DARYL GELLER

Analytic Pseudodifferential Operators for the Heisenberg Group and Local Solvability. (MN-37)

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by

Daryl Geller

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Dedicated to my mother, Libby, and
to the memory of my father, Samuel.

CONTENTS

Analytic Pseudodifferential Operators for the Heisenberg Group and Local Solvability

Introduction

Section One: Main Results

The main purpose of this book is to develop a calculus of pseudodifferential operators for the Heisenberg group $\mathbb{H}^n$, in the (real) analytic setting, and to apply this calculus to the study of certain operators arising in several complex variables. Our main new application is the following theorem (Theorem 10.2 and Corollary 10.3):

1. Suppose M is a smooth, compact CR manifold of dimension $2n + 1$. Suppose $\mathcal{U} \subset M$ is open and is a real analytic strictly pseudoconvex CR manifold. Further suppose:

(i) There is a smooth, bounded pseudoconvex domain $D \subset \mathbb{C}^{n+1}$ with boundary M. (D may be weakly pseudoconvex.)

Let S denote the Szegö projection (onto ker $\bar{\partial}_b$ in $L^2(M)$). Then:

(a) If $f \in L^2(M)$, $V \subset \mathcal{U}$, V open, and f is analytic on V, then Sf is analytic on V.

(b) Say $f \in L^2(M)$, $p \in \mathcal{U}$. Then there exists $\omega \in (\mathcal{E}')^{0,1}(M)$ with $\bar{\partial}_b^*\omega = f$ near p if and only if Sf is analytic near p.

(b)' Say $f \in L^2(M)$, $p \in \mathcal{U}$. Then there exists $\omega \in (\mathcal{E}')^{0,1}(M)$ with $\bar{\partial}_b^*\omega = f$ near p if and only if Sf can be extended to a holomorphic function on a neighborhood U of p in $\mathbb{C}^{n+1}$.

(c) Let V be an open subset of $\mathcal{U}$. If $f \in L^2(M)$ and $\bar{\partial}_b f$ is analytic on V, then (I-S)f is analytic on V.

Even if D is strictly pseudoconvex everywhere, the result is new for $n = 1$. (If $n > 1$, D strictly pseudoconvex, it follows from the results in [77], [81].)

The theorem we shall present is actually more general than what we have stated. For example, instead of (i), for (a), (b), (c) one need only assume:

(i)' The range of $\bar{\partial}_b : C^\infty(M) \to \Lambda^{0,1}(M)$ is closed in the C^∞ topology.

(This is weaker than (i) by the results of Kohn [58].)

Further, (a), (b) and (c) hold for $f \in \mathcal{E}'(\mathcal{U}) + L^2(M)$ (not just $f \in L^2$.) As we shall discuss, in many situations, (a), (b) and (c) hold for $f \in \mathcal{E}'(M)$.

The importance of knowing that S preserves analyticity was first recognized by Greiner-Kohn-Stein [35]. They proved the analogue of our result for $M = \mathbb{H}^n$. In the case of $\mathbb{H}^1$, $\bar{\partial}_b^*$ is the unsolvable operator of Lewy and (b) gives necessary and sufficient conditions on f for local solvability of the Lewy operator.

Greiner, Kohn and Stein showed that (b) follows readily from (a). In fact, from (i)', one sees that one can always globally solve $\bar{\partial}_b^* \omega_1 = (I-S)f$; so we need only understand when we can solve

$\bar{\partial}_b^* \omega_2 = Sf$ near p. If Sf is analytic near p, this can clearly be done by Cauchy-Kowalewski. If, on the other hand, $\bar{\partial}_b^* \omega_2 = Sf$ near p, and $\omega_2 \in E'(U)$, then $Sf = S(Sf - \bar{\partial}_b^* \omega_2)$ is analytic near p, by (a) for $E'(U) + L^2(M)$.

We prove (a) by use of the work of Henkin [43]. On strictly pseudoconvex domains, Kerzman-Stein [52] found a simple relationship between S and the Henkin projector H (which also projects onto ker $\bar{\partial}_b$). S is the product of H and the inverse of a singular integral operator. In order to prove that, in appropriate situations, inverses of singular integral operators also preserve analyticity, we need a calculus of analytic pseudodifferential operators. Boutet de Monvel and Kree [9] developed an analytic calculus which is suitable for dealing with elliptic operators on $\mathbb{R}^n$; we need a calculus which is suitable for dealing with analogous operators on $\mathbb{H}^n$.

The simplest operators that our calculus is intended to deal with — the analogues of the Laplacian on R^n — are the Folland-Stein operators $L_\alpha (\alpha \in \mathbb{C})$. These second-order differential operators on $\mathbb{H}^n$ are intimately connected to the Kohn Laplacian $\square_b$. Folland and Stein [20] showed that if $\alpha, -\alpha \notin \{n, n+2, \ldots,\}$, then

$$L_\alpha \phi_\alpha = \delta \tag{0.1}$$

where ϕ_α is homogeneous with respect to the parabolic dilations which are automorphisms of $\mathbb{H}^n$, and is (real) analytic away from

$0 \in \mathbb{H}^n$. They used this fact to study $\Box_b$ on nondegenerate CR manifolds. They speculated that there ought to be a calculus of pseudodifferential operators modelled on the L_α and parabolically homogeneous distributions on $\mathbb{H}^n$, just as the usual calculus is modelled on Δ and homogeneous distributions on $\mathbb{R}^n$. Such a calculus would then be appropriate for the intrinsic (non-isotropic) Sobolev and Lipschitz spaces on a nondegenerate CR manifold.

We present such a calculus here, and we do so in the analytic setting. Our calculus is analogous to the C^∞ calculus of Taylor [79], but our outlook is quite different from his, and our proofs — of necessity — are much more elaborate, since we are working in the analytic setting.

Besides application #1 above (to the Szegö projection) we obtain a number of other new results in this book. Here is a summary of our main results:

2. A very precise form of an analytic parametrix for $\Box_b$ on any nondegenerate analytic CR manifold (Theorem 9.6). From it, one can read off simultaneously the analytic regularity and the (nonisotopic) Sobolev and Lipschitz regularity for $\Box_b$;

3. An analytic calculus on $\mathbb{H}^n$, natural for dealing with $\Box_b$ and operators like it. Simple, explicit formulae for products and adjoints (Theorem 7.11). Simple and natural representation-theoretic conditions, analogous to ellipticity, for determining if operators in the calculus are analytic hypoelliptic, having

parametrices in the calculus (Theorem 8.1). The calculus may be transplanted to provide a very natural calculus on non-degenerate analytic CR manifolds, and more generally, on analytic contact manifolds.

4. Generalization of the theory of operators like the L_α beyond the study of differential operators, analogous to the way in which the usual theory of pseudodifferential operators introduces one to the notion of elliptic operators which are not differential operators. In this way we find a large new, natural class of analytic hypoelliptic operators (Theorem 8.1). Those which are not differential operators were not previously known to be analytic hypoelliptic. Our calculus is the first analytic calculus modelled on parabolic homogeneity, instead of the usual isotropic notion of homogeneity.

5. A characterization of the Fourier transform of the space $\{K \in S'(\mathbb{R}^n)$: K is homogeneous (with respect to a given dilation structure) and analytic away from $0\}$ (Theorem 1.3). (The dilations are to be of the type $D_r x = (r^{a_1}x_1,\ldots,r^{a_n}x_n)$ for $x \in \mathbb{R}^n$, $r > 0$, where $a_1,\ldots,a_n$ are positive rationals). Of course, $\hat{K}$ is homogeneous. In the isotropic case ($a_1=\ldots=a_n=1$) it is known that $\hat{K}$ must be analytic away from 0. This does not hold in general for other dilation weights. In the parabolic case, for instance, where $a_1 = 2$, $a_2=\ldots=a_n = 1$, with

$(t,x) \in \mathbb{R} \times \mathbb{R}^s = \mathbb{R}^n$, and with dual coordinates (λ,ξ), we have the well-known formula

$$[(|x|^2+it)^{-1}]^{\wedge} = (4\pi\lambda)^{-s/2} e^{-|\xi|^2/4\lambda} H(\lambda) = J_0(\lambda,\xi), \text{ say,} \qquad (0.2)$$

where H is the characteristic function of $(0,\infty)$. J_0 is the familiar kernel of the heat operator $\Delta_\xi + \partial/\partial\lambda$, and is not analytic at $\lambda = 0$.

Our characterization of $\{\hat{K} : K \text{ as above}\}$ is most easily understood in the case $a_1 = p$, $a_2 = \ldots = a_n = 1$, $p \in \mathbb{Q}$, $p > 1$ (Theorem 2.11).

In addition, we lay the groundwork for the following further studies:

6. Generalization of our calculus to a wide class of nilpotent Lie groups ("graded homogeneous groups"—see e.g. [12] for the definition). We see no difficulty in doing this, but it would then usually make more sense to work in the C^∞ setting, since analytic hypoelliptic operators on most other groups are rare.

7. Generalization to the study of operators like the L_α, but for $\alpha \in \pm\{n,n+2,\ldots\}$. In this case, one has not (0.1), but

$$L_\alpha \phi_\alpha = \delta - P_\alpha \qquad (0.3)$$

with ϕ_α, P_α homogeneous and analytic away from 0, and with $f \to f * P_\alpha$ $(f \in L^2)$ being the projection in $L^2(\mathbb{H}^n)$ onto $[L_\alpha S(\mathbb{H}^n)]^\perp$.

ϕ_α is called a relative fundamental solution for L_α.

When $\alpha = n$, L_α is the same as $\Box_b$ on functions on $\mathbb{H}^n$, P_α is the Szegö projection for $\mathbb{H}^n$, and (0.3) is one of the results of Greiner, Kohn and Stein alluded to above. As in (b) of application #1 above, for $f \in E'$, $L_\alpha g = f$ is locally solvable near $u \in \mathbb{H}^n$ if and only if $f*P_\alpha$ is analytic near u.

Peter Heller and this author have obtained a generalization of (0.3) to left-invariant differential operators on $\mathbb{H}^n$ which are "transversally elliptic." This generalization will be presented in a future paper. Historical references and further discussions are in Section Two of this Introduction.

Local Solvability and Analytic Pseudolocality of the Szegö Projection

We now explain in more detail our results on the Szegö projection. Again suppose that M is a smooth compact CR manifold of dimension 2n + 1; suppose $U \subset M$ is open, real analytic, and strictly pseudoconvex, and that the range of $\bar{\partial}_b : C^\infty(M) \to \Lambda^{0,1}(M)$ is closed in the C^∞ topology. Let S be the Szegö projection on M. Our hypotheses are too weak to imply that $S : C^\infty(M) \to C^\infty(M)$, so we cannot necessarily extend $S : E'(M) \to E'(M)$. What we shall show is that S is analytic pseudolocal on U when restricted to its "natural domain." That is: first we shall show $S : C^\infty(M) \to C^\infty(U)$ continuously. Fix $V \subset U$, V open, so that $S : C^\infty(M) \to C^\infty(V)$ continuously. We define $D(S)$, the domain of S, to be $\mathcal{E}'(\mathcal{V}) + L^2(M)$. We shall show that if $f \in \mathcal{D}(S)$, $W \subset \mathcal{U}$, W open, f analytic on W, then Sf is analytic on W. (By the results of Kohn [57], we may take $\mathcal{V} = M$ if $M = \partial D$, D a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, provided D is of finite type if $n = 2$, or D is of finite ideal type if $n > 2$.)

If $n > 1$, and M is in addition globally strictly pseudoconvex, analytic pseudolocality of S follows from the work of Treves [81] and Tartakoff [77]. Indeed, $\Box_b$ on (0,1)-forms has a good "Hodge theory." Let N invert $\Box_b$ on (0,1)-forms on the orthocomplement of its kernel. Kohn's formula

$$I - S = \bar{\partial}_b^* N \bar{\partial}_b \tag{0.4}$$

shows that the analytic pseudolocality of S may be deduced from that of N, which in turn follows from the results of [81], [77].

When $n = 1$, there is no good Hodge theory for $\Box_b$ on 1-forms, so this method cannot be used in this case, even if M is globally strictly pseudoconvex.

Our proof of analytic pseudolocality of S begins with a reduction to a local analogue, and it is in this reduction that we use the "closed range" hypothesis for $\bar{\partial}_b$. The local analogue is this: Say $p \in \mathcal{U}$. We shall show that p has a neighborhood $V \subset\subset \mathcal{U}$, and there exist operators

$$S_1 : \mathcal{E}'(V) \to \mathcal{D}'(V),\ K_1 : (\mathcal{E}')^{0,1}(V) \to \mathcal{D}'(V),\ S_1,K_1 \text{ analytic pseudolocal on } V \tag{0.5}$$

$$K_1\bar{\partial}_b = (I-S_1),\ \bar{\partial}_b S_1 = 0,\ S_1\bar{\partial}_b^* = 0 \text{ on } V \text{ mod analytic regularizing errors.} \tag{0.6}$$

Precisely: if $f \in \mathcal{E}'(V)$, $\omega \in (\mathcal{E}')^{0,1}(V)$, then

$$[K_1\bar{\partial}_b - (I-S_1)]f,\ \bar{\partial}_b S_1 f,\ S_1\bar{\partial}_b^*\omega \text{ are all analytic on } V. \tag{0.7}$$

If V were M and the equations in (0.6) were exact, then S_1 would have to equal S. In general, though, we can show this: say $p \in V_1 \subset\subset V$, V_1 open, $\zeta \in C_c^\infty(V)$, $\zeta = 1$ on V_1. Then we claim:

p has an open neighborhood $V_2 \subset\subset V_1$, so that if $f \in \mathcal{D}(S)$,

$$Sf - S_1(\zeta f) \text{ is analytic on } V_2. \tag{0.8}$$

This would then clearly show analytic pseudolocality of S on $\mathcal{U}$.

To illustrate the method of proof, let us examine the analogue in the C^∞ setting (replace "analytic" by "smooth" in (0.5), (0.7), (0.8); call the new statements (0.5)', (0.7)', (0.8)'). In (0.8)', we may as well assume V_1, supp ζ are as small as we like, by the pseudolocality of S_1. We may also assume $S_1 : \mathcal{E}'(M) \to \mathcal{D}'(V)$, $K_1 : (\mathcal{E}')^{0,1}(M) \to \mathcal{D}'(V)$ and (0.7)' holds for $f \in \mathcal{E}'(M)$, $\omega \in (\mathcal{E}')^{0,1}(M)$. Indeed, to achieve these relations we need only shrink V slightly and multiply the Schwartz kernels of S_1,K_1 by an appropriate smooth bump function ϕ

supported near the diagonal of $M \times M$. If supp ζ, V_1, V_2 are sufficiently small, the value of $S_1(\zeta f)$ on V_2 is unaltered if we change S_1 in this manner, so (0.8)' is unaffected. To establish (0.8)', we just have to look at $S_1 Sf$. On the one hand, by the closed range property of $\bar{\partial}_b$, we have $(I-S)f = \bar{\partial}_b^* u$ for some $u \in E'(M)$; thus, on V, $S_1 Sf = S_1(f-\bar{\partial}_b^* u) = S_1 f + g_1 = S_1(\zeta f) + g_2$ where g_1 is smooth on V, g_2 is smooth on V_1; while $S_1 Sf = (I-K_1\bar{\partial}_b)Sf + g_3 = Sf + g_3$, g_3 smooth on V. (0.8)' follows at once. (This part of the argument is similar to certain reasoning in [10].)

In the analytic setting, the bump function ϕ cannot be chosen to be analytic, of course; so we instead carry out this procedure with a sequence of special bump functions, due to Ehrenpreis. These bump functions, and the errors g_1, g_2, g_3 which result in the above argument, satisfy the conditions for analyticity (i.e. conditions like $|\partial^\alpha F| < CR^{\|\alpha\|}\alpha!$) for $\|\alpha\|$ less than or equal to a number N. We may then let $N \to \infty$ to establish (0.8).

To establish (0.6), since U is analytic and strictly pseudoconvex, we may in fact assume $U \subset M'$, $M' = \partial D'$, $D' \subset \mathbb{C}^{n+1}$ a smooth, bounded strictly <u>convex</u> domian. If $n > 1$, we are done by (0.4). If $n = 1$, we use the work of Henkin and a method of Kerzman-Stein.

Henkin [43] shows that there are operators R, H on M' so that

$$R\bar{\partial}_b = I-H, \quad \bar{\partial}_b H = 0. \tag{0.9}$$

H is the Henkin projection onto the kernel of $\bar{\partial}_b$ in L^2. These relations apparently give us part of (0.6). But H, the generalization from $\mathbb{C}$ of the Cauchy projection, need not be orthogonal. Thus we need not have $H\bar{\partial}_b^* = 0$ near p modulo an analytic regularizing error.

Kerzman and Stein, however, observed [52] that if S' is the Szegö projection on M', then $S'(I+H-H^*) = H$ (since $S'H = H$, $HS' = S' \Rightarrow S'H^* = S'$), so that

$$S' = H(I+H-H^*)^{-1} \tag{0.10}$$

($I + H - H^*$ is invertible on L^2 since $H - H^*$ is skew-adjoint.) Since $(I-S')(I-H) = I - S'$, we would like to obtain (0.6) from (0.9) by putting $K_1 = (I-S')R$. The problem, then, is to obtain analytic pseudolocality on $\mathcal{U}$ for S' from (0.10); and for this we need to develop an analytic calculus on the Heisenberg group.

Overview

Here now is an overview of our calculus. Let A be a classical pseudodifferential operator of order j on $\mathbb{R}^n$, so that

$$(Af)(u) = (2\pi)^{-n}\iint e^{-i(u-v)\cdot\xi}a(u,\xi)f(v)\,dvd\xi. \tag{0.11}$$

Here let us say the symbol $a(u,\xi) \sim \sum a^m(u,\xi)$ as $\xi \to \infty$, where each a^m is smooth in u and homogeneous of degree $j - m$ in ξ.

We may then also formally write

$$(Af)(u) = (K_u * f)(u) \tag{0.12}$$

where $$K_u(w) = (2\pi)^{-n}\int e^{-iw\cdot\xi} a(u,\xi)\,d\xi, \tag{0.13}$$

the inverse Fourier transform of a in the ξ variable. Let us call $K_u(w) = K(u,w)$ the <u>core</u> of the operator A; the <u>kernel</u> of A is then $K(u,u-v)$. We have

$$K_u(w) \sim \sum K_u^m(w) \quad \text{near} \quad w = 0 \tag{0.14}$$

where K_u^m is smooth in u and homogeneous of degree $k + m$ $(k=-n-j)$, at least if $k + m \notin \mathbb{Z}^+ = \{0,1,2,\ldots\}$. (If $k + m \in \mathbb{Z}^+$, $K_u^m(w)$ may in addition contain a "log term" of the form $p_u(w)\log|w|$, where p_u is a homogeneous polynomial in w of degree $k + m$.) A is a differential operator with analytic coefficients if all the K_u^m are supported at 0. A is elliptic if $\hat{K}_u^0(\xi) \neq 0$ for $u \in U$, $\xi \neq 0$; then it has a parametrix of the same type on any relatively compact open subset V of U.

We intend to develop a calculus on $\mathbb{H}^n$ which is analogous to (0.12), (0.14). In (0.12), * will be replaced by group convolution on $\mathbb{H}^n$, and in (0.14), the $K_u^m(w)$ will now be homogeneous in w (or homogeneous plus a "log term") in the <u>parabolic</u> sense. The condition analogous to ellipticity is that $\pi(K_u^0)$ be injective on Schwartz vectors for all non-trivial irreducible unitary representations π of $\mathbb{H}^n$.

The idea of working with operators of type (0.12) in the C^∞ category and in the nilpotent group situation is due to Folland-Stein [20]. Such operators, and generalizations thereof, were also studied by Rothschild-Stein [70] and Nagel-Stein [66]. Taylor [19], in the C^∞ setting on $\mathbb{H}^n$, defines "pseudo-differential operator" by (0.12), and then chooses to pass directly from these operators to define a new kind of symbol. We shall stay instead on the core level, although the Fourier transform will still play an integral role in our work. Also, we will work in the analytic setting. For 3-step nilpotent groups, in the C^∞ setting, a calculus based on cores was also recently constructed by Cummins [14].

Of course, the concept of "core" is far from new. We have introduced the new name since we intend to place absolute emphasis on the core, as opposed to the kernel or the symbol.

It is worth recalling some of the basic reasons why one usually prefers, in the standard situation on $\mathbb{R}^n$, to use definition (0.11) instead of (0.12), when (0.12) has a much simpler appearance. Two elementary reasons are:

(a) The Fourier transform converts convolution to multiplication, which is easier to handle.

(b) The Fourier transform converts the finding of convolution inverses (such as fundamental solutions for constant coefficient differential operators) to division.

Reason (a) loses much of its significance on a nilpotent group, where the Euclidean Fourier transform on the underlying manifold ($\mathbb{R}^n$) converts group convolution to a complicated analogue of multiplication, while the complicated group Fourier transform converts group convolution to multiplication. One could, then, write down formulas analogous to (0.11), but their complexity makes them undesirable to manipulate compared to (0.12). We shall see that, using convolution, we can obtain an analogue of the Kohn-Nirenberg product rule for $\mathbb{H}^n$ (and more generally, for graded homogeneous groups).

As for reason (b), even when one inverts, it is possible in many circumstances, to avoid use of the Fourier transforms. The problem can be reduced to inverting the principal part of the operator (in (0.12), (0.14), the operator with core equal to $K_u^o(w)$ near $w = 0$). The rest can be handled with a Neumann series, after a product rule is established. Thus, the initial problem to be solved is:

> Given K_1 homogeneous and analytic (resp. smooth) away from 0, can we find K_2 homogeneous and analytic (resp. smooth) away from 0, with $K_2 * K_1 = \delta$?

Say we are in the "smooth" setting, and for simplicity, say that K_1 has the same homogeneity as δ. Then, on graded homogeneous groups, a necessary and sufficient condition for the existence of K_2 is that the map $f \to K_1 * f$ be left invertible on L^2 ([12]).

The Fourier transform on the group can nevertheless be a valuable tool for proving K_2 exists. Thus, for certain groups, representation-theoretic conditions for the existence of K_2 are known. In [12], it was conjectured that, in the smooth setting, on homogeneous groups, the existence of K_2 is equivalent to this condition:

> $\pi(K_1)$ is injective on Schwartz vectors for all irreducible unitary representations π (except $\pi \equiv 1$).

This is the analogue of the Helffer-Nourrigat theorem [41] for singular integral operators. The conjecture was proved for $\mathbb{H}^n$ in [12], generalized to graded groups of step less than or equal to three, by Moukaddem [65], and has now been proved in full generality by Glowacki [32].

In the analytic setting, one could only expect to find such a K_2 on a small number of groups (the "H-groups" [62], to which the $\mathbb{H}^n$ belong). One of our objectives is to prove that, in the analytic setting on $\mathbb{H}^n$, the representation-theoretic criterion given above implies the existence of K_2 (Theorem 5.1).

Thus the Fourier transform will still play an important role in our work; but the methods we have discussed will enable us to limit its use to the study of homogeneous distributions. In conclusion, we will frequently take the Fourier transforms of homogeneous distributions, but we will never have to deal with complicated analogues of (0.11).

The Core Approach on $\mathbb{R}^n$

Since the approach to pseudodifferential operators using cores instead of symbols is unfamiliar even on $\mathbb{R}^n$, let alone on $\mathbb{H}^n$, at this point we wish to discuss the $\mathbb{R}^n$ situation in some detail—even though that is not part of the actual subject matter of this book. Many of the methods that can be used on $\mathbb{R}^n$ go through with only minor modification on $\mathbb{H}^n$. When it comes to inversion, however, new methods must be developed for $\mathbb{H}^n$, and that is the main subject matter of the book.

Using cores instead of symbols on $\mathbb{R}^n$, there is of course the annoyance of having to deal with convolution instead of multiplication, but there are certain real conceptual advantages. For instance, as Taylor [79] has pointed out, the Kohn-Nirenberg formula for products of pseudodifferential operators can be proved just by extending the proof that convolution is associative. Indeed, let $\mathcal{K}_1, \mathcal{K}_2$ be the operators with cores K_1, K_2; then

$$(\mathcal{K}_2\mathcal{K}_1 f)(u) = \int K_2(u,u-v)\int K_1(v,v-w)f(w)\,dw\,dv. \qquad (0.15)$$

In this expression we formally write

$$K_1(v,v-w) = \sum(v-u)^{\alpha}\partial_u^{\alpha}K_1(u,v-w)/\alpha! \tag{0.16}$$

to see that $K = K_2K_1$ should have core K, where

$$K_u = \sum_{\alpha}[(-w)^{\alpha}K_{2u}]*[\partial_u^{\alpha}K_{1u}]/\alpha! \tag{0.17}$$

One could then recover the Kohn-Nirenberg formula through (0.12). (A very similar formula holds on $\mathbb{H}^n$—see (0.28) below). This of course, would all be easiest to justify if K_{1u} were analytic in u.

The analytic pseudodifferential operators of Boutet de Monvel and Kree [6] are also most readily motivated if one looks at the cores. (The operators we discuss are slight modifications of those of Boutet de Monvel and Kree.) Roughly speaking, we look at cores K_u as in (0.14) where each $K_u^m(w)$ is analytic in u, analytic in w away from 0, and where K_u^m grows at most on the order of R^m on $|w| = 1$, for some R. Because of the homogeneity of the K_u^m, the series in (0.14) will now <u>converge</u> in a punctured ball about 0, and provide a very natural generalization of a power series.

A little notation will help to make things clearer. If $k \notin \mathbb{Z}^+$, let

$$\mathcal{A}\mathcal{K}^k = \{K \in \mathcal{D}' : K \text{ is homogeneous of degree } k \text{ and analytic away from } 0\}. \tag{0.18}$$

If $k \in \mathbb{Z}^+$, let

$$\mathcal{A}\mathcal{K}^k = \{K \in \mathcal{D}' : K = K' + p(x)\log|x|, \text{ with } K', p \text{ homogeneous of degree } k \text{ and analytic away from } 0, \text{ and } p \text{ a polynomial}\}. \tag{0.19}$$

If $U \subset \mathbb{R}^n$ is relatively compact and open, and $k \in \mathbb{C}$, we let $C^k(U)$ denote the space of all cores of the form

$$K_u(w) = \chi(w)\left[\sum_{m=0}^{\infty} K_u^m(w) + Q(u,w)\right] \tag{0.20}$$

where for some $r,R > 0$,

$$\chi \text{ is the characteristic function of the ball } B_r = \{w : |w| < r\}, \tag{0.21}$$

$$K_u^m \in \mathcal{A}\mathcal{K}^{k+m} \text{ for all } u \in U, \text{ and depends analytically on } u; \tag{0.22}$$

$$Q \text{ is analytic in } U \times B_r; \tag{0.23}$$

The series $\sum K^m(v,w)$ converges absolutely for v in a complexified neighborhood of $\bar{U}$ and for $0 < |w| \leqq r$, to a function holomorphic in v and analytic in w. (0.24)

For every compact $E \subset U$ there exists $C_E > 0$ so that

$$|\partial_u^\gamma \partial_w^\rho Q(u,w)| < C_E R^{|\gamma|+|\rho|}\rho!\gamma! \quad \text{for } u \in E,\ w \in B_r; \tag{0.25}$$

$$|\partial_u^\gamma \partial_w^\rho K^m(u,w)| < C_E R^{|\gamma|+|\rho|+m}\rho!\gamma! \text{ for } 1 \leqq |w| \leqq 2, \ u \in E,\ \gamma,\rho \text{ multi-indices.} \tag{0.26}$$

We write $K \sim \sum K_u^m$, and we call r the support radius of K. ($\mathcal{C}^k(U)$ can also be defined for U not relatively compact, by modifying the definition slightly.) By (0.26), K_u^m grows on the order of at most R^m on $|w| = 1$, so the convergence of the series legislated by (0.24) is actually a consequence of (0.26)—at least for w sufficiently small ($0<|w|<1/3R$ will do). The series in fact clearly has a holomorphic extension to $\sum K^m(v,\omega)$ where $(v,\omega) \in \mathcal{V} \times S$, $\mathcal{V}$ being a complexified neighborhood of $\overline{U}$, S being a truncated sector of the form $\{\omega \in \mathbb{C}^n$: $|\text{Im}\ \omega| < c|\text{Re}\ \omega|,\ |\omega| < b\}$ for some $c,b > 0$; (0.26) is virtually equivalent to this. Thus $\mathcal{C}^k(U)$ is an extremely natural space of cores.

The one arbitrary point in the definition is the use of the characteristic function χ in (0.20) to localize. One might consider using instead a C^∞ cutoff function ϕ, or the characteristic function χ' of an arbitrary neighborhood of 0. Recall, however, that $(\mathcal{K}f)(u) = (K_u * f)(u)$. We shall always "operate locally"— that is, we shall assume supp f has small diameter and we shall only be interested in the behavior of $\mathcal{K}f$ at points in or very near supp f. Operating locally, it cannot matter whether one uses χ, ϕ or χ' to localize; $\mathcal{K}f$ will be the same.

Operating locally, it is easy to believe that operators with cores in $\mathcal{C}^k(U)$ are analytic pseudolocal, and that an operator with core in $C^{k_1}(U)$ may be composed with one whose core is in $C^{k_2}(U)$, if their support radii are unequal, to produce one with core in $C^{k_1+k_2+n}(U)$. Operating locally, it is also easy to believe that if $V \subset\subset U$, V open, then an elliptic operator with core in $C^{k-n}(U)$ will have, on V, as a parametrix an operator with core in $C^{-k-n}(V)$. The parametrix inverts the operator up to an analytic regularizing operator $\mathcal{Q}$, with core $Q \sim 0$.

All these statements are true if one operates locally, and the reader can no doubt envision what some of the proofs might look like, using such techniques as (0.15)-(0.17). We still need to clarify several points about the calculus, but let us first pause to see how things look on $\mathbf{H}^n$.

The Core Approach on $\mathbf{H}^n$

$\mathbf{H}^n$ is the Lie group with underlying manifold $\mathbb{R} \times \mathbb{C}^n$ and multiplication $(t,z)\cdot(t',z') = (t+t'+2\,\mathrm{Im}\, z\cdot\bar{z}', z+z')$ where $z\cdot\bar{z}' = \sum z_j\bar{z}_j'$. The dilations $D_r(t,z) = (r^2t, rz)$ are automorphisms. We write $z = x+iy$. The left-invariant vector fields agreeing with $\partial/\partial x_j, \partial/\partial y_j, \partial/\partial t$ at 0 are $X_j = \partial/\partial x_j + 2y_j\partial/\partial t$, $Y_j = \partial/\partial y_j - 2x_j\partial/\partial t$, $T = \partial/\partial t$. The right-invariant analogues are $X_j^R = \partial/\partial x_j - 2y_j\partial/\partial t$, $Y_j^R = \partial/\partial y_j + 2x_j\partial/\partial t$, and T. Put $Z_j = (X_j - iY_j)/2$, $Z_j^R = (X_j^R - iY_j^R)/2$.

If $u = (t,z) \in \mathbb{H}^n$, we put $|u| = (t^2+|z|^4)^{1/4}$; then $|\,|$ is (parabolically) homogeneous of degree one and is analogous to the norm function on $\mathbb{R}^n$.

With $|\,|$ as above, and using parabolic homogeneity, we define AK^k as in (0.18)-(0.19) and $C^k(U)$ as in (0.20)-(0.26) for $U \subset \mathbb{H}^n$ open and relatively compact. The operator with core K is K where $(Kf)(u) = (K_u * f)(u)$ for $f \in C_c^\infty(U)$, and where we use group convolution.

Operating locally, analytic pseudolocality again holds for these operators. As for composition, we replace (0.16) by

$$K_1(v,vw^{-1}) = \sum_\alpha (vu^{-1})^\alpha \sigma_\alpha(\mathcal{U}) K_1(u,vw^{-1})/\alpha! \tag{0.27}$$

invoking Taylor's theorem for $\mathbb{H}^n$. Here $\mathcal{U} = (T, X_1^R, \ldots, X_n^R, Y_1^R, \ldots, Y_n^R)$ acting on the u variable, and for $\alpha \in (\mathbb{Z}^+)^{2n+1}$, $\sigma_\alpha(\mathcal{U})$ is the <u>symmetrization</u> of $\mathcal{U}^\alpha$, familiar from the Birkhoff-Poincaré-Witt theorem. That is, $\sigma_\alpha(\mathcal{U})$ is the coefficient of τ^α in the multinomial expansion of $(\alpha!/\|\alpha\|!)(\tau \cdot \mathcal{U})^{\|\alpha\|}$. (Here $\tau \in R^{2n+1}$, $\|\alpha\| = \alpha_1 + \ldots + \alpha_{2n+1}$.) If the X_j^R, Y_j^R commuted, $\sigma_\alpha(\mathcal{U})$ would be just $\mathcal{U}^\alpha$; as it is, however, the symmetrization of $X_1^R Y_1^R$ (for example) is $(X_1^R Y_1^R + Y_1^R X_1^R)/2$. (0.27) follows easily from right invariance and the observation that the right-invariant operator agreeing with ∂^α at 0 is $\sigma_\alpha(\mathcal{U})$.

Using (0.23), we see that the formal composition rule (0.17) should be replaced by:

$$K_u = \sum_\alpha [(-w)^\alpha K_{2u}]*[\sigma_\alpha(U)K_{1u}]/\alpha! \tag{0.28}$$

In this manner, we develop a natural analytic calculus which contains $\square_b$ and a parametrix for it, on nondegenerate analytic CR manifolds, in the following sense. A nondegenerate analytic CR manifold M is an example of what is called a contact manifold, and Darboux's theorem implies that M is locally analytically diffeomorphic to $\mathbb{H}^n$ by means of a specific kind of diffeomorphism—a contact transformation. One of our objectives, then, is to prove that after such a contact transformation, $\square_b$ and a parametrix for it lie in the analogue of our calculus for systems. This will be true under Kohn's hypothesis—if M is k-strongly pseudoconvex, we must be looking at $\square_b$ on (0,q) forms, where $q \neq k$ or $n - k$.

In the situation of (0.6), K_1 is a (left) relative parametrix for $\bar{\partial}_b$ on functions; it and S are locally in our calculus. We can also obtain a (left or right) analytic parametrix for $\square_b$ on functions, within the calculus.

The Core Approach on $\mathbb{R}^n$ and the Fourier Transform

Continuing with our description of the core approach on $\mathbb{R}^n$, let us explain in more detail the role played by the Fourier transform.

Given a sequence $K^m \in AK^{k+m}$, satisfying estimates

$$|\partial^\rho K^m(w)| < CR^{m+|\rho|}\rho! \quad \text{for} \quad 1 \leq |w| \leq 2 \tag{0.29}$$

we let $J^m = \hat{K}^m$. Boutet de Monvel and Kree [9] show that there are dual estimates

$$|\partial^\rho J^m(\xi)| \leqq C_o R_o^{m+|\rho|} \rho! m! \quad \text{for} \quad 1 \leqq |\xi| \quad 2. \tag{0.30}$$

We review the proof of this fact, since the method is very important for us. For simplicity we assume Re $k < 0$, in which case (as is known and as we shall verify in much greater generality later) $\hat{}: AK^k \to AK^{-n-k}$; further if $C_1, R_1 > 0$ there are C_2, R_2 depending only on C, R so that if $K \in AK^k$, $J = \hat{K}$ and

$$|\partial^\rho K(w)| < C_1 R_1^{|\rho|} \rho! \quad \text{for} \quad 1 \leqq |w| \leqq 2, \text{ all } \rho, \tag{0.31}$$

then $|\partial^\rho J(\xi)| < C_2 R_2^{|\rho|} \rho!$ for $1 \leqq |\xi| \leqq 2$, all ρ. (0.32)

In (0.29), note that if $|\alpha| = m$, we have estimates

$$|\partial^\rho (\partial^\alpha K^m)(w)| < C_3 R_3^{m+|\rho|} \rho! m!$$

and $\partial^\alpha K^m \in AK^k$. By (0.31), (0.32), this leads to

$$|\partial^\rho (\xi^\alpha J^m)(\xi)| < C_4 R_4^{m+|\rho|} \rho! m! \quad \text{if} \quad |\alpha| = m.$$

If the left side were $|(\xi^\alpha \partial^\rho J^m)(\xi)|$ instead, this would give us (0.30). It is, in fact, not hard to reach (0.30) from here.

One can reverse this argument up to a point. Starting with (0.30), if Re $k < 0$, one can deduce (0.29) for $|\rho| \geqq m$. The argument gives no estimate for $\partial^\rho K^m$, $|\rho| < m$. This, in fact, could hardly be expected, since if k is an integer, J^m

is not even, in general, well-determined by its values for $1 \leqq |\xi| \leqq 2$. J^m is very singular near 0 for m large, and we must specify its effect on test functions whose supports include 0. It is well known how one can do this. For instance, if $G \in C^\infty(\mathbb{R}^n - \{0\})$ is homogeneous of degree $-n$, one can define a distribution $J \in (AK^0)^\wedge$ agreeing with G away from 0 by

$$J(f) = \int_{|x|<1} G(x)[f(x)-f(0)]dx + \int_{|x|\geqq 1} G(x)f(x)dx$$

for $f \in C_c^\infty$. If instead G is homogeneous of degree $-n-\varkappa$, $\operatorname{Re} \varkappa \geqq 0$, one can similarly define a $J \in (AK^\varkappa)^\wedge$ agreeing with G away from 0, by use of similar integrals, in one of which one subtracts additional terms of the Taylor series of f at 0, and splits at $|x| = 1$. We can write $J = \Lambda_G$. It is then not hard to show that if the J^m are as in (0.30), $J^m = G^m$ away from 0, and $J^m = \Lambda_{G^m}$, then (0.29) follows even for $|\rho| < m$.

Our second point illustrates the assistance of the Fourier transform. Consider the formal composition law (0.17), as applied to elements of the C^k spaces. If $K_1 \sim \sum K_1^m$, $K_2 \sim \sum K_2^m$, the support radii are unequal, and K is the operator obtained by composing K_2 and K_1, then one might imagine K has core $K \sim \sum K^m$, where

$$K_u^m = \sum_{|\alpha|+a+b=m} [(-w)^\alpha K_{2u}^b] * [\partial_u^\alpha K_{1u}^a]/\alpha! \tag{0.33}$$

This is hardly possible, since if a and b are large, the convolution no longer makes sense, due to the divergence of the integral at ∞. It is necessary, then, to generalize convolution. Change notation, then; say $\mathrm{Re}(k_1+k_2) > -n$, and $K_\nu \in AK^{k_\nu}$ for $\nu = 1,2$; what replacement is available for K_2*K_1? One could choose K with $\hat{K} = \hat{K}_2\hat{K}_1$ away from 0; but it is also necessary to specify $\hat{K}$ near 0. In fact, if $\hat{K}_\nu = G_\nu$ away from 0, $\nu = 1,2$, we put

$$K_2 \underline{*} K_1 = (\Lambda_{G_2 G_1}) \tag{0.34}$$

with Λ as before. One can then see that (0.33) should be replaced by

$$K_u^m = \sum_{|\alpha|+a+b=m} [(-w)^\alpha K_{2u}^b] \underline{*} [\partial_u^\alpha K_{1u}^a]/\alpha! \tag{0.35}$$

In fact, it follows easily from our earlier comments about Λ that if K_u^m is as in (0.35) then the K_u^m satisfy estimates as in (0.26).

[It is less clear that K, the composition of K_2 and K_1, has a core K asymptotic to $\sum K^m$ (operating locally). One must see why, if Q_u^m is the difference between the right side of (0.35) and

$$\sum_{|\alpha|+a+b=m} [(-w)^\alpha \chi_2 K_{2u}^b] * [\partial_u^\alpha \chi_1 K_{1u}^a]/\alpha!$$

then Q_u^m is analytic in (u,w) and small enough that $\sum Q_u^m(w)$ converges to an analytic function of (u,w) ($u \in U$, $|w|$ small). (χ_1, χ_2 are characteristic function of balls about 0 of radii

r_1, r_2, with $r_1 \neq r_2$.) changing notation, this comes down to a study of

$$Q = K_2 * K_1 - \chi_2 K_2 * \chi_1 K_1 \tag{0.36}$$

near 0, if $K_\nu \in AK^{k_\nu}$ ($\nu = 1,2$), $\mathrm{Re}(k_1+k_2) > -n$. Say $r_1 > r_2$. If $|\gamma| > M = [\mathrm{Re}(k_1+k_2)] + n$, one can rather easily estimate $\partial^\gamma Q = (1-\chi_2)K_2 * \partial^\gamma K_1$ near 0, and thereby also prove Q is analytic near 0. It is then necessary to estimate the Taylor polynomial of Q about 0 of degree M. Since this is a polynomial, it is enough to estimate this on a small sphere about 0, and this can be done rather easily directly from (0.36).]

If $K_1 \sim \sum K_1^m$, $K_2 \sim \sum K_2^m$ are in the C^k spaces, we define the formal composition of formal series

$$(\sum K_2^m) \# (\sum K_1^m) = \sum K^m \tag{0.37}$$

with K^m as in (0.35).

Our third and last point about the core approach on $\mathbb{R}^n$ concerns the finding of parametrices for elliptic operators in the calculus, and how the Fourier transform is again of assistance. If $K_1 \sim \sum K_1^m$ is elliptic, we begin by composing with a core asymptotic to $(1/\hat{K}_1^0)^\vee$ to reduce (by (0.35)) to the case $K_1^0 \equiv \delta$. One then solves this case by use of a formal Neumann series, in which $\sum\limits_{m>0} K_1^m$ is repeatedly #'d with itself. To estimate the resulting terms, one uses the Fourier transform and the duality between the estimates (0.29) and (0.30). This requires some simple combinatorics. Once the formal inverse is found, the composition theorem is invoked to rigorously produce a parametrix.

The Core Approach on $\mathbb{H}^n$ and the Fourier Transforms

On $\mathbb{H}^n$, (0.35) should be replaced by

$$K^m = \sum_{|\alpha|+a+b=m} [(-w)^\alpha K^b_{2u}] \underline{*} [\sigma_\alpha(U) K^a_{1u}]/\alpha! \qquad (0.38)$$

where $\underline{*}$ is "generalized group convolution." (0.34), of course, cannot be used. One must first adapt Λ to the parabolic setting. Next note that $K_2 * T^N K_1$ makes sense for N large. Using F to denote Euclidean Fourier transforms on $\mathbb{R}^{2n+1} = \mathbb{H}^n$, and with (λ,p,q) dual to (t,x,y), put $G(\lambda,p,q) = F(K_2 * T^N K_1)/(-i\lambda)^N$ for $\lambda \neq 0$; this ought to be $F(K_2 \underline{*} K_1)$ for $\lambda \neq 0$. In fact G has a C^∞ extension for $(\lambda,\xi,\eta) \neq 0$, and one defines $K_2 \underline{*} K_1 = F^{-1}\Lambda_G$.

This description of $K_2 * K_1$ is the simplest, but it is not clear how to generalize this construction to other groups. An alternative procedure, however, will generalize to other groups, at least in the C^∞ setting; and in fact we shall be using this procedure on $\mathbb{H}^n$ (Chapter 7, Section 2). It is a "Poincaré-lemma" type of procedure; let us illustrate it on $\mathbb{R}^n$ in the C^∞ situation. Say $K_\nu \in K^{k_\nu}$ for $\nu = 1,2$. If N is sufficiently large, we can surely define $K_\alpha = K_2 * \partial^\alpha K_1$ for all $\alpha \in (\mathbb{Z}^+)^n$, $\|\alpha\| \geq N$; and surely $\partial^\beta K_\alpha = \partial^\alpha K_\beta$ for all α,β. Then we need only find a K with

$\partial K^{\alpha} = K_{\alpha}$ for all α; such a K, though only well-defined up to a polynomial, could serve as $K_2 * K_1$. In the analytic setting we would uniquely determine K by requiring $\hat{K} = \Lambda_G$ for some G. (For 3-step nilpotent groups, in the C^{∞} setting, a similar procedure was also recently followed by Cummins [14].)

Back on $\mathbb{H}^n$, the arguments needed to establish analytic pseudolocality, and the composition and adjoint theorems are given in Chapter 7. However, the main problem is in finding analytic parametrices under the hypothesis which is analogous to ellipticity--that $\pi(K_u^0)$ is injective on Schwartz vectors for all $u \in U$ and all non-trivial irreducible unitary representations π of $\mathbb{H}^n$. This necessitates that we obtain a complete understanding of $F(AK^k)$ (Chapters 1 and 2) as well as a partial understanding of the group Fourier transform of AK^k (Chapters 4 and 5). The two Fourier transforms will be used, as in the Euclidean case, to invert the principal core (Chapter 5) and then to estimate the terms of the Neumann series (Chapter 8).

Using parabolic homogeneity, neither step would be possible if we used standard Euclidean convolution. $\mathbb{H}^n$ convolution must be used. This is an indication of how subtle the problem is.

The Fourier Transform of AK^k

Again we write ^ for Euclidean Fourier transform; for the reasons indicated above, we seek to understand $(AK^k)\hat{}$.

A characterization of this space can be given for general dilations $D_r x = (r^{a_1}x_1,\dots,r^{a_n}x_n)$ on $\mathbb{R}^n$, $a_1,\dots,a_n$ positive rationals (Theorem 1.3). Our characterization is by far clearest, however, if $n = s + 1$, $(a_1,\dots,a_{s+1}) = (p,1,\dots,1)$, $p > 1$, and we restrict to this case here. We use coordinates $(t,x) \in \mathbb{R} \times \mathbb{R}^s$ with dual coordinates (λ,ξ). We define AK^k as in (0.18), (0.19), where now $|\;|$ is a fixed homogeneous norm function which is analytic away from 0. (Thus $|\;|$ is homogeneous of degree 1, $|x| \geqq 0$ for all x, and $|x| = 0 \Leftrightarrow x = 0$.)

If $K \in AK^k$, then $J = \hat{K}$ is homogeneous and smooth away from 0, so that it is determined away from 0 by two functions,

$$J_+(\xi) = J(1,\xi), \text{ and } J_-(\xi) = J(-1,\xi).$$

Looking at (0.2), one would suspect that J_+ and J_- must be restrictions to $\mathbb{R}^s$ of entire functions on $\mathbb{C}^s$. This turns out to be correct. Let us denote the entire extensions by $J_+(\zeta)$, $J_-(\zeta)$ for $\zeta \in \mathbb{C}^s$. The salient question is: what is the class of entire functions which arises in this way? Again, looking at (0.2), one might guess that if $p = 2$, then $J_+(\zeta)$ and $J_-(\zeta)$ must have exponential order 2. (An entire functions F on $\mathbb{C}^s$ is said to have exponential order m if for some B, C > 0, $|F(\zeta)| < Ce^{B|\zeta|^m}$ for all $\zeta \in \mathbb{C}^s$.) This also turns out to be correct. Further, if $p \in \mathbb{Q}$ is general, we shall show that J_+ and J_- must have exponential order q, where $(1/p) + (1/q) = 1$.

The behavior of $J_+(\xi)$, $J_-(\xi)$ as $\xi \to \infty$ ($\xi \in \mathbb{R}^s$) is more subtle. One might guess from (0.2) that these functions must have Gaussian decay when $p = 2$. This turns out to be not true in general, but it is true of an important subclass. To be precise, let us define

$$Z_q^q = Z_q^q(\mathbb{C}^s) = \{\text{entire functions } f \text{ on } \mathbb{C}^s \mid \text{ for some } B_1, B_2, C > 0 \text{ we have } |f(\xi)| < Ce^{B_1|\xi|^q} \text{ for } \xi \in \mathbb{C}^s, \text{ while } |f(\xi)| < Ce^{-B_2|\xi|^q} \text{ for } \xi \in \mathbb{R}^s\}.$$

We say $f \in Z_q^q(\mathbb{R}^s)$ if f is the restriction to $\mathbb{R}^s$ of a function $f \in Z_q^q(\mathbb{C}^s)$. This space was first investigated by Gelfand-Silov ([21], [22]); we shall say much more about it presently. For now, let us simply state that <u>if</u> J_+, J_- are two functions in $Z_q^q(R^s)$, <u>then</u> there exists $K \in \mathcal{AK}^k$ such that $\hat{K}(1,\xi) = J_+(\xi)$, $\hat{K}(-1,\xi) = J_-(\xi)$.

However, the converse is not true; if $K \in \mathcal{AK}^k$, $\hat{K}(1,\xi) = J_+(\xi)$ and $\hat{K}(-1,\xi) = J_-(\xi)$ then $J_+(\xi)$ and $J_-(\xi)$ need not decay so rapidly as $\xi \to \infty$. One observation which will be crucial to us in understanding the behavior of J_+ and J_- as $\xi \to \infty$ was made by Taylor [79], who studied homogeneous distributions which are only assumed to be smooth away from 0. The observation is that, since $\hat{K}(\lambda,\xi)$ is homogeneous, its behavior as $\lambda \to 0$ fixed $\xi \neq 0$ is reflected in its behavior as $\xi \to \infty$ for fixed $\xi \neq 0$. Now, for fixed $\lambda \neq 0$, $\hat{K}(\lambda,\xi)$ has a Taylor series expansion in λ

about $\lambda = 0$. This then implies that J_+ and J_- must have asymptotic expansions in ξ at infinity. It is rare, then, that $J_+(\xi)$ and $J_-(\xi)$ will decay rapidly at infinity; this will only happen when the aforementioned Taylor series vanish identically.

The assumption that K is analytic away from 0 implies, as we shall see, that there are in addition certain growth restrictions on the terms of the asymptotic series for J_+ and J_-. Precisely, we can describe a new space in which J_+ and J_- must lie. Let

$Z^q_{q,j}(\mathbb{C}^s) = \{$entire functions $f(\xi)$ on $\mathbb{C}^s \mid$ for some B,C,R,c:

(i) $|f(\zeta)| < Ce^{B|\zeta|^q}$ for all ζ

(ii) In the sector $S = \{\zeta = \xi + i\eta : |\eta| < c|\xi|\}$ there exist holomorphic functions $g_\ell(\zeta)$, homogeneous of degree $j - p\ell$, which satisfy $|g_\ell(\zeta)| < CR^\ell \ell!^{p-1}$ for $|\zeta| = 1, \zeta \varepsilon S$, and such that for all $L > 0$, if $|\zeta| > 1$, $\zeta \varepsilon S$, then

$$\left|f(\zeta) - \sum_{\ell=0}^{L-1} g_\ell(\zeta)\right| < CR^L L!^{p-1}|\zeta|^{\mathrm{Re}\, j-pL}.\} \qquad (0.39)$$

In the situation above, we write $f \sim \sum g_\ell$. The relation with Z^q_q is that $f \varepsilon Z^q_q \Leftrightarrow f \varepsilon Z^q_{q,j}$ and $f \sim 0$.

We say $f \varepsilon Z^q_{q,j}(\mathbb{R}^s)$ if f is the restriction to $\mathbb{R}^s$ of a $Z^q_{q,j}$ function on $\mathbb{C}^s$. Also, let $Q = p+s$.

The main result of Chapters 1 and 2 of this book is:

Theorem 2.11. Suppose $J_+, J_- \in C^\infty(\mathbb{R}^s)$ and $j \in \mathbb{C}$. Then there exists $K \in AK^{-Q-j}$ such that $\hat{K}(1,\xi) = J_+(\xi)$ and $\hat{K}(-1,\xi) = J_-(\xi)$ for all ξ if and only if:

$$J_+, J_- \in Z^q_{q,j}(\mathbb{R}^s) \text{ and if } J_+ \sim \sum g_\ell \text{ then } J_- \sim \sum (-1)^\ell g_\ell.$$

If this is the case, then

$$g_\ell(\xi) = (\partial_\lambda^\ell J)(0,\xi)/\ell!. \tag{0.40}$$

Let us explain one of the main ideas in the proof of Theorem 2.11. This idea is fully exploited in Chapter one. For $k \notin \mathbb{Z}^+$, let

$$K^k = \{\text{distributions } K \text{ which are homogeneous of degree } k \text{ and smooth away from } 0\}.$$

Say for simplicity that $p = 2$ and $-Q < \operatorname{Re} k < 0$. Then, not only does $\hat{}: K^k \to K^{-Q-k}$, but it does so continuously, in the following sense. Let $\| \ \|_\infty$ denote sup norm on the "unit sphere" $\{x: |x| = 1\}$, and let $\| \ \|_{C^N}$ denote the C^N norm on the unit sphere. Then for some N, we have

$$\|\hat{K}\|_\infty < C\|K\|_{C^N}$$

for all $K \in K^k$. Say now $K \in AK^k$. If $\alpha \in (\mathbb{Z}^+)^s$, let us write $|\alpha| = \alpha_1 + \ldots + \alpha_s$. Then, if $|\alpha|$ is even, one has that $\partial_t^{|\alpha|/2} x^\alpha K$ and $\partial_x^\alpha t^{|\alpha|/2} K$ are in AK^k. It is then not hard to see:

$$\|\lambda^{|\alpha|/2}\partial_\xi^\alpha \hat{K}\|_\infty < C\|\partial_t^{|\alpha|/2} x^\alpha K\|_{C^N} < C_1 R_1^{|\alpha|}(|\alpha|/2)! \quad (0.41)$$

$$\|\xi^\alpha \partial_\lambda^{|\alpha|/2}\hat{K}\|_\infty < C\|\partial_x^\alpha t^{|\alpha|/2} K\|_{C^N} < C_2 R_2^{|\alpha|}|\alpha|! \quad (0.42)$$

(0.41), and a variant for $|\alpha|$ odd, implies that if $J_+(\xi) = \hat{K}(1,\xi)$, then for ξ in any compact subset of $\mathbb{R}^s$,

$$|\partial_\xi^\alpha J_+(\xi)| < C_3 R_3^{|\alpha|}(|\alpha|!)^{1/2}$$

and similarly for J_-. This then implies easily that J_+ and J_- have extensions to entire functions of exponential order 2, as claimed before.

Further, for $|\xi| = 1$, (0.42) implies that as $\lambda \to 0$

$$|\partial_\lambda^m \hat{K}(\lambda,\xi)| < C_4 R_4^m (2m)!$$

which shows that $\hat{K}$ will not, in general, be analytic as $\lambda \to 0$. Rather, the terms of its Taylor expansion about $\lambda = 0$ in general grow on the order of $R^m m!$, for $|\xi| = 1$. This in turn implies, by homogeneity, that the terms of the asymptotic expansions of J_+ and J_- in general grow on the order of $R^m m!$, in the sense explained by Theorem 2.11 and the definition of $Z^2_{2,j}$.

Analytic Parametrices on $\mathbb{H}^n$

We briefly explain how our understanding of the Fourier transform of $\mathcal{A}\mathcal{K}^k$ helps us to find analytic parametrices for operators in our calculus. We write $\mathcal{F}$ for Euclidean Fourier transform on $\mathbb{H}^n$, thought of as $\mathbb{R}^{2n+1}$. We use coordinates

(t,x,y) on $\mathbb{H}^n$, with dual coordinates (λ,p,q).

The first step is to find a convolution inverse for the principal core, under natural hypotheses. We have:

Theorem (Part of Theorem 5.1). Say $K_1 \in AK^{k-2n-2}$ for some $k \in \mathbb{C}$. Then the following are equivalent:

(a) There exists $K_2 \in AK^{-k-2n-2}$ so that $K_2*K_1 = \delta$.

(b) The operator $f \to K_1*f$ $(f\in E')$ is analytic hypoelliptic.

(c) $\pi(K_1)$ is injective on Schwartz vectors for all irreducible unitary representations π of $\mathbb{H}^n$ (except $\pi \equiv 1$).

We explain the meaning of $\pi(K_1)$ precisely in Chapter five. Suffice it to say now, however, that as a consequence of (c), $FK_1(0,p,q) \neq 0$ for $(p,q) \neq 0$. Say $FK_1 = J_1$, $J_{1+}(p,q) = J_1(1,p,q)$, $J_{1+} \sim \sum g_\ell$ in $Z^2_{2,-k}$. By (0.40),

$$g_o(p,q) \neq 0 \text{ for } (p,q) \neq 0 \tag{0.43}$$

Using this, we construct a "first guess" K_2' so that if

$$K_o = K_2'*K_1 - \delta \tag{0.44}$$

then

$$FK_o(1,p,q) \in Z^2_2(\mathbb{R}^{2n}); \tag{0.45}$$

in other words, at least the asymptotic series for $FK_o(1,p,q)$ in $Z^2_{2,0}$ is zero. Part of what is involved in finding the needed

K_2' is to work on the level of asymptotic series, using (0.43). Next, by (0.44), it suffices for (a) to find $K_2'' \in AK^{-k-2n-2}$ with $K_2''*K_1 = -K_o$. It turns out that (0.45) says that K_o is of such a simple form that the group Fourier transform now becomes a very effective tool. It enables one to rapidly find K_2'', using the rest of the hypothesis (c).

Now say $K_1 \sim \sum K_{1u}^m \in C^{k-2n-2}(U)$ and K_{1u}^o satisfies the condition (c) above for all $u \in U$. Operating locally, to find an analytic parametrix in $C^{-k-2n-2}(V)$ (V a relatively compact open subset of U), we first use (c) $\Longrightarrow$ (a) of the above Theorem, and the rule (0.38) for composition to reduce to the case $K_{1u}^o(w) = \delta(w)$. The parametrix is constructed by use of a Neumann series, as in the Euclidean case; the terms are estimated by use of <u>both</u> the Euclidean and group Fourier transforms.

We have now completed our discussion of the main results and the main methods of this book. We turn next to a discussion of historical perspectives and future directions. After that, we will discuss ancillary results, and give a more detailed exposition of some of the methods which go into the main results.

<u>Prerequisites</u>

This book is long, but the proofs are very detailed, and the prerequisites are few. Thus it should be pleasant reading for a wide audience. We do not assume that the reader necessarily has any prior acquaintance with this area of analysis. In particular, it is certainly not necessary for the reader to have read any of this author's earlier papers on this subject. At a few points, however, we do refer to these and other papers for proofs of very standard or "obvious-looking" facts.

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Section Two: Historical Perspectives and Future Directions

Ideally, a pseudodifferential operator calculus on $\mathbb{H}^n$ should have three properties:

A. Analyticity: It should be an analytic calculus. It should contain analytic parametrices for operators K with principal core K_u^0 satisfying

$$\pi(K_u^0) \text{ is injective on Schwartz vectors for all non-trivial irreducible representations } \pi \text{ and all } u. \tag{0.46}$$

B. Homogeneity: It should be modelled on homogeneous distributions and group convolution on $\mathbb{H}^n$.

C. Relative Parametrices: Operators like the L_α in (0.3) should have relative parametrices in the calculus. In (0.3), we say ϕ_α is a relative fundamental solution. More generally, if K, with core in $C^k(\mathbb{H}^n)$, has principal core K_u^0 satisfying

$$\pi(K_u^0) \neq 0 \text{ for all one-dimensional non-trivial irreducible unitary representations } \pi \text{ and all } u \tag{0.47}$$

we would like K to have a relative parametrix in the calculus, an operator Φ with

$$K\Phi = \delta - P + Q \quad \text{(operating locally)},$$

where P projects in L^2 onto $[KS(\mathbb{H}^n)]^\perp$ and Q has core χQ, with χ, Q as in (0.21), (0.23).

There should also be generalizations to the situation where K operates on functions on a compact analytic contact manifold, and is locally in the calculus after an analytic contact transformation.

The calculus described in this book is the first to have both properties A and B. Further, as we explain below, we have, with Peter Heller, established C if K is a right-invariant differential operator.

Many researchers have proved results on property A and on property B, and there are a few results on property C. We now review some of the history of each, and attempt to place our work in perspective.

A. After Kohn's breakthrough [54], proving the hypoellipticity of $\Box_b$ under his condition Y(q), it was a number of years before analytic hypoellipticity was established if the manifold is in addition real analytic. This was done independently, and virtually simultaneously, by Tartakoff [77], [78] and Treves [81], in remarkable papers. Métivier [63] adapted Treves' methods, and in particular proved analytic hypoellipticity of differential operators satisfying (0.46). An analytic calculus which contains these differential operators was developed by Sjöstrand [72].

Treves, Métivier and Sjöstrand did construct analytic parametrices; we wish to retract, with apologies to them, a statement made near the beginning of our announcement [30]. (The statement was: "We study subelliptic operators, for which one rarely knows how to construct an analytic parametrix.") The parametrices of these authors, however, did not satisfy property B; in particular, Métivier's parametrices were only known to lie in the $OpS_{1/2,1/2}$ spaces. (Operating locally, our calculus is also contained in $OpS_{1/2,1/2}$.) Further, the operators in our calculus which are not differential operators are not covered by the earlier theorems. (Indeed, these operators are not classical analytic pseudodifferential operators. Further, the operators $\pi(K_u^o)$, mentioned in (0.46), for representations π on $L^2(\mathbb{R}^n)$, are not Grušin operators. For these two reasons, the theorems of [81] and [63] do not apply to these operators.)

Using the FBI transform, Treves, Baouendi, Rothschild and others have obtained a series of fine results about extendability of CR functions and analyticity of CR mappings, for weakly pseudoconvex real analytic domains. (A few references are: Baouendi-Chang-Treves [3], Baouendi-Jacobowitz-Treves [4], Baouendi-Bell-Rothschild [6], Baouendi-Rothschild [5].) Our techniques do not deal with the weakly pseudoconvex case. Perhaps some of our methods of dealing with general dilation weights $(a_1,\ldots,a_n)$ may ultimately be of some assistance in understanding the analytic regularity of the Kohn Laplacian