JACK FREDERICK CONN

Non-Abelian Minimal Closed Ideals of Transitive Lie Algebras. (MN-25)

PRINCETON LEGACY LIBRARY

NON-ABELIAN MINIMAL CLOSED IDEALS
OF TRANSITIVE LIE ALGEBRAS

by

Jack F. Conn

Princeton University Press

and

University of Tokyo Press

Princeton, New Jersey

1981

Published in Japan Exclusively

by University of Tokyo Press

in other parts of the world by

Princeton University Press

Printed in the United States of America

by Princeton University Press, Princeton, New Jersey

Library of Congress Cataloging in Publication Data will

be found on the last printed page of this book

Table of Contents

Preface

Apart from their inherent interest as algebraic structures, transitive Lie algebras play an essential role in any study of the integrability problem for transitive pseudogroup structures on manifolds. This monograph presents, in an essentially self-contained way, work on the structure of transitive Lie algebras and their non-abelian minimal closed ideals. Many of the results contained here have simple differential-geometric interpretations, and bear directly upon the integrability problem.

It is a pleasure to acknowledge the constant encouragement and advice given me by Hubert Goldschmidt and Donald C. Spencer; this work owes much to their suggestions and trenchant criticism. Victor Guillemin generously provided an unpublished manuscript which afforded some essential new results. I would also like to thank Vivian Davies whose typing is responsible for the attractive appearance of the finished manuscript. The Institute for Advanced Study provided gracious hospitality to the author during a portion of this research, which was partially supported by Grant MCS77-18723 from the National Science Foundation.

Jack F. Conn

Pasadena, California

Introduction

Transitive pseudogroups of local diffeomorphisms preserving geometric structures on manifolds have been studied by many authors; the origins of this subject are classical, and may be said to lie in the works of Sophus Lie and Élie Cartan. The structure of such a pseudogroup Γ acting on a manifold X is reflected in the structure of the Lie algebra of formal infinitesimal transformations of Γ, that is to say, those formal vector fields on X which are formal solutions to the linear partial differential equation which defines the infinitesimal transformations of Γ. The Lie algebras of formal vector fields obtained in this way provide examples of what are now known as transitive Lie algebras; such Lie algebras are, in general, infinite-dimensional.

The study of transitive Lie algebras was first placed on a strictly algebraic basis by the paper ([16]) of V. W. Guillemin and S. Sternberg. Subsequent work of Guillemin ([11]) characterized transitive Lie algebras as linearly compact topological Lie algebras which satisfy the descending chain condition on closed ideals, and established the existence of a Jordan-Hölder decomposition in such Lie algebras. This latter result is a weak analogue of the Levi decomposition for finite-dimensional Lie algebras. Several authors have since adopted this abstract algebraic viewpoint for the study of transitive algebras; one result of their work has been the rigorous and progressively simplified proofs ([12], [14], [15], [21], [23], [29]), in the category of transitive Lie algebras, of the classification of the infinite-dimensional primitive Lie pseudogroups given by E. Cartan ([3]). We shall make use of this classification in the present work.

Transitive Lie algebras have been studied also to provide insight into the behavior of the integrability problem for transitive pseudogroup structures. A precise formulation of this problem may be found in ([20]); for surveys of the principal results concerning this problem, we refer the reader to ([8]) and the introduction of ([9]). The role played by real transitive Lie algebras and their non-abelian minimal closed ideals in the integrability problem was elucidated by H. Goldschmidt and D. C. Spencer ([9]). In our present work, we give a complete algebraic description of the structure of these non-abelian minimal closed ideals. Our study was undertaken as a tool for the investigation of the integrability problem, and is an essential element in the proof of Conjecture III of ([9]) as outlined in ([8]) and in greater detail in the introduction to ([31]). The proof of this conjecture implies, in particular, that the integrability problem is solved for all transitive Lie pseudogroups acting on $\mathbb{R}^n$ which contain the translations, a fortiori for all flat pseudogroups. In an attempt to prove Conjecture I of ([9]) following the outline suggested there, we found that the geometry of pseudo-complex structures (induced structures on real submanifolds of complex n-space $\mathbb{C}^n$) was expressed in the structure of non-abelian minimal closed ideals of complex type in real transitive Lie algebras. From this observation, we were able to construct simple counterexamples to Conjectures I and II of ([9]) involving such closed ideals; these counterexamples have appeared in our note ([4]). Our presentation follows through Section four the outline given in §13 of ([9]); this part of the present work contains our results on the structure of non-abelian minimal closed ideals of real type which are used by Goldschmidt in ([31]) to prove Conjecture I of ([9]) for these closed ideals. In a sequel to ([31]), Goldschmidt will present a proof of Theorem 9 of

([8]) which relies on our description of non-abelian minimal closed ideals of complex type in terms of pseudo-complex structures given in Section five; in this way, the proof of Conjecture III of ([9]) will be completed.

In this work, we shall view transitive Lie algebras from an abstract viewpoint as topological Lie algebras, following the work of Guillemin and Sternberg ([11] , [16]) mentioned above. Let K be a field of characteristic zero, endowed with the discrete topology (even when K is equal to $\mathbb{R}$ or $\mathbb{C}$). A transitive Lie algebra is a linearly compact topological Lie algebra over K which possesses a fundamental subalgebra, that is, an open subalgebra L^0 containing no ideals of L except $\{0\}$; this is equivalent ([11]) to requiring that L satisfy the descending chain condition on closed ideals. Any finite-dimensional Lie algebra L over K becomes a transitive Lie algebra when endowed with the discrete topology, since $\{0\}$ is then a fundamental subalgebra of L. However, in the infinite-dimensional examples the topology plays a more essential role. If Γ is a transitive pseudogroup acting on a manifold X, and L is the Lie algebra of formal infinitesimal transformations of Γ at a point $x \in X$, then the isotropy subalgebra of L, that is, the subalgebra of formal vector fields in L which vanish at x, is a fundamental subalgebra L^0 of L. Conversely, it is a theorem of H. Goldschmidt ([6]) that any transitive Lie algebra L over $\mathbb{R}$ and fundamental subalgebra $L^0 \subset L$ can be realized in this way. The abstract viewpoint of Guillemin and Sternberg which we adopt is thus seen to be completely consistent with the differential-geometric viewpoint.

As we mentioned above, Guillemin proved ([11]) that a Jordan-Hölder decomposition can be introduced in any transitive Lie algebra L. Such a decomposition consists of a finite descending chain

$$L = I_0 \supset I_1 \supset \cdots \supset I_n = \{0\}$$

of closed ideals of L, such that, for each integer p with $0 \leq p \leq n-1$, either

(i) The quotient I_p/I_{p+1} is abelian; or

(ii) The quotient I_p/I_{p+1} is non-abelian, and is a minimal closed ideal of L/I_{p+1}.

Guillemin also showed that the number and type of quotients of type (ii), both as topological Lie algebras and as topological L-modules, is independent of the choice of Jordan-Hölder sequence for L. The existence of such a decomposition had been conjectured (in the category of transitive pseudogroups) by E. Cartan. The quotient of a transitive Lie algebra by a closed ideal is again a transitive Lie algebra, since it also satisfies the descending chain condition on closed ideals; therefore, each of the quotients I_p/I_{p+1}, in a Jordan-Hölder sequence for a transitive Lie algebra L is a closed ideal in a transitive Lie algebra L/I_{p+1}. Quotients of type (i); that is, closed abelian ideals of transitive Lie algebras, have been extensively studied as part of the work of Goldschmidt and Spencer ([9], [10]). We shall concentrate here upon the structure of quotients of type (ii), that is, non-abelian minimal closed ideals of transitive Lie algebras. The investigations of Goldschmidt and Spencer cited above reduce the integrability problem for a transitive pseudogroup Γ to a series of questions concerning the structure of the quotients I_p/I_{p+1}, both as topological Lie algebras and topological L-modules, appearing in a Jordan-Hölder sequence for the Lie algebra L of formal infinitesimal transformations of Γ. As a consequence, our results bear directly upon the integrability problem for transitive pseudogroup structures.

We now describe the main results of this work; to simplify our outline, we assume, unless otherwise specified, that all Lie algebras considered below are defined over the field $\mathbb{R}$ of real numbers. Many of our results are obtained for linearly compact topological Lie algebras without the assumption of transitivity. For the sake of clarity, we make several preliminary observations before beginning our outline itself.

Let L be a linearly compact topological Lie algebra, and suppose that I is a non-abelian minimal closed ideal of L. Then it is known ([11]) that I possesses a unique maximal closed ideal J; moreover, the quotient I/J is a non-abelian simple transitive Lie algebra R. The commutator ring K_R of R, that is, the algebra of $\mathbb{R}$-linear mappings $c : R \to R$ such that, for all $\xi, \eta \in R$,

$$c([\xi, \eta]) = [c(\xi), \eta]$$

is, according to ([11]), actually a field which is a finite algebraic extension of $\mathbb{R}$. Thus, the field K_R is equal to $\mathbb{R}$ or to $\mathbb{C}$; we shall, then, say that the non-abelian minimal closed ideal I of L is of real or complex type, respectively. The simple real transitive Lie algebra R may be viewed naturally as a transitive Lie algebra over its commutator field K_R, and every real-linear derivation of R is actually K_R-linear. Unless R is finite-dimensional, it need not be true that every derivation of R is inner. However, the space Der(R) of derivations of R has a natural structure of transitive Lie algebra over K_R, and the adjoint representation of R allows us to identify R with a closed ideal of finite codimension in Der(R). For n an integer ≥ 0, consider the local algebra

$$F = K_R[[x_1, \cdots, x_n]]$$

of formal power series in n indeterminates over K_R (when $n = 0$, we mean that $F = K_R$); endow F with the Krull topology. The maximal ideal F^0 of F consists of those formal series which vanish at the origin; the powers $\{(F^0)^\ell\}_{\ell \geq 1}$ of F^0 comprise a fundamental system of neighborhoods of zero in F, which is a linearly compact topological algebra. The space Der(F) of derivations of F has a natural structure of transitive Lie algebra over K_R, with the Lie bracket given by the usual commutator of derivations; the stabilizer

$$\mathrm{Der}^0(F) = \{\xi \in \mathrm{Der}(F) \mid \xi(F^0) \subset F^0\}$$

of F^0 is a fundamental subalgebra of Der(F). There are natural structures of topological Lie algebra over K_R and topological Der(F)-module on the tensor product

$$\mathrm{Der}(R) \otimes_{K_R} F \ ;$$

the Hausdorff completion $\mathrm{Der}(R) \hat{\otimes}_{K_R} F$ of this space inherits linearly compact structures of topological Lie algebra and topological Der(F)-module. Furthermore, the transitive Lie algebra Der(F) acts by derivations on the Lie algebra $\mathrm{Der}(R) \hat{\otimes}_{K_R} F$. We can, then, form the semi-direct product

$$(\mathrm{Der}(R)\ \hat{\otimes}_{K_R} F) \oplus \mathrm{Der}(F)\ ,$$

which is a transitive Lie algebra over K_R, and $R\ \hat{\otimes}_{K_R} F$ is then a non-abelian minimal closed ideal in this Lie algebra.

We come now to the actual outline of our results on the structure of non-abelian minimal closed ideals. Although our results are of greater interest and novelty in the case of ideals of complex type, it will be convenient to treat the real case first. We maintain the notational conventions of the previous paragraph.

Assume that the non-abelian minimal closed ideal I of L is of real type. Then the normalizer

$$N = N_L(J)$$

in L of the maximal closed ideal J of I is a subalgebra of finite codimension in L, as is proved in ([11]). Set $n = \dim(L/N)$, and

$$F = \mathbb{R}[[x_1, \ldots, x_n]]\ .$$

In Theorem 4.2 we prove that there exists a morphism of real topological Lie algebras

$$\Phi : L \to (\mathrm{Der}(R)\ \hat{\otimes}_{\mathbb{R}} F) \oplus \mathrm{Der}(F)\ ,$$

such that the restriction of Φ *to* I *is an isomorphism*

$$\Phi|_I : I \to R\ \hat{\otimes}_{\mathbb{R}} F\ .$$

The kernel of Φ is equal to the commutator of I in L, and the projection $\pi(\Phi(L))$ of $\Phi(L)$ onto Der(F) is a transitive closed subalgebra of Der(F), in the sense that

$$\pi(\Phi(L)) + \mathrm{Der}^0(F) = \mathrm{Der}(F) .$$

Guillemin proved in ([11]) that I and R $\hat{\otimes}_{\mathbb{R}}F$ are isomorphic as abstract Lie algebras; our proof of Theorem 4.2 consists mainly of a close examination of Guillemin's work, combined with the observation (Lemma 2.6) that the topology of Der(R) as a transitive Lie algebra coincides with the weak topology Der(R) inherits as a subspace of the continuous linear transformations of R.

We now assume that the non-abelian minimal closed ideal I of L is of complex type. As above, the normalizer $N = N_L(J)$ of J in L is a subalgebra of finite codimension n in L. In Section five, we show that I may be viewed naturally as a complex topological Lie algebra with J a maximal closed complex ideal, and that L acts, via the adjoint representation, on I by continuous complex-linear mappings. This action of L on I may be complexified to a representation of the complex Lie algebra $L_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} L$ on I; the normalizer N'' of J under this representation is a complex subalgebra of $L_{\mathbb{C}}$ of finite (complex) codimension $m \leq n$, since N'' must contain $N_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} N$. We then associate to L and N'' a (unitary) monomorphism of complex local algebras

$$\varphi^* : \mathbb{C}[[z_1, \dots , z_m]] \to \mathbb{C}[[x_1, \dots , x_n]] ,$$

whose image $\varphi^*(\mathbb{C}[[z_1, \dots , z_m]])$ we call H. Using the natural identification

$$\mathbb{C}[[x_1, \dots , x_n]] \simeq \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[[x_1, \dots , x_n]],$$

we allow $\mathrm{Der}(\mathbb{R}[[x_1, \dots , x_n]])$ to act on $\mathbb{C}[[x_1, \dots , x_n]]$ and denote by A the stabilizer

$$A = \{\xi \in \mathrm{Der}(\mathbb{R}[[x_1, \ldots, x_n]]) \mid \xi(H) \subset H\}$$

of H; then A is a transitive Lie subalgebra of $\mathrm{Der}(\mathbb{R}[[x_1, \ldots, x_n]])$ in the sense that

$$\mathrm{Der}(\mathbb{R}[[x_1, \ldots, x_n]]) = A + \mathrm{Der}^0(\mathbb{R}[[x_1, \ldots, x_n]]),$$

and

$$A^0 = A \cap \mathrm{Der}^0(\mathbb{R}[[x_1, \ldots, x_n]])$$

is a fundamental subalgebra of A. Upon defining the semi-direct product

$$(\mathrm{Der}\,(R) \,\hat{\otimes}_{\mathbb{C}} H) \oplus A$$

as before, we obtain a real transitive Lie algebra in which $R \,\hat{\otimes}_{\mathbb{C}} H$ forms a non-abelian minimal closed ideal. In Theorem 5.2, we prove that there exists a morphism of real topological Lie algebras

$$\psi : L \to (\mathrm{Der}(R) \,\hat{\otimes}_{\mathbb{C}} H) \oplus A$$

such that the restriction of ψ to I is an isomorphism

$$\psi|_I : I \to R \,\hat{\otimes}_{\mathbb{C}} H .$$

The kernel of ψ is equal to the commutator of I in L, and the projection $\pi(\psi(L))$ of $\psi(L)$ onto A is a transitive subalgebra of A, in the sense that

$$A = \pi(\psi(L)) + A^0 .$$

We also associate to L and N'' a Hermitian mapping

$$\mathscr{L} : (N''/N_{\mathbb{C}}) \times (N''/N_{\mathbb{C}}) \to (L_{\mathbb{C}}/(N'' + \overline{N''}))$$

which we call the Levi form of I. The vanishing of $\mathscr{L}$ is shown in Proposition 5.4 to be equivalent to the existence of an isomorphism

$$f : \mathbb{R}[[x_1, \ldots, x_n]] \to \mathbb{R}[[x_1, \ldots, x_n]]$$

of real local algebras such that the complexification

$$f_{\mathbb{C}} : \mathbb{C}[[x_1, \ldots, x_n]] \to \mathbb{C}[[x_1, \ldots, x_n]] ,$$

when restricted to H, is an isomorphism of H onto the subring $\mathbb{C}[[x_1, \ldots, x_m]]$ of $\mathbb{C}[[x_1, \ldots, x_n]]$. In Proposition 5.6, we prove that if L is a real transitive Lie algebra with fundamental subalgebra L^0 and abelian subalgebra V such that

$$L = L^0 \oplus V,$$

then the Levi form of each non-abelian quotient of complex type occurring in a Jordan-Hölder sequence for L must vanish. This proposition is an essential result for the proof of Conjecture III of ([9]). A key step in the proof of Proposition 5.6 is provided by an unpublished result of Guillemin, which states that if I is a non-abelian minimal closed ideal of a transitive Lie algebra L (over any field of characteristic zero) and $N_L(J)$ is the normalizer in L of the maximal closed ideal J of I, then $N_L(J)$ contains every fundamental subalgebra L^0 of L. We have reproduced Guillemin's result here as Proposition 4.5.

Returning to our discussion of ideals of complex type, the monomorphism

$$\varphi^* : \mathbb{C}[[z_1, \ldots, z_m]] \to \mathbb{C}[[x_1, \ldots, x_n]] ,$$

which we construct from L and N'', may be viewed geometrically as the pullback mapping associated to the formal expansion φ at $0 \in \mathbb{R}^n$ of an embedding

$$\tilde{\varphi} : U \to \mathbb{C}^m , \qquad \tilde{\varphi}(0) = 0$$

of a neighborhood U of 0 in $\mathbb{R}^n$ as a generic real submanifold of $\mathbb{C}^m$. Using results of Goldschmidt ([6]) on the analytic realization of transitive Lie algebras, one can show that φ can be chosen to be convergent, in which case $\tilde{\varphi}(U)$ becomes a real-analytic generic real submanifold of $\mathbb{C}^m$; the proof of this will appear in a separate publication. Moreover, the local biholomorphic mappings of $\mathbb{C}^m$ which preserve $\tilde{\varphi}(U)$ restrict to form a transitive analytic pseudogroup Γ on $\tilde{\varphi}(U)$; this places strong restrictions on the structure of the real submanifold $\tilde{\varphi}(U) \subset \mathbb{C}^m$. The subring $H \subset \mathbb{C}[[x_1, \ldots, x_n]]$ then corresponds to the restrictions to $\tilde{\varphi}(U)$ of the formal holomorphic functions at $0 \in \mathbb{C}^m$. The analytic infinitesimal transformations of Γ are the restrictions to $\tilde{\varphi}(U)$ of those holomorphic vector fields defined near $\tilde{\varphi}(U)$ in $\mathbb{C}^m$ which are tangent to $\tilde{\varphi}(U)$; the real subalgebra $A \subset \mathrm{Der}(\mathbb{R}[[x_1, \ldots, x_n]])$ corresponds to the formal infinitesimal transformations at 0 of Γ. We show that the Levi form of I can be identified with the Levi form, in the differential-geometric sense, of the real submanifold $\tilde{\varphi}(U)$. In these terms, Proposition 5.4 asserts that the Levi form of I vanishes if and only if $\tilde{\varphi}(U)$ can be chosen to be an n-dimensional real hyperplane of $\mathbb{C}^m$; indeed, this proposition accomplishes our analytic constructions explicitly in the special case in which I has vanishing Levi form. We make no appeal to these analytic constructions in the formal development of our work, although many of our results were conceived through such geometric considerations. We provide informal geometric interpretations in Section five to many of our formal statements.

One consequence of our structural descriptions, Theorems 4.2 and 5.2, is that if I is a non-abelian minimal closed ideal of a linearly compact topological Lie algebra L, then there exists a closed subalgebra L' of L such that

$$L = L' \oplus I .$$

This is proved in Corollaries 4.5 and 5.1.

To conclude our introduction, we describe the organization of this work. Section one is a compendium of results on transitive and linearly compact Lie algebras, containing those facts of which we make use in the remainder of this book. A serious effort was made to keep this section self-contained; the reader may, nonetheless, find it valuable to consult the references ([11], [16], [24]) for additional information.

In Section two, we consider derivations of transitive Lie algebras. In contrast to the finite-dimensional case, it is <u>not</u>, in general, true that every derivation of a simple transitive Lie algebra is inner. Let K be a field of characteristic zero, let $n \geq 1$ be an integer, and L be a closed Lie subalgebra of $\mathrm{Der}(K[[x_1, \dots, x_n]])$ such that

$$L + \mathrm{Der}^0(K[[x_1, \dots, x_n]]) = \mathrm{Der}(K[[x_1, \dots, x_n]]) ;$$

set $L^0 = L \cap \mathrm{Der}^0(K[[x_1, \dots, x_n]])$. In Proposition 2.1 we show that if D is a derivation of L into $\mathrm{Der}(K[[x_1, \dots, x_n]])$ such that

$$D(L^0) \subset \mathrm{Der}^0(K[[x_1, \dots, x_n]]) ,$$

then D is induced by the adjoint action of a unique element of $\mathrm{Der}^0(K[[x_1, \dots, x_n]])$; in particular, if D is a derivation of L such that $D(L^0) \subset L^0$, then D results from the adjoint action of a unique element of the normalizer of L^0 in $\mathrm{Der}^0(K[[x_1, \ldots, x_n]])$. The significance of the latter result rests in the fact that L^0 is of finite codimension in its normalizer. We apply this result to the classical simple transitive infinite-dimensional Lie algebras L defined in Section one; in Theorem 2.1, we prove that for the unique primitive realizations of the classical algebras as transitive Lie subalgebras of $\mathrm{Der}(K[[x_1, \ldots, x_n]])$, the first Hochschild-Serre cohomology group

$$H^1(L, \mathrm{Der}(K[[x_1, \ldots, x_n]])$$

vanishes. As a corollary, we deduce that the space Der(L) of derivations of L has a natural structure of transitive Lie algebra in which L is identified, via the adjoint representation, with a closed ideal of finite codimension. These results hold for any simple transitive Lie algebra L over a field K which is either algebraically closed or equal to the field of real numbers $\mathbb{R}$, by virtue of the known classification of such Lie algebras. For these algebras L, we establish, in Section three, the isomorphism

$$\mathrm{Der}(L \hat{\otimes}_{K_L} K_L[[x_1, \cdots, x_n]]) \simeq (\mathrm{Der}(L) \hat{\otimes}_{K_L} K_L[[x_1, \cdots, x_n]]) \oplus \mathrm{Der}(K_L[[x_1, \cdots, x_n]]).$$

This endows the space $\mathrm{Der}(L \hat{\otimes}_{K_L} K_L[[x_1, \ldots, x_n]])$ with a structure of transitive Lie algebra in which $L \hat{\otimes}_{K_L} K_L[[x_1, \ldots, x_n]]$ forms a non-abelian minimal closed ideal. The results of the remainder of the book are essentially independent of Section three, which is included only for completeness. Most of the material in Section two was previously known, but our proofs are particularly elementary and may be new.

In Section four we have included, for the benefit of the reader, most of the results on closed ideals and Jordan-Hölder decompositions contained in Guillemin's paper ([11]). This section culminates in Theorem 4.2, which is the first of our topological structure theorems for non-abelian minimal closed ideals described above. Section five contains our results on non-abelian minimal closed ideals of complex type; these results have been outlined previously.

§1. Preliminaries

Throughout this section, we denote by K an arbitrary field of characteristic zero which is endowed with the discrete topology.

Let V be a Hausdorff topological vector space over K. Then V is said to be linearly compact if:

(i) V is complete; and

(ii) There exists a fundamental system $\{V_\alpha\}$ of neighborhoods of 0 in V such that each V_α is a vector subspace of finite codimension in V.

An example of such a space is provided by the local algebra

$$F = K[[x_1, \ldots, x_n]]$$

of formal power series over K in n indeterminates $x_1, \ldots, x_n$. Denote by F^0 the maximal ideal of F, which consists of those formal power series whose constant term vanishes. If F^ℓ denotes the $(\ell + 1)$ - rst power of F^0, then the ideals $\{F^\ell\}_{\ell \geq 0}$ form a fundamental system of neighborhoods of 0 for the Krull topology on F; endowed with this topology, F becomes a linearly compact topological vector space over K. Furthermore, in this topology, multiplication in the algebra F is a continuous mapping $F \times F \to F$; thus we see that F is a linearly compact topological algebra over K. If V is a finite-dimensional vector space over K, then V is linearly compact in the discrete topology. Moreover, since a finite-dimensional Hausdorff vector space satisfying (ii) above is necessarily discrete, the discrete topology provides the only linearly compact structure on a finite-dimensional space. In the sequel, a finite-dimensional vector space will often be implicitly endowed with the discrete topology.

The properties of linearly compact spaces which we shall require are subsumed in the following proposition.

<u>Proposition</u> 1.1. Let V and W be linearly compact topological vector spaces over K.

(i) If V' is a closed subspace of V, then both V' and V/V' are linearly compact.

(ii) A subspace V' of V is open if and only if V' is closed and of finite codimension in V.

(iii) If V' and V'' are closed subspaces of V, then the sum V' + V'' is closed in V.

(iv) The topological direct sum $V \oplus W$ is linearly compact.

(v) If $\varphi : V \to W$ is a continuous linear mapping, then the image $\varphi(V)$ is a closed subspace of W, and φ is an open mapping of V onto $\varphi(V)$. In particular, any continuous linear bijection $\varphi : V \to W$ is a topological isomorphism.

(vi) (Closed graph theorem). A linear mapping $\varphi : V \to W$ is continuous if and only if the graph of φ is closed in $V \times W$.

(vii) If $\{V_\alpha\}$ is a family of linearly compact spaces, then the product $\prod_\alpha V_\alpha$ is linearly compact. If $\{(V_\alpha, f_{\alpha\beta})\}$ is an inverse system of linearly compact vector spaces and continuous linear mappings, then the projective limit $\varprojlim V_\alpha$ is linearly compact.

(viii) (Chevalley's theorem). Let U be an open subspace of V, and let