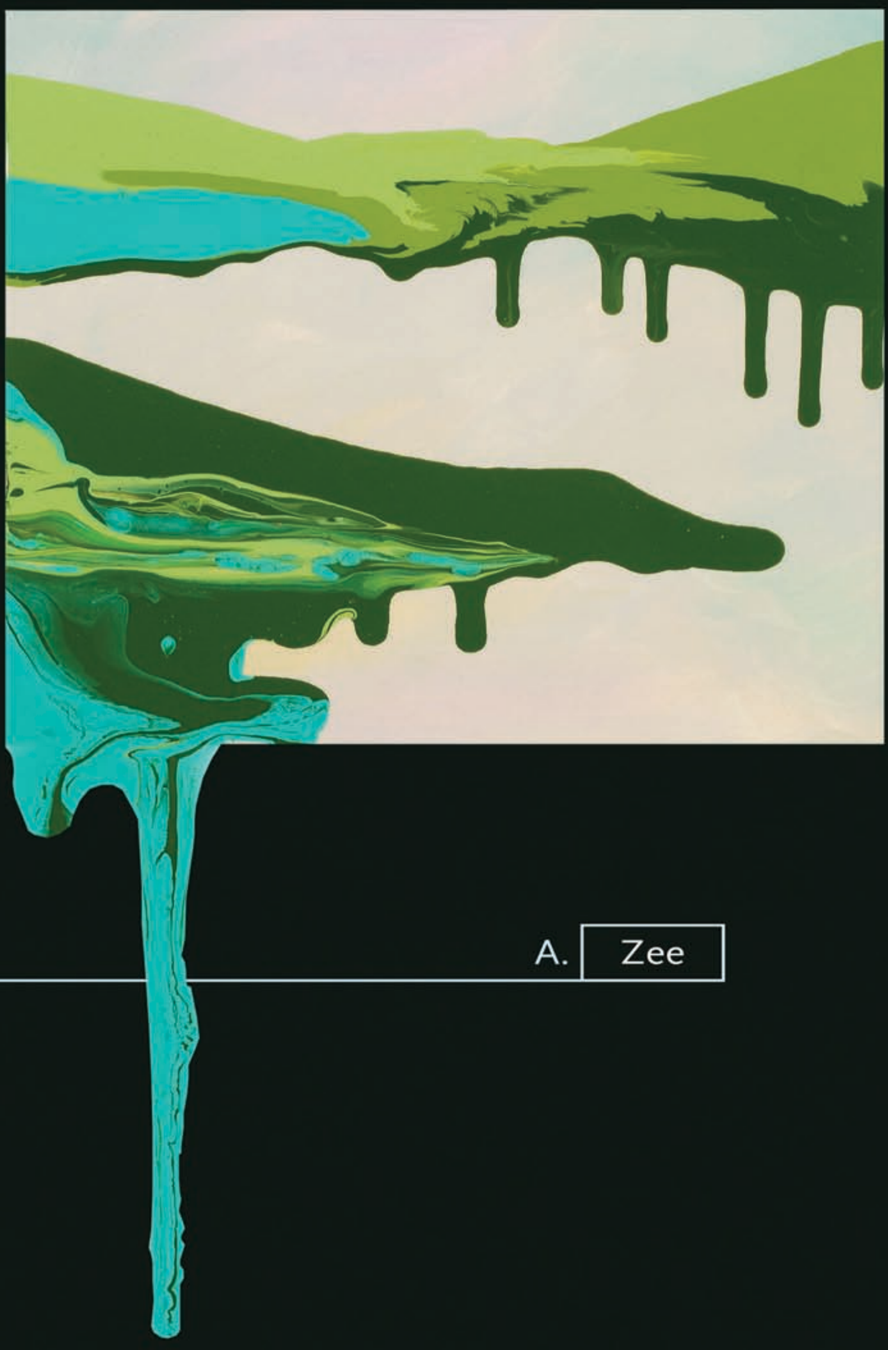
Einstein Gravity in a Nutshell



A. Zee

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Summary: "This unique textbook provides an accessible introduction to Einstein's general theory of relativity, a subject of breathtaking beauty and supreme importance in physics. With his trademark blend of wit and incisiveness, A. Zee guides readers from the fundamentals of Newtonian mechanics to the most exciting frontiers of research today, including de Sitter and anti–de Sitter spacetimes, Kałuza-Klein theory, and brane worlds. Unlike other books on Einstein gravity, this book emphasizes the action principle and group theory as guides in constructing physical theories. Zee treats various topics in a spiral style that is easy on beginners, and includes anecdotes from the history of physics that will appeal to students and experts alike. He takes a friendly approach to the required mathematics, yet does not shy away from more advanced mathematical topics such as differential forms. The extensive discussion of black holes includes rotating and extremal black holes and Hawking radiation. The ideal textbook for undergraduate and graduate students, Einstein Gravity in a Nutshell also provides an essential resource for professional physicists and is accessible to anyone familiar with classical mechanics and electromagnetism. It features numerous exercises as well as detailed appendices covering a multitude of topics not readily found elsewhere. Provides an accessible introduction to Einstein's general theory of relativity Guides readers from Newtonian mechanics to the frontiers of modern research Emphasizes symmetry and the Einstein-Hilbert action Covers topics not found in standard textbooks on Einstein gravity Includes interesting historical asides Features numerous exercises and detailed appendices Ideal for students, physicists, and scientifically minded lay readers Solutions manual (available only to teachers) "— Provided by publisher.
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To WW and Max

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Preface

Not simple, but as simple as possible

> Physics should be made as simple as possible,
> but not any simpler.
>
> —A. Einstein

Einstein gravity should be made as simple as possible, but not any simpler.

My goal is to make Einstein gravity* as simple as possible. I believe that Einstein's theory should be readily accessible to those who have mastered Newtonian mechanics and a modest amount of classical mathematics. To underline my point, I start with a review of $F = ma$.

Seriously, what do you need to know to read this book? Only some knowledge of classical mechanics and electromagnetism! So I fondly imagine, perhaps unrealistically. More importantly, you need to be possessed of what we theoretical physicists call sense—physical, mathematical, and also common.

I wrote this book in the same spirit as my *Quantum Field Theory in a Nutshell*.[1] In his *Physics Today* review of that book, Zvi Bern wrote this lovely sentence aptly capturing my pedagogical philosophy: "The purpose of Zee's book is not to turn students into experts—it is to make them fall in love with the subject." I might extend that to "fall in love with the subject so that they might desire to become experts." Here I am echoing William Butler Yeats, who said, "Education is not the filling of a pail, but the lighting of a fire."

* Also known as general relativity.

A portion of this book can be used for an undergraduate course. I have done it, and I provide a detailed course outline later in this preface.

Accessible is not to be equated with dumbed-down or watered-down. Also, accessible is not necessarily the same as elementary: in the last parts of the book, I include some topics far beyond the usual introductory treatment.

My strategy to make Einstein gravity as simple as possible has two prongs. The first is the emphasis on symmetry. As some readers may know, I have written an entire book[2] on the role of symmetry in physics, and I absolutely love how symmetry guides us in constructing physical theories, a notion that started with Einstein gravity, in fact. The second is the extensive use of the action principle. The action is invariably simpler than the equations of motion and manifests the inherent symmetry much more forcefully. I can hardly believe that some well-known textbooks on Einstein's theory barely mention the Einstein-Hilbert action. Symmetry and the action principle constitute the two great themes of theoretical physics.

To get a flavor of what the book is about, you might want to glance at the recaps first; there is one at the end of each of the ten parts of the book.

How difficult is Einstein gravity?

> Any intelligent student can grasp it without too much trouble.
>
> —A. Einstein, referring to his theory of gravity

When Arthur Eddington returned from the famous 1919 solar eclipse expedition that observed light from a distant star bending in agreement with Einstein gravity, somebody asked him if it were true that only three people understood Einstein's theory. Eddington replied, "Who is the third?" The story, apocryphal[3] or not, is one of many[4] that gives Einstein's theory its undeserved reputation of being incomprehensible.

I believe that in some cases, people like to persist in believing that Einstein's theory is beyond them. A renowned philosopher who is clearly well above average in intelligence (and who understands things that I find impossible to understand) once told me that he was tired of popular accounts of general relativity and that he would like to finally learn the subject for real. He also emphasized to me that he had taken advanced calculus[5] in college, as if to say that he could handle the math. I replied that, for a small fee, my impecunious graduate student could readily teach him the essence of general relativity in six easy lessons. I never heard from the renowned philosopher again. I was happy and he was happy: he could go on enunciating philosophical profundities about relative truths[6] and physical reality.

The point of the story is that it is not that difficult.

For whom is this book intended

Experience with my field theory textbook suggests that readers of this book will include the following overlapping groups: students enrolled in a course on general relativity, students and others indulging in the admirable practice of self-study, professional physicists in other research specialties who want to brush up, and readers of popular books on Einstein gravity who want to fly beyond the superficial discussions these books (including my own[7]) offer. My comments below apply to some or all of these groups.[8]

Personally, I feel special sympathy for those studying the subject on their own, as I remember struggling[9] one summer during my undergraduate years with a particularly idiosyncratic text on general relativity, the only one I could find in São Paulo back in those antediluvian times. That experience probably contributed to my desire to write a textbook on the subject. From the mail I have received regarding *QFT Nut,* I have been pleasantly surprised, and impressed, by the number of people out there studying quantum field theory on their own. Surely there are even more who are capable of self-studying Einstein gravity. All power to you! I wrote this book partly with you in mind.

Serious students of physics know that one can't get far without doing exercises. Some of the exercises lead to results that I will need later.

Quite naturally, I have also written this book with an eye toward quantum field theory and quantum gravity. While I certainly do not cover quantum gravity, I hope that the reader who works through this book conscientiously will be ready for more specialized monographs[10] and the vast literature out there.

So, I prevaricated a little earlier. In the latter part of the book, occasionally you will need to know more than classical mechanics and electromagnetism. But, to be fair, how do you expect me to talk about Hawking radiation, a quintessentially quantum phenomenon, in chapter VII.3? Indeed, how could we discuss natural units in the introduction if you have never heard of quantum mechanics? For the readers with only a nodding acquaintance with quantum mechanics, the good news is that for the most part, I only ask that you know the uncertainty principle.

I do not doubt that some readers will encounter difficult passages. That's because I have not made the book "any simpler"!

In the preface to the second edition of my quantum field theory book, I mentioned that Steve Weinberg and I, each referring to his own textbook, each said, "I wrote the book that I would have liked to learn from." So this is the book I would have liked as an undergrad* eager to learn Einstein gravity. I would have liked having at least a flavor of what the latest

* In a letter to the editors of *Physics Today* in 2005, A. Harvey and E. Schucking wrote that, in view of the "monumental lip service" paid to Einstein in the physics community, "it is a scandal" that Einstein gravity is still not regularly taught to undergraduates. I find it even more of a scandal that many physics professors proudly profess ignorance of Einstein gravity, saying that it is irrelevant to their research. Yes, maybe, but this is akin to being proudly ignorant of Darwinian evolution because it is irrelevant to whatever you are doing.

excitement was all about. In this spirit, I offer chapter X.6 on twistors, for example, trusting the reader to be sophisticated enough to know that all one should expect to get from a single textbook chapter is an entry key to the research literature rather than a complete account of an emerging area.

The importance of feeling amazed, and amused

I am amazed that students are not amazed.

The action principle amazed Feynman when he first heard about it. In learning theoretical physics, I was, and am, constantly amazed. But in teaching, I am amazed that students are often not amazed. Even worse, they are not amused.

Perhaps it is difficult for some students to be amazed and amused when they have to drag themselves through miles of formalism. So this exhortation to be amazed is related to my attempt to keep the formalism to an absolute minimum in my textbooks and to get to the physics.

To paraphrase another of my action heroes, students should be required to gasp and laugh[11] periodically. Why study Einstein gravity unless you have fun doing it?

As much fun as possible

Bern started his review of my quantum field theory textbook thus:

> When writing a book on a subject in which a number of distinguished texts already exist, any would-be author should ask the following key question: What new perspectives can I offer that are not already covered elsewhere? . . . perhaps foremost in A. Zee's mind was how to make *Quantum Field Theory in a Nutshell* as much fun as possible.

Good question! My answer remains the same. I want to make Einstein gravity as much fun as possible.

Sidney Coleman, my professor in graduate school and thesis advisor, once advised me that theoretical physics is a "gentleman's diversion." I was made to understand that I should avoid doing long sweaty calculations. This book reflects some of that spirit. Thus, in chapter VI.1, instead of deriving Einstein's field equation as a true Confucian scholar would, I try to get to it as quickly as possible by a method I dub "winging it southern California style." Similarly, in chapter VI.2, I get to cosmology as quickly as possible.

This invariably brings me to the dreaded topic of drudgery in general relativity. Many theory students in my generation went into particle physics rather than general relativity to avoid the drudgery of spending an entire day calculating the Riemann curvature tensor. I did.[12] But that was the old days. Nowadays, students of general relativity can use ready-made symbolic manipulation programs[13] to do all the tedious work. I strongly urge you, however, to write your own programs, as I did, rather than open a can. It also goes without saying that you should calculate the Riemann curvature tensor from scratch at least a few times to know how all the cogs fit together.

You make the discoveries

My pedagogical philosophy is to let students discover certain things on their own. Some of these lessons evolved into what I call extragalactic fables. For example, in part IV, I let the extragalactic version of you discover electrodynamics and gravity. In chapter IV.3, you discover that gravity affects the flow of time.

I also whet your appetite by anticipating. For example, I mention the Einstein-Rosen bridge already in chapter I.6. In working out the shortest distance between two points in chapter II.2, I mention that you will encounter the same equations when you study motion around black holes. In part II, I note that the peculiar replacement of a simple equation by a more complicated looking equation foreshadows Einstein's deep insight about gravity to be discussed in part V.

The return of Confusio

Readers of *QFT Nut* might be pleased to hear that Confusio makes a return appearance, together with other characters, such as the Smart Experimentalist. Some other friends of mine, for example the Jargon Guy, also show up. Here I am alluding to what Einstein referred[14] to as "more or less dispensable erudition."

An outline of this book

This book appears to start at a rather low level, with a review of Newtonian mechanics in part I. The reason is that I want to treat two topics more thoroughly than usual: rotations and coordinate transformations. A good understanding of these two elementary subjects allows us to jump to the Lorentz group and curved spacetime later. My pedagogical approach is to beat 2-dimensional rotations to death. Depending on how mechanics is taught, students typically miss, or fail to grasp, some of the material in the chapter on tensors. I repeat the discussion of tensors under various guises and in different contexts. One of my students who read the book points to various places where I appear to repeat myself, but I told her that it is better to hear some key point for the third time[15] than not to have understood it at all. A respected senior colleague and pioneer in Einstein gravity said to me that a good teacher is someone who never says anything worth saying only once.

I devote part II to a discussion of the all-important action principle, because I believe that it provides the quickest, and the most fundamental, way to Einstein gravity (and to quantum field theory). Part III is devoted to special relativity but, in contrast to some elementary treatments, the emphasis is on geometry and completion, not on a collection of paradoxes. In part IV, as was mentioned earlier, I let you discover electromagnetism and gravity, and so the treatment is somewhat nonstandard. Thus, even if you feel that you already know special relativity, you might want to take a quick look at part III and part IV.

Many readers probably pick up this book because of a burning desire to learn Einstein gravity. These readers would have already mastered Newtonian mechanics and special relativity, and they could probably cut to the chase and skip directly to part V. To them, the first four parts may appear to be a rather leisurely preparation for Einstein gravity. Still, I would counsel skimming, rather than skipping, the first four parts. At the very least, parts I–IV set down the conventions and notation. More importantly, they offer up the ideology of this text, an ideology that can be simply stated: action!

While I appear to start slow in parts I–III, I am actually setting things up so that we can go fast in parts V and VI. For example, all the discussion about coordinate transformation and curved spaces is to prepare the reader for a quick plunge into curved spacetime in chapter V.1. Similarly, the action principle enables the geodesic equation to be introduced early on, in part II, so that it is "ready to trot" when needed in part V. In considering whether to sign up for my course that grew into this book, some students ask how fast I will be zooming through special relativity to get to the "good stuff." But special relativity is good stuff! In particular, it is essential to understand special relativity as the geometry of spacetime* before moving on to general relativity.

The essence of Einstein gravity is explained in parts V and VI. The rest of the book contains what may be regarded as applications of the theory as developed in part VI. Part X contains extras that some might consider beyond the scope of an introductory text. The title is thus something of a misnomer, but to please my publisher, I am obliged to keep up a running joke I started with my field theory book. A better title might be *Gravity from Newton to the Brane World*.

The role of appendices

As a textbook writer, I am torn between being concise and being complete. One way out is to place numerous topics in appendices to various chapters. Some are fun, such as Einstein's derivation of $E = mc^2$ in his 1946 Haifa lectures (see chapter III.6), which, unfortunately, is in danger of being forgotten and which I much prefer to his 1905 derivation. Another example is Weyl's shortcut to the Schwarzschild solution (see chapter VI.3). Some are results I will need later, but often much later. For example, I talk about the speed of sound in an appendix to chapter III.6, but I won't need it until I get to the cosmic microwave background. Some appendices are peripheral or technical. When possible, I try to give an intuitive and heuristic understanding before launching into a long development, such as

* A multitude of books treat special relativity, but while they all get the job done, they differ widely in conceptual clarity. Besides the geometrical view of special relativity, I also want to emphasize the Lorentz action as leading to a unified approach to both massive and massless particles.

the treatment of Fermi normal coordinates. Some are for enrichment. In sum, the use of appendices represents my effort to appeal to a broad range of readers with enormously different levels of knowledge and sophistication. The reader should not feel obliged, upon first reading this book, to study all the appendices. Each should exercise his or her own judgment.

Still, a book this size is inevitably incomplete, and so it comes down to the author's choice (of course). So many beautiful results, so little space and time! I regard certain topics, though important, as better covered in more specialized tomes, such as gravitational lensing, and prefer to include some topics not discussed in several standard textbooks, such as anti de Sitter spacetime, brane worlds, and twistors.

The most incomprehensible thing about some physics textbooks

> The most incomprehensible thing about the physical world is that it is comprehensible.
>
> —A. Einstein

The most incomprehensible thing about some physics textbooks is that they are incomprehensible.

They manage to render the easily comprehensible into the nearly incomprehensible. Some textbook writers are simplifiers, others are what I call complicators. In defiance of Einstein's exhortation, many authors strive to make physics as complicated as possible, or so it seems to me. In the research literature, the cause of obscurity may be unintentional or intentional: either the author has not understood the issues involved completely (often laudably so, when the author is at the cutting edge), or the author wants to impress upon the reader the profundity of his or her paper by resorting to obfuscations. But in a textbook?

My task, and hope, in my textbooks is to make physics as simple as possible, as the "old man" with his toy[16] said. Having written both a textbook and a couple of popular books, I am perhaps qualified to express my opinions here. Popular books attempt to make physics simpler than it really is, thus in some sense deceiving the reader. Textbooks are different: they must make the reader work to master the subject. But making the reader work is not the same as making the reader suffer by rendering simple things obscure.

No bijective maps in this book

I am puzzled by students who profess no trouble with the physics but moan* about the math. All the "grown-ups" would say the opposite. The pros regard Riemannian geometry,

* Indeed, many of the postings on the sites of online booksellers regarding general relativity texts lament the difficulty of the math. At the other extreme, a few, by misguided individuals in my opinion, complain about the lack of rigor.

which is after all totally logical and algorithmic, as easy, but continue to lose sleep over Einstein's theory. Regarding the math, I can say, with only slight exaggeration, that mastery of the index notation and the chain rule almost suffices. Indeed, any serious student with a future in theoretical physics should be continually puzzled by the physics but not at all by the math.

Einstein did not say that physics should be made simple. Of course, physics is not simple, and understanding Einstein's theory does require effort. Surely you have heard that Einstein gravity involves curved spacetime, so there is no getting around learning the language needed to describe curvature. My strategy is to introduce math only when necessary, and then to illustrate the key concepts with plenty of examples. I dislike the Red Army[17] approach, and so I do not start by defining bundles on the tangent plane. I bring in the math gently and sneak in curvature early on via the familiar change of coordinates.

As for rigor, I will let yet another of my action heroes speak. "I'll differentiate any function, even the freaking delta function, as many times as I darn well please." So if you have to differentiate, just differentiate until the expression you are differentiating starts bleating for mercy. The trick is to know when it is absolutely necessary to be rigorous (which is seldom—I would never say never).

I respectfully submit that this book is not for those who want rigor.

While I realize the need for and the benefit of precise definition, for the most part I simply plead membership in the Feynman[18] "Shut up and calculate" school of physics.[19] Thus, I won't trouble your sleep with assertions such as "A bijective differentiable map of a manifold, whose inverse is also differentiable, is called a diffeomorphism." Regarding statements like this, I think that another Einstein quote may be apropos: "We should take care not to make the intellect our god; it has, of course, powerful muscles, but no personality."[20] Yet another relevant quote: "The people in Göttingen sometimes strike me, not as if they wanted to help one formulate something clearly, but instead as if they wanted only to show us physicists how much brighter they are than we."[21] Alas, "the people in Göttingen" have now gone off and multiplied,* and some even live in our midst. Precise definitions are indeed necessary occasionally, but by and large, they don't do much good in theoretical physics. Some things are better left undefined. In this connection, also keep in mind the distinction between true clarity and false clarity.[22] For example, I consider the insistence on saying "pseudo-Riemannian manifolds" in a book of this level false clarity at best.

As I was putting the finishing touches on this book, I read about some notes[23] Feynman scribbled to himself before teaching some course: "First figure out why you want the student to learn the subject and what you want them to know, and the method will result more or less by common sense." Well said! As it turned out, that was the method I followed when writing this book.

If you feel that bijection is indispensable for your existential essence, then I also respectfully submit that this book is not for you.

* One tribe is known to look at "old fashioned" indices with contempt. Only coordinate-free notations[24] are good enough for them.

But of course I am not against mathematics. For instance, I am all for differential forms (see chapters IX.7 and IX.8). However, when faced with a new formalism, I tend to be practical and ask, “For the time invested in learning it, what is the payoff?” How significant is it for the physics?

Teaching from this book and self-studying

It would be ideal to teach a leisurely year-long course based on this book. But I have also taught Einstein gravity at the University of California, Santa Barbara, as a scandalously short one-quarter undergraduate course consisting of only 29 lectures. The students allegedly knew the action principle and special relativity, but I was appropriately skeptical. Here is the actual course plan.

Lecture 1 gives an overview. Lectures 2–6 cover chapters I.5 and I.6, starting with the notion of a metric and illustrated with numerous examples, including the Poincaré half plane, and ending with locally flat coordinates and a count of the components contained in the curvature tensor. Lectures 7 and 8 cover part II, and lectures 9 and 10 part III. In lectures 11 and 12, I let the students discover electromagnetism and gravity and derive how gravity affects the flow of time. Lectures 13–15 introduce the equivalence principle and cover part V up to chapter V.3, ending with closed, flat, and open universes.

The second half of the course proceeds as follows:

Lecture 16: the geodesic equation reduced to Newton’s equation, gravitational redshift, spherically symmetric spacetime with time dependence

Lecture 17: the motion of particles and light in static spherically symmetric spacetime

Lecture 18: covariant differentiation, the geometrical picture

Lecture 19: to Einstein’s field equation as quickly as possible

Lecture 20: the Riemann curvature tensor and its symmetry properties

Lecture 21: the Einstein-Hilbert action

Lecture 22: the cosmological constant and the expanding universe

Lecture 23: Schwarzschild metric, with precession of planets and radar echo delay described in words and pictures

Lecture 24: the energy momentum tensor

Lecture 25: general proof of energy momentum conservation

Lecture 26: the Einstein tensor and the Bianchi identity

Lecture 27: black holes in various coordinates

Lecture 28: the causal structure of spacetime

Lecture 29: Hawking radiation and a grand review

So it is entirely possible to cover the bulk of this book in a one-quarter course! I did it. Students were expected to do some reading and to fill in some gaps on their own. Of course, instructors could deviate considerably from this course plan, emphasizing one topic at the expense of another. They might also wish to challenge the better students by assigning the appendices and some later chapters.

Here I come back to those I applauded earlier for self-studying Einstein gravity. Some of you might want to know which chapters to read. The answer is of course that you should read them all, in an ideal world. But if you want to get "there" quickly, I suggest the following. You are on your own regarding the first three parts: it all depends on what you already know. So try starting with part IV and see how often you need to refer back to an earlier chapter. Part V is indispensable, particularly the equivalence principle and the tour of curved spacetimes. You need to understand the covariant derivative, but you could skip the somewhat heavier appendices in chapter V.6. After the covariant derivative, you are ready for the heart of the matter, Einstein's field equation, in chapter VI.1. The rest of part VI forms the core of a traditional course on general relativity, but my emphasis is somewhat less on working out orbits in detail. That's it! You would have then reached a certain level of mastery of Einstein gravity. You could then regard the rest of the book, parts VII–X, as a buffet of topics that you could browse at your leisure. Part X contains more speculative topics, including some that may not be of lasting value. Be warned!

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I wrote the bulk of this book in Santa Barbara, California, but some parts were written while visiting the Academia Sinica in Taipei, the Republic of China. I deeply appreciate the generous hospitality of Maw-kuen Wu, the director of the Institute of Physics. Bits and pieces were also written in Munich, Germany, and in Beijing, the People's Republic of China.

My editor at Princeton University Press, Ingrid Gnerlich, has always been a pleasure to talk to and work with. She has listened patiently to my ranting and raving for years. For my copyeditor, I am delighted to have, once again, Cyd Westmoreland, who worked on both editions of my *Quantum Field Theory in a Nutshell*. I am much impressed by the meticulous

work of Peter Strupp and Princeton Editorial Associates. I also thank Craig Kunimoto for his indispensable computer help.

I am deeply grateful to my wife Janice for her loving support and encouragement throughout the writing of this book. As this book was nearing completion, she gave birth to our son Max.

Notes

1. Hereafter referred to as *QFT Nut*.
2. A. Zee, *Fearful Symmetry*. Hereafter, *Fearful*.
3. See chapter VI.3.
4. Chaim Weizmann, the first president of Israel and a chemist, once crossed the ocean with Albert Einstein on the same liner, and Einstein tried to explain the theory of relativity to him. When asked about this later, Weizmann said something like "I did not understand his theory, but he certainly convinced me that he did."
5. For the record, I took a philosophy course in college. To further emphasize that I am not totally lacking in "philosophical credentials," I was once invited by a philosophy professor to lecture, thanks to one of my popular books, to an auditorium full of philosophers. I like philosophers.
6. Einstein once said that he should have called his work "invariance theory" and lamented his use of the word "relative."
7. A. Zee, *An Old Man's Toy*. Hereafter, *Toy/Universe*.
8. In my introduction to Feynman's book on quantum electrodynamics, I wrote about three different kinds of readers of that book. Only part 0 of this book will be comprehensible to the first kind. See R. P. Feynman, *QED: The Strange Theory of Light and Matter,* with a new introduction by A. Zee, Princeton Science Library, 2006.
9. An undergrad friend had also deluded me into thinking that it was salutary to read Einstein in the original German!
10. Read J. Polchinski, *String Theory,* for example.
11. *QFT Nut,* p. 473.
12. For the record, I started my research career with John Wheeler, studying gravitational wave emission from neutron stars. For Wheeler's influence on his students, see Charles W. Misner, "John Wheeler and the Recertification of General Relativity as True Physics," in *General Relativity and John Archibald Wheeler,* ed. I. Ciufolini and R. Matzner, Springer, 2010.
13. See my remarks in chapter IX.9, for example.
14. A. Einstein, *Autobiographical Notes*, Open Court, 1999.
15. In any case, if you think that I talk too much about tensors, you could simply feel smugly superior to those poor souls who never get it.
16. See *Toy/Universe*. Also see figure 2b in the prologue to book two.
17. I learned this terminology (which, I should clarify, referred to the Russian, not the Chinese, version) in a conversation with Steve Weinberg about textbooks. It has something to do with lining up all the tanks first.
18. A colleague who got his doctorate at Caltech told me the following story. He was examined by a committee consisting of Feynman and a bunch of lesser lights. One of the lesser lights posed a question to my friend, who proceeded to answer it perfectly, outlining the calculation necessary and explaining the physical significance of the result. The lesser light then opined ominously, "You should have also said . . . " and hereforth issued from his mouth a long string of highfalutin hundred-dollar words. Feynman turned to the lesser light and announced to the rest of the room, "But that's exactly what he said!"

    Here is a totally gratuitous Feynman story that has nothing to do with the discussion at hand. During the exam, Feynman asked a question about quantum mechanics that the student was unable to answer. Feynman exploded, saying something like "Quantum mechanics was invented in the 1920s and it's now 1972; you really should have mastered quantum mechanics by now!" A committee member turned to Feynman and said softly, "Dick, Dick, it's now 1973."
19. A colleague told me his retort to Feynman: "Shut up and contemplate." Of course, Feynman is capable of doing both. Contrary to myth, Feynman won the national Putnam mathematics competition. Here we are talking about people who can only talk and not calculate.

20. The quote is possibly apocryphal.
21. Quoted in C. Reid, *Hilbert,* Springer, 1996, p. 142.
22. As one of my professors, an exceedingly distinguished theoretical physicist, used to say, the main purpose of all the talk about tangent bundles and pullback is to frighten young children. This is not entirely true, but, oh well.
23. R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics,* volume III, Addison Wesley (Commemorative issue 2004), p. xi.
24. I am certainly not against coordinate-free notations. In physics, the only issue is which notation is best suited for the job at hand. Coordinate-free notations are great for proving general theorems but are not so good for calculating. In this connection, I might regale the reader with a story. At a recent Santa Barbara conference on black holes, dS, AdS, gravity dual, and so on—in short, the latest hot stuff—I was chatting at lunch with two leading young researchers, up and coming stars, not some aging curmudgeons with congealed opinions. When I mentioned how some people clamored for index-free notations, one of these two leading lights basically said to please get those people out of her sight. The other told me a more illuminating story. During grad school, to deepen his understanding of Einstein gravity, he enrolled in a course taught by a famous mathematician. As it happened, he was the only student able to do the problems in the final exam involving actual calculations: he did them by first using old fashioned indices and then translating back into the abstract notation used in the course.

    The index-free notation in Einstein gravity is somewhat analogous to using vectors without committing to any specific coordinate choice. For example, one can prove easily that $\vec{L} = \vec{r} \times \vec{p}$ is conserved, but try to do the spinning top on an oscillating inclined plane without setting up coordinates! The difference between the uninitiated and the misinformed is that the uninitiated is not acquainted with a particular formalism, while the misinformed insists that only the particular formalism he or she likes is any good.

Part 0 Setting the Stage

Prologue

Three Stories

Story 1: The drowning beauty and the scrawny lifeguard

Since I started my quantum field theory text[1] with a story, possibly apocryphal, about Feynman in a quantum mechanics class, I feel compelled to start this text also by telling a story, possibly true,[2] about Feynman. The movie opens on a gorgeous southern California beach. We zoom in on a lifeguard, noticeably scrawnier than the other lifeguards. But on the other hand, we soon discover that he is considerably smarter. Egads, it is Dick Feynman, in the days before Baywatch! Perched on his high chair, he has been watching an attractively curvaceous swimmer with great interest, plotting how he could win the girl's affection, all the while solving a field theory problem in his head. Suddenly, he notices that the girl is splashing about frantically. She is going under! Must be a cramp! An action hero is as an action hero does: Feynman jumps down from his lookout and goes into action.*

The other lifeguards are already proceeding in a straight line (starting from point F, the lifeguard station, in figure 1, going along the dotted line) toward the girl (at point G). That would be the path of least distance. But no, Feynman has already calculated the path that would allow him to reach the girl in the least amount of time. Time counts more than space here: least time trumps least distance. Our hero (like other humans) can run much faster, even on a soft sandy beach, than he can swim. So the rescuer should spend more time running before plunging into the sea. A simple high school level calculation (exercise 1) shows Feynman the best path to take (see the solid line in figure 1). Our hero beats the other guys and gets to the eternally grateful girl first!

* "Physics is where the action is." See chapter III.2.

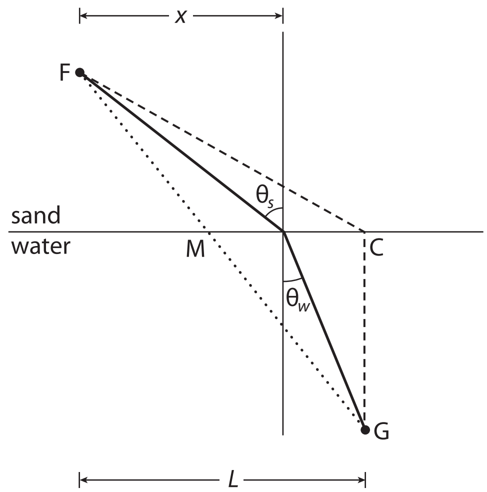


Figure 1 The best possible path for Feynman to follow to get to the drowning girl is along the solid lines from F to G.

But you don't have to calculate to see that there is an optimal path. Only a cretin would follow the third path (the dashed line) shown in the figure!

In the 17th century, Fermat discovered that light, just like Feynman, also follows a least time principle, and as a result "bends" as it enters from one medium (say, air) into another (say, water). To read these very words, you have, or rather your saintly mother has, cleverly positioned in your eyes a blob of watery substance (known to the cognoscenti as a lens) that you squeeze just so, using tiny muscles, to bend light to your advantage and bring the ambient light bouncing off these words on the printed page into focus. Your mother, as the product of eons of evolution, was oh so clever, giving you eyes. As we speak (so to speak), you are using precisely this phenomenon of light bending to save the light entering your eyes some time, a phenomenon known as refraction, and to gain yourself some knowledge about physics and the universe—an activity evolution applauds: reading this book could conceivably boost your reproductive advantage.

We all know that light travels in a straight line, but we also notice easily that when light enters water from air, it bends (as shown in figure 1 with "sand" replaced by "air"). Indeed, that explains why people standing in swimming pools appear to have comically short legs,* a phenomenon you can test by sticking a pencil in a glass of water.

It has also been known ever since Euclid† that the shortest path between two points is a straight line. Ergo, if light is always in a hurry to get from one point to another, it

* If you can't explain that, see figure 7.1 in *Fearful*. See also the common mirage shown in figure 7.2: on a hot day, the highway beneath a distant car appears to be wet, but is in fact dry. This mirage shows that light only cares about the local, not the global, minimum in time of transit.

† Babies have no need for Euclid; as soon as they can crawl, they move toward the obscure objects of their desire along a straight line.

wants to move in a straight line. Fermat and others realized that the bending of light could be explained if light moves more slowly in water than in air. Indeed, if light were really stupid, it would move in a straight line through point M to get from F to G, just like the other lifeguards.

Story 2: An ant and her honey

When I was a kid, I was challenged by a puzzle about an ant and a drop of honey. An ant located on the outside of a cylindrical glass of radius R and a vertical distance d below the rim, sees, never mind how, or perhaps smells, a drop of honey directly opposite her, but on the inside of the glass (see figure 2a). The ant wants to get to the honey in the shortest possible time,[3] crawling at some constant speed.

The solution depends on a cute trick. Imagine that the glass is made of paper. Tear out the bottom and cut the cylindrical glass down some vertical line. Lay the paper down flat, as shown in figure 2b. Further, imagine the paper to be double-sheeted, so the side with the drop of honey could be folded out, as shown in figure 2c. Now clearly, the path of shortest distance between the ant and the honey is a straight line, with distance $\sqrt{(\pi R)^2 + (2d)^2}$. The path is also indicated in figure 2b, with the segment inside the glass indicated by a dotted line. A really dumb ant would go up vertically to the rim of the glass, then move along the rim to a point above the honey, and then go down (or along a number of similar paths equal in distance to the one just described).

This puzzle contains two of the themes central to this book: the shortest path between two points and curvature, intrinsic and extrinsic.

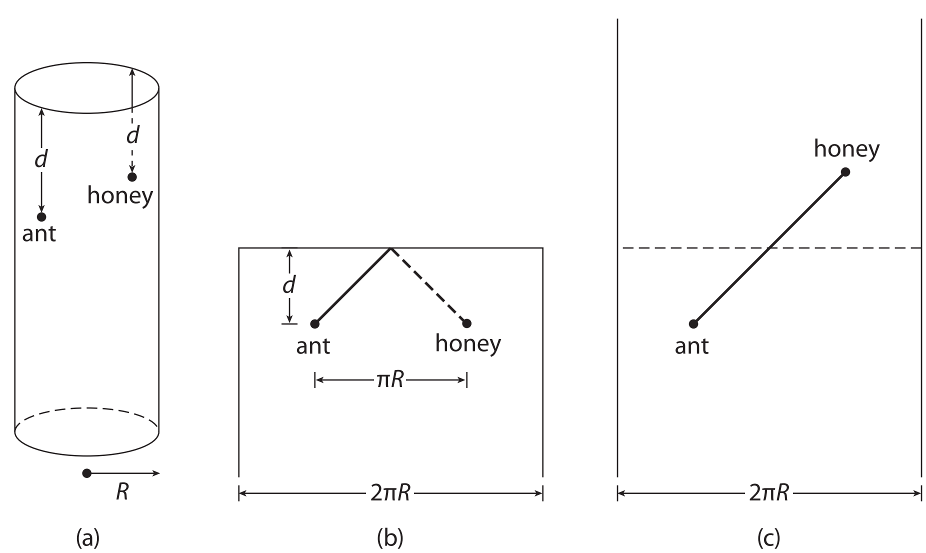


Figure 2 The best possible path for the ant to follow to get to her honey.

Draw circles and triangles on a flat piece of paper. Then roll the paper up into a cylinder. The radius and circumference of a circle maintain the same value as when flat: the paper is neither stretched nor compressed in any way. Similarly, the three angles in the triangle remain the same. A cylinder has extrinsic curvature, but zero intrinsic curvature: it is intrinsically flat. In contrast, the sphere is intrinsically curved: there is no way to construct a sphere from a flat piece of paper without stretching and compressing the paper.

The proverbial guy and gal in the street think that cylinders are curved, but you and the ant* know better. The uninitiated are talking about extrinsic curvature, regarding how the 2-dimensional surface of a cylinder is embedded into an external 3-dimensional Euclidean space.

Imagine a civilization of mites living on some curved surface. The mites are much smaller than the characteristic radius of the curvature of the surface. Once they learn how to measure the distance along any path (by pacing off the steps they have to take, for instance) they are ready for geometry. They could define the straight line between two points P_1 and P_2 as the path of least distance. Eventually, the mite professors of geometry could determine whether the world of mites is curved without getting out of their world to take a look. For example, with enough government funding, the professors could organize teams of mites to draw small circles of any desired radius by finding the set of all points a fixed distance from a given point P. Then they can measure the circumference of the circle and compute

$$R = \lim_{\text{radius}\to 0} \frac{6}{(\text{radius})^2}\left(1 - \frac{\text{circumference}}{2\pi\ \text{radius}}\right) \tag{1}$$

as the circle shrinks to zero. For flat space, R vanishes everywhere. Thus, a nonvanishing value of R gives the mites a measure of the intrinsic curvature at P—of how the geometry of their world differs† from Euclid's flat geometry. (The factor of 6 provides a convenient normalization to match another definition of R to be given later.) Another measure would be the extent that the sum of the angles enclosed by a triangle deviates from π.

Our mites are not interested in the extrinsic curvature, since they cannot get off the surface to take a look. Similarly, we are only interested in the intrinsic curvature of our universe, not in the extrinsic curvature, since we cannot get out‡ of the universe to take a look.

* Ants will eventually find the shortest path to food if the starting point is the location of the colony, but you need a whole colony of them to do so. Their trick is to lay down pheromone on the path as they go along and to prefer to follow paths with the stronger pheromone. It is crucial that the pheromone evaporates at some fixed rate and that ants often wander off the beaten paths to try out nearby paths. (Moral: wander off the beaten paths!) We explore this variational principle in chapter II.2. A multitude of physicists may also eventually solve the mystery of quantum gravity. The paths correspond to published papers, the strength of the pheromone to the prestige of the authors and the number of citations received, and so on and so forth. Not a perfect analogy by any means.

† Early in the 20th century, a distinguished professor, Sir Arthur Eddington, did precisely that, defining a straight line by the trajectory of light. See chapter VI.3.

‡ There exist some wild speculations that our universe is embedded in a much larger spacetime, but even in these theories, it does not appear that their proponents can get out of our universe, at least not until after this book is published. See chapter X.2.

Story 3: Dueling thinkers

Professor Vicious and Dr. Nasty have been at each other's throats for decades. Theoretical physicists are forever fighting over who did what when. They are constantly bickering, telling each other (as the joke goes), "Nyah, nyah, what you did is trivial and wrong, and I did it first!"

Of course, the fight for credit goes on in every field, but in theoretical physics, it is almost a way of life, since ideas are by nature ethereal. And the stakes are high: the victor gets to go to Stockholm, while the loser is consigned to the dustbin of history, a history largely written by the victor with the help of an army of idolaters and science writers.

We are finally going to settle matters between Vicious and Nasty once and for all. We place the two of them at two ends of a long hall, Vicious at $x = 0$ and Nasty at $x = L$.

We now tell Vicious and Nasty to solve the basic mystery of why the material world comes in three copies.[4] As soon as they figure it out, they are to push a button in front of them. When the button is pushed, a pulse of light is flashed to the middle of the room where, at $x = L/2$, our experimental colleague, an electronics wiz, has set up a screen. When the screen detects the arrival of a light pulse, all kinds of bells and whistles are rigged to go off. In particular, if, and only if, two light pulses arrive at the screen at precisely the same instant, a huge imperial Chinese gong will be bonged.

"Fair is fair, any and all priority claims will be settled," we tell Vicious and Nasty. "Now go to work and solve the mystery of the family problem: why do quarks and leptons come in three sets?" The dueling duo immediately assume the Rodinesque pose of the deep thinker and lock themselves in a think to the death.

Meanwhile, you are sitting on a train, moving smoothly relative to the dueling thinkers. Denote the time and space coordinates in your rest frame by t' and x'. In the Newtonian universe, time is absolute, and so we have $t' = t$. In your frame, Vicious and Nasty are moving by according to $x' = vt$ and $x' = L + vt$, respectively, but you are sitting at $x' = 0$. Of course, in the duelists' frame, you are the one who appears to be moving, gliding by at $x = -vt$ (see figure 3).

Some time passes, and all of a sudden we all hear a loud bong of the gong. "The best possible outcome, you solved the problem simultaneously!" we exclaim joyously with much relief. "You guys are equally smart and you should go to Stockholm together!"

The arrangement is electronically fool-proof. We won't have either of them gloating, "I did it first!" Peace shall reign on earth. But guess what?

A Swede is sitting next to you. He, too, heard the gong. That's the whole point of the gong: you either heard it or you didn't. It is all admissible in a court of law. Now, not only is the Swede on the Committee, but he also happens to be an intelligent Swede. He reasons as follows.

The two thinkers are gliding by as described by $x' = vt'$. When Professor Vicious pushed the button, she sent forth a multitude of photons surging toward the screen at the speed of light c. But the screen was also moving forward, away from the surging photons. Of

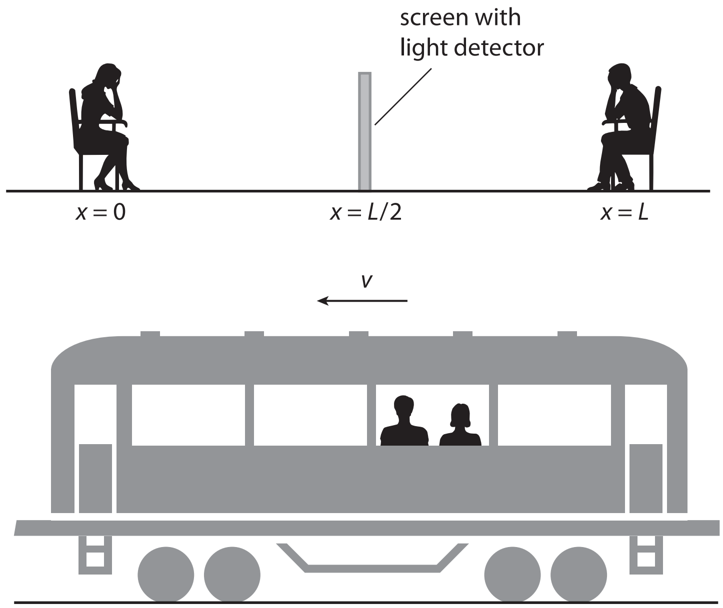


Figure 3 Professor Vicious versus Doctor Nasty.

course, light moves at the maximum allowed speed in the universe, and it soon catches up with the screen. The opposite is true for Dr. Nasty. The screen is moving toward the photons he sent forth. Thus, to reach the screen, his photons have less distance to cover than Vicious's photons.

Hence, reasons the Swede, for the two bunches of photons to reach the screen at the same time and so cause the gong to bong, the photons sent out by Vicious must have gotten going earlier. Thus, Vicious solved the problem first. With malicious glee, the Swede solemnly intones, "After Professor Vicious is awarded the Nobel Prize, she will kindly help us stuff Dr. Nasty into the dustbin of history!"

As Vicious[5] enjoys her fleeting immortality, we bemoan or toast, as our taste might be, the fall of simultaneity. Nasty, trying to climb out of the dustbin, insists that he and Vicious had been sitting still, thinking hard, and it was the Swede who was moving. Since the gong had bonged, Nasty is absolutely sure that he and Vicious hit their buttons at the same instant and so he is entitled to half the prize, while the Swede is equally sure that Vicious hit her button before Nasty hit his.

The very notion of simultaneity depends on the observer!

Meanwhile, another Swede, also on the Committee, is moving by on another train described in the duelists' frame by $x = vt$. You can fill in the rest.

Young Einstein has bent the stately flow of time out of shape. Albert himself thought up this gedanken experiment—I have merely added a few dramatic details—showing that the constancy of the speed of light necessarily has to alter our notion of simultaneity in time.

In theoretical physics, we say, "Mind-boggler in, mind-boggler out!" We feed the mind-boggling fact that the speed of light does not depend on the observer into the wondrous machinery of logic and out pops another mind-boggling fact, namely that simultaneity is in the mind of the beholder. Making up one gedanken experiment after another, Einstein showed that our common sense notion of time must be modified.

Exercises

1 Derive Snell's law: $\sin\theta_w/\sin\theta_a = c_w/c_a < 1$, where c_w and c_a denote the speed of light in water and in air, respectively.

2 Suppose the ant is outside a hemispherical bowl and the drop of honey is inside the bowl directly across from her. Find the shortest distance.

3 What happens if the ant can crawl faster on the outside of the glass than on the inside?

Notes

1. *QFT Nut*.
2. R. P. Feynman, *QED: The Strange Theory of Light and Matter,* with a new introduction by A. Zee, Princeton Science Library, 2006.
3. A colleague told me that this reminded him, at least superficially, of the umveg test (http://www.guidehorse.com/intellig.htm) for assessing intelligence in horses.
4. I am referring to the fact that quarks and leptons come in three families.
5. In his autobiography, Michael Faraday wrote of his conception of scientists: "My desire to escape from trade, which I thought vicious and selfish, and to enter into the service of Science, which I imagined made its pursuers amiable and liberal. . . . " Do I detect in the word "imagined" a trace of cynical disillusion?

Introduction

A Natural System of Units, the Cube of Physics, Being Overweight, and Hawking Radiation

Planck gave us natural units

Max Planck* is properly revered for his profound contribution to quantum mechanics. But he is also much loved for his second greatest contribution to physics: in a far-reaching and insightful paper, he gave us a natural system of units.

Once upon a time, we used some English king's feet to measure lengths.† Einstein recognized that with the universal speed of light c, we no longer need separate units for length and time. Even the proverbial guy and gal in the street understand that henceforth, we could measure length in lightyears.

We and another civilization, be they in some other galaxy, would now be able to agree on a unit of distance, if we could only communicate to them what we mean by one year or one day. Therein lies the rub: our unit for measuring time derives from how fast our home planet spins and revolves around its star. Only homeboys would know. How could we possibly communicate to a distant civilization this period of rotation we call a day, which is merely an accident of how some interstellar debris came together to form the rock we call home?

* In his personal life, Planck suffered terribly. He lost his first wife, then his son in action in World War I, then both daughters in childbirth. In World War II, bombs totally demolished his house, while the Gestapo tortured his other son to death for trying to assassinate Hitler.

† Notions we take for granted today still had to be thought up by someone. Maxwell, in his magnum opus on electromagnetism, proposed that the meter be tied to the wavelength of light emitted by some particular substance, adding that such a standard "would be independent of any changes in the dimensions of the earth, and should be adopted by those who expect their writings to be more permanent than that body." The various eminences of our subject could be quite sarcastic.

Newton's discovery of the universal law of gravity brought another constant G into physics. Comparing the kinetic energy $\frac{1}{2}mv^2$ of a particle of mass m in a gravitational potential with its potential energy $-GMm/r$ and canceling off m, we see that the combination* GM/c^2 has dimensions of length. In other words, having two universal constants c and G at hand allows us to measure masses in terms of our unit for length (or equivalently time), or lengths in terms of our unit for mass.

Planck with his constant $\hbar$ made a monumental contribution to physics by noting that the quantum world gives us for free a fundamental set of units that physicists call natural units.

Three big names, three basic principles, three natural units

To see how, note that Heisenberg's uncertainty principle tells us that $\hbar$ divided by the momentum Mc is a length. Equating the two lengths GM/c^2 and $\hbar/Mc$, we see that the combination $\hbar c/G$ has dimensions of mass squared. In other words, the three fundamental constants G, c, and $\hbar$ allow us to define a mass,[1] known rightfully as the Planck mass

$$M_{\mathrm{P}} = \sqrt{\frac{\hbar c}{G}} \tag{1}$$

We can immediately define, with Heisenberg's help, a Planck length

$$l_{\mathrm{P}} = \frac{\hbar}{M_{\mathrm{P}} c} = \sqrt{\frac{\hbar G}{c^3}} \tag{2}$$

and, with Einstein's help, a Planck time

$$t_{\mathrm{P}} = \frac{l_{\mathrm{P}}}{c} = \sqrt{\frac{\hbar G}{c^5}} \tag{3}$$

Einstein, Newton, and Heisenberg—three big names, three basic principles, three natural units to measure space, time, and energy by. We have reduced the MLT system to nothing! We no longer have to invent or find some unit, such as the good king's foot, to measure the universe with. We measure mass in units of M_{P}, length in units of l_{P}, and time in units of t_{P}. Another way of saying this is that in these natural units, $c = 1$, $G = 1$, and $\hbar = 1$. The natural system of units is understood no matter where your travels might take you, within this galaxy or far beyond.

Newton small, so Planck huge, and the Mother of All Headaches

The Planck mass works out to be 10^{19} times the proton mass M_p. That humongous number 10^{19}, as we will see, is responsible for the Mother of All Headaches plaguing fundamental

* You will learn shortly what this combination means physically.

physics today.[2] That M_P is so gigantic compared to the known particles can be traced back to the extreme feebleness of gravity: G is tiny, so M_P is enormous.

As the Planck mass is huge, the Planck length and time are teeny. If you insist on contaminating the purity of natural units by manmade ones, t_P comes out to be $\sim 5.4 \times 10^{-44}$ second, the Planck length $l_P \sim 1.6 \times 10^{-33}$ centimeter, and the Planck mass $M_P \sim 2.2 \times 10^{-5}$ gram!

It is important to realize how profound Planck's insight was. Nature herself, far transcending any silly English king or some self-important French revolutionary committee, gives us a set of units to measure her by. We have managed to get rid of all manmade units. We needed three fundamental constants, each associated with a fundamental principle, and we have precisely three!

This suggests that we have discovered all* the fundamental principles that there are. Had we not known about the quantum, then we would have to use one manmade unit to describe the universe, which would be weird. From that fact alone, we would have to go looking for quantum physics.

The cube of physics

Here is a nifty summary of all of physics as a cube (see figure 1). Physics started with Newtonian mechanics at one corner of the cube, and is now desperately trying to get to the opposite corner, where sits the alleged Holy Grail. The three fundamental constants, c^{-1}, $\hbar$, and G, characterizing Einstein, Planck or Heisenberg, and Newton, label the three axes. As we turned on one or the other of three constants (in other words, as each of these constants came into physics), we took off from the home base of Newtonian mechanics.† Much of 20th century physics consisted of getting from one corner of the cube to another. Consider the bottom face[3] of the cube. When we turned on c^{-1} we went from Newtonian mechanics to special relativity. When we turned on $\hbar$, we went from Newtonian mechanics to quantum mechanics. When we turned on both c^{-1} and $\hbar$, we arrived at quantum field theory, in my opinion the greatest monument of 20th century physics.

Newton himself had already moved up the vertical axis from Newtonian mechanics to Newtonian gravity by turning on G. Turning on c^{-1}, Einstein took us from that corner to Einstein gravity, the main subject of this book.‡ All the Stürm und Drang of the past few decades is the attempt to cross from that corner to the Holy Grail of quantum gravity, when (glory glory hallelujah!) all three fundamental constants are turned on.§

* These days, fundamental principles are posted on the physics archive with abandon. There might be hundreds by now.

† By this I mean the three laws, $F = ma$ and so on, not including the law of universal gravitation.

‡ The corner with $c^{-1} = 0$ but $\hbar \neq 0$ and $G \neq 0$ is relatively unpublicized and generally neglected. It covers phenomena described adequately by nonrelativistic quantum mechanics in the presence of a gravitational field. Two fascinating experiments in this area are: (1) dribbling neutrons like basketballs, and (2) interfering a neutron beam with itself in a gravitational field.[4]

§ This statement carries a slight caveat, which we will come to in chapter VII.3.

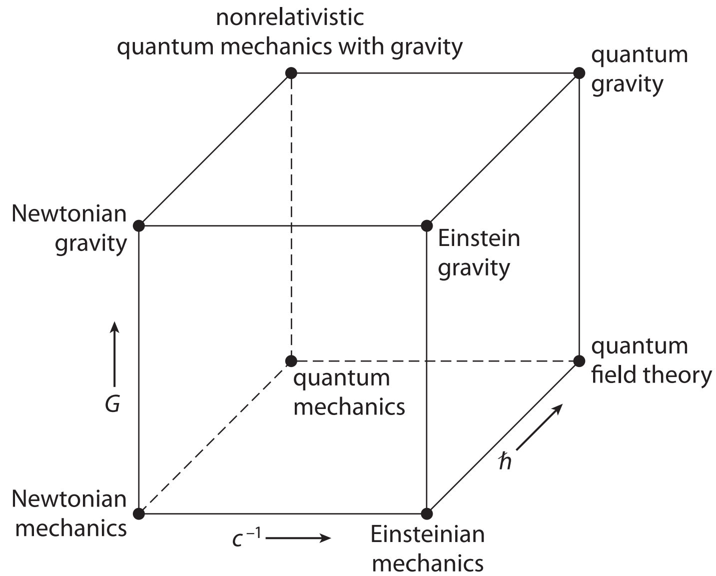


Figure 1 The cube of physics.

In our everyday existence, we are aware of only two corners of this cube, because these three fundamental constants are either absurdly small or absurdly large compared to what humans experience.

The universe's obesity index

As the obesity epidemic sweeps over the developed countries, one government after another has issued some kind of obesity index, basically dividing body weight by size. As we have seen, for an object of mass M, the combination GM/c^2 is a length that can be compared to the characteristic size of the object. So, Nature has her own obesity index for any object, from electron to galaxy. Indeed, as is well known, John Michell in 1783 and the Marquis Pierre-Simon Laplace in 1796 pointed out that even light cannot escape from an object excessively massive for its size.

More precisely, consider an object of mass M and radius R. A particle of mass m at the surface of this object has a gravitational potential energy $-GMm/R$ and kinetic energy $\frac{1}{2}mv^2$. Equating these two energies gives the escape velocity $v_{\text{escape}} = \sqrt{2GM/R}$. Setting v_{escape} to c tells us that if $2GM > Rc^2$, not even light can escape, and the object is a black hole.[5] Remarkably, even though the physics behind the argument* is not correct in detail (as we now know, we should not treat light as a Newtonian corpuscle with a tiny mass), this

* This often cited Newtonian argument actually does not establish the existence of black hole defined as an object from which nothing could escape. The escape velocity refers to the initial speed with which we attempt to fling something into outer space. In the Newtonian world, we could certainly escape from any massive planet in a rocket with a powerful enough engine.

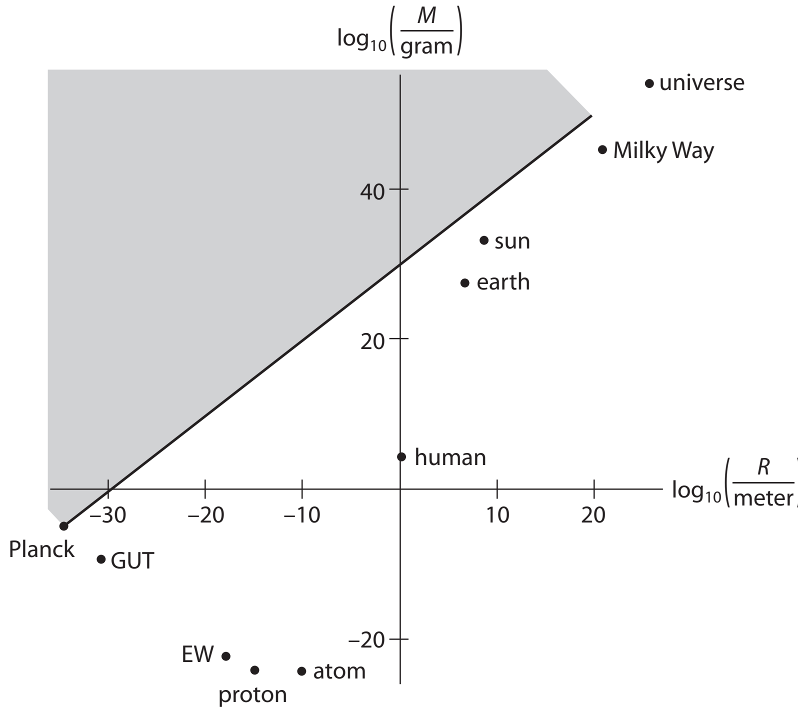


Figure 2 A plot of M versus R for various objects in the universe. EW stands for electroweak and GUT for grand unified theory. The shaded area represents the "black hole" regime with $2GM > R$.

criterion, including the factor of 2, turns out to hold in Einstein's theory. Figure 2 shows a plot of M versus R for various objects in the universe.

Hawking radiation

Unless you have been hiding out in the jungles of New Guinea, you would have heard that in an extremely influential paper, Stephen Hawking, building on the earlier work of Jacob Bekenstein and others, and working in collaboration with Gary Gibbons, pointed out this purely classical argument needs to be amended when quantum effects are included: black holes evaporate and radiate particles.

In fact, the temperature of the radiation, known as the Hawking temperature T_H of the black hole, can be estimated by using dimensional analysis. You may be puzzled,* since there are two masses in the problem, the mass M of the black hole and the Planck mass M_P. With two masses, any function of M/M_P is admissible, and so dimensional analysis appears to be inapplicable. Indeed, we need one more piece of information. The key is that

* I was talking to a distinguished condensed matter physicist just the other day, and he was puzzled about precisely this point. So your unspoken question may be widespread.

Newton's constant G is a multiplicative measure of the strength of gravity. In Einstein's theory as well as in Newton's, the gravitational field around an object of mass M can only depend on the combination of GM. Let us now set c and $\hbar$ (but not G) to 1. The combination GM is a length and hence an inverse mass. On the other hand, Boltzmann and the founding fathers of statistical mechanics had long ago revealed to us that temperature, a highly mysterious concept at one time, is merely the average energy[6] of the microscopic constituents of macroscopic matter. Hence temperature has the dimensions of energy, that is, of a mass in units with $c = 1$.

It follows immediately that $T_H \sim \frac{1}{GM}$. This "sophisticated" dimensional analysis captures an essential piece of physics: the radiation is explosive! As the black hole radiates energy, M goes down and T_H goes up, and thus the black hole radiates faster. The radiative mass loss accelerates. Certainly not something you want to see in the kitchen: an object that gets hotter as it loses energy.

In chapter VII.3, we will see that the overall numerical constant can be determined in a couple of lines of algebra, so that

$$T_H = \frac{\hbar c^3}{8\pi GM} \tag{4}$$

We have restored c and $\hbar$ by high school dimensional analysis using everyday unnatural units. It is gratifying to see that indeed, with $\hbar = 0$ and quantum effects turned off, $T_H = 0$, and the black hole does not radiate.

Thermodynamics states that entropy S is given by $dE = TdS$. Here E is just the mass of the black hole. Integrating $\frac{dS}{dM} = \frac{1}{T_H} \sim GM$, we obtain

$$S \sim GM^2 \sim \left(\frac{M}{M_P}\right)^2 \tag{5}$$

Note that, as expected, S is dimensionless.

Using the fact that the black hole has radius $R \sim GM$ and hence surface area $A \sim R^2$, we conclude that

$$S \sim \frac{R^2}{G} \sim \frac{A}{l_P^2} \tag{6}$$

You should be shocked, shocked, shocked. Most theoretical physicists were, and are.

Not shocked?

Normally, the entropy of a system is extensive, that is, proportional to its volume. Somehow, a black hole has an entropy proportional to its surface area rather than to its volume. This fact has led to the so-called holographic principle. Many fundamental physicists believe that this mysterious property of black holes holds the key to quantum gravity.

All of this merely from dimensional analysis!

Notes

1. Some readers might wonder why we do not use the mass of the electron m_e. In modern particle physics, the electron may not always have had the mass it has now, and in fact it might have been massless in the early universe. The masses of elementary particles depend on quantum field theoretic notions known as spontaneous symmetry breaking and the Higgs mechanism. We should express m_e in terms of M_P, not M_P in terms of m_e. In different areas of physics, different units are used: for example, the size of the hydrogen atom might be used as a length unit.
2. I return to this problem in due course, in chapter X.8, for example.
3. This face, regarded as a square, was discussed in the very first section of the first chapter in *QFT Nut*.
4. See appendix 5 to chapter X.8; for more details, see J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics,* pp. 110 and 133.
5. Named by John Wheeler almost 200 years later.
6. The Boltzmann constant k, which is merely a conversion factor between energy units and the markings on some tubes containing mercury known as degrees, has been set to 1.

Prelude

Relativity Is an Everyday and Ancient Concept

Butterflies will fly indifferently toward every side

Relativity is all about the notion that you are as good as the next guy, or to put it relatively, the other guy is as good as you.

More seriously, relativity expresses the fact that the laws of physics as deduced by two observers in uniform motion with respect to each other must be the same.

We physicists believe in the fundamental principle that physics should not depend on the physicist, unlike some other academic disciplines we need not name, in which the truth can vary according to the practitioner.

The proverbial guy in the street thinks that relativity started with Albert Einstein (1879–1955), but you know better, of course. Surely, some smart human had an inkling of it as soon as sufficiently smooth transport* became available, perhaps even the proverbial "cave man"† drifting downriver on a log watching his buddies moving by. Galileo Galilei (1564–1642) first[1] explicitly stated the principle of relativity. In *Dialogue Concerning the Two Chief World Systems* (first published in 1632) the character Salviati says:

> Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the

* Of course, we are on a spinning rock orbiting a star in a rotating galaxy hurtling toward its neighbor at high speed, but our transport is so smooth that we didn't notice it for the longest time.

† Or a Sung dynasty poet in a boat; see *Fearful*, p. 52.

vessel beneath; and, in throwing something to your friend, you need throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction. When you have observed all these things carefully (though doubtless when the ship is standing still everything must happen in this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still. In jumping, you will pass on the floor the same spaces as before, nor will you make larger jumps toward the stern than toward the prow even though the ship is moving quite rapidly, despite the fact that during the time that you are in the air the floor under you will be going in a direction opposite to your jump. In throwing something to your companion, you will need no more force to get it to him whether he is in the direction of the bow or the stern, with yourself situated opposite. The droplets will fall as before into the vessel beneath without dropping toward the stern, although while the drops are in the air the ship runs many spans. The fish in their water will swim toward the front of their bowl with no more effort than toward the back, and will go with equal ease to bait placed anywhere around the edges of the bowl. Finally the butterflies and flies will continue their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship, from which they will have been separated during long intervals by keeping themselves in the air. And if smoke is made by burning some incense, it will be seen going up in the form of a little cloud, remaining still and moving no more toward one side than the other. The cause of all these correspondences of effects is the fact that the ship's motion is common* to all the things contained in it, and to the air also. That is why I said you should be below decks; for if this took place above in the open air, which would not follow the course of the ship, more or less noticeable differences would be seen in some of the effects noted.[2]

That† is so beautifully stated! Much better than most popular physics books on the market (see figure 1).

Galileo's ship was updated to Einstein's train‡ and later to rocket ships and other space vehicles. Let's use Einstein's train, moving smoothly along the x-axis with velocity u (see figure 2). Let an event occur at the point (x, y, z) at time t for the observer on the train (call her Ms. Unprime) and at the point (x', y', z') at time t' for the observer on the ground (Mr. Prime). We are of course utilizing the profound and brilliant insight of Galileo's contemporary René Descartes (1596–1650) that geometry can be reduced to algebra by associating three numbers with each point in space. The Galilean transformation states that

$$t' = t \tag{1}$$

* The phrase "common to all the things contained in it" will play a starring role when we get to Einstein's equivalence principle, as we will see in part V.

† Galileo intended this passage as a refutation of the argument that the earth could not rotate since otherwise objects would fall toward the west.

‡ The historian Peter Galison has pointed out that in the period leading up to 1905, the year Einstein proposed his theory of special relativity, high speed trains and the telegraph linked the cities of Europe, and an increasingly technological society was preoccupied with clock synchronization among other things.[3]

Figure 1 Galileo's vision: butterflies fly normally in a cabin on a smoothly moving ship.

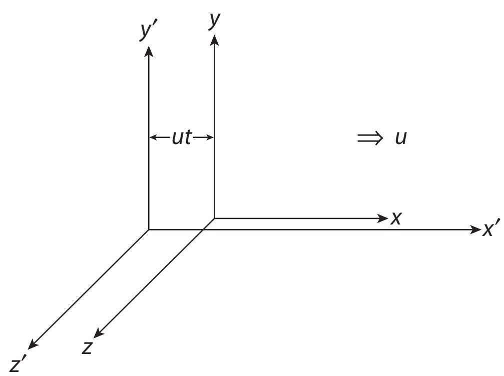


Figure 2 Galilean transformation.

and

$$x' = x + ut, \qquad y' = y, \quad \text{and} \quad z' = z \tag{2}$$

with u the constant relative velocity between the two observers.

We simply differentiate: $\frac{dx'}{dt'} = \frac{dx'}{dt} = \frac{dx}{dt} + u$. Thus, if Ms. Unprime tosses a ball forward with speed v, Mr. Prime sees the ball moving forward with speed $v' = v + u$, in accordance with everyday observation, as known to you, me, and Salviati. We have derived the Galilean law* for the addition of velocities:

$$v' = v + u \tag{3}$$

Differentiating again, we obtain the ball's acceleration $a' = \frac{dv'}{dt} = \frac{dv}{dt} = a$. Since Newton's law of motion $F = ma$ involves acceleration, we conclude that Newtonian mechanics is invariant under the Galilean transformation, as Salviati told us.

* Which you can verify these days at any major airport with a moving sidewalk.

Special relativity in one minute

Special relativity can be simply summarized. (Of course, we will be going through it in much greater detail later.) Maxwell's laws of electromagnetism turned out not to be invariant under the Galilean transformation. The speed of light c is determined by how fast an electric field can turn into a magnetic field and vice versa and so does not depend on the observer. In total defiance of (3), Maxwell had

$$c \neq c + u \tag{4}$$

In the high noon showdown between Maxwell and Galileo, Maxwell won. The Galilean transformation had to be replaced by the Lorentzian transformation involving that universal constant of Nature, c for celeritas.* The relations (1) and (2) between space and time were modified.

General relativity in 30 seconds

That was special relativity in 60 seconds. But then we could ask, what would happen if u were not constant, if Salviati's ship encountered a storm, as it were? In deriving $a' = a$, we used $\frac{du}{dt} = 0$, but if that were not so, we would have

$$a' = a + \frac{du}{dt} \tag{5}$$

Multiply this by m, the mass of the ball Ms. Unprime tossed forward, to obtain $ma' = ma + m\frac{du}{dt}$. Mr. Prime, invoking Newton's law, thus sees an additional force $m\frac{du}{dt}$ acting on the ball.

What could that force possibly be? The answer to that question will lead us to curved spacetime and Einstein gravity.[4]

Truth is not relative

Later in life, Einstein moaned that he should have called his work "invariant theory" instead of "relativity theory." Had he been more judicious in his choice of words, you, I, and Einstein would have been spared the spectacle of eminent humanities scholars asserting that "Truth is relative" since "There is no absolute truth: Einstein proved it so." Of course, you know that Einstein said exactly the opposite. Physics must be invariant and true.

Notes

1. Perhaps some historian will track down others before Galileo.
2. Galileo, *Dialogue Concerning the Two Chief World Systems*, trans. S. Drake, University of California Press, 1953, pp. 186–187.
3. See P. Galison, *Einstein's Clocks, Poincaré's Maps: Empires of Time*, W. W. Norton, 2004.
4. Nitpickers, please! It's what I could say in 30 seconds!

* Einstein used V in his 1905 paper.

BOOK ONE

From Newton to the Gravitational Redshift

Part I From Newton to Riemann: Coordinates to Curvature

1.1 Newton's Laws

The foundational equation of our subject

> For in those days I was in the prime of my age for invention and minded Mathematicks & Philosophy more than at any time since.
>
> —Newton describing his youth in his memoirs

Let us start with one of Newton's laws, which curiously enough is spoken as $F = ma$ but written as $ma = F$. For a point particle moving in D-dimensional space with position given by $\vec{x}(t) = (x^1(t), x^2(t), \cdots, x^D(t))$, Mr. Newton taught us that

$$m\frac{d^2x^i}{dt^2} = F^i \tag{1}$$

with the index* $i = 1, \cdots, D$. For $D \leq 3$ the coordinates have traditional "names": for example, for $D = 3$, x^1, x^2, x^3 are often called, with some affection, x, y, z, respectively.

Bad notation alert! In teaching physics, I sometimes feel, with only slight exaggeration, that students are confused by bad notation almost as much as by the concepts. I am using the standard notation of x and t here, but the letter x does double duty, as the position of the particle, which more strictly should be denoted by $x^i(t)$ or $\vec{x}(t)$, and as the space coordinates x^i, which are variables ranging from $-\infty$ to ∞ and which certainly are independent of t.

The different status between x and t in say (1) is particularly glaring if $N > 1$ particles are involved, in which case we write $m\frac{d^2x^i{}_a}{dt^2} = F^i{}_a$ or $m\frac{d^2\vec{x}_a}{dt^2} = \vec{F}_a$ with $x^i{}_a(t)$ for $a = 1, 2, \cdots, N$. But certainly t_a is a meaningless concept in Newtonian physics. In the Newtonian universe, t is the time ticked off by a universal clock, while $\vec{x}_a(t)$ is each particle's private business. We will have plenty more to say about this point. Here $x^i{}_a(t)$ are $3N$ functions of t, but there are still only 3 x^i.

* See appendix 2.

Some readers may feel that I am overly pedantic here, but in fact this fundamental inequality of status between x and t will come to a head when we get to the special theory of relativity. (I now drop the arrow on $\vec{x}$.) Perhaps Einstein as a student was bothered by this bad notation. One way to remedy the situation is to use q (or q_a) to denote the position of particles, as in more advanced treatments. But here I bow to tradition and continue to use x.

Have differential equation, will solve

After Newton's great insight, we "merely" have to solve some second order differential equations.

To understand Newton's fabulous equation, it's best to work through a few examples. (I need hardly say that if you do not already know Newtonian mechanics, you are unlikely to be able to learn it here.)

A priori, the force F^i could depend on any number of things, but from experience we know that in many simple cases, it depends only on x and not on t or $\frac{dx}{dt}$. As physicists unravel the mysteries of Nature, it becomes increasingly clear that fundamental forces are derived from an underlying quantum field theory and that they have simple forms. Complicated forces often merely result from some approximations we make in particular situations.

Example A

A particle in 1-dimensional space tied to a spring oscillates back and forth.

The force F is a function of space. Newton's equation

$$m\frac{d^2x}{dt^2} = -kx \tag{2}$$

is easily solved in terms of two integration constants: $x(t) = a\cos\omega t + b\sin\omega t$, with $\omega = \sqrt{\frac{k}{m}}$. The two constants a and b are determined by the initial position and initial velocity, or alternatively* by the initial position at $t = 0$ and by the final position at some time $t = T$. Energy, but not momentum, is conserved.

Example B

We kick a particle in 1-dimensional space at $t = 0$.

The force F is a function of time. This example allows me to introduce the highly useful Dirac[1] delta function, or simply delta function.[2] By the word "kick" we mean that the time scale τ during which the force acts is much less than the other time scales we are

* See part II.

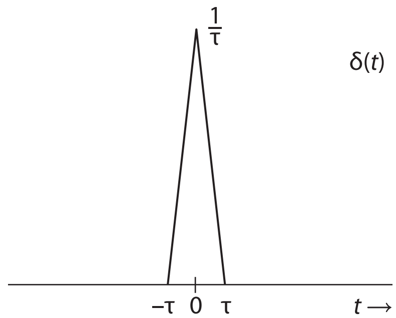


Figure 1 The delta function, which could be thought of as an infinitely sharp spike, is strictly speaking not a function, but the limit of a sequence of functions.

interested in. Thus, take $F(t) = w\delta(t)$, where the function $\delta(t)$ rises sharply just before $t = 0$, rapidly reaches its maximum at $t = 0$, and then sharply drops to 0. Because we included a multiplicative constant w, we could always normalize $\delta(t)$ by

$$\int dt\ \delta(t) = 1 \tag{3}$$

As we will see presently, the precise form of $\delta(t)$ does not matter. For example, we could take $\delta(t)$ to rise linearly from 0 at $t = -\tau$, reach a peak value of $1/\tau$ at $t = 0$, and then fall linearly to 0 at $t = \tau$. For $t < -\tau$ and for $t > \tau$, the function $\delta(t)$ is defined to be zero. Take the limit $\tau \to 0$, in which this function is known as the delta function. In other words the delta function is an infinitely sharp spike. See figure 1.

The δ function is somehow treated as an advanced topic in mathematical physics, but in fact, as you will see, it is an extremely useful function that I will use extensively in this book, for example in chapters II.1 and III.6. More properties of the δ function will be introduced as needed.

Integrating

$$\frac{d^2x}{dt^2} = \frac{w}{m}\delta(t) \tag{4}$$

from some time $t_- < 0$ to some time $t_+ > 0$, we obtain the change in velocity $v \equiv \frac{dx}{dt}$:

$$v(t_+) - v(t_-) = \frac{w}{m} \tag{5}$$

Note that in this example, neither energy nor momentum is conserved. The lack of conservation is easy to understand: (4) does not include the agent administering the kick. In general, a time-dependent force indicates that the description is not dynamically complete.

Example C

A planet approximately described as a point particle of mass m goes around its sun of mass $M \gg m$.

This is of course the celebrated problem Newton solved to unify celestial and terrestrial mechanics, previously thought to be two different areas of physics. His equation now reads

$$m\frac{d^2\vec{r}}{dt^2} = -GMm\frac{\vec{r}}{r^3} \tag{6}$$

where we use the notation $\vec{r} = (x, y, z)$ and $r = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x^2 + y^2 + z^2}$.

John Wheeler has emphasized the interesting point that while Newton's law (1) tells us how a particle moves in space as a function of time, we tend to think of the trajectory of a particle as a curve fixed in space. For example, when we think of the motion of a planet around the sun, we think of an ellipse rather than a spiral around the time axis. Even in Newtonian mechanics, it is often illuminating to think in terms of a spacetime picture rather than a picture in space.[3]

Newton and his two distinct masses

> By thinking on it continually.
> —Newton (reply given when asked how he discovered the law of gravity)

Conceptually, in (6), m represents two distinct physical notions of mass. On the left hand side, the inertial mass measures the reluctance of the object to move. On the right hand side, the gravitational mass measures how strongly the object responds to a gravitational field. The equality of the inertial and the gravitational mass was what Galileo tried to verify in his famous apocryphal experiment dropping different objects from the Leaning Tower of Pisa. Newton himself experimented with a pendulum consisting of a hollow wooden box, which he proceeded to fill with different substances, such as sand and water. In our own times, this equality has been experimentally verified[4,5] to incredible accuracy.

That the same m appears on both sides of the equation turns out to be one of the greatest mysteries in physics before Einstein came along. His great insight was that this unexplained fact provided the clue to a deeper understanding of gravity. At this point, all we care about this mysterious equality is that m cancels out of (6), so that $\ddot{\vec{r}} = -\kappa\frac{\vec{r}}{r^3}$, with $\kappa \equiv GM$.

Celestial mechanics solved

Since the force is "central," namely it points in the direction of $\vec{r}$, a simple symmetry argument shows that the motion is confined to a plane, which we take to be the (x-y) plane. Set $z = 0$ and we are left with

$$\ddot{x} = -\kappa x/r^3 \quad \text{and} \quad \ddot{y} = -\kappa y/r^3 \tag{7}$$

I have already, without warning, switched from Leibniz's notation to Newton's dot notation

$$\dot{x} \equiv \frac{dx}{dt} \quad \text{and} \quad \ddot{x} \equiv \frac{d^2x}{dt^2} \tag{8}$$

Since this is one of the most beautiful problems[6] in theoretical physics, I cannot resist solving it here in all its glory. Think of this as a warm-up before we do the heavy lifting of learning Einstein gravity. Also, later, we can compare the solution here with Einstein's solution.

Evidently, we should change from Cartesian coordinates (x, y) to polar coordinates (r, θ). We will do it by brute force to show, in contrast, the elegance of the formalism we will develop later. Differentiate

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \tag{9}$$

twice to obtain first

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} \quad \text{and} \quad \dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta} \tag{10}$$

and then

$$\begin{aligned} &\ddot{x} = \ddot{r}\cos\theta - 2\dot{r}\sin\theta\,\dot{\theta} - r\cos\theta\,\dot{\theta}^2 - r\sin\theta\,\ddot{\theta} \\ \text{and} \quad &\ddot{y} = \ddot{r}\sin\theta + 2\dot{r}\cos\theta\,\dot{\theta} - r\sin\theta\,\dot{\theta}^2 + r\cos\theta\,\ddot{\theta} \end{aligned} \tag{11}$$

(Note that in each pair of these equations, the second could be obtained from the first by the substitution $\theta \to \theta - \frac{\pi}{2}$, so that $\cos\theta \to \sin\theta$, and $\sin\theta \to -\cos\theta$.)

Multiplying the first equation in (7) by $\cos\theta$ and the second by $\sin\theta$ and adding, we obtain, using (11),

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\kappa}{r^2} \tag{12}$$

On the other hand, multiplying the first equation in (7) by $\sin\theta$ and the second by $\cos\theta$ and subtracting, we have

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \tag{13}$$

I remind the reader again that we are doing all this in a clumsy brute force way to show the power of the formalism we are going to develop later.

After staring at (13) we recognize that it is equivalent to

$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \tag{14}$$

which implies that

$$\dot{\theta} = \frac{l}{r^2} \tag{15}$$

for some constant l. Inserting this into (12), we have

$$\ddot{r} = \frac{l^2}{r^3} - \frac{\kappa}{r^2} = -\frac{dv(r)}{dr} \tag{16}$$

where we have defined

$$v(r) = \frac{l^2}{2r^2} - \frac{\kappa}{r} \tag{17}$$

Multiplying (16) by $\dot{r}$ and integrating over t, we have

$$\int dt\ \frac{1}{2}\frac{d}{dt}\dot{r}^2 = \int dt\ \dot{r}\ddot{r} = -\int dt \frac{dr}{dt}\frac{dv(r)}{dr} = -\int dr \frac{dv(r)}{dr}$$

so that finally

$$\frac{1}{2}\dot{r}^2 + v(r) = \epsilon \tag{18}$$

with ϵ an integration constant.

This describes a unit mass particle moving in the potential $v(r)$ with energy ϵ. Plot $v(r)$. Clearly, if ϵ is equal to the minimum of the potential $v_{\text{min}} = -\frac{\kappa^2}{2l^2}$, then $\dot{r} = 0$ and r stays constant. The planet follows a circular orbit of radius l^2/κ. If $\epsilon > v_{\text{min}}$ the orbit is elliptical, with r varying between r_{min} (perihelion) and r_{max} (aphelion) defined by the solutions to $\epsilon = v(r)$. For $\epsilon > 0$ the planet is not bound and should not even be called a planet.

We have stumbled across two conserved quantities, the angular momentum l and the energy ϵ per unit mass, seemingly by accident. They emerged as integration constants, but surely there should be a more fundamental and satisfying way of understanding conservation laws. We will see in chapter II.4 that there is.

Orbit closes

One fascinating apparent mystery is that the orbit closes. In other words, as the particle goes from r_{min} to r_{max} and then back to r_{min}, θ changes by precisely 2π. To verify that this is so, solve (18) for $\dot{r}$ and divide by (15) to obtain $\frac{dr}{d\theta} = \pm(r^2/l)\sqrt{2(\epsilon - v(r))}$. Changing variable from r to $u = 1/r$, we see, using (17), that $2(\epsilon - v(r))$ becomes the quadratic polynomial $2\epsilon - l^2u^2 + 2\kappa u$, which we can write in terms of its two roots as $l^2(u_{\text{max}} - u)(u - u_{\text{min}})$. Since u varies between u_{min} and u_{max}, we are led to make another change of variable from $u = u_{\text{min}} + (u_{\text{max}} - u_{\text{min}})\sin^2\zeta$ to ζ, so that ζ ranges from 0 to $\frac{\pi}{2}$. Thus, as the particle completes one round trip excursion in r, the polar angle changes by (note that $u_{\text{min}} = 1/r_{\text{max}}$ and $u_{\text{max}} = 1/r_{\text{min}}$)

$$\begin{aligned}\Delta\theta &= 2\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{l\,dr}{r^2\sqrt{2(\epsilon - v(r))}} = 2\int_{u_{\text{min}}}^{u_{\text{max}}} \frac{l\,du}{\sqrt{2\epsilon - l^2u^2 + 2\kappa u}} \\ &= 2\int_{u_{\text{min}}}^{u_{\text{max}}} \frac{du}{\sqrt{(u_{\text{max}} - u)(u - u_{\text{min}})}} = 4\int_0^{\frac{\pi}{2}} d\zeta = 2\pi\end{aligned} \tag{19}$$

That this integral turns out to be exactly 2π is at this stage nothing less than an apparent miracle. Surely, there is something deeper going on, which we will reveal in chapter I.4. Note also that the inverse square law is crucial here. Incidentally, the change of variable here indicates how the Newtonian orbit* (and also the Einsteinian orbit, as we will see in part VI) could be determined. See exercise 2.

Bad notation alert! In (1), the force on the right hand side should be written as $F^i(x(t))$ (in many cases). In C, the gravitational force exists everywhere, namely $F(x)$ exists as a function, and what appears in Newton's equation is just $F(x)$ evaluated at the position of the particle $x(t)$. In contrast, in A, with a mass pulled by a spring, $F(x)$ does not make sense, only $F(x(t))$ does. The force exerted by the spring does not pervade all of space, and hence is defined only at the position of the particle $x(t)$, not at any old x. I can practically hear the reader chuckling, wondering what kind of person I could be addressing here, but believe me, I have encountered plenty of students who confuse these two basic concepts: spatial coordinates and the location of particles. I may sound awfully pedantic, but when we get to curved spacetime, it is often important to be clear that certain quantities are defined only on so-called geodesic curves, while others are defined everywhere in spacetime.

A historical digression on the so-called Newton's constant

> Wouldn't we be better off with the two eyes we now have plus a third that would tell us what is sneaking up behind? . . . With six eyes, we could have precise stereoscopic vision in all directions at once, including straight up. A six-eyed Newton might have dodged that apple and bequeathed us some levity rather than gravity.
>
> —George C. Williams[7]

Physics textbooks by necessity cannot do justice to physics history. As you probably know, in the *Principia,* Newton (1642–1727) converted his calculus-based calculations to geometric arguments,[8] which most modern readers find rather difficult to follow. Here I want to mention another curious point: Newton never did specifically define what we call his constant G. What he did with $ma = GMm/r^2$ was to compare the moon's acceleration with the apple's acceleration: $a_{\text{moon}} R^2_{\text{lunar orbit}} = GM_{\text{earth}} = a_{\text{apple}} R^2_{\text{radius of earth}}$. But to write $GM_{\text{earth}} = a_{\text{apple}} R^2_{\text{radius of earth}}$, he had to prove what is sometimes referred to as the first of Newton's two "superb theorems," namely that with the inverse square law the gravitational force exerted by a spherical mass distribution acts as if the entire mass were concentrated in a point at the center of the distribution. (See exercise 4.) Even with his abilities, Newton had to struggle for almost 20 years, the length of which contributed to the bitter priority fight he had with Hooke on the inverse square law, with Newton claiming that he had the law a long time before publication. You should be able to do it faster by a factor of $\sim 10^4$ as an exercise.

* On the old one pound note, a portrait of Newton together with his orbits appears on the back. Amusingly, the artist felt compelled to put the sun at the center, rather than one of the foci, of the ellipse.

Knowing the moon's period and $R_{\text{lunar orbit}}$, Newton could calculate a_{moon}. Since $R_{\text{radius of earth}}$ had been known since antiquity, he was thus able to calculate a_{apple} and obtained agreement* with Galileo's measurement of a_{apple}. This of course represents one of the most magnificent advances in physics history, with Newton unifying[9] the previously disparate subjects of celestial and terrestrial mechanics in one stroke. I don't have space to dwell on this here, but I do want to call your attention to the fact that Newton did not need to know G and M_{earth} to perform his feat.

Indeed, G was not measured until 1798 by Henry Cavendish (1731–1810) using equipment built and designed by his friend John Michell (1724–1793), now of black hole fame, who died before he could carry out the experiment.

Needless to say, what I have presented here should only be regarded as a comic book version of history.

Appendix 1: Where is hell?

You will find it in this appendix, sort of.

Curiously, contrary to what some textbooks and popular books stated, Cavendish's goal was not to measure G, but M_{earth} and hence the earth's density. Why this was of more interest to physicists of the time than G is in itself another interesting tidbit in physics history.

I mentioned that Newton had two superb theorems and that the first triggered his feud with Hooke. His second superb theorem states that there is no gravitational force inside a spherical shell.[10] Are you curious why Newton would even attack such a problem? An erroneous calculation had convinced him that the earth was much less dense than the moon, which led his friend Edmond Halley (1656–1742), who by the way published the *Principia* at his expense, to propose the hollow earth theory.[11] Witness the popularity of the idea in science fiction, notably Jules Verne's *Journey to the Center of the Earth* (1864). The idea may seem absurd to us, but at that time, a location for hell had to be found, and leading physicists gave serious thought to this pressing problem. Every epoch in physics has its own top ten problems.

So now we understand Cavendish's interest in M_{earth} and hence in the density of the earth rather than in G. Some textbooks give the impression that people easily obtained M_{earth} by multiplying the density of rock and the volume of the earth. Not so easy if you think that the earth might be hollow! We learn from Newton's second theorem that there is no gravitational force in hell, so the usual portrayal of the leaping flames can't be right!

Appendix 2: Fear of indices

Occasionally, a student or two would profess, unaccountably, a "fear of indices." In fact, there is nothing to fear.[12] At this stage, just stand back and admire how clever the invention of indices is. Instead of giving names to each coordinate axis, such as x, y, and z, we could pass fluidly between different dimensions by writing x^i, with $i = 1, 2, \cdots, D$. The length of the alphabet we use does not limit us, and we could easily go beyond 26 dimensions.

When we get to Einstein's theory, there will be a flood of indices, and we will have to distinguish between upper and lower indices. In Newtonian mechanics, there is no significance to whether we write the index as a superscript or a subscript. Have no fear: we will discuss each of these features of indices when the need arises. At this point, we merely note that a quantity can carry more than one index. In the text, we wrote ${x^i}_a$, with $i = 1, 2, \cdots, D$ labeling the different spatial directions, and $a = 1, 2, \cdots, N$ labeling the different particles. We will encounter more examples as we go along.

* Newton's first try did not lead to excellent agreement, because the value for the earth's equatorial radius was off. Just a reminder that physics never progresses as smoothly as textbooks say.

With only slight exaggeration, we could say that the invention of indices represents one of the really clever ideas[13] in the history of physics and mathematics, almost a "magic trick" that enables us to deal with as many particles in as many spatial dimensions as we like with the mere addition of some subscripts and superscripts.

Exercises

1 Show that for some suitably smooth function $f(x)$, the integral $\int_{-\infty}^{\infty} dx\delta(x)f(x) = f(0)$. Then show that $\delta(ax) = \delta(x)/|a|$ by evaluating the integral $\int_{-\infty}^{\infty} dx\delta(ax)f(x)$ for some smooth function $f(x)$.

2 Determine the orbit $r(\theta)$ by changing variable from r to $u = 1/r$. We will need the result of this exercise later.

3 Newton thought that light consists of "corpuscles." Calculate the deflection of light by the sun, applying what you learned in the text to the case $\epsilon > 0$. Note that the mass of these minute "particles of light" drops out in Newtonian theory anyway. We will need this result to compare with Einstein's theory later in chapter VI.3.

4 Prove Newton's first superb theorem: the gravitational force exerted by a spherical mass distribution acts as if the entire mass were concentrated in a point at the center of the distribution.

5 Prove Newton's second superb theorem.

6 Suppose engineers can build a straight tunnel connecting two cities on earth. Then we could have a free unpowered "gravity express"[14] by simply dropping a railroad car into the tunnel, allowing it to fall from one city to the other. Use Newton's two superb theorems to calculate the transit time.

Notes

1. Also introduced by Cauchy, Poisson, Hermite, Kirchoff, Kelvin, Helmholtz, and Heaviside. See J. D. Jackson, *Am. J. Phys.* 76 (2008), pp. 707–709.
2. Rigorous mathematicians go berserk at physicists' use of the word "function" here; they prefer to call it a distribution, defined as the limit of a function. But working physicists do not give a flying barf about such niceties. In any case, I do not personally know a theoretical physicist suffering any harm by calling $\delta(t)$ a function.
3. Consider a game of tennis. Compare a hard drive down the line and a soft lob high over the net. In both cases, we are to solve Newton's law $\frac{d^2x}{dt^2} = 0$, $\frac{d^2y}{dt^2} = -g$, with the boundary conditions $x(0) = 0$, $x(T) = L$, and $y(0) = y(T) = 0$. (The problem is so elementary that we won't bother to explain the notation, that y denotes the vertical direction, that $y = 0$ is the ground, that T is the time of flight before the ball hits the ground, that L is the length of the tennis court, and so on and so forth. You might want to draw your own figure.) The solution is $x = Lt/T$, $y = \frac{1}{2}g(T - t)t$. Note that the two types of tennis shots are governed by the same equation and the same L. Hence we obtain the same solution, but keep in mind that T is small in the case of the hard drive and that T is large in the case of the soft lob. Now eliminate t to obtain y as a function of x, namely $y(x) = \frac{1}{2}gT^2(1 - \frac{x}{L})\frac{x}{L}$, a parabola in both cases (of course). But compare the curvature of the two parabolas: we have $\frac{d^2y}{dx^2} = -g(T/L)^2$, very small in the case of the hard drive (small T) and very large in the case of the lob (large T). The hard drive down the line barely skimming over the net, and the soft lob climbing lazily high up into the sky, look and feel totally different pictured in space. In contrast, consider y as a function of t. We also have two parabolas (of course), namely $y(t) = \frac{1}{2}g(T - t)t$, as given earlier. Now compare the curvature of the two parabolas: we have $\frac{d^2y}{dt^2} = -g$, the same in both cases. The curvature of the ball's trajectory in spacetime is universal (universal gravity, get it?). But we tend to see in our mind's eye the two parabolas $y(x)$ in space, one for the hard drive and one for the lob, which look quite different, rather than the parabolas $y(t)$ in spacetime, which have the same curvature. I learned this long ago from John Wheeler.
4. Currently to one part in 10^{13}. The modern round of experiments started with Loránd Eötvös in 1885 and continues with the Eöt-Wash experiment led by E. Adelberger in our days.

5. The equality of the gravitational and inertial mass of the neutron has also been verified to good accuracy using neutron interferometry.
6. For Newton's letter to Halley about Hooke on the inverse square, see P. J. Nahin, *Mrs. Perkins's Electric Quilt,* Princeton University Press, 2009.
7. G. C. Williams, *The Pony Fish's Glow,* Basic Books, 1997, p. 128.
8. S. Chandrasekhar, *Newton's Principia for the Common Reader,* Oxford University Press, 2003.
9. *Fearful,* pp. 74–75.
10. For a popular account, see *Toy/Universe*.
11. N. Kollerstrom, "The Hollow World of Edmond Halley," *J. Hist. Astronomy* 23 (1992), p. 185.
12. Surely most readers are familiar with indices. My son the biologist informs me that even biologists use indices routinely; for example, on p. 20 of *Genetics and Analysis of Quantitative Traits* by M. Lynch and B. Walsh, indices appear without explanation or apology.
13. A colleague told me to mention that indices are crucial in computer programming, something that many readers can relate to.
14. *Toy/Universe,* p. xxix.

I.2 Conservation Is Good

An integrability condition

Conservation has been important to physics from day one.[1] In this chapter, we discuss the origin of various conservation laws in Newtonian mechanics.

The most important case is when the force F^i depends only on x and can be written in the form

$$F^i(x) = -\frac{\partial V(x)}{\partial x^i} \tag{1}$$

for $i = 1, 2, \cdots, D$. As we all learned, $V(x)$ is called the potential.

Suppose such a function $V(x)$ exists; then a clever person might have the insight to multiply each of Newton's equations

$$m\frac{d^2x^i}{dt^2} = F^i = -\frac{\partial V(x)}{\partial x^i} \tag{2}$$

by $\frac{dx^i}{dt}$ to obtain the D equations

$$m\frac{d^2x^i}{dt^2}\frac{dx^i}{dt} = -\frac{\partial V(x)}{\partial x^i}\frac{dx^i}{dt}, \qquad \text{with } i = 1, \cdots, D \tag{3}$$

He or she would then recognize that the sum of these D equations could be written as

$$\frac{d}{dt}\left[\frac{1}{2}m\sum_i\left(\frac{dx^i}{dt}\right)^2 + V(x)\right] = 0 \tag{4}$$

which we could verify by explicit differentiation. Lo and behold, the total energy, defined by

$$E = \frac{1}{2}m\sum_i\left(\frac{dx^i}{dt}\right)^2 + V(x) \tag{5}$$

is conserved. It does not change in time.

For $D = 1$, (1) holds automatically: $V(x)$ is simply given by $-\int^x dx' F(x')$. For $D > 1$, the D equations in (1), namely $F^i(x) = -\frac{\partial V(x)}{\partial x^i}$, imply the consistency or integrability condition

$$\frac{\partial F^i(x)}{\partial x^j} = \frac{\partial F^j(x)}{\partial x^i} \tag{6}$$

(Since derivatives commute, both sides of (6) are equal to $-\frac{\partial^2 V(x)}{\partial x^i \partial x^j}$.) Thus, given $F^i(x)$, we merely have to check to see whether (6) holds. If not, then V does not exist. If yes, then we could integrate $F^i(x) = -\frac{\partial V(x)}{\partial x^i}$ for each i to determine V.

Apples do not fall down

Suppose $V(r)$ depends only on $r \equiv \left(\sum_{i=1}^{D}(x^i)^2\right)^{\frac{1}{2}}$. In other words, the potential does not pick out any preferred direction. We take this for granted nowadays, but it represents one of the most astonishing insights of physics.[2] Newton realized that the apple did not fall down, but toward the center of the earth.

Differentiating $r^2 = \sum_{i=1}^{D}(x^i)^2$, we obtain $r dr = \sum_i x^i dx^i$ (an "identity," which we will use again and again in this text) or $\frac{\partial r}{\partial x^j} = \frac{x^j}{r}$, so that

$$F^i = -\frac{x^i}{r}V'(r) \quad \text{and} \quad \frac{\partial F^i(x)}{\partial x^j} = -\frac{1}{r}[\delta^{ij}V'(r) + \frac{x^i x^j}{r^2}(-V'(r) + rV''(r))]$$

which is manifestly symmetric under $i \leftrightarrow j$.

Here we have introduced the Kronecker delta δ^{ij}, defined by

$$\delta^{kj} = 1 \text{ if } k = j, \qquad \delta^{kj} = 0 \text{ if } k \neq j \tag{7}$$

(which we can think of as an ancestor of the Dirac delta function[3] introduced in chapter I.1).

The important point is not the somewhat involved expression for $\frac{\partial F^i(x)}{\partial x^j}$, but that it is a linear combination of δ^{ij} and $x^i x^j$. We haven't talked about tensors yet (see chapter I.4), but this result could have been anticipated by a "what else can it be?" type of argument. Not having any preferred direction, we could only construct an object with indices i and j out of δ^{ij} and $x^i x^j$. We could have seen immediately that the integrability condition (6) holds.

Note that this discussion holds for any value of D.

Conservation of angular momentum

Suppose the force in (2) points toward the center, so that it has the form $F^i = f(r)x^i$ (with $f(r) = -V'(r)/r$, as we just saw). Then we obtain angular momentum conservation immediately. To see this, multiply Newton's equation (2)

$$m\frac{d^2x^i}{dt^2} = f(r)x^i \tag{8}$$

by x^j, so that $m\frac{d^2x^i}{dt^2}x^j = f(r)x^i x^j$. Subtract from this the same equation but with i and j interchanged. Regardless of the function $f(r)$, we find

$$x^j \frac{d^2x^i}{dt^2} - x^i \frac{d^2x^j}{dt^2} = 0 \tag{9}$$

But this is the same as

$$\frac{d}{dt}\left(x^j \frac{dx^i}{dt} - x^i \frac{dx^j}{dt}\right) = 0 \tag{10}$$

Clever, eh? I am constantly amazed by how brilliant early physicists were.

The quantity $l^{ij} \equiv \left(x^j \frac{dx^i}{dt} - x^i \frac{dx^j}{dt}\right)$, the angular momentum per unit mass, is conserved. Recall that in the preceding chapter, this fact seemingly fell out when we changed to polar coordinates. Note also that the argument given here holds for any $D \geq 2$.

Exercise

1 Let N particles interact according to

$$m_a \frac{d^2x_a^i}{dt^2} = -\frac{\partial V(x)}{\partial x_a^i} \tag{11}$$

with $a = 1, \cdots, N$. Suppose $V(x_1, \cdots, x_N)$ depends only on the differences $x_a^i - x_b^i$, with $a, b = 1, \cdots, N$. Show that the total momentum $\sum_a m_a \frac{dx_a^i}{dt}$ is conserved.

Notes

1. *Fearful.*
2. I once explained this point to humanists using Einstein's terminology by saying that "The words up and down have no place in the Mind of the Creator." See A. Zee, *New Lit. Hist.* 23 (1992), pp. 815–838. See also web.physics.ucsb.edu/jatila/supplements/zee lecture.pdf.
3. In the sense that $\delta(x - y)$ is zero for $x \neq y$.

I.3 Rotation: Invariance and Infinitesimal Transformation

Rotation in the plane

My pedagogical strategy for this chapter is to take something you know extremely* well, namely rotations in the plane, present it in a way possibly unfamiliar to you, and go through it slowly in great detail, "beating the subject to death," so to speak.

I have already mentioned that Monsieur Descartes had the clever idea of reducing geometry to algebra. Put down Cartesian coordinate axes so that a point P is labeled by two real numbers (x, y). Suppose another observer (call him Mr. Prime) puts down coordinate axes rotated by angle θ with respect to the axes put down by the first observer (call her Ms. Unprime) but sharing the same origin O. Elementary trigonometry tells us that the coordinates (x, y) and (x', y') assigned by the two observers to the same point P are related by† (see figure 1)

$$x' = \cos\theta\, x + \sin\theta\, y, \qquad y' = -\sin\theta\, x + \cos\theta\, y \tag{1}$$

The distance from P to the origin O of course has to be the same for the two observers. According to Pythagoras, this requires $\sqrt{x'^2 + y'^2} = \sqrt{x^2 + y^2}$, which you can check using (1).

Introduce the column vectors $\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{r}\,' = \begin{pmatrix} x' \\ y' \end{pmatrix}$ and the rotation matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{2}$$

so that we can write (1) more compactly as $\vec{r}\,' = R(\theta)\vec{r}$.

* If you don't know rotations in the plane extremely well, then perhaps you are not ready for this book. A nodding familiarity with matrices and linear algebra is among the prerequisites.

† For example, by comparing similar triangles in the figure, we obtain $x' = (x/\cos\theta) + (y - x\tan\theta)\sin\theta$.

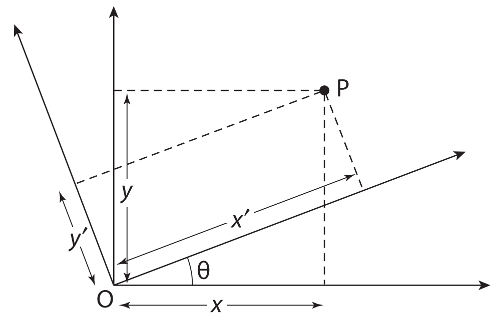


Figure 1 The same point P is labeled by (x, y) and (x', y'), depending on the observer's frame of reference.

As you recall from a course on mechanics, we can either envisage rotating the physical body we are studying or rotating the observer. We will consistently rotate the observer.

We have already used the word "vector." A vector is a physical quantity (for example the velocity of a particle in the plane) consisting of two real numbers, so that if Ms. Unprime represents it by $\vec{p} = \begin{pmatrix} p^1 \\ p^2 \end{pmatrix}$, then Mr. Prime will represent it by $\vec{p}' = R(\theta)\vec{p}$. In short, a vector is something that transforms like the coordinates $\begin{pmatrix} x \\ y \end{pmatrix}$ under rotation.

Given two vectors $\vec{p} = \begin{pmatrix} p^1 \\ p^2 \end{pmatrix}$ and $\vec{q} = \begin{pmatrix} q^1 \\ q^2 \end{pmatrix}$, the scalar or dot product is defined by $\vec{p}^T \cdot \vec{q} = p^1 q^1 + p^2 q^2$. Here T stands for transpose and $\vec{p}^T$ the row vector (p^1, p^2). By definition, rotations leave $\vec{p}^2 \equiv \vec{p}^T \cdot \vec{p} = (p^1)^2 + (p^2)^2$ invariant. In other words, if $\vec{p}' = R(\theta)\vec{p}$, then $\vec{p}'^2 = \vec{p}^2$. Since this works for any vector $\vec{p}$, including the case in which $\vec{p}$ happens to be the sum of two vectors $\vec{p} = \vec{u} + \vec{v}$, and since $\vec{p}^2 = (\vec{u} + \vec{v})^2 = \vec{u}^2 + \vec{v}^2 + 2\vec{u}^T \cdot \vec{v}$, rotation also leaves the dot product between two arbitrary vectors invariant: the invariance of $\vec{p}^2$ implies that $\vec{u}'^T \cdot \vec{v}' = \vec{u}^T \cdot \vec{v}$.

Since $\vec{u}' = R\vec{u}$ (to unclutter things, we often suppress the θ dependence in $R(\theta)$) and so $\vec{u}'^T = \vec{u}^T R^T$, we now have $\vec{u}^T \cdot \vec{v} = \vec{u}'^T \cdot \vec{v}' = (\vec{u}^T R^T) \cdot (R\vec{v}) = \vec{u}^T \cdot (R^T R)\vec{v}$. (The transpose M^T of a matrix M is of course obtained by interchanging the rows and columns of M.) As this holds for any two vectors $\vec{u}$ and $\vec{v}$, we must have the matrix equation

$$R^T R = I \tag{3}$$

where, as usual, I denotes the identity or unit matrix: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Indeed, we could verify (3) explicitly:

$$R(\theta)^T R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

Matrices that satisfy (3) are called orthogonal.

Taking the determinant of (3), we obtain $(\det R)^2 = 1$, that is, $\det R = \pm 1$. The determinant of an orthogonal matrix may be -1 as well as $+1$. In other words, orthogonal matrices also include reflection matrices, such as $\mathcal{P} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, a reflection in the y-axis.

To focus on rotations, let us exclude reflections by imposing the condition (since $\det \mathcal{P} = -1$)

$$\det R = 1 \tag{5}$$

Matrices with unit determinant are called special.

We define a rotation as a matrix that is both orthogonal and special, that is, a matrix that satisfies both (3) and (5). Thus, the rotation group of the plane consists of the set of all special orthogonal 2 by 2 matrices and is known as $SO(2)$.

Note that matrices of the form $\mathcal{P}R$ for any rotation R are also excluded by (5), since $\det(\mathcal{P}R) = \det \mathcal{P} \det R = (-1)(+1) = -1$. In particular, a reflection in the x-axis $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, which is the product of $\mathcal{P}$ and a rotation through 90°, is also excluded.

Act a little bit at a time

The Norwegian physicist Marius Sophus Lie (1842–1899) had the almost childishly obvious but brilliant idea that to rotate through, say, 29°, you could just as well rotate through a zillionth of a degree and repeat the process 29 zillion times. To study rotations, it suffices to study rotation through infinitesimal angles. Shades of Newton and Leibniz! A rotation through a finite angle could always be obtained by performing infinitesimal rotations repeatedly. As is typical with many profound statements in physics and mathematics, Lie's idea is astonishingly simple. Replace the proverb "Never put off until tomorrow what you have to do today" by "Do what you have to do a little bit at a time."

When the angle is small enough, the rotation is almost the identity, that is, no rotation at all. Thus, we can write

$$R(\theta) \simeq I + A \tag{6}$$

where A denotes some infinitesimal matrix.

Now suppose we have never seen (2). Indeed, suppose we have never even heard of sine and cosine. Instead, let us define rotations as the set of linear transformations on 2-component objects $\vec{u}' = R\vec{u}$ and $\vec{v}' = R\vec{v}$ that leave $\vec{u}^T \cdot \vec{v}$ invariant. Following Lie, we solve this condition on R, namely (3) $R^T R = I$, by considering an infinitesimal transformation $R(\theta) \simeq I + A$. Since by assumption, A^2 can be neglected relative to A, $R^T R \simeq (I + A^T)(I + A) \simeq (I + A^T + A) = I$. We thus obtain $A^T = -A$, namely that A must be antisymmetric. But there is basically only one 2-by-2 antisymmetric matrix:

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{7}$$

In other words, the solution of $A^T = -A$ is $A = \theta \mathcal{J}$ for some real number θ. Thus, rotations close to the identity have the form $R = I + \theta \mathcal{J} + O(\theta^2) = \left(\begin{smallmatrix} 1 & \theta \\ -\theta & 1 \end{smallmatrix}\right) + O(\theta^2)$. The antisymmetric matrix $\mathcal{J}$ is known as the generator of the rotation group.

An equivalent way of saying this is that for infinitesimal θ, the transformation $x' \simeq x + \theta y$ and $y' \simeq y - \theta x$ (you could verify that (1) indeed reduces to this to leading order in

θ) obviously satisfies the Pythagorean condition $x'^2 + y'^2 = x^2 + y^2$ to first order in θ. Or, write $x' = x + \delta x$, $y' = y + \delta y$ and solve $x\delta x + y\delta y = 0$.

Alternatively, simply draw figure 1 for θ infinitesimal. Since we know the transformation is linear, we could determine the matrix R in (6) by looking at the figure to see what happens to the two points $(x = 1,\ y = 0)$ and $(x = 0,\ y = 1)$ under an infinitesimal rotation.

Now recall the identity $e^x = \lim_{N\to\infty}(1 + \frac{x}{N})^N$ (which you can easily prove by differentiating both sides). Then, for a finite (that is, not infinitesimal) angle θ, we have

$$R(\theta) = \lim_{N\to\infty} R\left(\frac{\theta}{N}\right)^N = \lim_{N\to\infty}\left(1 + \frac{\theta\mathcal{J}}{N}\right)^N = e^{\theta\mathcal{J}} \tag{8}$$

The first equality represents Lie's profound idea. For the last equality, we use the identity just mentioned, which amounts to the definition of the exponential.

Some readers may not be familiar with the exponential of a matrix. Given a well-behaved function f with a power series expansion, we can define $f(M)$ for an arbitrary matrix M using that power series. For example, define $e^M \equiv \sum_{n=0}^{\infty} M^n/n!$; since we know how to multiply and add matrices, this series makes perfect sense. (Whether or not any given series converges is of course another issue.) We must be careful, however, in using various identities that may or may not generalize. For example, the identity $e^a e^a = e^{2a}$ for a a real number, which we could prove by applying the binomial theorem to the product of two series (square of a series in this case) generalizes immediately. Thus, $e^M e^M = e^{2M}$. But for two matrices M_1 and M_2 that do not commute with each other, $e^{M_1}e^{M_2} \neq e^{M_1+M_2}$.

This provides an alternative but of course equivalent path to our result. To leading order, we have every right to write $R\left(\frac{\theta}{N}\right) = 1 + \frac{\theta\mathcal{J}}{N} \simeq e^{\frac{\theta\mathcal{J}}{N}}$ and thus $R(\theta) = R\left(\frac{\theta}{N}\right)^N = e^{\theta\mathcal{J}}$.

Finally, we easily check that the formula $R(\theta) = e^{\theta\mathcal{J}}$ reproduces (2) for any value of θ. We simply note that $\mathcal{J}^2 = -I$ and separate the exponential series into even and odd terms. Thus

$$\begin{aligned} e^{\theta\mathcal{J}} &= \sum_{n=0}^{\infty}\theta^n\mathcal{J}^n/n! = \left(\sum_{k=0}^{\infty}(-1)^k\theta^{2k}/(2k)!\right) I + \left(\sum_{k=0}^{\infty}(-1)^k\theta^{2k+1}/(2k+1)!\right)\mathcal{J} \\ &= \cos\theta\ I + \sin\theta\ \mathcal{J} = \cos\theta\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \sin\theta\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} \end{aligned} \tag{9}$$

which is precisely $R(\theta)$ as given in (2). Note this works because $\mathcal{J}$ plays the same role as i in the identity $e^{i\theta} = \cos\theta + i\sin\theta$.

Poor Lie, he never made it into the 20th century.

Two approaches to rotation

Notice that I actually gave you two different approaches to rotation. Let us summarize the two approaches. In the first approach, applying trigonometry to figure 1, we write down (1) and hence (2). In the second approach, we specify what is to be left invariant by rotations and hence define rotations by the condition (3) that rotations must satisfy. Lie then tells us that it suffices to solve (3) for infinitesimal rotations. We could then build up rotations through finite angles by multiplying infinitesimal rotations together, thus also arriving at (2).

It might seem that the first approach is much more direct. One writes down (2) and that is that. The second approach appears more roundabout and involves some "fancy math." It might even provoke an adherent of the first "more macho" approach to wisecrack, "Why, with a bit of higher education, sine and cosine are not good enough for you any more? You have to go around doing fancy math!" The point is that the second approach generalizes to higher dimensional spaces (and to other situations) much more readily than the first approach does, as we will see presently. Dear reader, in going through life, you would be well advised to always separate fancy but useful math from fancy but useless math.

Before we go on, let us take care of one technical detail. We assumed that Mr. Prime and Ms. Unprime set up their coordinate systems to share the same origin O. We now show that this condition is unnecessary if we consider two points P and Q (rather than one point, as in our discussion above) and study how the vector connecting P to Q transforms.

Let Ms. Unprime assign the coordinates $\vec{r}_{\mathrm{P}} = (x, y)$ and $\vec{r}_{\mathrm{Q}} = (\tilde{x}, \tilde{y})$ to P and Q, respectively. Then Mr. Prime's coordinates $\vec{r}^{\,\prime}_{\mathrm{P}} = (x', y')$ for P and $\vec{r}^{\,\prime}_{\mathrm{Q}} = (\tilde{x}', \tilde{y}')$ for Q are then given by $\vec{r}^{\,\prime}_{\mathrm{P}} = R(\theta)\vec{r}_{\mathrm{P}}$ and $\vec{r}^{\,\prime}_{\mathrm{Q}} = R(\theta)\vec{r}_{\mathrm{Q}}$. Subtracting the first equation from the second and defining $\Delta x = \tilde{x} - x$, $\Delta y = \tilde{y} - y$, and the corresponding primed quantities, we obtain

$$\begin{pmatrix} \Delta x' \\ \Delta y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \tag{10}$$

Rotations leave the distance between the points P and Q unchanged: $(\Delta x')^2 + (\Delta y')^2 = (\Delta x)^2 + (\Delta y)^2$. You recognize of course that this is a lot of tedious verbiage stating the perfectly obvious, but I want to be precise here. Of course, the distance between any two points is left unchanged by rotations. (This also means that the distance between P and the origin is left unchanged by rotations; ditto for the distance between Q and the origin.)

Invariance and geometry

> There is no royal road to geometry.
> —Euclid's advice to a prince

> Let no one unversed in geometry enter here.
> —Plato's motto, carved over the entrance to his academy

Let us take the two points P and Q to be infinitesimally close to each other and replace the differences $\Delta x'$, Δx, and so forth by differentials dx', dx, and so forth. Indeed, 2-dimensional Euclidean space is defined by the distance squared between two nearby points:

$$ds^2 = dx^2 + dy^2 \tag{11}$$

Rotations are defined as linear transformations* $(x, y) \to (x', y')$, such that

$$dx^2 + dy^2 = dx'^2 + dy'^2 \tag{12}$$

The whole point is that this now makes no reference to the origin O (and whether Mr. Prime and Ms. Unprime even share the same origin).

The column $d\vec{x} = \begin{pmatrix} dx^1 \\ dx^2 \end{pmatrix} \equiv \begin{pmatrix} dx \\ dy \end{pmatrix}$ is defined as the basic or ur-vector, the template for all other vectors. To repeat, a vector is defined as something that transforms like $d\vec{x}$ under rotations.

So, a vector is defined by how it transforms. An array of two numbers $\vec{p} = \begin{pmatrix} p^1 \\ p^2 \end{pmatrix}$ is a vector if it transforms according to $\vec{p}' = R(\theta)\vec{p}$.

Sometimes it is very helpful, in order to understand what something is, to be given an example of something that is not. As a simple example, given a $\vec{p}$, then $\begin{pmatrix} ap^1 \\ bp^2 \end{pmatrix}$ is definitely not a vector if $a \neq b$. (You could easily write down more outrageous examples, such as $\begin{pmatrix} (p^1)^2 p^2 \\ (p^1)^3 + (p^2)^3 \end{pmatrix}$. That ain't no vector!) You will work out further examples in exercise 1. An array of numbers is not a vector unless it transforms in the right way.[1]

Oh, about the advice Euclid gave to the prince who wanted to know a quick way of mastering geometry. Mr. E is also telling you that, to master the material covered in this book, there is no way other than to cogitate over the material until you get it and to work through as many exercises as possible.

From the plane to higher dimensional space

The reader who has wrestled with Euler angles in a mechanics course knows that the analog of (2) for 3-dimensional space is already quite a mess. In contrast, Lie's approach allows us, as mentioned above, to immediately jump to D-dimensional Euclidean space, defined by specifying the distance squared between two nearby points (compare this with (11)), as given by the obvious generalization of Pythagoras' theorem:

$$ds^2 = \sum_{i=1}^{D} \left(dx^i\right)^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \cdots + \left(dx^D\right)^2 \tag{13}$$

This is as good a place as any to say a word about indices. As I said in chapter I.1, in my experience teaching, there are always a couple of students confounded by indices. Dear reader, if you are not, you could simply laugh and skip to the next paragraph. Indices provide a marvelous notational device to save us from having to give names to individual elements belonging to a set. (For example, consider all humans h^i now alive, with $i = 1, 2, \cdots, P$ where P denotes the population size.) Take a look at the 19th century physics literature, before the use of indices became widespread. I am always amazed by

* Indeed, most, but not all, of the readers[2] of this book are constantly rotating between two coordinate systems.

the fact that, for example, Maxwell could see through the morass of the electromagnetic equations written out component by component.

Rotations are defined as linear transformations $d\vec{x}' = Rd\vec{x}$ that leave ds unchanged. The preceding discussion allows us to write this condition as $R^T R = I$. As before, we want to focus on rotations by imposing the additional condition $\det R = 1$. The set of D-by-D matrices R that satisfy these two conditions forms the simple orthogonal group $SO(D)$, which is just a fancy way of saying the rotation group in D-dimensional space.

Lie in higher dimensions

The power of Lie now shines through when we want to work out rotations in higher dimensional spaces. All we have to do is satisfy the two conditions $R^T R = I$ and $\det R = 1$.

So let us follow Lie and write $R \simeq I + A$. Then $R^T R = I$ is solved by requiring $A = -A^T$, namely that A must be antisymmetric. But it is very easy to write down all possible antisymmetric D-by-D matrices! For $D = 2$, there is basically only one: the $\mathcal{J}$ introduced earlier. For $D = 3$, there are basically three of them:

$$\mathcal{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{J}_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{14}$$

Any 3-by-3 antisymmetric matrix can be written as $A = \theta_x \mathcal{J}_x + \theta_y \mathcal{J}_y + \theta_z \mathcal{J}_z$, with three real numbers θ_x, θ_y, and θ_z. At this point, you can verify that $R \simeq I + A$, with A as given here, satisfies the condition $\det R = 1$.

The three matrices $\mathcal{J}_x$, $\mathcal{J}_y$, $\mathcal{J}_z$ are known as the generators of the 3-dimensional rotation group $SO(3)$. They generate rotations, but are of course not to be confused with rotations, which are by definition 3-by-3 orthogonal matrices with determinant equal to 1.

The upshot of this whole discussion is that any 3-dimensional rotation (not necessarily infinitesimal) can be written as $R(\theta) = e^A$ and is thus characterized by three real numbers. As I said, those readers who have suffered through the rotation of a rigid body in a course on mechanics must appreciate the simplicity of studying the generators of infinitesimal rotations and then simply exponentiating them.

Index notation and rotations

Some readers will find this obvious, but others might find it helpful if we derive the condition $R^T R = I$ explicitly once again using the index notation. I prefer to go slow here, since we will need some of the same formalism later when we get to special relativity. Once the reader feels sure-footed, we could then dispense with indices.

Let me start by reminding the reader that a D-by-D matrix M carries two indices and has entries M^{ij}, with the standard convention that the first index labels the rows, the second the column (for $i, j = 1, 2, \cdots, D$). For example, for $D = 2$, $M = \begin{pmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{pmatrix}$, and M^{12} is the entry in the first row and the second column, whereas M^{21} is the entry in the second row and the first column. Note that the transpose of a matrix M is given by $(M^T)^{ji} \equiv M^{ij}$. Thus, if $\vec{v}$ is a column vector with entries v^j, then the entries of the column vector $\vec{u} = M\vec{v}$ are given by $u^i = \sum_j M^{ij} v^j$. For A and B two D-by-D matrices, the product AB is defined as the matrix with the entries $(AB)^{ij} = \sum_k A^{ik} B^{kj}$. (If everything here is news to you, see the first footnote in this chapter.)

Under a rotation,

$$dx'^i = \sum_j R^{ij} dx^j = R^{i1} dx^1 + R^{i2} dx^2 + \cdots + R^{iD} dx^D \tag{15}$$

(I have written the sum out explicitly for the benefit of the rare reader afflicted by fear of indices.) Also, as was mentioned in chapter I.1, at this stage it is completely arbitrary whether we write upper or lower indices.

Let us pause and recall the Kronecker delta symbol δ^{ij} introduced in (I.2.7), defined to be equal to $+1$ if $i = j$ and 0 otherwise, and which we can also think of as a D-by-D unit matrix. We will be encountering the highly useful Kronecker delta often in this book. For example, $\sum_j A^j B^j = \sum_k \sum_j \delta^{kj} A^k B^j$. Since δ^{kj} vanishes unless k is equal to j, the double sum on the right hand side collapses to the single sum on the left hand side. In other words, the Kronecker delta allows us to write a single sum as a double sum. It seems like a really silly thing to do, but as we will see presently, it is an extremely useful trick that we use quite often in this book.

We now determine how the matrix R must be restricted for it to be a rotation. The statement that $ds^2 = \sum_{i=1}^D (dx^i)^2$ as defined in (13) is left unchanged by the rotation implies that (with all indices running over $1, \cdots, D$)

$$\sum_i (dx'^i)^2 = \sum_i \sum_k \sum_j R^{ik} dx^k R^{ij} dx^j = \sum_j (dx^j)^2 = \sum_k \sum_j \delta^{kj} dx^k dx^j \tag{16}$$

In the last step, we used what we just learned.

Since the infinitesimals dx^i can take on arbitrary values, to have the second term equal to the last term in (16), we must equate the coefficients of $dx^k dx^j$ and demand that

$$\sum_i R^{ik} R^{ij} = \delta^{kj} = \sum_i (R^T)^{ki} R^{ij} = (R^T R)^{kj} \tag{17}$$

Indeed, we obtain $R^T R = I$ just as in (3), but now in D-dimensional space for any D.

We end this section with a trivial remark. So far in this chapter, we have written the column vectors as columns. But columns take up so much space, and so for typographical convenience (editors must be placated!) we will henceforth write the entries of a column vector as $d\vec{x} = (dx^1, dx^2, \cdots, dx^D)$, a practice we will indulge in throughout this book. (If we want to be insufferably pedantic, we could put in a T for transpose: the column ur-vector $d\vec{x} = (dx^1, dx^2, \cdots, dx^D)^T$.)

Einstein's repeated index summation

Observe that in all those sums in (16) the indices to be summed over always appear twice, that is, they are repeated. For example, in the second term in (16), $\sum_i \sum_k \sum_j R^{ik} dx^k R^{ij} dx^j$, the indices i, k, and j all appear repeated. Thus, we could adopt the so-called repeated index summation convention proposed by Albert Einstein himself: omit the pesky summation symbol and agree that if an index is repeated, then it is to be summed over. For example, $dx'^i = \sum_j R^{ij} dx^j$ can now be written as $dx'^i = R^{ij} dx^j$: in the expression on the right hand side, the index j appears twice and is thus to be summed over.* In contrast, i is a "free" index and does not appear twice in the same expression. Notice that free indices must match on opposite sides of any equation. It is rightly said that one of Einstein's greatest contribution to physics is the repeated index summation convention.† When we get to Einstein gravity, we will meet lots of indices to be summed over, and it would be silly to keep on writing the summation symbol.

Vector fields

The vectors we encounter may well vary in space. For example, the flow velocity in a fluid in general would depend on where we are. We are then dealing with a vector field $\vec{V}(\vec{x})$. Again, consider two observers studying the same vector field. Mr. Prime would see

$$\vec{V}'(\vec{x}') = R\vec{V}(\vec{x}) \tag{18}$$

with $\vec{x}' = R\vec{x}$ of course. In other words, the two observers are studying the same vector field at the same point P. See figure 2. As another example, the familiar electric $\vec{E}(\vec{x})$ and magnetic fields $\vec{B}(\vec{x})$ are both vector fields.

Physics should not depend on the observer

Let me stress again why physicists constantly talk about vectors. The laws of physics often involve the statement that one vector is equal to another, for example, Newton's law states $m\vec{a} = \vec{F}$. Applying a rotation matrix $R(\theta)$, we obtain $mR(\theta)\vec{a} = R(\theta)\vec{F}$. If $\vec{F}$ transforms like a vector, then $m\vec{a}' = \vec{F}'$. Ms. Unprime and Mr. Prime see the same Newton's law, and more generally, the same laws of physics!

This statement, while self-evident, is profound, and in some sense, it is what makes physics possible. Physics should not depend on the physicist. Ms. Unprime and Mr. Prime

* When a pair of repeated indices, such as j here, is summed over, they are often said to be contracted with each other. In a tiny abuse of terminology, people also say that R^{ij} is contracted with dx^j.

† It appeared only in his later work. In 1905, Einstein did not even use vector notation! In one system, the coordinates were denoted by x, y, z, in the other, by ξ, η, ζ; the components of the force acting on the electron were called X, Y, Z. To modern eyes, his notation was a horrific mess.

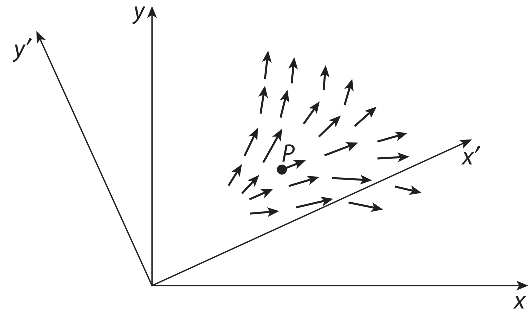


Figure 2 Two observers studying the same vector field.

see different accelerations $\vec{a}$ and $\vec{a}'$, and different forces $\vec{F}$ and $\vec{F}'$, but the same Newton's law. We say that Newton's law is invariant—that is, it does not change—under rotation.*

We should also remind ourselves that mass is an example of a scalar: a physical quantity that does not change under rotation. If it does change, Newton's law would not be invariant under rotation and one observer would be preferred over another, which is unacceptable. Physics rests on the democratic ideal.

Let me remind you that the gravitational force in the planetary problem studied in chapter I.1 is derived from what is sometimes called a central potential, namely one without a preferred direction: $F^i(x) = -\frac{\partial}{\partial x^i} V(r) = -\frac{x^i}{r} V'(r)$. Hence, $\vec{F}$ is proportional to $\vec{x}$ and so a fortiori transforms like a vector.

At this point, it may be worthwhile to be a bit more pedantic and professorial. Some authors give long-winded speeches about covariance versus invariance, and take great pain to distinguish the two. We should too. The equation $m\vec{a} = \vec{F}$ is covariant, that is, the two sides transform the same way under rotations. The physics expressed by Newton's second law is, however, invariant, that is, independent of observers related by a rotation. If physics depends on how you tilt your head, we are in trouble. Physics does not, but the way physics is expressed, in terms of equations, does.

Here is the profound and trivial statement. Under a certain set of transformations, a purportedly fundamental equation is said to be covariant if the two sides of the equation transform in the same way. If so, then that transformation is known as a symmetry of physics.[3] Physics is said to be invariant under that transformation. As we will see, both sides of Einstein's field equation transform in the same way, as tensors, under what are known as general coordinate transformations. I will explain what a tensor is in the next chapter. I will allow myself the luxury of using the words invariance and covariance interchangeably and simply trust you to be discerning.

Since we can always move the quantity on the right hand side of an equation to the left hand side, we can rewrite a physical law of the form $\vec{u} = \vec{v}$ in the form $\vec{w} \equiv \vec{u} - \vec{v} = 0$. Physics students sometimes joke that they could already write down the ultimate

* The reader who has already been exposed to the special theory of relativity knows that this notion of invariance represents the essence of Einstein's insight. We will of course have a great deal more to say about that!

equation of physics, namely $\mathcal{X} = 0$, whatever $\mathcal{X}$ is. Thus, the statement of invariance merely expresses the mathematically obvious fact that if $\vec{w} = 0$, then $R(\theta)\vec{w} = 0$. (Strictly speaking, the 0 on the right hand side should be written as $\vec{0}$, but we don't want to be that pedantic!)

Descartes versus Euclid

I remember how excited I was when I learned about analytic geometry. Surely you were excited too. What a genius, that Descartes! Henceforth, we could prove geometric theorems by doing algebra. After Descartes,[4] physics can no longer live without the concept of coordinates,* but he also managed to obscure what was once obvious to Euclid. We now must also insist on invariance. Indeed, the notion of invariance is at the heart of what we mean by geometry.

For example, suppose somebody hands you a formula for the area of a triangle with vertices at (a_1, b_1), (a_2, b_2), (a_3, b_3). You better insist that the formula is invariant under rotation. In fact, this requirement, plus the requirement that the area should scale as the square of the separation between the three vertices, suffices to determine the formula. This simple example rings in the central motif of this book.

Appendix 1: Differential operators rather than matrices

Here I have to divide readers into the haves and the have-nots, but only temporarily. What I will say may sound difficult, but really, it amounts to not much more than a notational triviality.

If you have studied quantum mechanics, you would know that the generators $\mathcal{J}$ of rotation studied here are related to angular momentum operators. You would also know that in quantum mechanics, observables are represented by hermitean operators. However, in our discussion, the $\mathcal{J}$s come out naturally as antisymmetric matrices and are thus antihermitean. To make them hermitean, we multiply them by some multiples of i.

If you have not studied quantum mechanics, then the preceding would sound like gibberish to you, but do not worry. Simply take the attitude that, hey, it is a free country, and we can always invite ourselves to define a new set of physical quantities by multiplying an existing set of physical quantities by some constant. Heck, we could multiply by $\sqrt{17i}$ if we want.

Even though here we are nowhere near quantum mechanics, we will bow to customary usage and define $J_x \equiv -i\mathcal{J}_x$ and so forth. From (14) we see that, for example, J_z acting on the column vector (x, y, z) gives $i(y, -x, 0)$. Thus, instead of using matrices, we could also represent J_z by $i(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})$, since $J_z x = i(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})x = iy$, $J_z y = i(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})y = -ix$, and $J_z y = i(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})z = 0$. Note that J_z is precisely the z-component of the angular momentum operators in quantum mechanics. We can naturally pass back and forth between matrices and differential operators. We will not make use of this differential representation until a later chapter.

* Regarding the argument (which I mentioned in a footnote in the preface) between those who live with coordinates and those who live coordinate free, I would say that the proof of angular momentum conservation, which I already gave, not once, but twice in the two preceding chapters using coordinates, provides an example in favor of the latter group: $\frac{d}{dt}\vec{l} = \frac{d}{dt}(\vec{r} \times \vec{p}) = m\frac{d}{dt}(\vec{r} \times \frac{d\vec{r}}{dt}) = m\frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} + m\vec{r} \times \frac{d^2\vec{r}}{dt^2} = 0$ for rotationally symmetric potentials. While this indeed looks simpler than the two previous discussions, the former group could also say that this requires learning "considerable formal math," such as the cross product and its various properties.

Appendix 2: Rotations in higher dimensional space

Here we discuss rotations in D-dimensional Euclidean space. As you have no doubt heard, Einstein combined space and time into a 4-dimensional spacetime. Thus, what you will learn here about $SO(4)$ will be put to good use.* If you prefer, you could skip this discussion and come back to it later.

Start with a D-by-D matrix with 0 everywhere. Generalize (14). Stick a 1 into the mth row and nth column, and a (-1) into the nth row and mth column. Call this matrix $J_{(mn)}$. We put the subscripts (mn) in parentheses to emphasize that (mn) labels the matrix. They are not indices to tell us which element of the matrix we are talking about. As explained before, we define $J_{(mn)} = -i\mathcal{J}_{(mn)}$ so that explicitly

$$J^{ij}_{(mn)} = -i(\delta^{mi}\delta^{nj} - \delta^{mj}\delta^{ni}) \tag{19}$$

To repeat, in the symbol $J^{ij}_{(mn)}$, the indices i and j indicate respectively the row and column of the entry $J^{ij}_{(mn)}$ of the matrix $J_{(mn)}$, while the indices m and n, which I put in parentheses for pedagogical clarity, indicate which matrix we are talking about. The first index m on $J_{(mn)}$ can take on D values, and then the second index n can take on only $(D-1)$ values since, obviously, $J_{(mm)} = 0$. Also, since $J_{(nm)} = -J_{(mn)}$, we require $m > n$ to avoid double counting. Thus, there are only $\frac{1}{2}D(D-1)$ real antisymmetric D-by-D matrices $J_{(mn)}$, and A could be written as a linear combination of them: $A = i\sum_{m>n}\theta_{mn}J_{(mn)}$, where θ_{mn} denote $\frac{1}{2}D(D-1)$ real numbers. (As a check, for $D=2$ and 3, $\frac{1}{2}D(D-1)$ equals 1 and 3, respectively.) The matrices $J_{(mn)}$ are known as the generators of the group $SO(D)$.

Notice a notational peculiarity: for $SO(3)$, the Js could be labeled with one index rather than two indices. The reason is simple. In this case, the indices m, n take on 3 values, and so we could write $J_x = J_{23}$, $J_y = J_{31}$, and $J_z = J_{12}$. We will, as we do here, often pass freely between the index sets (123) and (xyz). In general, rotations are labeled by the plane they occur in, say the (m-n) plane spanned by the mth and nth axes. In 3-dimensional space, and only in 3-dimensional space, a plane is uniquely specified by the vector perpendicular to it. Thus, a rotation commonly spoken of as a rotation around the z-axis is better thought of as a rotation in the (1-2) plane, that is, the (x-y) plane. (In this connection, note that the $\mathcal{J}$ in (7) appears as the upper left 2-by-2 block in $\mathcal{J}_z$ in (14).) In contrast, for $SO(4)$ it makes no sense to speak of a rotation around, say, the third axis.

The reader who has studied some group theory knows that the essence of the group is captured by the extent to which the multiplication of two group elements does not commute. For rotations, everyday observations show that $R(\theta)R(\theta')$ is in general quite different from $R(\theta')R(\theta)$. See figure 3.

Following Lie, we could try to capture this essence by focusing on infinitesimal rotations. Let $R_1 \simeq I + A$ and $R_2 \simeq I + B$. Then $R_1R_2 \simeq (I+A)(I+B) \simeq I + A + B + AB + O(A^2, B^2)$ (where rather pedantically we have indicated that to the desired order if we keep AB, we should also keep terms of order $O(A^2, B^2)$, but we will see immediately that they are irrelevant). If we multiply in the other order, we simply interchange A and B, thus $R_2R_1 \simeq (I+A)(I+B) \simeq I + B + A + BA + O(A^2, B^2)$. Hence, R_1R_2 and R_2R_1 differ by the amount $[A, B] \equiv AB - BA$, a quantity known as the commutator between A and B.

More formally, given two matrices X and Y, to measure how they differ from each other, we could ask how $X^{-1}Y$ differs from the identity. If $X = Y$, then this product is equal to the identity. Now, the inverse of a matrix $I + A$ infinitesimally close to the identity is easy to determine: it is just $I - A$, since $(I-A)(I+A) = I + O(A^2)$. Thus, let us calculate $(R_2R_1)^{-1}R_1R_2$:

$$\begin{aligned}(R_2R_1)^{-1}R_1R_2 &= [I - (B + A + BA + O(A^2, B^2))][I + A + B + AB + O(A^2, B^2)] \\ &= I + [A, B] + \cdots \end{aligned} \tag{20}$$

For $SO(3)$, for example, A is a linear combination of the J_is, known as the generators of the Lie algebra. Thus, we could write $A = i\sum_i \theta_i J_i$ and similarly $B = i\sum_j \theta'_j J_j$. Hence $[A, B] = i^2\sum_{ij}\theta_i\theta'_j[J_i, J_j]$, and so it suffices to calculate the commutators $[J_i, J_j]$.

Recall that for two matrices M_1 and M_2, $(M_1M_2)^T = M_2^T M_1^T$. Transpose reverses the order. Thus $([J_i, J_j])^T = -[J_i, J_j]$. In other words, the commutator $[J_i, J_j]$ is itself an antisymmetric 3-by-3 matrix and thus could be written as a linear combination of the J_ks:

$$[J_i, J_j] = ic_{ijk}J_k \tag{21}$$

* Higher dimensional rotation groups often pop up in the most unlikely places in theoretical physics. For example, $SO(4)$ is relevant for a deeper understanding of the spectrum of the hydrogen atom.[5]

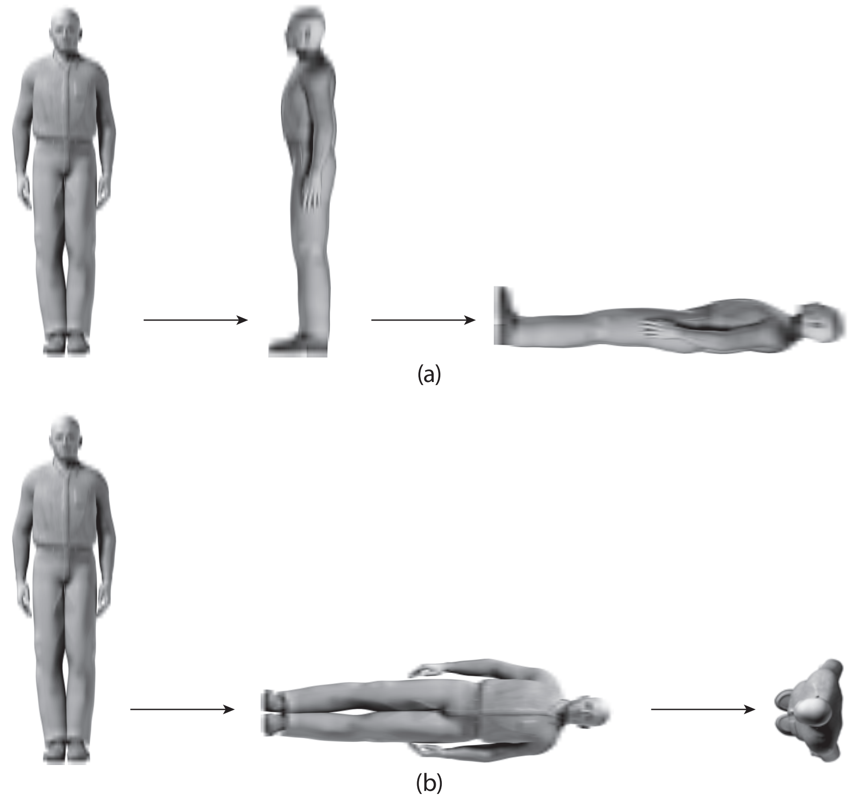


Figure 3 A marine recruit in a boot camp is standing and facing north. When the drill sergeant shouts, "Rotate by 90° eastward around the vertical axis" our recruit turns to face east. Suppose the sergeant next shouts, "Rotate by 90° westward around the north-south axis." Our recruit ends up lying down on his back with his head pointing west, his feet pointing east. But what would happen if the sergeant reverses his two commands? You could easily verify that our recruit now ends up lying down on his left elbow, with his head pointing north. The order matters. For this reason, the study of rotations has been a *bête noire* for generations of physics students.

for a set of real (convince yourself of this!) numbers c_{ijk}. The summation over k is implied by the repeated index summation convention.

By explicit computation using (14), we find

$$[J_x, J_y] = iJ_z \tag{22}$$

You should work out the other commutators or argue by cyclic substitution $x \to y \to z \to x$. The three commutation relations may be summarized by

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{23}$$

We define the totally antisymmetric symbol ϵ_{ijk} by saying that it changes sign upon the interchange of any pair of indices (and hence it vanishes when any two indices are equal) and by specifying that $\epsilon_{123} = 1$. In other words, we found that $c_{ijk} = \epsilon_{ijk}$.

Lie's great insight is that the preceding discussion holds for any group whose elements are labeled by a set of continuous parameters (such as θ_i, $i = 1, 2, 3$ in the case of $SO(3)$), groups now known as Lie groups. Expanding the group elements around the origin, we arrive at (20) and hence the structure (21) for any continuous group. The set of all commutation relations of the form (21) is said to define a Lie algebra, with c_{ijk} referred to as the structure constants of the algebra. The matrices J_i are called the generators of the Lie algebra. The idea is that by studying the Lie algebra, we go a long way toward understanding the group.

You should now work out (exercise 4), starting from (19), the Lie algebra for $SO(D)$:

$$[J_{(mn)}, J_{(pq)}] = i(\delta_{mp} J_{(nq)} + \delta_{nq} J_{(mp)} - \delta_{np} J_{(mq)} - \delta_{mq} J_{(np)}) \tag{24}$$

This may look rather involved to the uninitiated, but in fact it is quite simple. First, the right hand side, a linear combination of the Js, as required by the general argument above, is completely fixed by the first term by noting that the left hand side is antisymmetric under three separate interchanges: $m \leftrightarrow n$, $p \leftrightarrow q$, and $(mn) \leftrightarrow (pq)$. Next, all those Kronecker deltas just say that if the two sets (mn) and (pq) have no integer in common, then the commutator vanishes. If they do have an integer in common, you simply "cross off" that integer. This is best explained by using $SO(4)$ as an example. We have $[J_{(12)}, J_{(34)}] = 0$, $[J_{(12)}, J_{(14)}] = iJ_{(24)}$, $[J_{(23)}, J_{(31)}] = -iJ_{(21)} = iJ_{(12)}$, and so forth. The first of these relations says that rotations in the (1-2) plane and in the (3-4) plane commute, as you might expect. Do write down a few more and you will get it.

Exercises

1 Suppose we are given two vectors $\vec{p}$ and $\vec{q}$ in ordinary 3-dimensional space. Consider this array of three numbers: $\begin{pmatrix} p^2q^3 \\ p^3q^1 \\ p^1q^2 \end{pmatrix}$. Prove that it is not a vector, even though it looks like a vector. (Check how it transforms under rotation!) In contrast, $\begin{pmatrix} p^2q^3 - p^3q^2 \\ p^3q^1 - p^1q^3 \\ p^1q^2 - p^2q^1 \end{pmatrix}$ does transform like a vector. It is in fact the vector cross product $\vec{p} \times \vec{q}$.

2 Show that the product of two delta functions $\delta(x)\delta(y)$ is invariant under rotation around the origin.

3 Using (14) show that a rotation around the x-axis through angle θ_x is given by

$$R_x(\theta_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix}$$

Write down $R_y(\theta_y)$. Show explicitly that $R_x(\theta_x)R_y(\theta_y) \neq R_y(\theta_y)R_x(\theta_x)$.

4 Calculate $[J_{(mn)}, J_{(pq)}]$.

5 Given a 3-vector $\vec{p}$, show that the quantity $\vec{p}^i \vec{p}^j$ when averaged over the direction of $\vec{p}$ is given by $\frac{1}{4\pi} \int d\theta d\varphi \cos\theta \, \vec{p}^i \vec{p}^j = \frac{1}{3} \vec{p}^2 \delta^{ij}$.

Notes

1. Outside of physics, people often erroneously call any array of numbers a vector. Of course, people are free to call anything anything, so let's not quibble about the word "erroneously."
2. I say "most, but not all," because it is conceivable that you are a native speaker of Guugu Yimithirr. See G. Deutscher, *Through the Language Glass,* H. Holt and Co., 2010, p. 161.
3. The intellectual precision of our definition of symmetry is necessary lest we make the same mistake as the ancient Greeks. See *Fearful,* pp. 11–12 and figure 2.2.
4. According to one story, take it or leave it, Descartes was lying in bed when he noticed a fly buzzing around the room. He then realized that he could fix the fly's position given how far the fly was from two intersecting walls and the ceiling.
5. For example, J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics,* pp. 265–268.

I.4 Who Is Afraid of Tensors?

A tensor is something that transforms like a tensor

Long ago, an undergrad who later became a distinguished condensed matter physicist came to me after a class on group theory and asked me, "What exactly is a tensor?" I told him that a tensor is something that transforms like a tensor. When I ran into him many years later, he regaled me with the following story. At his graduation, his father, perhaps still smarting from the hefty sum he had paid to the prestigious private university his son attended, asked him what was the most memorable piece of knowledge he acquired during his four years in college. He replied, "A tensor is something that transforms like a tensor."

But this should not perplex us. A duck is something that quacks like a duck. Mathematical objects could also be defined by their behavior. We already saw in the preceding chapter that a vector is defined by how it transforms: $V'^i = R^{ij} V^j$. Consider a collection of "mathematical entities" T^{ij} with $i, j = 1, 2, \cdots, D$ in D-dimensional space. If they transform under rotations according to

$$T^{ij} \to T'^{ij} = R^{ik} R^{jl} T^{kl} \tag{1}$$

then we say that T transforms like a tensor, and hence is a tensor. (Here we are using the Einstein summation convention introduced in the previous chapter: The right hand side actually means $\sum_{k=1}^{D} \sum_{l=1}^{D} R^{ik} R^{jl} T^{kl}$ and is a sum of D^2 terms.) Indeed, we see that we are just generalizing the transformation law of a vector.

Fear of tensors

In my experience teaching, a couple of students are invariably confused by the notion of tensors. The very word "tensor" apparently make them tense. Dear reader, if you are not one of these unfortunates, so much the better for you! You could zip through this chapter. But to allay the nameless fear of the tensorphobe, I will go slow and be specific.

Think of the tensor T^{ij} as a collection of D^2 mathematical entities that transform into linear combinations of one another. To help the reader focus, I will often specialize to $D = 3$. Compounded and intertwined with their fear of tensors, the unfortunates mentioned above are also unaccountably afraid of indices, as mentioned in chapter I.1. For them, let us list T^{ij} explicitly for $D = 3$. There are $3^2 = 9$ of them: T^{11}, T^{12}, T^{13}, T^{21}, T^{22}, T^{23}, T^{31}, T^{32}, T^{33}. That's it, 9 objects that transform into linear combinations of one another. For example, (1) says that $T'^{21} = R^{2k}R^{1l}T^{kl} = R^{21}R^{11}T^{11} + R^{21}R^{12}T^{12} + R^{21}R^{13}T^{13} + R^{22}R^{11}T^{21} + R^{22}R^{12}T^{22} + R^{22}R^{13}T^{23} + R^{23}R^{11}T^{31} + R^{23}R^{12}T^{32} + R^{23}R^{13}T^{33}$. This shows explicitly, as if there were any doubt to begin with, that T'^{21} is given by a particular linear combination of the 9 objects. That's all: the tensor T^{ij} consists of 9 objects that transform into linear combinations of themselves under rotations.

We could generalize further and define* 3-indexed tensors, 4-indexed tensors, and so forth by such transformation laws as $W'^{ijn} = R^{ik}R^{jl}R^{nm}W^{klm}$. Here we will focus on 2-indexed tensors, and if we say tensor without any qualifier, we often, but not always, mean a 2-indexed tensor. With this definition, we might say that a vector is a 1-indexed tensor and a scalar is a 0-indexed tensor, but this usage is not common. A scalar transforms as a tensor with no index at all, namely $S' = S$; in other words, a scalar does not transform.

Tensor field

In the preceding chapter, we introduced the notion of a vector field $V^i(\vec{x})$, nothing more or less than a vector function of position. That it is a vector means that it transforms according to $V'^i(\vec{x}') = R^{ij}V^j(\vec{x})$. Now consider the derivative of this vector field $\frac{\partial V^j(\vec{x})}{\partial x^k}$, which we will call $W^{kj}(\vec{x})$.

Use the fact that $\vec{x}' = R\vec{x}$ implies $\vec{x} = R^{-1}\vec{x}' = R^T\vec{x}'$ and thus $\frac{\partial x^k}{\partial x'^h} = (R^T)^{kh} = R^{hk}$. (The O in the rotation group $SO(D)$ is crucial: the inverse of a rotation is its transpose.) Then

$$\frac{\partial}{\partial x'^h} = \frac{\partial x^k}{\partial x'^h}\frac{\partial}{\partial x^k} = R^{hk}\frac{\partial}{\partial x^k} \tag{2}$$

Thus

$$W'^{hi}(\vec{x}') \equiv \frac{\partial V'^i(\vec{x}')}{\partial x'^h} = R^{hk}\frac{\partial}{\partial x^k}(R^{ij}V^j(\vec{x})) = R^{hk}R^{ij}\frac{\partial V^j(\vec{x})}{\partial x^k} = R^{hk}R^{ij}W^{kj}(\vec{x}) \tag{3}$$

Comparing with (1) we see that $W^{kj}(\vec{x})$ transforms like a tensor and, hence, is a tensor. Indeed, it is a tensor field.

Notice that a tensor T^{ij} transforms as if it were composed of two vectors v^iw^j, that is, T^{ij} and v^iw^j transform in the same way. (Compare $v^iw^j \to v'^iw'^j = R^{ik}v^kR^{jl}w^l = R^{ik}R^{jl}v^kw^l$ with (1).) It is important to recognize that only in exceptional cases does a tensor T^{ij} happen to be equal to v^iw^j for some v and w. In general, a tensor cannot be

* Our friend the Jargon Guy tells us that the number of indices carried by a tensor is known as its rank. (The Jargon Guy is a new friend of the author; he did not appear in *QFT Nut*.)

written in the form $v^i w^j$. Our tensor field $W^{kj}(\vec{x})$ offers a ready example: in general, it is not equal to some vector U^k multiplied by $V^j(\vec{x})$.

Also, note in our example that the differential operator $\frac{\partial}{\partial x^k}$ transforms (2) like a vector. For example, if $\phi'(x') = \phi(x)$ transforms like a scalar, then $\frac{\partial \phi}{\partial x^k}$ transforms like a vector. Indeed, that's why you have encountered the notation $\vec{\nabla}$ for the gradient in an elementary physics course. This remark will be important later when we revisit Newton's inverse square law in chapter II.3. Do exercise 1 now.

Representation theory

Go back to the 9 objects T^{ij} that form a tensor. Mentally arrange them in a column

$$\begin{pmatrix} T^{11} \\ T^{12} \\ \vdots \\ T^{33} \end{pmatrix}$$

The linear transformation on the 9 objects can then be represented by a 9-by-9 matrix $\mathcal{D}(R)$ acting on this column. (Here we are going painfully slowly because of common confusion on this point. Some authors refer to this column as a 9-component "vector," which is a horrible abuse of terminology. We reserve the word "vector" for something that transforms like a vector $V'^i = R^{ij} V^j$. It is not true that any old collection of stuff arranged in a column is a vector. Don't call anything with feathers a duck!)

For every rotation, specified by a 3-by-3 matrix R, we could thus associate a 9-by-9 matrix $\mathcal{D}(R)$ transforming the 9 objects T^{ij} linearly among themselves. We say that the 9-by-9 matrix $\mathcal{D}(R)$ represents the rotation matrix R in the sense that

$$\mathcal{D}(R_1)\mathcal{D}(R_2) = \mathcal{D}(R_1 R_2) \tag{4}$$

Multiplication of $\mathcal{D}(R_1)$ and $\mathcal{D}(R_2)$ mirrors the multiplication of R_1 and R_2, as it were. The tensor T is said to furnish a 9-dimensional representation of the rotation group $SO(3)$. The 9-by-9 matrices $\mathcal{D}(R)$ represent R. Notice that with this jargon, the vector furnishes a 3-dimensional representation of the rotation group, known as the defining or fundamental representation.

Reducible versus irreducible

Let us now pose the central question of representation theory. Given these 9 entities T^{ij} that transform into each other, consider the 9 independent linear combinations that we can form out of them. Is there a subset among them that only transform into each other? A secret in-club, as it were.

A moment's thought reveals that there is indeed an in-club. Consider $A^{ij} \equiv T^{ij} - T^{ji}$. Under a rotation,

$$A^{ij} \to A'^{ij} = T'^{ij} - T'^{ji} = R^{ik}R^{jl}T^{kl} - R^{jk}R^{il}T^{kl}$$
$$= R^{ik}R^{jl}T^{kl} - R^{jl}R^{ik}T^{lk} = R^{ik}R^{jl}(T^{kl} - T^{lk}) = R^{ik}R^{jl}A^{kl} \quad (5)$$

I have again gone painfully slow here, but it is obvious, isn't it? We just verified in (5) that A^{ij} transforms like a tensor and is thus a tensor. Furthermore, this tensor changes sign upon interchange of its two indices ($A^{ij} = -A^{ji}$) and is said to be antisymmetric. The transformation law (1) treats the two indices democratically, without favoring one over the other, and thus preserves the antisymmetric character of a tensor: if $A^{ij} = -A^{ji}$, then $A'^{ij} = -A'^{ji}$ also.

Let us count. The index i in A^{ij} could take on D values; for each of these values, the index j could take on only $D-1$ values (since the D diagonal elements $A^{ii} = 0$ for $i = 1, 2, \cdots, D$, no Einstein repeated index summation here); but to avoid double counting (since $A^{ij} = -A^{ji}$) we should divide by 2. Hence, the number of independent components in A is equal to $\frac{1}{2}D(D-1)$. For example, for $D = 3$, we have the 3 objects: A^{12}, A^{23}, and A^{31}. The attentive reader would recall that we did the same counting in the previous chapter.

Obviously, the same goes for the symmetric combination $S^{ij} \equiv T^{ij} + T^{ji}$. You could verify as a trivial exercise that $S'^{ij} = R^{ik}R^{jl}S^{kl}$. A tensor S^{ij} that does not change sign upon interchange of its two indices ($S^{ij} = S^{ji}$) is said to be symmetric. Evidently, the symmetric tensor S has more components than the antisymmetric tensor A. In addition to the components S^{ij} with $i \neq j$, S also has D diagonal components, namely $S^{11}, S^{22}, \cdots, S^{DD}$. Thus, the number of independent components in S is equal to $\frac{1}{2}D(D-1) + D = \frac{1}{2}D(D+1)$.

For $D = 3$, the number of components in A and S are $\frac{1}{2} \cdot 3 \cdot 2 = 3$ and $\frac{1}{2} \cdot 3 \cdot 4 = 6$, respectively. (For $D = 4$, the number of components in A and S are 6 and 10, respectively.) Thus, in a suitable basis, the 9-by-9 matrix referred to above actually breaks up into a 3-by-3 block and a 6-by-6 block. We say that the 9-dimensional representation is reducible: it could be reduced to smaller representations.

But we are not done yet. The 6-dimensional representation is also reducible. To see this, note

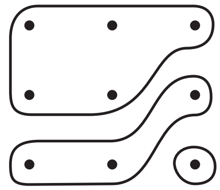
$$S'^{ii} = R^{ik}R^{il}S^{kl} = (R^T)^{ki}R^{il}S^{kl} = (R^{-1})^{ki}R^{il}S^{kl} = \delta^{kl}S^{kl} = S^{kk} \quad (6)$$

where we have used the O in $SO(D)$. (Here we are using repeated index summation: the indices i and k are both summed over.) In other words, the linear combination $S^{11} + S^{22} + \cdots + S^{DD}$, the trace of S, transforms into itself, that is, does not transform at all. It is a loner forming an in-club of one. The 6-by-6 matrix describing the linear transformation of the 6 objects S^{ij} breaks up into a 1-by-1 block and a 5-by-5 block. See figure 1.

Again, for the sake of the beginning student, let us work out explicitly the 5 objects that furnish the representation 5 of $SO(3)$. First define a traceless symmetric tensor $\tilde{S}$ by

$$\tilde{S}^{ij} = S^{ij} - \delta^{ij}(S^{kk}/D) \quad (7)$$

(The repeated index k is summed over.) Explicitly, $\tilde{S}^{ii} = S^{ii} - D(S^{kk}/D) = 0$, and $\tilde{S}$ is traceless. Specialize to $D = 3$. Now we have only 5 objects, namely $\tilde{S}^{11}, \tilde{S}^{22}, \tilde{S}^{12}, \tilde{S}^{13}, \tilde{S}^{23}$.



$9 \to 5+3+1$

Figure 1 How the collection of 9 objects T^{ij} splits up. The figure is meant to be schematic: the dots do not represent the original 9 objects, but linear combinations of them, and the positions of the dots are not meaningful.

We do not count $\tilde{S}^{33}$ separately, since it is equal to $-(\tilde{S}^{11}+\tilde{S}^{22})$. Under an $SO(3)$ rotation, these 5 objects transform into linear combinations of one another, as we just explained.

Let us be specific: the object $\tilde{S}^{13}$, for example, transforms into $\tilde{S}'^{13} = R^{1k}R^{3l}\tilde{S}^{kl} = R^{11}R^{31}\tilde{S}^{11} + R^{11}R^{32}\tilde{S}^{12} + R^{11}R^{33}\tilde{S}^{13} + R^{12}R^{31}\tilde{S}^{21} + R^{12}R^{32}\tilde{S}^{22} + R^{12}R^{33}\tilde{S}^{23} + R^{13}R^{31}\tilde{S}^{31} + R^{13}R^{32}\tilde{S}^{32} + R^{13}R^{33}\tilde{S}^{33} = (R^{11}R^{31} - R^{13}R^{33})\tilde{S}^{11} + (R^{11}R^{32} + R^{12}R^{31})\tilde{S}^{12} + (R^{11}R^{33} + R^{13}R^{31})\tilde{S}^{13} + (R^{12}R^{32} - R^{13}R^{33})\tilde{S}^{22} + (R^{12}R^{33} + R^{13}R^{32})\tilde{S}^{23}$, where in the last equality, we used $\tilde{S}^{ij} = \tilde{S}^{ji}$ and $\tilde{S}^{33} = -(\tilde{S}^{11}+\tilde{S}^{22})$. Indeed, $\tilde{S}^{13}$ transforms into a linear combination of $\tilde{S}^{11}$, $\tilde{S}^{22}$, $\tilde{S}^{12}$, $\tilde{S}^{13}$, $\tilde{S}^{23}$.

To summarize, what we found is that if, instead of the basis consisting of the 9 entities T^{ij}, we use the basis consisting of the 3 entities A^{ij}, the single entity S^{kk} (remember repeated index summation!), and the 5 entities $\tilde{S}^{ij}$, the 9-by-9 matrix $\mathcal{D}(R)$ (that represents rotation in the sense of (4)) breaks up into a 3-by-3 matrix, a 1-by-1 matrix, and a 5-by-5 matrix "stacked on top of each other." This is represented schematically as

$$\mathcal{D}(R) = (\text{9-by-9 matrix}) \to \left[\begin{array}{c|c|c} \text{(3-by-3 block)} & 0 & 0 \\ \hline 0 & \text{(1-by-1 block)} & 0 \\ \hline 0 & 0 & \text{(5-by-5 block)} \end{array}\right] \tag{8}$$

Note that once we chose the new basis, this decomposition holds true for all rotations. (For the readers who know their linear algebra, the technical statement is that there exists a similarity transformation that block-diagonalizes $\mathcal{D}(R)$ for all R. Incidentally, we will encounter plenty of similarity transformations later.)

More generally, the D^2 representation furnished by a general 2-indexed tensor decomposes into a $\frac{1}{2}D(D-1)$-dimensional representation, a $(\frac{1}{2}D(D+1)-1)$-dimensional representation, and a 1-dimensional representation. We say that in $SO(3)$, $9 = 5+3+1$. (In $SO(4)$, $16 = 9+6+1$.)

You might have noticed that in this entire discussion we never had to write out R explicitly in terms of the 3 rotation angles and how the 5 objects $\tilde{S}^{11}, \cdots, \tilde{S}^{23}$ transform into one another in terms of these angles. It is only the counting that matters. You might regard that as the difference between mathematics and arithmetic.

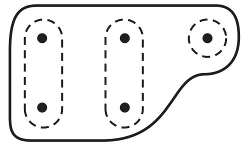


Figure 2 Under $SO(3)$, the 5 objects inside the solid line transform into linear combinations of each other, but under the smaller group of transformations $SO(2)$, the objects inside each of the 3 dashed lines transform into linear combinations of each other. The 5 breaks up as $5 \to 2 + 2 + 1$. As in figure 1, this figure is meant to be schematic.

Restriction to a subgroup

You definitely do not have to master group theory[1] to read this book, but it would be useful for you to learn a few basic concepts and to be able to count. For instance, the notion of a subgroup. Consider the group $SO(2)$ that we studied to exhaustion, consisting of rotations around the z-axis, say. Evidently, $SO(2)$ is a subgroup of $SO(3)$ in that its elements are all elements of $SO(3)$ and form a group all by themselves. The components of the 3-vector V^i could be split into two sets: (V^1, V^2) and V^3. Under a rotation around the z-axis, (V^1, V^2) transform as a 2-vector and V^3 as a scalar. We say that upon restriction to the subgroup $SO(2)$, the irreducible representation 3 breaks up into the representations 2 and 1 of the subgroup, a decomposition we write as $3 \to 2 + 1$. All the group theoretic results we need in this book could be obtained by explicit listing and simple counting.

Look at the 5 objects, S^{11}, S^{22}, S^{12}, S^{13}, S^{23}, that furnish the representation 5 of $SO(3)$. Now consider a restriction to the subgroup $SO(2)$. In other words, we restrict ourselves to rotations around the z-axis, that is, rotations under which $V^3 \to V'^3 = V^3$, namely rotations with $R^{33} = 1$ and R^{13}, R^{23}, R^{31}, R^{32} all vanishing. Since $SO(2)$ does not touch the index 3, we conclude immediately that the combination $S^{11} + S^{22} = -S^{33}$ does not transform, or in other words, it transforms as a singlet under $SO(2)$. Similarly, the pair (S^{13}, S^{23}) transforms as a doublet, since the index 3 is "invisible" to $SO(2)$: the group transforms the indices 1 and 2 into each other, while leaving the index 3 alone. Indeed, we see that our earlier expression for S'^{13} collapses to $S'^{13} = R^{11}S^{13} + R^{12}S^{23}$, as expected. Finally, you can verify that the remaining combinations $(S^{12}, S^{11} - S^{22})$ transform like a doublet. These results could be summarized by saying that, upon restriction to the subgroup $SO(2)$, the irreducible representation 5 of the group $SO(3)$ breaks up as $5 \to 2 + 2 + 1$. See figure 2.

Tensors in Newtonian mechanics

Let us give another example, particularly apt for a book on gravity, of a Newtonian tensor. Consider two nearby particles moving in a potential. Denote their trajectories by $\vec{x}(t)$

and $\vec{y}(t)$, respectively, determined by $\frac{d^2x^i}{dt^2} = -\partial^i V(\vec{x})$ and $\frac{d^2y^i}{dt^2} = -\partial^i V(\vec{y})$. (I am also testing whether there are any readers who do not understand thoroughly the concept of notational freedom.) We want to know how the separation vector $\vec{s} \equiv \vec{y} - \vec{x}$ changes with time, keeping terms to leading order in $\vec{s}$:

$$\frac{d^2s^i}{dt^2} = \frac{d^2y^i}{dt^2} - \frac{d^2x^i}{dt^2} = -\partial^i[V(\vec{y}) - V(\vec{x})] = -\partial^i[V(\vec{x}+\vec{s}) - V(\vec{x})] \simeq -\partial^i\partial^j V(\vec{x})s^j$$

The object $\mathcal{R}^{ij}(\vec{x}) \equiv \partial^i\partial^j V(\vec{x})$ is manifestly a tensor if $V(\vec{x})$ is a scalar. For example, verify that $\mathcal{R}^{ij} = GM(\delta^{ij}r^2 - 3x^ix^j)/r^5$ for the gravitational potential $V(\vec{x}) = -GM/r$. Note that $\mathcal{R}^{ij}$ is a symmetric traceless tensor. Since $\mathcal{R}^{ii} = \partial^i\partial^i V(\vec{x}) = \vec{\nabla}^2 V$, the tracelessness merely reaffirms the fact that the $1/r$ potential satisfies Laplace's equation $\vec{\nabla}^2 V = 0$. Also, $\mathcal{R}^{ij}$ is manifestly not the product of two vectors, but it transforms as if it were.

Let us see how rotational covariance works in the equation

$$\frac{d^2s^i}{dt^2} = -\mathcal{R}^{ij}s^j \tag{9}$$

The right hand side has to be linear in the vector $\vec{s}$. Since the left hand side transforms like a vector, the right hand side must also: indeed, it is given by a tensor $\mathcal{R}$ contracted* with a vector $\vec{s}$. A tensor is needed on the right hand side.

Imagine yourself falling toward a spherical planet or star. With no loss of generality, let your location at some instant be $(0, 0, r)$ along the z-axis. The tensor $\mathcal{R}$ written out as a matrix is then diagonal and is given by (for example, $\mathcal{R}^{33} = GM(\delta^{33}r^2 - 3x^3x^3)/r^5 = GM(1-3)/r^3$)

$$\mathcal{R} = \frac{GM}{r^3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{10}$$

Thus, the sign of $\frac{d^2\vec{s}}{dt^2}$ depends on the orientation of $\vec{s}$.

To see why this is so and to understand what tensors are all about, imagine surrounding yourself with a circular arrangement of balls lying in the (x-z) plane (see figure 3a) and initially at rest in your frame. Using (9) and (10), we can now write down how the separation between two balls along different directions changes.

Since we are going to specify the direction, we will denote the separation simply by s. Along the z-axis, s grows according to (see (9)) $\frac{d^2s}{dt^2} = -\mathcal{R}^{33}s = +2\frac{GM}{r^3}s$. The plus sign indicates that the two balls move away from each other. In contrast, along the x-axis, s decreases according to $\frac{d^2s}{dt^2} = -\mathcal{R}^{11}s = -\frac{GM}{r^3}s$. The two balls approach each other. (Similarly for two balls aligned along the y-axis.) (Note that acting on $\vec{s}$ on the right hand side of (9) by a tensor makes it possible for $\frac{d^2s}{dt^2}$ to change sign depending on the orientation of $\vec{s}$.)

Inspecting figure 3a, you see why. Look at it as an observer on the planet. In the first case, one of the two balls, being closer to the planet, is falling faster than the other. Thus, they

* When a pair of repeated indices, such as j in (9), is summed over, they are often said to be contracted with each other (as mentioned in a footnote in the preceding chapter) in the sense that this index no longer appears in the result, as shown by the left hand side of (9).

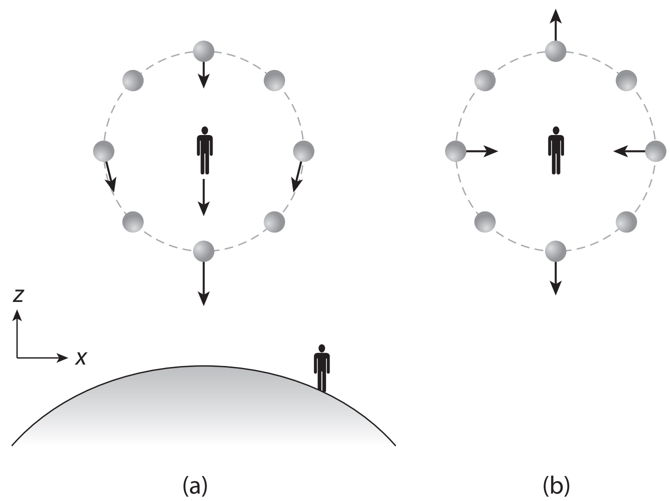


Figure 3 A falling ring of balls as seen by an observer on the planet (a), and as seen by an observer falling with the balls (b).

are moving away from each other. In the second case, the two balls are coming closer due to spherical symmetry: they are both heading toward the center of the planet. As Newton pointed out, objects do not fall down to earth, but toward the center of the earth.

In your rest frame (figure 3b) as you fall along with the balls, however, you see a tidal force acting on the circular ring (or a spherical shell if you prefer) of balls. The force appears to stretch the ring in the z-direction and to squeeze it in the orthogonal direction. When we come to Einstein's prediction of gravitational waves in chapter IX.4, we will see that gravitational waves act on the detector according to equations analogous to (9) and (10). Note also for future reference that the tidal force $\mathcal{R}^{ij}(\vec{x}) \equiv \partial^i \partial^j V(\vec{x})$ involves two derivatives acting on the gravitational potential $V(\vec{x})$.

Invariant tensors

In D-dimensional space, define the antisymmetric symbol $\varepsilon^{ijk\cdots n}$ carrying D indices to have the following properties:

$$\varepsilon^{\cdots l \cdots m \cdots} = -\varepsilon^{\cdots m \cdots l \cdots} \quad \text{and} \quad \varepsilon^{12\cdots D} = 1 \tag{11}$$

In other words, the antisymmetric symbol ε flips sign upon the interchange of any pair of indices. It follows that ε vanishes when two indices are equal. (Note that the second property listed is just normalization.) Since each index can take on only values $1, 2, \cdots, D$, the antisymmetric symbol for D-dimensional space must carry D indices as already noted. For example, for $D = 2$, $\varepsilon^{12} = -\varepsilon^{21} = 1$, with all other components vanishing. For $D = 3$, $\varepsilon^{123} = \varepsilon^{231} = \varepsilon^{312} = -\varepsilon^{213} = -\varepsilon^{132} = -\varepsilon^{321} = 1$, with all other components vanishing (as was already noted in the preceding chapter).

Using the Kronecker delta and the antisymmetric symbol, we can write the defining properties of rotations $R^T R = I$ and $\det R = 1$ as

$$\delta^{ij} R^{ik} R^{jl} = \delta^{kl} \tag{12}$$

and

$$\varepsilon^{ijk\cdots n} R^{ip} R^{jq} R^{kr} \cdots R^{ns} = \varepsilon^{pqr\cdots s} \det R = \varepsilon^{pqr\cdots s} \tag{13}$$

respectively. In (13) we used the definition of $\det R$. (Verify this for $D = 2$ and 3.)

Referring to (1), we see that we can describe δ^{ij} and $\varepsilon^{ijk\cdots n}$ as invariant tensors: they transform into themselves. For the rest of this text, we will often use, implicitly or explicitly, the notion of invariant tensors.

For example, for $SO(3)$, using (13) you can show that $\varepsilon^{ijk} A^i B^j \equiv C^k$ defines a vector $\vec{C} = \vec{A} \times \vec{B}$, the familiar cross product. Various identities follow. Consider, for example,

$$\varepsilon^{ijk} \varepsilon^{lnk} = \delta^{il} \delta^{jn} - \delta^{in} \delta^{jl} \tag{14}$$

To prove this, simply note that both sides transform as invariant tensors with four indices, and the symmetry properties (such as under $i \leftrightarrow j$) of the two sides match. Contracting with A^j, B^l, and C^n, we obtain an identity you might recognize: $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$.

Closing of Newtonian orbits once again

We can now go back to the apparent mystery in chapter I.1, that the Newtonian orbits in a $1/r$ potential close. Out of the conserved angular momentum vector $\vec{l} = \vec{r} \times \vec{p} = \vec{r} \times \dot{\vec{r}}$ (we are using the notation of chapter I.1; we have effectively set the mass to unity and hence the second equality) we can form the Laplace-Runge-Lenz vector $\vec{\mathcal{L}} \equiv \vec{l} \times \dot{\vec{r}} + \frac{\kappa}{r}\vec{r}$. Computing the time derivative $\dot{\vec{\mathcal{L}}}$, you can verify (see exercise 4) that $\vec{\mathcal{L}}$ is conserved for an inverse square central force. When $\dot{\vec{r}}$ is perpendicular to $\vec{r}$, which occurs at perihelion and aphelion, the vector $\vec{\mathcal{L}}$ points in the direction of $\vec{r}$. We could take the constant vector $\vec{\mathcal{L}}$ to point toward the perihelion, and thus the position of the perihelion does not change. Hence the orbit closes.

This result does not hold in Einstein gravity. The precession of the perihelion of Mercury, which we will discuss in chapter VI.3, is of course one of the classic tests of general relativity.

Appendix: Two lemmas for future use

There is a lot more we could say about tensors, but let me mention two simple lemmas that we will happen to need later.

Let S^{ij} and A^{ij} be two arbitrary and unrelated tensors, symmetric and antisymmetric, respectively. Then $S^{ij} A^{ij} = 0$. (See exercise 5.)

Tensors can have all kinds of symmetry properties, which you can explore on your own and in the exercises. For example, a totally antisymmetric 3-indexed tensor T^{ijk} is such that T flips sign under the interchange of any pair of indices (for example, $T^{ijk} = -T^{jik} = +T^{jki}$). A multi-indexed tensor can also have symmetry properties under the interchange of a specific pair, or may have no symmetry at all. Consider, for example, a tensor G^{kij} symmetric under the interchange of the first pair of indices only, that is, $G^{kij} = G^{ikj}$. To be pedantic and absolutely clear, sometimes I like to put a space or a dot between the indices, thus $G^{ki\ j}$ or $G^{ki\cdot j}$ to separate the "special" pair from the other indices. For example, our tensor could happen to be $G^{ki\cdot j} = \partial^k\partial^i W^j(\vec{x})$ for some vector field W^j.

Given $G^{ki\cdot j}$, define $H^{k\cdot ij} \equiv G^{ki\cdot j} + G^{kj\cdot i}$. (Note that $H^{k\cdot ij} = H^{k\cdot ji}$ by definition, but $H^{i\cdot kj}$ is in general not equal to $H^{k\cdot ij}$.) Then we can solve for G in terms of H:

$$G^{ki\cdot j} = \frac{1}{2}(H^{k\cdot ij} + H^{i\cdot jk} - H^{j\cdot ki}) \tag{15}$$

(See exercise 8.)

Exercises

1 Define $\vec{\nabla} \equiv \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \cdots, \frac{\partial}{\partial x^D}\right)$. Show that if ϕ is a scalar, then $(\vec{\nabla}\phi)^2 = \vec{\nabla}\phi\cdot\vec{\nabla}\phi = \sum_k \left(\frac{\partial\phi}{\partial x^k}\right)^2$ and $\nabla^2\phi$ transform like a scalar. The Laplacian is defined by

$$\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial(x^1)^2} + \frac{\partial^2}{\partial(x^2)^2} + \cdots + \frac{\partial^2}{\partial(x^D)^2}$$

2 Show that the symmetric tensor S^{ij} is indeed a tensor.

3 Show that the infinitesimal volume element d^3x is a scalar.

4 Show that the Laplace-Runge-Lenz vector is conserved.

5 Show that $S^{ij}A^{ij} = 0$ if S^{ij} is a symmetric tensor and A^{ij} an antisymmetric tensor.

6 Let T^{ijk} be a totally antisymmetric 3-indexed tensor. Show that T has $\frac{1}{3!}D(D-1)(D-2)$ components. Identify the one component for $D = 3$.

7 Consider for $SO(3)$ the tensor T^{ijk} from exercise 6. Show that it transforms as a scalar.

8 Prove the lemma in (15).

9 Verify (13) for $D = 2$ and 3.

Note

1. For a concise introduction to some of the group theory needed in theoretical physics, see *QFT Nut*, appendix B.

I.5 From Change of Coordinates to Curved Spaces

Euclidean spaces described with different coordinates

In discussing rotations in chapter I.3, I emphasized that Euclid is defined by Pythagoras. That the square of the distance between two neighboring points in 2-dimensional Euclidean space with coordinates (x, y) and $(x + dx, y + dy)$ is given by $ds^2 = dx^2 + dy^2$ defines what we mean by Euclidean space.

But even the familiar Euclidean space can look unfamiliar. You know well that in many physics problems, one set of coordinates is often much more convenient than another. Indeed, in discussing Newton's planetary orbit problem in chapter I.1, we changed from Cartesian* coordinates (x, y) to polar coordinates (r, θ), with $x = r\cos\theta$ and $y = r\sin\theta$. Differentiating, we have $dx = dr\cos\theta - r\sin\theta\, d\theta$ and $dy = dr\sin\theta + r\cos\theta\, d\theta$, so that

$$ds^2 = dx^2 + dy^2 = (dr\cos\theta - r\sin\theta\, d\theta)^2 + (dr\sin\theta + r\cos\theta\, d\theta)^2 = dr^2 + r^2 d\theta^2 \tag{1}$$

We are free to make any coordinate transformation we feel like. Consider the most general transformation $x = f(u, v)$, $y = g(u, v)$. Then $dx = f_u(u, v)du + f_v(u, v)dv$ where $f_u \equiv \frac{\partial f}{\partial u}$ and so on, and $dy = g_u(u, v)du + g_v(u, v)dv$. Just plug in to obtain $ds^2 = dx^2 + dy^2 = (f_u^2 + g_u^2)du^2 + (f_v^2 + g_v^2)dv^2 + 2(f_u f_v + g_u g_v)du dv$. With a gunky choice of f and g you will end up with a mess of a coordinate system that would only make your life miserable. (Note that even the innocuous change $x = u + v$ and $y = v$ leads to $ds^2 = du^2 + 2dv^2 + 2du dv$ with the rather unpleasant $du dv$ cross term.) Of course, it was discovered long ago that by choosing $f(u, v) = u\cos v$ and $g(u, v) = u\sin v$, we can get rid of the cross term. By now probably all the nice choices for f and g have already been published by someone.

* When I was in high school, I got the erroneous impression that the notion of coordinates originated with Descartes. In fact, by the time of Ptolemy, astronomers in the West certainly had latitudes and longitudes. In China, Chang Heng, roughly a contemporary of Ptolemy, was said to have derived, by watching a woman weaving, a system of coordinates to map heaven and earth with. The Chinese words for latitudes and longitudes, "jing" and "wei," are just the terms for warp and weft in weaving.

I presume that you also know how to go from Cartesian coordinates (x, y, z) in 3-dimensional Euclidean space E^3 to spherical coordinates (r, θ, φ), with $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$, $z = r \cos\theta$. The more-than-familiar (and who can blame you if you have been in it all your life?) E^3 could be described by either $ds^2 = dx^2 + dy^2 + dz^2$ in Cartesian coordinates or by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \tag{2}$$

in spherical coordinates.

From Latin to Greek

We can systematize and generalize this to D-dimensional space easily enough. In previous chapters, I used Latin letters for the index on the coordinates. I now switch, for later convenience, from Latin to Greek and call the coordinates $x^\mu = (x^1, x^2, \cdots, x^D)$. Then, for Euclid's spaces E^D, Pythagoras said that $ds^2 = \sum_{\mu=1}^{D} (dx^\mu)^2$.

We write this in the fancier form $ds^2 = \sum_{\mu=1}^{D} \sum_{\nu=1}^{D} g_{\mu\nu} dx^\mu dx^\nu$ by introducing a D-by-D matrix g whose diagonal elements are all equal to one and whose other elements are all zero, the famous matrix known far and wide as the identity matrix. To repeat, the indices μ, ν run over $1, 2, \cdots, D$, and $g_{\mu\nu}$ is defined by $g_{\mu\mu} = 1$ and $g_{\mu\nu} = 0$ if $\mu \neq \nu$. (In other words, it is just the Kronecker delta introduced in chapter I.3: $g_{\mu\nu} = \delta_{\mu\nu}$.) Thus, in the double sum for ds^2, the terms with $\mu \neq \nu$ drop out and we are left with $ds^2 = \sum_{\mu=1}^{D} (dx^\mu)^2$.

Now a word on notation. In the chapter on rotation, I have already introduced this expression for ds^2, and furthermore, the repeated index summation convention. Einstein suggested that between us friends we could omit the cumbersome summation symbol and agree that if an index is repeated, then it is to be summed over. Thus, we suppress the double summation $\sum_{\mu=1}^{D} \sum_{\nu=1}^{D}$ and write simply $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. Here μ and ν are both repeated and hence summed over. Unless there is a risk of confusion, no more summation symbols!

The metric

The matrix $g_{\mu\nu}$ is called the metric, a word meaning measure, as in geometry, the science of measuring the earth. We use the metric to measure space. This step of introducing a metric for Euclidean spaces seems like one of those totally senseless moves that certain academics like and publish. In the discussion just given, the metric is simply the identity matrix.

But as soon as we change coordinates, the metric is no longer so simple. As we have already noted in (1), with polar coordinates, the plane E^2 is described by a metric with $g_{rr} = 1$, $g_{\theta\theta} = r^2$, and $g_{r\theta} = 0 = g_{\theta r}$. With spherical coordinates, E^3 is described by a metric with $g_{rr} = 1$, $g_{\theta\theta} = r^2$, $g_{\varphi\varphi} = r^2 \sin^2\theta$, with all other entries zero, as in (2).

In both examples, the metric is not given by the identity matrix. Furthermore, the metric $g_{\mu\nu}(x)$ varies from point to point. For example, $g_{\varphi\varphi}$ depends on both r and θ. Note, however, that for these examples, the metric is diagonal. (That is why polar and spherical coordinates are so popular!) In general, the metric $g_{\mu\nu}$ need not be diagonal (as shown in the example $ds^2 = du^2 + 2dv^2 + 2dudv$, for which $g_{uu} = 1,\ g_{vv} = 2,\ g_{uv} = g_{vu} = 1$). However, in this text, for the sake of simplicity, we will mostly stick to metrics that are diagonal. Furthermore, since $dx^\mu dx^\nu = dx^\nu dx^\mu$, the metric is symmetric under interchange of indices: $g_{\mu\nu} = g_{\nu\mu}$. It goes without saying that the reader encountering all this for the first time should verify everything I say.

Lower indices appear

The attentive reader might have noticed that lower indices have sneakily appeared! The metric $g_{\mu\nu}$ carries lower indices, while dx^μ carries an upper index. When I taught Einstein gravity, the appearance of upper and lower indices invariably confused some students. In this text, I will try to motivate the point of introducing upper and lower indices, more from a utilitarian, rather than a profoundly mathematical, point of view. My strategy is to introduce this business of two kinds of indices in stages.

At this stage, the motivation, to put it bluntly, is that we just feel like it. But this caprice immediately leads to a useful rule. In the Einstein repeated index summation, we will insist that when we sum over a pair of repeated indices, one of them must be upstairs, the other downstairs. This is manifestly, and trivially, satisfied by the only example $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$ we have encountered thus far. The whole business of two kinds of indices may seem unnecessary at this point, but later, you will see that the distinction between upper and lower indices becomes essential, or at least highly useful.

A word about terminology: Some authors refer to $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$ as the square of the line element, reserving the term metric for the object $g_{\mu\nu}(x)$ contained in the line element. I find it convenient to abuse terminology and simply refer to both as the metric.

Let me mention one trivial point, but one with the potential for confusing beginners. Some years ago, when I surveyed the students in my class for points of confusion, one student told me that for quite a while he did not realize that $g_{\rho\mu}(x)dx^\rho dx^\mu$, $g_{\zeta\psi}(x)dx^\zeta dx^\psi$, and so on, all denote the same thing! Perhaps this is because the summation symbol has been suppressed: the same student could recognize that $\sum_{\mu=1}^{D}\sum_{\nu=1}^{D} g_{\mu\nu}(x)dx^\mu dx^\nu = \sum_{\rho=1}^{D}\sum_{\mu=1}^{D} g_{\rho\mu}(x)dx^\rho dx^\mu = \sum_{\zeta=1}^{D}\sum_{\psi=1}^{D} g_{\zeta\psi}(x)dx^\zeta dx^\psi$.

Change of coordinates, curved space, and curved spacetime

We all know that in Euclidean 3-space, if we restrict r to be equal to a, we would find ourselves on the surface of a sphere of radius a. In other words, the set of points at a distance a from the origin form a sphere with radius a.

This procedure gives us an easy way to determine the metric on a sphere. Simply take the metric (2) $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$ and set $r = a$. Then $dr = 0$, so that ds^2 collapses to $a^2(d\theta^2 + \sin^2\theta d\varphi^2)$. From 3-dimensional flat space we have "lost" the coordinate r and gone to a 2-dimensional curved space with coordinates $x^\mu = (x^1, x^2) = (\theta, \varphi)$. Without loss of generality, we can take a as the unit of distance and set $a = 1$. So, on the unit 2-dimensional sphere S^2

$$ds^2 = d\theta^2 + \sin^2\theta d\varphi^2 \tag{3}$$

with a metric given by $g_{11} = g_{\theta\theta} = 1$, $g_{22} = g_{\varphi\varphi} = \sin^2\theta$, and $g_{12} = g_{\theta\varphi} = g_{21} = g_{\varphi\theta} = 0$.

The take-home message here is that curved space is just a skip and a hop away from the familiar change of coordinates. This is fortunate for students of physics: when you learned to change coordinates, you were actually also learning about curved spaces. We are now going to develop a general formalism for changing coordinates. Even though you already know how to change coordinates, it pays to learn this formalism, because we can also use it to study curved space and curved spacetime (which, as you have surely heard, plays a central role in Einstein gravity).

Change of coordinates, curved space, and curved spacetime: basically the same deal, as you will see.

How do we know whether a space is curved or not?

This raises an exceedingly interesting and crucial question: given a space with the metric $g_{\mu\nu}(x)$, how do we know whether it is curved or flat?

A complicated looking metric does not necessarily mean that the space is curved, since somebody could have simply chosen an especially gunky coordinate system. It could be flat space in disguise. To forcefully bring home this point, I invite you to consider $ds^2 = (1 + u^2)du^2 + (1 + 4v^2)dv^2 + 2(2v - u)du dv$ and $ds^2 = (1 + u^2)du^2 + (1 + 2v^2)dv^2 + 2(2v - u)du dv$. One describes flat space, the other a space that at some points is violently curved. Which is which?

Puzzled, you reply: "How could I possibly tell?"

That's in fact the correct answer at this stage of this discussion. The two metrics I just gave you look almost identical except for one single $2 \to 4$. In one of the most famous episodes in mathematics, Carl Friedrich Gauss (1777–1855) solved this problem for 2-dimensional spaces. His work was then generalized by his student Bernhard Riemann (1826–1866). Later, in chapter VI.1, given any metric in any number of dimensions, you will be able to calculate, and even better, to train the computer to calculate, something called the Riemann curvature tensor, which will tell you once and for all if the space is flat or curved. No more thinking involved! Gauss and Riemann did it for you.

But for now, let me ask you to think about two simple examples in good old 2-dimensional space, for which our intuition is allegedly pretty good. We know that $ds^2 = dr^2 + r^2 d\theta^2$ describes flat space. Consider

$$ds^2 = d\rho^2 + \sin^2\rho \, d\theta^2 \tag{4}$$

Is the space being described flat or curved? Or consider the space described by

$$ds^2 = \cos^2 \rho \, d\rho^2 + \sin^2 \rho \, d\theta^2 \tag{5}$$

Is it flat or curved? You should think about this before reading on. The answers are given in appendix 1.

Remember the civilization of mites in the prologue? You are in the same position as the mite professors of geometry: they can measure the distance between infinitesimally separated points, and from that they have to figure out whether their world is curved. We will face the same problem as the mites when we get to cosmology in parts V and VI.

The logic of differential geometry

Differential geometry, as developed by Gauss and Riemann, tells us that given the metric, we can calculate the curvature. The logic goes as follows. The metric tells you the distance between two nearby points. Integrating, you can obtain the distance along any curve joining two points, not necessarily nearby. Find the curve with the shortest distance. By definition, this curve is the "straight line" between these two points. Once you know how to find the "straight line" between any two points, you can test all of Euclid's theorems to see whether our space is flat. For example, as described in the prologue, the mite geometers could now draw a small circle around any point, measure its circumference, and see if it is equal to 2π times the radius. (See appendix 1.) Thus, the metric can tell us about curvature.

Take an everyday example: given an airline table of distances, you can deduce that the world is curved without ever going outside. If I tell you the three distances between Paris, Berlin, and Barcelona, you can draw a triangle on a flat piece of paper with the three cities at the vertices. But now if I also give you the distances between Rome and each of these three cities, you would find that you can't extend the triangle to a planar quadrangle (figure 1). So the distances between four points suffice to prove that the world is not flat. But the metric tells you the distances between an infinite number of points.

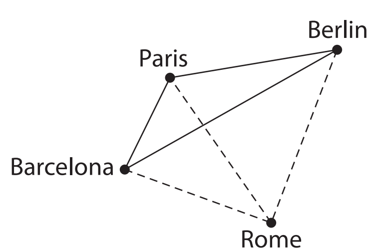


Figure 1 The distances between four cities suffice to prove that the world is not flat.