

function is said to be 'effectively calculable' if its values can be found by some purely mechanical process. Although it is fairly easy to get an intuitive grasp of this idea it is nevertheless desirable to have some definite, mathematically expressible definition. Such a definition was first given by Gödel at Princeton in 1934 (Gödel [2], 26) following in part an unpublished suggestion of Herbrand, and has since been developed by Kleene (Kleene [2]). We shall not be concerned much here with this particular definition. Another definition of effective calculability has been given by Church (Church [3], 356-358) who identifies it with λ -definability. The author has recently suggested a definition corresponding more closely to the intuitive idea (Turing [1], see also Post [1]). It was said above "a function is effectively calculable if its values can be found by some purely mechanical process." We may take this statement literally, understanding by a purely mechanical process one which could be carried out by a machine. It is possible to give a mathematical description, in a certain normal form, of the structures of these machines. The development of the idea leads to the author's definition of a computable function, and an identification of computability³ with effective calculability. (³We shall use the expression 'computable function' to mean a function calculable by a machine, and let 'effectively calculable' refer to the intuitive idea without particular identification with any one of these definitions. We do not restrict the values taken by a computable function to be natural numbers; we may for instance have computable propositional functions.) It is not difficult though somewhat laborious, to prove these three definitions equivalent (Kleene [3], Turing [2]). In the present paper we shall make considerable use of Church's identification of effective calculability with λ -definability, or, what comes to the same, of the identification with computability and one of the equivalence theorems. In most cases where we have to deal with an effectively calculable function we shall introduce the corresponding W. F. F. with some such phrase as "the function f is effectively calculable, let F be a formula λ -defining it" or "let F be a formula such that $F(\underline{n})$ is convertible to... whenever $\underline{n}$ represents a positive integer". In such cases there is no difficulty in seeing how a machine could in principle be designed to calculate the values of the function concerned, and assuming this done the equivalence theorem can be applied. A statement as to what the formula F actually is may be omitted. We may introduce immediately on this basis a W. F. F. ω with the property that $\omega(\underline{m}, \underline{n})$ conv r if r is the greatest positive integer for which m' divides n , if any, and is 1 if there is none. We also introduce Dt with the properties: $Dt(\underline{n}, \underline{n})$ conv 3; $Dt(\underline{n} + \underline{m}, \underline{n})$ conv 2; $Dt(\underline{n}, \underline{n} + \underline{m})$ conv 1. There is another point to be made clear in connection with the point of view we are adopting. It is intended that all proofs that are given should be regarded no more critically than proofs in classical analysis.

The subject matter, roughly speaking, is constructive systems of logic, but as the purpose is directed towards choosing a particular constructive system of logic for practical use; an attempt at this stage to put our theorems into constructive form would be putting the cart before the horse. Those computable functions which take only the values 0 and 1 are of particular importance since they determine and are determined by computable properties, as may be seen by replacing '0' and '1' by 'true' and 'false'. But besides this type of property we may have to consider a different type, which is roughly speaking, less constructive than the computable properties, but more so than the general predicates of classical mathematics. Suppose we have a computable function of the natural members taking natural numbers as values, then corresponding to this function there is the property of being a value of the function. Such a property we shall describe as 'axiomatic'; the reason for using this term is that it is possible to define such a property by giving a set of axioms, the property to hold for a given argument if and only if it is possible to deduce that it holds from the axioms. Axiomatic properties may also be characterized in this way. A property ψ of positive integers is axiomatic if and only if there is a computable property ϕ of two positive integers such that $\psi(x)$ is true if and only if there is a positive integer y such that $\phi(x, y)$ is true. Or again ψ is axiomatic if and only if there is a W. F. F. F such that $\psi(n)$ is true if and only if $F(\underline{n})$ conv 1. By a number theoretic theorem⁴

Alan Turing's Systems of Logic

THE PRINCETON THESIS

Edited and introduced by Andrew W. Appel

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Preface

Alan M. Turing, after his great result in 1936 discovering a universal model of computation and proving his incompleteness theorem, came to Princeton in 1936–38 and earned a PhD in mathematics. Before 1936 there were no universal computers. By 1955 there was not only a theory of computation, but there were real universal (“von Neumann”) computers in Philadelphia, Cambridge (Massachusetts), Princeton, Cambridge (England), and Manchester. The new field of computer science had a remarkably short gestation.

The great engineers who built the first computers are well known: Konrad Zuse (Z3, Berlin, 1941); Tommy Flowers (Colossus, Bletchley Park, 1943); Howard Aiken (Mark I, Harvard, 1944); Prosper Eckert and John Mauchley (ENIAC, University of Pennsylvania, 1946).

But computer science is not just the construction of hardware. Who were the creators of the intellectual revolution underlying the theory of computers and computation?

Turing is very well known as a founder and pioneer of this discipline. In 1936 at the age of twenty-four he discovered the universal model of computation now known as the Turing machine; in 1938 he developed the notion of “oracle relativization”; in 1939–45 he was a principal figure in breaking the German Enigma ciphers using computational devices (though not “Turing machines”); in 1948 he invented the LU-decomposition method in numerical computation; in 1950 he foresaw the field of artificial intelligence and made

remarkably accurate predictions about the future of computing and computers. And, of course, he famously committed suicide in 1954 after prosecution and persecution for practicing homosexuality in England.

But as significant as Turing is for the foundation of computer science, he was not the only scholar whose work in the 1930s led to the birth of this field.

In Fine Hall,¹ home in the 1930s of the Princeton Mathematics Department and the newly established Institute for Advanced Study, were mathematicians whose students would form a significant part of the new fields of computer science and operations research.

This volume presents the manuscript of Alan Turing's PhD thesis. It is accompanied by two introductory essays that explore both the work and the context of Turing's stay in Princeton. My essay elucidates the significance of Turing's work (and that of his adviser, Alonzo Church) for the field of computer science; Solomon Feferman's essay describes its significance for mathematics. Feferman also explains how to relate some of Turing's 1938 terminology to more current usage in the field. But on the whole, the notation and terminology in this field have been fairly stable: "Systems of Logic Based on Ordinals" is still readable as a mathematical and philosophical work.

Andrew W. Appel
Princeton, New Jersey

1 Fine Hall was built in 1930, named for the mathematician Henry Burchard Fine. During the 1930s it housed the Mathematics Department of Princeton University and the mathematicians (e.g., Gödel and von Neumann) and physicists (e.g., Einstein) of the Institute for Advanced Study. In 1939, the Institute moved to its own campus about a mile away from Princeton University's central campus. In 1969, the University's Mathematics Department moved to the new Fine Hall on the other side of Washington Road. The old building was renamed Jones Hall, in honor of its original donors, and now houses the departments of East Asian Studies and Near Eastern Studies.



OSWALD VEBLEN, chairman of the Princeton University Mathematics Department and first professor at the Institute for Advanced Study. His students include Alonzo Church (PhD 1927), and his PhD descendants through Philip Franklin (Princeton PhD 1921) via Alan Perlis (Turing Award 1966) include David Parnas, Zohar Manna, Kai Li, Jeannette Wing, and 500 other computer scientists. Veblen has more than 8000 PhD descendants overall. He helped oversee the development of the pioneering ENIAC digital computer in the 1940s.

(Photographer unknown, from the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA.)



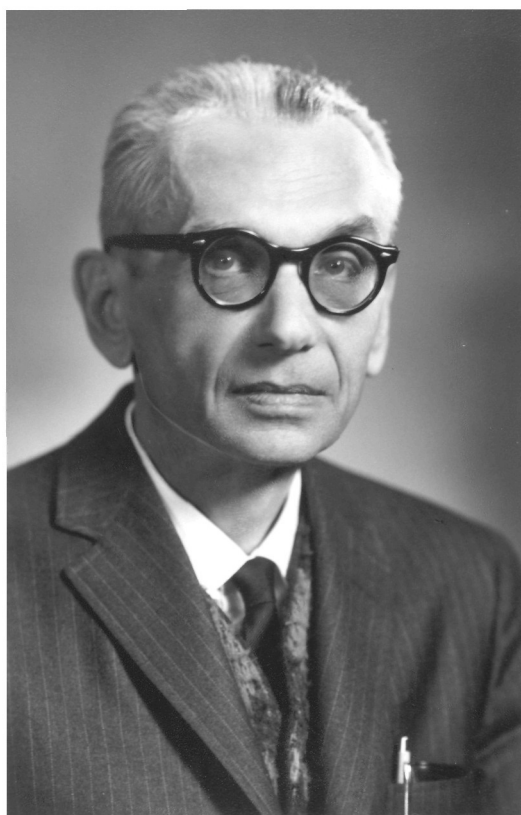
ALONZO CHURCH, professor of mathematics, whose students include Alan Turing, Leon Henkin, Stephen Kleene, Martin Davis, Michael Rabin (Turing Award 1976), Dana Scott (Turing Award 1976), and Barkley Rosser, and whose PhD descendants include 3000 other mathematicians and computer scientists, among them Robert Constable, Edmund Clarke (Turing Award 2007), Allen Emerson (Turing Award 2007), and Les Valiant (Turing Award 2010).

(Photo from the Alonzo Church Papers. Department of Rare Books and Special Collection. Princeton University Library.)



SOLOMON LEFSHETZ, professor of mathematics, whose students include John McCarthy (Turing Award 1971), John Tukey, Ralph Gomory, and Richard Bellman (inventor of dynamic programming), and whose 6181 PhD descendants include John Nash (Nobel Prize 1994), Marvin Minsky (Turing Award 1969), Manuel Blum (Turing Award 1995), Barbara Liskov (Turing Award 2008), Gerald Sussman, Shafi Goldwasser, Umesh and Vijay Vazirani, Persi Diaconis, and Peter Buneman.

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KURT GÖDEL, visitor to the Institute in 1933, 1934, and 1935, and professor at the Institute from 1940, had no students but had an enormous influence on the fields of mathematics and computer science. His 1931 incompleteness result—that it will never be possible to enumerate in logic the true statements of mathematics—stunned mathematicians and philosophers with its unexpectedness. His methods—the numerical encoding of syntax and the numerical processing of logic—set the stage for many techniques of computer science. Major results of Church, Kleene, Turing, and von Neumann clearly and explicitly owe much to Gödel.

(Photo from the Kurt Gödel Papers, the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA, on deposit at Princeton University.)



JOHN VON NEUMANN, at Princeton University from 1930 and professor at the Institute for Advanced Study from 1933, had only a few students (including the pioneer in parallel computer architecture Donald Gillies), but also had an enormous influence on the development of physics, mathematics, logic, economics, and computer science. In 1931 he was the first to recognize the significance of Gödel's work, and toward 1950 he brought Turing's ideas of program-as-data to the engineering of the first stored-program computers. Stored-program computers are called "von Neumann machines," and essentially all computers today are von Neumann machines.

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