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Nonparametric Comparative Statics and Stability

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PRINCETON UNIVERSITY PRESS
PRINCETON, NEW JERSEY

Published by Princeton University Press, 41 William Street,
Princeton, New Jersey 08540
In the United Kingdom: Princeton University Press,
Chichester, West Sussex

Library of Congress Cataloging-in-Publication Data

Nonparametric comparative statics and stability / by Douglas Hale ...
[et al.].
p. cm.
Includes bibliographical references and index.
ISBN 0-691-00690-3 (cloth : alk. paper)
1. Economics, Mathematical. 2. Macroeconomics—Mathematical models. 3. Equilibrium (Economics). I. Hale, Douglas, 1946–
HB135.N664 1999
330'.01'51--dc21 98-55311
CIP

This book has been composed in Times Roman with Helvetica Display

The paper used in this publication meets the minimum requirements of
ANSI/NISO Z39.48-1992 (R1997) (*Permanence of Paper*)

http://pup.prınceton.edu

Printed in the United States of America

1 3 5 7 9 10 8 6 4 2

To Yvonne M. M. Bishop

Two True Scholars: David F. Lady and Dorothy E. Lady

Carol

Shirley

Contents

Preface

The fundamental issue addressed in this book concerns the degree to which subject matter such as economics can be considered a science, i.e., a source of refutable hypotheses. Such a concern is not limited to the realm of scholarly debate, but arrives with ever-increasing importance in an age when many governmental and commercial policies are derived from computer-based renditions of conceptual models from economics and elsewhere.

Our particular experience with applied analysis includes the energy market forecasting systems developed by the Energy Information Administration (EIA) of the U.S. Department of Energy over the decades since the 1973–1974 oil embargo. The issues at stake in the use of these systems include the degree of expected dependency upon foreign sources of energy supply; the effect on energy production and costs of adopting, or removing, the regulation of energy markets; and the consequences of undertaking alternative polices designed to ameliorate the environmental problems associated with the production and consumption of many energy products. The outcomes of policy issues such as these have substantial material significance for all of us; further, the design of many specific policies depends upon exactly how the data about the relationships at issue are used. Given this, the EIA has maintained an aggressive interest in reviewing and determining the operational quality of its policy analysis modeling.

The particular circumstances that led to this book can be traced in part to the EIA model quality control program. In light of the importance of modeling to the energy policy debate, a symposium was held at the University of Colorado in 1980, with attendees invited from a broad range of disciplines (proceedings in Greenberg and Maybee [1981]). The papers and discussions at the symposium covered the technical issues that needed to be resolved, and promising approaches for resolving them, in order to understand the basis for the expected accuracy of the results from large computer-based forecasting systems. Many of the results reported here stem from investigations prompted by this symposium. Some of the examples presented in Chapter 4 were initially sponsored by the EIA as part of its continuing interest in model quality control.

But the subject matter at issue is not limited to energy, or economics, but instead, potentially can by applied to any inquiry. Even in disciplines traditionally viewed as "fully quantitative," there can be problems with the accuracy of measurement and questions about what types of conclusions can be drawn nevertheless. An extensive bibliography on topics in (as called here) nonparametric comparative statics and stability may be found at the end of the book. In reviewing this, the reader can note instances of application in many scientific areas, including artificial intelligence, biology, chemistry, ecology, energy (as already noted), and large physical systems. Further, there has been a continuously active interest in the mathematical underpinnings of the analysis. We believe that some of the frames of reference for the analysis here that seem particular to economics, i.e., optimization and equilibrium, have rather straightforward analogues in other disciplines; e.g., the balance conditions for equilibrium correspond to double-entry concepts in accounting and conservation laws in physics. As a result, although we come to many issues within the framework of our own background in economics, it is hoped that the approaches taken and the results achieved can serve as a resource in the study of similar problems in other disciplines. We believe that the issue of "nonparametrics" must necessarily be present to one degree or another in any applied science.

The technical content of the book made this a very difficult book to edit. The quality of the editing job done by Lyn Grossman cannot be overstated. Production editor Molan Chun Goldstein directed the various tasks involved in actually producing the book. The authors wish to particularly acknowledge this support. Of course the ultimate responsibility for any remaining errors remains with the authors.

This book is specially dedicated to the memory of John Maybee. John was a friend and colleague of each of us for many decades and contributed to the preparation of this manuscript. John passed away before the book was completed. John's characterization of the structure of determinants in terms of cycles and chains is critical for the arguments in the book. He combined a rare ability to make esoteric mathematical arguments accessible to nonspecialists with good-natured humor. He is sorely missed.

Nonparametric Comparative Statics and Stability

1

Nonparametric Analysis

1.1 INTRODUCTION

In the early 1950s, the federal government decided to stockpile strategic materials to protect against the possibility of hostilities with the USSR. Planners constructed a massive 400×400 input-output model of the U.S. economy, entailing the estimation of a very large number of input-output coefficients, mainly based on data from the 1947 Census of Manufactures, to assess the effects of alternative inventory holdings. After several years of data gathering and analysis, they tested the resulting model, using the final bill of goods for the economy for 1951. Among other findings from that test was that the calculated domestic steel requirement for producing the 1951 final bill of goods was 40% more than the capacity of the U.S. steel industry in 1951, a physical impossibility. What had gone wrong was that relative prices of different grades of steel (alloy, stainless, carbon) had changed between 1947 and 1951, leading to a change in the output mix of the steel industry. Price changes in the steel industry had also led to changes in steel-using technology, and substitution of other metals for steel, by the customers of the steel industry. In the space of four years, the adaptiveness of the American economy had clearly revealed the volatility of the input-output coefficients underlying the Defense Department's economic planning models.[1] Problems with the volatility of input-output and other such coefficients continue to be commonplace in economic research.[2]

Since the quantitative particulars of the interrelationships that constitute economic phenomena are so often volatile and transitory, it is natural to inquire about aspects of the interrelationships that might be more stable and robust. This book explores what can be said about the collective outcome of interdependent quantitative phenomena when the precise nature and magnitudes of their separate influences are not known. Although the particular case of economic phenomena motivated the original work in this area, the problem of inference with limited quantitative information is endemic to scientific inquiry. Scientific explanations of observable phenomena are based on structural relations:

$$F(y;\alpha,e)=(f_1,\ldots,f_n),$$

where the f_i are functions linking the phenomena to be explained (the endogenous variables), $y=(y_1\ldots y_n)$, to conditioning numbers (parameters and exogenous variables, collectively, the data), $\alpha=(\alpha_1,\ldots,\alpha_m)$, determined outside the theory, and unobserved random disturbances, e. The random disturbance, e, is suppressed in most of this book.[3] Scientific predictions are derived under the assumption that observed values of the phenomena, y, are equilibrium values, $y^* = y^*(\alpha,e)$, defined by

$$F(y^*;\alpha,e)=(f_1,\ldots f_n)=0_n \tag{1.1}$$

Much of the daily work of scientists involves making observations and conducting experiments to estimate the form of the equation systems such as (1.1), the values of conditioning numbers, and the distribution of the unobserved disturbances. The result of a successful research program is a complete and internally consistent explanation of the phenomena. Comparison of its predictions to data can then test the validity of the theory.

1.2 QUANTITATIVE ANALYSIS

Quantitative analysis in economics has traditionally focused on comparative statics: the problem of computing changes in the equilibrium values of endogenous variables induced by changes in the data. Analysis of the local direction of change in economic magnitudes in response to changes in technology, resource endowments, people's preferences, and public policy naturally results in locally linear systems of equations under appropriate differentiability assumptions (see Samuelson 1947). In this book, most of the systems we analyze are local comparative statics models. In recent years, the comparative statics problem has been reformulated by Milgrom and Shannon (1994), Milgrom (1994), and Milgrom and Roberts (1990, 1994). This approach is directed at establishing conditions necessary and sufficient for global qualitative comparative statics results. Because of its generality, the approach is less useful as a device for identifying the specific comparative statics results that economic models can generate. The precise links between this approach and that taken in this book have not been completely established. However, in at least one case, that involving the maximization hypothesis, the correspondence between the approach taken here and the Milgrom-Shannon monotonicity theorem is easily established. See the discussion in Chapter 5.

Comparative Statics

The analysis is initiated by noting that in the neighborhood of an equilibrium, y^*, the changes induced in y^* by changes in α can be written in differential form as

$$\Sigma_j(\partial f_i/\partial y_j)dy_j^* + \Sigma_k(\partial f_i/\partial \alpha_k)d\alpha_k = 0, \tag{1.2}$$

where $i, j = 1, \ldots, n$ and $k = 1, \ldots, m$. If only one exogenous variable changes, the most common case in economic modeling, the differential system becomes

$$\Sigma_j(\partial f_i/\partial y_j)dy_j^*/d\alpha = -\partial f_i/\partial \alpha \qquad i = 1, \ldots, n. \tag{1.3}$$

Determining the change in the equilibrium values of phenomena with respect to a change in an exogenous variable, $dy_j^*/d\alpha$, is the subject matter of comparative statics ("statics" because time does not explicitly enter (1.1)–(1.3). Define the square $n \times n$ matrix, $A = [a_{ij}] = [\partial f_i/\partial y_j]$, the $n \times 1$ vector, $x = (dy_i^*/d\alpha)$, and the $n \times 1$ vector, $b = [b_j] = [-\partial f_i/\partial \alpha]$. The local comparative statics problem then can be written as

$$Ax = b, \tag{1.4}$$

where x is to be determined. The matrix A is called the *Jacobian matrix*, corresponding to a solution to the system (1.3).

A theory is locally scientific in the sense of Popper ([1934] 1959) if for a given, potentially observable b-vector, a particular x-vector could never arise as a solution to (1.4). The theory would be "refuted" if the particular x-vector were in fact observed. From the standpoint of refutable hypotheses, the content of a theory is represented by the characteristics of its Jacobian matrix.

Dynamics

The equilibrium y^* defined by (1.1) is often interpreted as the stationary state associated with a dynamic adjustment process operating on the phenomena, y, over time, t. Formally, the adjustment process is

$$\dot{y} \equiv dy/dt = g(y; \alpha, e) = (g_1, \ldots g_n), \tag{1.5}$$

where dy/dt is the time derivative of y. When the rate of change of each y_i is increasing with its "distance" from equilibrium as measured by f_i, the adjustment mechanism can be written as

$$\dot{y} \equiv dy/dt = g(f) = (g_i(f_i)). \tag{1.6}$$

A linear approximation of (1.6) in the neighborhood of y^* is obtained by a first-order Taylor series, $dy_i/dt = (dg_i/df_i)\Sigma_j(\delta f_i/\delta y_j)(y_j - y_j^*)$, evaluated at the equilibrium y^*. Writing this expression in matrix form yields

$$\dot{y} \equiv dy/dt = DA(y - y^*), \tag{1.7}$$

where D is a diagonal matrix with $d_{ii} = dg_i/df_i > 0$ and A is as defined in (1.4). Global stability analysis is concerned with determining conditions that ensure that (1.5) or (1.6) devolves to zero in the limit, for arbitrary initial conditions. Linear approximation stability analysis seeks conditions on D and A ensuring that (1.7) devolves to zero in the limit, in a neighborhood of y^*.[4]

1.3 NONPARAMETRIC ANALYSIS

The quantitative approach to scientific explanation breaks down when the theory's underlying equations, conditioning numbers, and unobserved disturbances are only vaguely known. In the natural sciences, where the relationships are presumed immutable in time and space, nascent theories may reveal the import of novel data sets that may not be available for decades. In the social sciences, both the underlying relationships and the magnitudes of the conditioning numbers may change with place and time. People change, institutions change, and the technology changes: careful observation and estimation may still yield only provisional approximations of a transient reality.[5] The demands of quantitative analysis are often simply not feasible. The validity of plausible inferences from theories, mental models, and computer programs can all be rejected on the basis of quantitative information, but the problem of inference with limited quantitative information remains. Social scientists in particular have little hope of ever achieving precise knowledge of people and their organizations.

Quantitative results in economic applications consequently have limited predictive power. Little is known about the actual form of the underlying relationships, and controlled field experiments to resolve magnitudes are seldom feasible. Even in the linearized structure of (1.4) or (1.7), precise quantitative information is typically absent. Formally, the magnitudes of the entries of A and b in (1.4) and D in (1.7) are not completely known. Given this, the immediate issue becomes, What *can* be safely assumed to be known? For the purposes of this book, any state of knowledge about the nature of (1.4) of (1.7) that is less than fully quantified will be termed *nonparametric*.

A basic difficulty in performing scientific work outside of a fully quantitative environment is that the set of nonparametric information available to researchers can come in a seemingless endless variety of forms. Depending on his or her progress, a researcher may know only which variables appear in individual relationships (i.e., which entries of A in (1.4) are zero and which are not), or the direction of influence of parameters on variables (i.e., the signs $[+, -, 0]$ of the entries of A in (1.4)), or the relative magnitudes of some of the entries of A (i.e., a ranking of the entries of A in (1.4)). There is no single nonparametric environment. Researchers have pursued two related approaches to this curse of riches.

One approach focuses on the types of information that are likely to be available to researchers concerning the entries of the Jacobian matrix. The classic example, developed in Chapter 2, is to assume researchers know only whether entries are greater than, less than, or equal to zero. This emphasis on sign information arose historically in economics because economists are most secure in their beliefs about which variables appear in relationships and the nature of their direct influence, i.e., whether $\partial f_i/\partial y_j$ and $\partial f_i/\partial \alpha$ are zero, positive, or negative. An analysis based upon sign pattern information alone is termed a *qualitative analysis*. Thus, a qualitative analysis deals with the matrices in equations (1.4) and (1.7) under the assumption that sign pattern information is available concerning the Jacobian matrix A and the vector b. Chapter 2 presents the results available for qualitative analyses. Sometimes a researcher may assume to know additional information about the matrix's entries, such as their relative sizes or bounds upon their magnitudes. Chapter 3 organizes sign pattern information analysis with these additional categories of information into a hierarchy analogous to that of measurement scales. Results are derived that show how the different categories of information about the entries of the Jacobian matrix can lead to definitive conclusions about the entries of the inverse Jacobian matrix. The other approach, developed in Chapters 5–8 is to hypothesize underlying principles, such as maximization or stability, governing the Jacobian matrix of the systems described by (1.4) and (1.7). These principles, combined with qualitative information, can sometimes yield definite results.

In all of these the fundamental mathematical questions are the same. First, under what conditions, given the information assumed to be available about the entries of the Jacobian matrix and the vector b (in (1.4)), can we solve (partially or completely) for the sign pattern of the vector x? And second, when can the same information enable us to determine the stability of the differential equation system (1.7)?

1.4 AN EXAMPLE

It might be helpful to consider an example of a qualitative analysis. In a simple full-employment economy, current output, X, is fixed at a level X_f. Total real output is divided among investment, consumption, and government expenditure, all expressed in real (inflation-adjusted) dollars. Equivalently, output can be viewed as the sum of real savings, consumption, and taxes. Investment, I, is assumed to decrease with the interest rate, i. Consumption, C, is assumed to increase with disposable income, X_d, which is defined as output less taxes, i.e., $X_d = X - T$. For the purposes of this example, taxes and government expenditures are assumed to be exogenous variables set by government policy, i.e., they can be chosen independently of other economic variables.

In the money market, it is assumed that the money supply, M is set by the central bank. The real money supply is M/P, where P is the price level. Demand for real money balances consists of transactions demand, kX, where k is a constant and X, as above, is real output; and speculative or liquidity demand $L(i)$, a decreasing function of the interest rate. In equilibrium, the demand for real money balances equals the supply of real balances. The equilibrium equations governing this simply economy, based upon market clearing in the goods and money markets, are

$$G + I(i) + C(X_d) = X$$

and

$$M/P = kX + L(i),$$

where $X = X_f$ and $X_d = X - T$.

Differentiating the two equilibrium equations totally with respect to all endogenous and exogenous variables yields

$$\begin{bmatrix} dI/di & 0 \\ -dL/di & -\dfrac{M}{P^2} \end{bmatrix} \begin{bmatrix} di \\ dP \end{bmatrix} = \begin{bmatrix} -dG + (dC/dX_d)dT \\ -dM/P \end{bmatrix}.$$

The differentials dG, dT, and dM are policy changes selected by the government or the central bank and result in changes in the interest rate and the price level, di and dP, respectively. The signs $(+, 0, -)$ of the derivatives are established by the assumptions made earlier. The qualitative system corresponding to (1.4) is

$$\begin{bmatrix} - & 0 \\ + & - \end{bmatrix} \begin{bmatrix} di \\ dP \end{bmatrix} = \begin{bmatrix} -dG + (dC/dX_d)dT \\ -dM/P \end{bmatrix}.$$

Inverting the coefficient matrix and solving yields

$$\begin{bmatrix} di \\ dP \end{bmatrix} = \begin{bmatrix} - & 0 \\ - & - \end{bmatrix} \begin{bmatrix} -dG + (dC/dX_d)dT \\ -dM/P \end{bmatrix}.$$

Consider what happens if government expenditures are increased ($dG > 0$), while taxes and the money supply are held constant ($dT = dM = 0$):

$$\begin{bmatrix} di \\ dP \end{bmatrix} = \begin{bmatrix} - & 0 \\ - & - \end{bmatrix} \begin{bmatrix} - \\ 0 \end{bmatrix} = \begin{bmatrix} + \\ + \end{bmatrix}.$$

Thus, this system is fully sign solvable—the signs of both di and dP are determined by the sign pattern information given. An increase in real government expenditures in this economic model gives rise to an increase in the interest rate and an increase in the price level, assuming taxes and the money supply are held fixed. Further, it is easy to verify that changing taxes or changing the money supply, holding the other exogenous variables fixed, also leads to a fully sign solvable system. Thus, within this model, increasing government expenditure increases the interest rate and the price level (both di/dG and dP/dG are positive). Increasing taxes reduces both the interest rate and the price level (both di/dT and dP/dT are negative). Increasing the money supply increases prices ($dP/dM > 0$), but does not affect the interest rate ($di/dM = 0$). To say that this simple economy is sign solvable means that, assuming that equilibrium is reestablished, it is possible to deduce the direction of change in all the economic variables as government policy changes, *independent of the magnitudes of the influences expressed by the Jacobian matrix so long as the directions of the influences (i.e., signs of the entries) are those assumed.*

The comparative statics approach assumes that a new equilibrium is established, given the changes in government policy. But in order to guarantee that equilibrium will be reestablished following an exogenous change, the dynamic adjustment mechanism must be stable. The dynamic adjustment mechanism for this economy is based on two assumptions. If investors attempt to invest more than savers save, the interest rate increases. Rising interest rates are assumed to choke off lower return investment opportunities. Similarly, if the supply of money exceeds what people wish to hold, the price level goes up; in effect, too much money is chasing too few goods. In terms of (1.5) the equations are

$$di/dt = b_1(I(i) - S);$$

and

$$dP/dt = b_2(M/P - kX - L(i)),$$

where the b's are positive constants measuring the "speeds of adjustment" of P and I; i.e., the b's determine how rapidly i and P adjust to disturbances to equilibrium. In the linear approximation form of (1.6) the equations become

$$di/dt = b_1\{(dI/di)(i - i^*)\}$$

and

$$dP/dt = b_2\{-(dL/di)(i - i^*) - (M/P^2)(P - P^*)\},$$

where * indicates an equilibrium value. In qualitative terms, the linear differential equation system corresponding to (1.7) becomes

$$\begin{bmatrix} di/dt \\ dP/dt \end{bmatrix} = \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \begin{bmatrix} - & 0 \\ + & - \end{bmatrix} \begin{bmatrix} (i - i^*) \\ (P - P^*) \end{bmatrix}.$$

It can be shown that this is a sign stable system: any matrix with the above sign pattern has both characteristic roots with negative real parts. Thus if equilibrium is disturbed by a policy change in G, T, or M, i and P will converge asymptotically to new equilibrium values, with qualitative changes as described above.

This example captures the analytic goals, and the main topics, of the material presented in this book. A model of phenomena is proposed, but something less that a full specification of its quantitative attributes is assumed to be known. Given this, what can be said about the comparative statics, and stability, of its solution? The example was contrived to present a case for which sign pattern information about the Jacobian matrix was sufficient to determine the sign pattern of its inverse and its stability. For larger models (or even different sign patterns for this smaller model) and/or different categories of information, the conditions for resolving the analytic issues at stake can be more complicated. The purpose of this book is to contribute to an understanding of the circumstances under which less than fully quantitative, i.e., nonparametric, information can be used to resolve the issues of comparative statics and stability of mathematical models of phenomena.

1.5 ORGANIZATION OF THE BOOK

This book is divided into eight chapters. Chapters 2–4 examine what can be learned about the comparative statics and stability of systems defined by equations (1.1) through (1.7) by using only information about the entries of the Jacobian matrix and the b vector. Chapter 2 examines the classic qualitative case, where only sign pattern information is available to the researcher. The chapter contains necessary and sufficient conditions for full and partial sign solvability. Necessary and sufficient conditions are also given for sign stability. These results have been available for some time and set the stage for the extensions of the analysis presented in the remainder of the book. A mathematical appendix to the chapter reports results from matrix analysis and stability theory particularly useful in analyzing signed systems.

Chapter 3 extends the analysis to cover other cases for which sign pattern information is not sufficient, but which can nevertheless be solved nonparametrically. The strategy of analysis is to develop a hierarchy of information typologies for the entries of the Jacobian matrix, and then find conditions under which (some or all of) the elements of the inverse Jacobian matrix can be signed. Some of the procedures used are developed algorithmically. Chapter 4 contains examples.

Chapters 5–8 examine the same issues when the Jacobian matrix must be consistent with principles governing the system being studied. By this means particular quantitative matrices that are consistent with sign and other nonparametric restrictions, but are not consistent with the basic principles governing the system can be eliminated from consideration. Chapter 5 is our first demonstration of how augmenting information about the entries of the Jacobian matrix with information about the nature of the equilibrium system itself can lead to definitive results. In this chapter the assumption that the equilibrium equations arise from an optimization problem is shown to have pervasive implications for sign solvability. For an unconstrained maximization, as is the case with firms maximizing profit, the assumption of a regular maximum implies that the matrix A in (1.4) is negative definite. When the equations arise from constrained maximization, such as individual choice under a limited budget, the matrix A is negative definite under constraint. This additional information is shown to permit definite conclusions where sign information alone would not.

Chapter 6 explores how invoking the correspondence principle, a system-wide property, can lead to sign solvability. The correspondence principle hypothesizes that equilibrium is stable in the sense that the qualitative matrix DA in (1.7) has characteristic roots with real parts negative, and then uses this (quantitative) information to derive comparative statics results. The additional scope for unambiguous solutions of the qualitative system (1.4) under the correspondence principle is shown to be fairly limited.

Chapters 7 and 8 deal with how system-wide properties tying all the equations together coupled with qualitative information can lead to definitive results. In particular, the quantitative restrictions imposed on models of competitive economies by Walras's law and homogeneity are incorporated into the analysis of general equilibrium systems, in which the pattern of substitutes and complements is assumed to be known. Chapter 7 considers how these system-wide restrictions affect our understanding of comparative statics. Chapter 8 examines the implications for stability. The theme of this book is that definite conclusions about a model's predictions can be reached in many fields of scientific inquiry even though complete quantitative information is not available. In the least informed cases, sign information and other ordinal information can be sufficient for unambiguous predictions in nontrivial models. When they are not, the methods discussed in Chapter 3 identify how additional information might be used to resolve ambiguity. These insights have obvious implications for the design of data research programs.

In addition to information about specific components, scientific models are held together by underlying principles. Maximization, stability, and equilibrium can all impose additional restrictions that can resolve uncertainties relative to a purely qualitative environment. Other principles, taken from outside economies, may also lead to definitive results beyond those reported here. Our hope is that researchers in other fields will build upon this work to establish when a mathematical representation of reality makes unambiguous predictions, i.e., has scientific content.

2

Qualitative Comparative Statics and Stability

2.1 INTRODUCTION

We begin our analysis with a study of purely qualitative systems, i.e., systems in which the *only* information assumed to be available is sign pattern information concerning the entries of the matrix A and the vector b in (1.4). We first take up the problem of sign solvability. We then investigate sign stability. The appendix to this chapter contains the mathematical results that are necessary for the reader to follow the formal arguments.

2.2 SIGN SOLVABILITY—BACKGROUND

Definitions

A vector x determined by a system of linear equations $Ax = b$ is said to be *fully sign solvable* if for any matrix B with the sign pattern of A and vector c with the sign pattern of b,

$$By = c$$

implies that y has the sign pattern of x. For example,

$$\begin{pmatrix} - & + \\ - & - \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} - \\ 0 \end{pmatrix}$$

is sign solvable with $x_1 > 0$, $x_2 < 0$.

A vector x determined by a system of linear equations as above is said to be *partially sign solvable* if there is a partition of $x = (x^1, x^2)$ where x^1 is sign solvable and x^2 is not. For example,

$$\begin{pmatrix} - & 0 \\ - & - \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} - \\ - \end{pmatrix}$$

is sign solvable for $x_1 > 0$, but not for x_2.

Formally, for any scalar a, define $\operatorname{sgn} a = 0$ if and only if $a = 0$; $\operatorname{sgn} a = 1$ if and only if $a > 0$; and $\operatorname{sgn} a = -1$ if and only if $a < 0$. Given a real matrix $A = [a_{ij}]$ then $\operatorname{sgn} A = [\operatorname{sgn} a_{ij}]$. We now introduce the set $Q_A = \{B \mid \operatorname{sgn} B = \operatorname{sgn} A\}$. Thus, Q_A is the set of all matrices with the same sign pattern as A.

The issues of full and/or partial sign solvability arise in connection with systems of the form

$$Ax = b, \tag{2.1}$$

where A and b are specified only in terms of their sign patterns $(+, -, 0)$. The system (2.1) so specified is said to be fully *sign solvable* if and only if the sign of every entry of x is determined. Thus, for any matrix B in Q_A and any vector c such that $\operatorname{sgn} c = \operatorname{sgn} b$, the entries of the solutions y to $By = c$ will have the same sign as the corresponding entries in x. Partially signed systems are defined similarly.

Qualitative Economics

A traditional starting point for qualitative analysis is the discussion provided for economists in a brief initial section of Samuelson's *Foundations of Economic Analysis* (1947, 23–29) called "A Calculus of Qualitative Relations." In this section, Samuelson confronts the issue of whether or not a comparative statics analytic framework (i.e., an equation system's Jacobian matrix and b matrix) can yield definite results given only "a general feeling for the direction of things" (i.e., the signs of the matrix's entries). Then, as now, the question is compelling, since the quantification of the derivatives associated with a model's solution may not be possible on the basis of theoretical principles, and even if accomplished in practice, will often be transitory and error prone. Samuelson considered a few small examples to illustrate the analytic problem, and then despaired. The conclusion reached was that the chances that the conditions for a successful analysis would be satisfied were simply too small for the possibility to be taken seriously.

The specific example he cited concerned the computation of a single parameter sensitivity. Using Cramer's rule, this sensitivity can be expressed as the ratio of two determinants, that of the Jacobian matrix itself and that of the same matrix with the appropriate column replaced with the right-hand side of the linear system being manipulated. For $n \times n$ matrices, there are $n!$-many terms being summed to the value of each determinant. Accordingly, for (say) $n = 10$, each determinant will involve the sum of over three million terms. From Samuelson's perspective, "Regarded simply as a problem in probability, the chance that a run of this length should always have one sign is about one out of one with a million zeros after it." As a result, the issue is abandoned in favor of processing information about the Jacobian matrix in addition to the signs of its entries.

There is no doubt that the conditions for a purely qualitative analysis are restrictive, and to that degree "improbable." And further, it is clear, as Samuelson pointed out, that other information usually would be needed to resolve qualitative questions (e.g., the maximization hypothesis). Nevertheless, the issue at stake that motivates the consideration of a purely qualitative analysis remains important. It is unfortunately true that sometimes only the signs of sensitivities within a model's system of relationships can be safely assumed. Given this, a thorough review of how a qualitative analysis could be performed remains interesting, if only to have it in hand against a rare chance to use it. And besides, the situation is not so bad as Samuelson supposed. For a variety of reasons, opportunities for a successful qualitative analysis of applied models are not impossibly rare.[1]

The most important circumstance that serves to moderate the "improbability" of a successful qualitative analysis is the fact that the Jacobian matrices corresponding to applied models can have many zero entries. The conditions for a successful analysis are only that the nonzero terms in the expansions of determinants have the same sign, and there may be far fewer of these than $n!$ for an n-variable system. Further, as discussed in Chapter 3, even if a qualitative analysis fails, the number of "wrong" signed terms may be small. As a result, through qualitative analysis, researchers can identify precisely what extraqualitative information they would need to reach unambiguous conclusions. In addition, conditions necessary and sufficient for full or partial sign solvability in the purely qualitative case are still valid when quantitative information is available as well. Thus, resolving (or at least confronting) the purely qualitative problems of full and partial sign solvability is an essential first step in handling the more common analytical problem in which other as well as qualitative information is present.

2.3 THE ALGORITHMIC APPROACH TO STRONG SIGN SOLVABILITY

The Standard Form Algorithm

Lancaster (1962) published the first formal attempt to develop necessary and sufficient conditions for sign solvability. Lancaster conjectured that if a system could be shown to yield qualitative results, then it could