1.41. THEOREM. *Let L_1 be a closed nonsingular trajectory of a dynamical system in the plane. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every point P_1 in the δ-neighborhood of L_1, at least one of the half-trajectories $f^+(P_1)$ or $f^-(P_1)$ is contained in the ε-neighborhood of L_1.*

Let T be the period of L_1. Then $L_1 = L_{0T} = \{f(P_0, t)\}$, $0 \leqq t \leqq T$, $P_0 \in L_1$, and the discussion in 1.3 assures the existence of a small neighborhood $(L_1)_{\varepsilon_2}$ of L_1. We may assume that $\varepsilon_2 \leqq \varepsilon$. Also, $S(P_0, \varepsilon_2)$ is a small neighborhood of P_0. Let $S(P_0, \delta_2)$ be a corresponding $S(P_0, \delta)$-neighborhood as in 1.12. Next, let $\gamma > 0$ be such that for every $P \in S(P_0, \gamma)$ we have

$$(1.4) \qquad \varrho(f(P_0, t), \ f(P, t)) < \delta_2 \quad \text{for} \quad 0 \leqq t \leqq T.$$

Take $P_1 \in S(P_0, \gamma)$ on the normal NP_0N'. Since $f(P_0, T) = P_0$, then $f(P_1, T) \in S(P_0, \delta_2)$, and the trajectory $f(f(P_1, T), t)$ cuts the normal segment NN' before leaving $S(P_0, \varepsilon_2)$. Let $P_2 = f(P_1, T_1)$, $T_1 \neq 0$, be the first such intersection following P_1.

Let C be the closed curve formed by the arc $f(P_1, t)$, $0 \leqq t \leqq T_1$ and by the segment P_2P_1 of the normal to L_1 at P_0. The curve C together with L_1 form a ring-shaped region Γ contained in the ε_2-neighborhood $(L_1)_{\varepsilon_2}$ of L_1.

If $P_2 = P_1$, both half-trajectories $f^+(Q_0)$ and $f^-(Q_0)$ lie in $(L_1)_{\varepsilon_2} \subset (L_1)_\varepsilon$ for every point Q_0 in the ring-shaped region Γ bounded by L_1 and $f(P_1, t)$.

Let $P_2 \neq P_1$. We distinguish two arrangements on the normal at P_0, viz., (1) $P_0P_2P_1$ and (2) $P_0P_1P_2$.

In the first case, for every $Q_0 \in \Gamma$, the positive half-trajectory $f^+(Q_0)$ cannot leave the region Γ, and in the second case, the negative half-trajectory $f^-(Q_0)$, $Q_0 \in \Gamma$, cannot leave Γ.

Let d be the distance between the arcs $f(P_0, t)$, $0 \leqq t \leqq T$, and $f(P_1, t)$, $0 \leqq t \leqq T_1$, in case $P_1 \neq P_0$ lies on NP_0 and let d' be the distance between these arcs for a choice of $P_1 \neq P_0$ on $N'P_0$. Let the ring-shaped regions corresponding to these choices of P_1, be Γ and Γ'. Then both d and d' are positive. Take $0 < \delta < \min(d, d')$. Then $(L_1)_\delta \subset \Gamma \cup \Gamma' \subset (L_1)_{\varepsilon_2} \subset (L_1)_\varepsilon$, and this δ fulfills the requirements of our theorem.

1.411. The above theorem implies that integral curves in the plane cannot approach a periodic solution arbitrarily close and then recede both for $t \to +\infty$ and for $t \to -\infty$. However, this may occur in R^3, for example.

1.42. DEFINITION. *We shall say that a half-trajectory $f^+(Q_0)$ $[f^-(Q_0)]$ approaches a trajectory Λ spirally if for any point $P_0 \in \Lambda$ and an arbitrarily small segment $P_1P_0P_2$ of the normal to Λ at the point P_0, there exists t_0 such that the half-trajectory $f(Q_0, t)$, $t > t_0$ $[t < t_0]$ intersects $P_1P_0P_2$ infinitely many times in such a way that either all the points of intersection lie on P_1P_0 or all of them lie on P_0P_2.*

A trajectory L is said to approach a trajectory Λ spirally if a half-trajectory of L approaches Λ spirally.

1.43. THEOREM. *If a closed integral curve L is contained in the ω-limit set Ω of some trajectory $f(P_0, t)$, then $\Omega = L$ and, if $f(P_0, t)$ is not closed, the half-trajectory $f^+(P_0)$ approaches L spirally.*

Take a point $Q_0 \in L$. Since $L \subset \Omega$, there exists a sequence $P_j = f(P_0, t_j)$, $j = 1, 2, \ldots$, such that $t_j \to +\infty$ and $P_j \to Q_0$. Consider a sequence $\varepsilon_j > 0$, such that $P_{j-1} \notin (L)_{\varepsilon_j}$. By Theorem

1.41, for each ε_j there exists a δ_j such that for $P_{m_j} \epsilon (L)_{\delta_j}$ either $f^-(P_{m_j})$ or $f^+(P_{m_j})$ lies in $(L)_{\varepsilon_j}$. We may take $m_1 < m_2 < \ldots$. Since $P_{j-1} \notin (L)_{\varepsilon_j}$ we have $f^+(P_{m_j}) \subset (L)_{\varepsilon_j}$. We note that $\varepsilon_j \to 0$. Since, by the above, the half-trajectory $f^+(P_0)$ remains outside $(L)_{\varepsilon_j}$ for only a finite duration $0 \leqq t \leqq T_{\varepsilon_j} < t_{m_j}$, we have $\Omega \subset L$ and hence $\Omega = L$. Also, there exists a j_0 such that every neighborhood $(L)_{\varepsilon_j}$, $j \geqq j_0$ is a small neighborhood of L. Hence L is approached spirally by $f^+(P_0)$.

1.44. The mode of approach to a nonsingular ω-limit point. Let $P_0 \epsilon \Omega(f^+(Q_0))$ be a nonsingular point. Let $S(P_0, \varepsilon)$ be a small neighborhood of P_0 (cf. 1.11 and 1.12). Since P_0 is an ω-limit point of $f^+(Q_0)$, there exists arbitrarily near P_0 a point $Q_1 \epsilon S(P_0, \varepsilon)$ of intersection of this half-trajectory and the segment NN' (cf. 1.11) of the normal to $f(P_0, t)$ at P_0.

1.441. *If Q_1 lies on NP_0, then the successive intersections $Q_1Q_2Q_3, \ldots$ of $f^+(Q_0)$ with NN' all lie on NP_0, are arranged on NP_0 in that order, and tend to P_0.*

Write $Q_2 = f(Q_1, t_1)$. The arc $C_{12} = f(Q_1, t)$, $0 \leqq t \leqq t_1$ and the subsegment Q_1Q_2 of the segment NN' form a closed curve C. We shall see that

1.442. *If $P_0 \notin f(Q_0, t)$ then the closed curve C above separates the point P_0 and every half-trajectory $f(Q_1, t)$, $t \leqq -\delta < 0$. Moreover, no trajectory can pass from the domain containing the point P_0 into the domain containing the half-trajectory $f(Q_1, t)$, $t \leqq -\delta < 0$, with increasing t. The domains referred to above are the exterior and the interior of C.*

The set of points not on C is decomposed by C into two domains D^+ and D^-. Since no trajectory cuts across the arc C_{12}, a trajectory can pass from one of these domains into the other only by cutting across the segment Q_1Q_2 of the normal. Since, moreover, a trajectory cutting across Q_1Q_2 must agree in direction with $f(Q_1, t)$ at Q_1, the half-trajectories $f^-(Q_1)$ and $f^+(Q_2)$ except for the points Q_1 and Q_2, lie in the different domains, say D^- and D^+ respectively, and no trajectory can pass from D^+ into D^- with increasing t.

To prove 1.441, it suffices to show that $Q_2 \epsilon Q_1P_0$. If $Q_2 \epsilon P_0N'$, then by Lemma 1.12, $f^+(Q_2)$ in coming close to P_0 would have to cut across NN' in the wrong direction, i.e., from D^+ into D^- with increasing t. If $Q_2 \epsilon NQ_1$, then $P_0 \epsilon D^-$ and hence is bounded away from $f^+(Q_2)$.

To complete the proof of 1.442, we need only to observe that in view of 1.441, $P_0 \in D^+$.

1.45. The mode of approach to an arc of a trajectory contained in an ω-limit set. Let $P_0 \in \Omega(f^+(Q))$ be a nonsingular point not on $f^+(Q)$, and let $S(P_0, \varepsilon)$ be a small neighborhood of P_0. Use the notation of Lemma 1.12 and of Section 1.44.

We observe first that the whole half-trajectory $f^+(P_0)$ lies in D^+ and hence is bounded away from the (negative) half-trajectory $f(Q_1, t)$, $t \leqq -\delta < 0$.

Next, consider an arc $L_{01} = f(P_0, t)$, $0 \leqq t \leqq t_0$. For all Q_i sufficiently close to P_0, say for $i \geqq i_0$, the whole arc $L_{i1} = f(Q_i, t)$, $0 \leqq t \leqq t_0$ lies in a small neighborhood of L_{01}. Thus each of these arcs L_{i1} in cutting a normal to L_{01}, cuts it in the same direction as L_{01}, and L_{i1} tend to L_{01} as i tends to $+\infty$.

1.46. LEMMA. *Consider a trajectory $f(p, t)$. Let $f(Q_0, t) \subset \Omega(f(p, t))$. If $P_0 \in \Omega(f(Q_0, t))$, then either $P_0 \in f(Q_0, t)$ or P_0 is a singular point (or both).*

If $P_0 \in \Omega(f(Q_0, t))$, then $P_0 \in \Omega(f(p, t))$. If $P_0 \notin f(Q_0, t)$ and if P_0 is a nonsingular point, then in view of 1.442, $f^+(p)$ will enter the region D^+ containing the ω-limit point P_0 and will be bounded away from every point of the half-trajectory $f(Q_0, t)$, $t \leqq -\delta < 0$. Hence no point on this half-trajectory can be an ω-limit point of $f^+(p)$, which is a contradiction.

1.47. THEOREM. *If the ω-limit set of a trajectory $f(p, t)$ is bounded and contains no singular points, then it consists of exactly one closed trajectory L. If $f(p, t)$ is not a closed trajectory, then it spirals toward L as t tends to $+\infty$.*

Let $Q_0 \in \Omega(f(p, t))$ and let $P_0 \in \Omega(f(Q_0, t))$. (Here $\Omega(f(Q_0, t))$ is not empty in view of the boundedness of $\Omega(f(p, t)))$. Since P_0 is nonsingular, then $P_0 \in f(Q_0, t)$ by Lemma 1.46, and hence $f(Q_0, t)$ is a closed curve by Theorem 1.2. The conclusions of our theorem follow at once from Theorem 1.43.

1.471. If a trajectory L is not closed, then it may have a limit cycle for $t \to +\infty$ as well as for $t \to -\infty$. In view of the discussion in 1.45, if both of the limit cycles exist, they are distinct.

1.48. THEOREM. *If L_1 consists of the ω-limit points of a trajectory L and is not a closed trajectory, then the set of the ω-limit points of L_1, if not empty, consists of singular points.*

This theorem follows at once from Theorem 1.2 and Lemma 1.46.