Group Theory

Birdtracks, Lie's, and Exceptional Groups

PREDRAG CVITANOVIĆ

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Predrag Cvitanović

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dedicated to the memory of
Boris Weisfeiler and William E. Caswell

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Contents

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The appellation “birdtracks” is due to Bernice Durand, who entered my office, saw the blackboard and asked, puzzled: “What is this? Footprints left by birds scurrying along a sandy beach?”

I am grateful to Dorte Glass for typing most of the manuscript, to the Aksel Tovborg Jensens Legat, Kongelige Danske Videnskabernes Selskab, for financial support which made the transformation from hand-drawn to drafted birdtracks possible, and, most of all, to the good Anders Johansen for drawing some 5,000 birdtracks — without him this book would still be but a collection of squiggles in my filing cabinet. Carol Monsrud and Cecile Gourgues helped with typing the early version of this manuscript.

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I would love to thank W. E. “Bill” Caswell for a perspicacious observation, and Boris Weisfeiler for delightful discussions at the Institute for Advanced Study, but it cannot be done. In 1985, while hiking in the Andes, Boris was kidnapped by the Chilean state security, then tortured and executed by Nazis at Colonia Dignidad, Chile. Bill boarded flight AA77 on September 11, 2001, and was flown into the Pentagon by a no-less-charming group of Islamic fanatics. This book is dedicated to Boris and Bill.

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Group Theory

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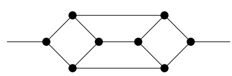
Chapter One

Introduction

This monograph offers a derivation of all classical and exceptional semisimple Lie algebras through a classification of "primitive invariants." Using somewhat unconventional notation inspired by the Feynman diagrams of quantum field theory, the invariant tensors are represented by diagrams; severe limits on what simple groups could possibly exist are deduced by requiring that irreducible representations be of integer dimension. The method provides the full Killing-Cartan list of all possible simple Lie algebras, but fails to prove the existence of F_4, E_6, E_7 and E_8.

One simple quantum field theory question started this project; what is the group-theoretic factor for the following Quantum Chromodynamics gluon self-energy diagram



$$= ? \tag{1.1}$$

I first computed the answer for $SU(n)$. There was a hard way of doing it, using Gell-Mann f_{ijk} and d_{ijk} coefficients. There was also an easy way, where one could doodle oneself to the answer in a few lines. This is the "birdtracks" method that will be developed here. It works nicely for $SO(n)$ and $Sp(n)$ as well. Out of curiosity, I wanted the answer for the remaining five exceptional groups. This engendered further thought, and that which I learned can be better understood as the answer to a different question. Suppose someone came into your office and asked, "On planet Z, mesons consist of quarks and antiquarks, but baryons contain three quarks in a symmetric color combination. What is the color group?" The answer is neither trivial nor without some beauty (planet Z quarks can come in 27 colors, and the color group can be E_6).

Once you know how to answer such group-theoretical questions, you can answer many others. This monograph tells you how. Like the brain, it is divided into two halves: the plodding half and the interesting half.

The plodding half describes how group-theoretic calculations are carried out for unitary, orthogonal, and symplectic groups (chapters 3–15). Except for the "negative dimensions" of chapter 13 and the "spinsters" of chapter 14, none of that is new, but the methods are helpful in carrying out daily chores, such as evaluating Quantum Chromodynamics group-theoretic weights, evaluating lattice gauge theory group integrals, computing $1/N$ corrections, evaluating spinor traces, evaluating casimirs, implementing evaluation algorithms on computers, and so on.

The interesting half, chapters 16–21, describes the "exceptional magic" (a new construction of exceptional Lie algebras), the "negative dimensions" (relations between bosonic and fermionic dimensions). Open problems, links to literature, software and other resources, and personal confessions are relegated to the epilogue, monograph's Web page birdtracks.eu. The methods used are applicable to field-theoretic model building. Regardless of their potential applications, the results are sufficiently intriguing to justify this entire undertaking. In what follows we shall forget about quarks and quantum field theory, and offer instead a somewhat unorthodox introduction to the theory of Lie algebras. If the style is not Bourbaki [29], it is not so by accident.

There are two complementary approaches to group theory. In the *canonical* approach one chooses the basis, or the Clebsch-Gordan coefficients, as simply as possible. This is the method which Killing [189] and Cartan [43] used to obtain the complete classification of semisimple Lie algebras, and which has been brought to perfection by Coxeter [67] and Dynkin [105]. There exist many excellent reviews of applications of Dynkin diagram methods to physics, such as refs. [312, 126].

In the *tensorial* approach pursued here, the bases are arbitrary, and every statement is invariant under change of basis. Tensor calculus deals directly with the invariant blocks of the theory and gives the explicit forms of the invariants, Clebsch-Gordan series, evaluation algorithms for group-theoretic weights, *etc.*

The canonical approach is often impractical for computational purposes, as a choice of basis requires a specific coordinatization of the representation space. Usually, nothing that we want to compute depends on such a coordinatization; physical predictions are pure scalar numbers ("color singlets"), with all tensorial indices summed over. However, the canonical approach can be very useful in determining chains of subgroup embeddings. We refer the reader to refs. [312, 126] for such applications. Here we shall concentrate on tensorial methods, borrowing from Cartan and Dynkin only the nomenclature for identifying irreducible representations. Extensive listings of these are given by McKay and Patera [234] and Slansky [312].

To appreciate the sense in which canonical methods are impractical, let us consider using them to evaluate the group-theoretic factor associated with diagram (1.1) for the exceptional group E_8. This would involve summations over 8 structure constants. The Cartan-Dynkin construction enables us to construct them explicitly; an E_8 structure constant has about $248^3/6$ elements, and the direct evaluation of the group-theoretic factor for diagram (1.1) is tedious even on a computer. An evaluation in terms of a canonical basis would be equally tedious for $SU(16)$; however, the tensorial approach illustrated by the example of section 2.2 yields the answer for all $SU(n)$ in a few steps.

Simplicity of such calculations is one motivation for formulating a tensorial approach to exceptional groups. The other is the desire to understand their geometrical significance. The Killing-Cartan classification is based on a mapping of Lie algebras onto a Diophantine problem on the Cartan root lattice. This yields an exhaustive classification of simple Lie algebras, but gives no insight into the associated geometries. In the 19th century, the geometries or the invariant theory were the central question, and Cartan, in his 1894 thesis, made an attempt to identify the primitive invariants. Most of the entries in his classification were the classical groups $SU(n)$, $SO(n)$, and $Sp(n)$. Of the five exceptional algebras, Cartan [44] identified G_2 as the group of octonion isomorphisms and noted already in his thesis that E_7 has a skew-symmetric quadratic and a symmetric quartic invariant. Dickson characterized E_6 as a 27-dimensional group with a cubic invariant. The fact that the orthogonal, unitary and symplectic groups were invariance groups of real, complex, and quaternion norms suggested that the exceptional groups were associated with octonions, but it took more than 50 years to establish this connection. The remaining four exceptional Lie algebras emerged as rather complicated constructions from octonions and Jordan algebras, known as the *Freudenthal-Tits construction.* A mathematician's history of this subject is given in a delightful review by Freudenthal [130]. The problem has been taken up by physicists twice, first by Jordan, von Neumann, and Wigner [173], and then in the 1970s by Gürsey and collaborators [149, 151, 152]. Jordan *et al.*'s effort was a failed attempt at formulating a new quantum mechanics that would explain the neutron, discovered in 1932. However, it gave rise to the Jordan algebras, which became a mathematics field in itself. Gürsey *et al.* took up the subject again in the hope of formulating a quantum mechanics of quark confinement; however, the main applications so far have been in building models of grand unification.

Although beautiful, the Freudenthal-Tits construction is still not practical for the evaluation of group-theoretic weights. The reason is this: the construction involves $[3 \times 3]$ octonionic matrices with octonion coefficients, and the 248-dimensional defining space of E_8 is written as a direct sum of various subspaces. This is convenient for studying subgroup embeddings [291], but awkward for group-theoretical computations.

The inspiration for the primitive invariants construction came from the axiomatic approach of Springer [314, 315] and Brown [34]: one treats the defining representation as a single vector space, and characterizes the primitive invariants by algebraic identities. This approach solves the problem of formulating efficient tensorial algorithms for evaluating group-theoretic weights, and it yields some intuition about the geometrical significance of the exceptional Lie groups. Such intuition might be of use to quark-model builders. For example, because $SU(3)$ has a cubic invariant $\epsilon^{abc}q_a q_b q_c$, Quantum Chromodynamics, based on this color group, can accommodate 3-quark baryons. Are there any other groups that could accommodate 3-quark singlets? As we shall see, G_2, F_4, and E_6 are some of the groups whose defining representations possess such invariants.

Beyond its utility as a computational technique, the primitive invariants construction of exceptional groups yields several unexpected results. First, it generates in a somewhat magical fashion a triangular array of Lie algebras, depicted in figure 1.1. This is a classification of Lie algebras different from Cartan's classification; in this new classification, all exceptional Lie groups appear in the same series (the bottom line of figure 1.1). The second unexpected result is that many groups and group representations are mutually related by interchanges of symmetrizations and antisymmetrizations and replacement of the dimension parameter n by $-n$. I call this phenomenon "negative dimensions."

For me, the greatest surprise of all is that in spite of all the magic and the strange diagrammatic notation, the resulting manuscript is in essence not very different from Wigner's [345] 1931 classic. Regardless of whether one is doing atomic, nuclear, or particle physics, all physical predictions ("spectroscopic levels") are expressed in terms of Wigner's $3n$-j coefficients, which can be evaluated by means of recursive or combinatorial algorithms.

Parenthetically, this book is *not* a book about diagrammatic methods in group theory. If you master a traditional notation that covers all topics in this book in a uniform way, more elegantly than birdtracks, more power to you. I would love to learn it.

						0 0	3 $\mathbf{A}_1$ 2
					0 0	1 $\mathbf{U}(1)$ 1	8 $\mathbf{A}_2$ 3
				0 0	0 1	3 $\mathbf{A}_1$ 3	14 $\mathbf{G}_2$ 7
			0 0	0 1	2 $2\mathbf{U}(1)$ 2	9 $\mathbf{3A}_1$ 4	28 $\mathbf{D}_4$ 8
		0 0	0 2	3 $\mathbf{A}_1$ 5	8 $\mathbf{A}_2$ 8	21 $\mathbf{C}_3$ 14	52 $\mathbf{F}_4$ 26
	0 0	0 1	2 $2\mathbf{U}(1)$ 3	8 $\mathbf{A}_2$ 6	16 $\mathbf{2A}_2$ 9	35 $\mathbf{A}_5$ 15	78 $\mathbf{E}_6$ 27
0 0	1 $\mathbf{U}(1)$ 2	3 $\mathbf{A}_1$ 4	9 $\mathbf{3A}_1$ 8	21 $\mathbf{C}_3$ 14	35 $\mathbf{A}_5$ 20	66 $\mathbf{D}_6$ 32	133 $\mathbf{E}_7$ 56
3 $\mathbf{A}_1$ 3	8 $\mathbf{A}_2$ 8	14 $\mathbf{G}_2$ 14	28 $\mathbf{D}_4$ 28	52 $\mathbf{F}_4$ 52	78 $\mathbf{E}_6$ 78	133 $\mathbf{E}_7$ 133	248 $\mathbf{E}_8$ 248

Figure 1.1 The “Magic Triangle” for Lie algebras. The “Magic Square” is framed by the double line. For a discussion, consult chapter 21.

Chapter Two

A preview

The theory of Lie groups presented here had mutated greatly throughout its genesis. It arose from concrete calculations motivated by physical problems; but as it was written, the generalities were collected into introductory chapters, and the applications receded later and later into the text.

As a result, the first seven chapters are largely a compilation of definitions and general results that might appear unmotivated on first reading. The reader is advised to work through the examples, section 2.2 and section 2.3 in this chapter, jump to the topic of possible interest (such as the unitary groups, chapter 9, or the E_8 family, chapter 17), and birdtrack if able or backtrack when necessary.

The goal of these notes is to provide the reader with a set of basic group-theoretic tools. They are not particularly sophisticated, and they rest on a few simple ideas. The text is long, because various notational conventions, examples, special cases, and applications have been laid out in detail, but the basic concepts can be stated in a few lines. We shall briefly state them in this chapter, together with several illustrative examples. This preview presumes that the reader has considerable prior exposure to group theory; if a concept is unfamiliar, the reader is referred to the appropriate section for a detailed discussion.

2.1 BASIC CONCEPTS

A typical quantum theory is constructed from a few building blocks, which we shall refer to as the *defining space* V. They form the defining multiplet of the theory — for example, the "quark wave functions" q_a. The group-theoretical problem consists of determining the symmetry group, *i.e.*, the group of all linear transformations

$$q'_a = G_a{}^b q_b \qquad a, b = 1, 2, \ldots, n\,,$$

which leaves invariant the predictions of the theory. The $[n \times n]$ matrices G form the *defining representation* (or "rep" for short) of the invariance group $\mathcal{G}$. The conjugate multiplet $\overline{q}$ ("antiquarks") transforms as

$$q'^a = G^a{}_b q^b\,.$$

Combinations of quarks and antiquarks transform as *tensors*, such as

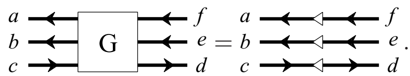
$$\begin{aligned} p'_a q'_b r'^c &= G_{ab}{}^c{}_{,\,d}{}^{ef} p_f q_e r^d\,, \\ G_{ab}{}^c{}_{,\,d}{}^{ef} &= G_a{}^f G_b{}^e G_d{}^c \end{aligned}$$

(distinction between $G_a{}^b$ and $G^a{}_b$ as well as other notational details are explained in section 3.2). Tensor reps are plagued by a proliferation of indices. These indices can either be replaced by a few collective indices:

$$\alpha = \left\{ \begin{matrix} c \\ ab \end{matrix} \right\}, \quad \beta = \left\{ \begin{matrix} ef \\ d \end{matrix} \right\},$$

$$q'_\alpha = G_\alpha{}^\beta q_\beta \,, \tag{2.1}$$

or represented diagrammatically:



(Diagrammatic notation is explained in section 4.1.) Collective indices are convenient for stating general theorems; diagrammatic notation speeds up explicit calculations.

A polynomial

$$H(\overline{q}, \overline{r}, \ldots, s) = h_{ab\ldots}{}^{\ldots c}\, q^a r^b \ldots s_c$$

is an invariant if (and only if) for any transformation $G \in \mathcal{G}$ and for any set of vectors $q, r, s, \ldots$ (see section 3.4)

$$H(\overline{Gq}, \overline{Gr}, \ldots Gs) = H(\overline{q}, \overline{r}, \ldots, s) \,. \tag{2.2}$$

An invariance group is defined by its *primitive invariants*, *i.e.*, by a list of the elementary "singlets" of the theory. For example, the orthogonal group $O(n)$ is defined as the group of all transformations that leaves the length of a vector invariant (see chapter 10). Another example is the color $SU(3)$ of QCD that leaves invariant the mesons $(q\bar{q})$ and the baryons (qqq) (see section 15.2). A complete list of primitive invariants *defines* the invariance group via the invariance conditions (2.2); only those transformations, which respect them, are allowed.

It is not necessary to list explicitly the components of primitive invariant tensors in order to define them. For example, the $O(n)$ group is defined by the requirement that it leaves invariant a symmetric and invertible tensor $g_{ab} = g_{ba}$, $\det(g) \neq 0$. Such definition is basis independent, while a component definition $g_{11} = 1, g_{12} = 0, g_{22} = 1, \ldots$ relies on a specific basis choice. We shall define all simple Lie groups in this manner, specifying the primitive invariants only by their symmetry and by the basis-independent algebraic relations that they must satisfy.

These algebraic relations (which I shall call *primitiveness conditions*) are hard to describe without first giving some examples. In their essence they are statements of irreducibility; for example, if the primitive invariant tensors are δ_b^a, h_{abc} and h^{abc}, then $h_{abc}h^{cbe}$ must be proportional to δ_a^e, as otherwise the defining rep would be reducible. (Reducibility is discussed in section 3.5, section 3.6, and chapter 5.)

The objective of physicists' group-theoretic calculations is a description of the spectroscopy of a given theory. This entails identifying the levels (irreducible multiplets), the degeneracy of a given level (dimension of the multiplet) and the level splittings (eigenvalues of various casimirs). The basic idea that enables us to carry this program through is extremely simple: a hermitian matrix can be diagonalized. This fact has many names: Schur's lemma, Wigner-Eckart theorem, full reducibility of unitary reps, and so on (see section 3.5 and section 5.3). We exploit it by constructing invariant hermitian matrices M from the primitive invariant tensors. The M's have collective indices (2.1) and act on tensors. Being hermitian, they can be diagonalized

$$CMC^\dagger = \begin{pmatrix} \lambda_1 & 0 & 0 & & \cdots \\ 0 & \lambda_1 & 0 & & \\ 0 & 0 & \lambda_1 & & \\ & & & \lambda_2 & \\ \vdots & & & & \ddots \end{pmatrix},$$

and their eigenvalues can be used to construct projection operators that reduce multiparticle states into direct sums of lower-dimensional reps (see section 3.5):

$$\mathbf{P}_i = \prod_{j \neq i} \frac{M - \lambda_j \mathbf{1}}{\lambda_i - \lambda_j} = C^\dagger \begin{pmatrix} \begin{array}{cc|} \ddots & \vdots \\ \cdots & 0 \\ \hline \end{array} & \cdots & 0 \\ \vdots & \boxed{\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 1 \end{array}} & \vdots \\ 0 & \cdots & \begin{array}{|cc} \hline 0 & \cdots \\ \vdots & \ddots \end{array} \end{pmatrix} C\,. \tag{2.3}$$

An explicit expression for the diagonalizing matrix C (Clebsch-Gordan coefficients or *clebsches,* section 4.2) is unnecessary — it is in fact often more of an impediment than an aid, as it obscures the combinatorial nature of group-theoretic computations (see section 4.8).

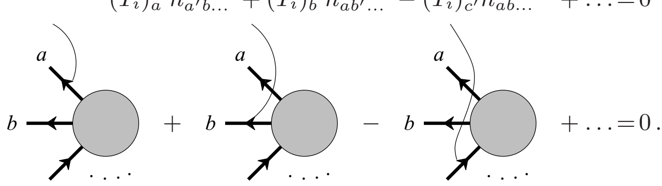
All that is needed in practice is knowledge of the characteristic equation for the invariant matrix M (see section 3.5). The characteristic equation is usually a simple consequence of the algebraic relations satisfied by the primitive invariants, and the eigenvalues λ_i are easily determined. The λ_i' s determine the projection operators $\mathbf{P}_i$, which in turn contain all relevant spectroscopic information: the rep dimension is given by $\mathrm{tr}\,\mathbf{P}_i$, and the casimirs, 6-$j$'s, crossing matrices, and recoupling coefficients (see chapter 5) are traces of various combinations of $\mathbf{P}_i$'s. All these numbers are *combinatoric*; they can often be interpreted as the number of different colorings of a graph, the number of singlets, and so on.

The invariance group is determined by considering infinitesimal transformations

$$G_a{}^b \simeq \delta^a_b + i\epsilon_i (T_i)^b_a\,.$$

The generators T_i are themselves clebsches, elements of the diagonalizing matrix C for the tensor product of the defining rep and its conjugate. They project out the adjoint rep and are constrained to satisfy the *invariance conditions* (2.2) for infinitesimal transformations (see section 4.4 and section 4.5):

$$(T_i)^{a'}_a h_{a'b\ldots}{}^{c\ldots} + (T_i)^{b'}_b h_{ab'\ldots}{}^{c\ldots} - (T_i)^c_{c'} h_{ab\ldots}{}^{c'\ldots} + \ldots = 0$$



(2.4)

As the corresponding projector operators are already known, we have an explicit construction of the symmetry group (at least infinitesimally — we will not consider discrete transformations).

If the primitive invariants are bilinear, the above procedure leads to the familiar tensor reps of classical groups. However, for trilinear or higher invariants the results are more surprising. In particular, all exceptional Lie groups emerge in a pattern of solutions which I will refer to as a *Magic Triangle.* The flow of the argument (see chapter 16) is schematically indicated in figure 2.1, with the arrows pointing to the primitive invariants that characterize a particular group. For example, E_7 primitives are a sesquilinear invariant $q\bar{q}$, a skew symmetric qp invariant, and a symmetric $qqqq$ (see chapter 20).

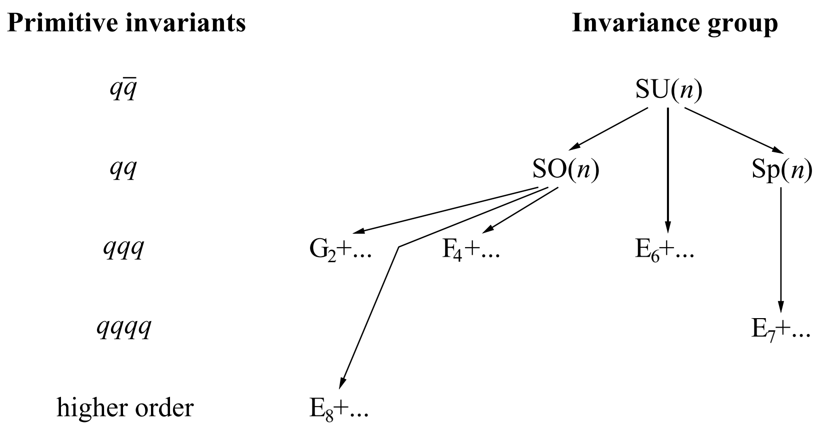


Figure 2.1 Additional primitive invariants induce chains of invariance subgroups.

The strategy is to introduce the invariants one by one, and study the way in which they split up previously irreducible reps. The first invariant might be realizable in many dimensions. When the next invariant is added (section 3.6), the group of invariance transformations of the first invariant splits into two subsets; those transformations that preserve the new invariant, and those that do not. Such decompositions yield Diophantine conditions on rep dimensions. These conditions are so constraining that they limit the possibilities to a few that can be easily identified.

To summarize: in the primitive invariants approach, all simple Lie groups, classical as well as exceptional, are constructed by (see chapter 21)

1. defining a symmetry group by specifying a list of *primitive invariants*;
2. using *primitiveness* and *invariance* conditions to obtain algebraic relations between primitive invariants;
3. constructing *invariant matrices* acting on tensor product spaces;
4. constructing *projection operators* for reduced rep from characteristic equations for invariant matrices.

Once the projection operators are known, all interesting spectroscopic numbers can be evaluated.

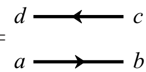
The foregoing run through the basic concepts was inevitably obscure. Perhaps working through the next two examples will make things clearer. The first example illustrates computations with classical groups. The second example is more interesting; it is a sketch of construction of irreducible reps of E_6.

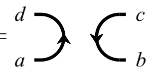
2.2 FIRST EXAMPLE: $SU(n)$

How do we describe the invariance group that preserves the norm of a complex vector? The *list of primitives* consists of a single primitive invariant,

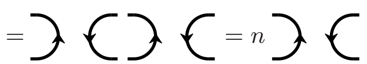
$$m(p,q) = \delta_b^a p^b q_a = \sum_{a=1}^{n} (p_a)^* q_a \, .$$

The Kronecker δ^a_b is the only primitive invariant tensor. We can immediately write down the two *invariant matrices* on the tensor product of the defining space and its conjugate,

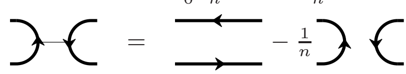
$$\text{identity}: \mathbf{1}^{a\,c}_{d,b} = \delta^a_b \delta^c_d =$$


$$\text{trace}: T^{a\,c}_{d,b} = \delta^a_d \delta^c_b =$$


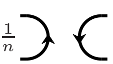
The *characteristic equation* for T written out in the matrix, tensor, and birdtrack notations is

$$T^2 = nT$$
$$T^{a\,f}_{d,e} T^{e\,c}_{f,b} = \delta^a_d \delta^f_e \delta^e_f \delta^c_b = n\, T^{a\,c}_{d,b}$$


Here we have used $\delta^e_e = n$, the dimension of the defining vector space. The roots are $\lambda_1 = 0$, $\lambda_2 = n$, and the corresponding *projection operators* are

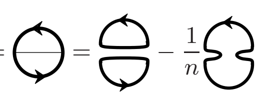
$$SU(n) \text{ adjoint rep:} \qquad \mathbf{P}_1 = \frac{T - n\mathbf{1}}{0-n} = \mathbf{1} - \frac{1}{n}T$$


(2.5)

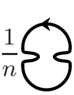
$$U(n) \text{ singlet:} \qquad \mathbf{P}_2 = \frac{T - 0\cdot\mathbf{1}}{n-0} = \frac{1}{n}T = \frac{1}{n}$$


Now we can evaluate any number associated with the $SU(n)$ adjoint rep, such as its dimension and various casimirs.

The *dimensions* of the two reps are computed by tracing the corresponding projection operators (see section 3.5):

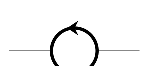
$$SU(n) \text{ adjoint: } d_1 = \mathrm{tr}\,\mathbf{P}_1 =$$


$$= \delta^b_b \delta^a_a - \frac{1}{n}\delta^b_a \delta^a_b = n^2 - 1$$

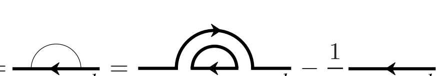
$$\text{singlet: } d_2 = \mathrm{tr}\,\mathbf{P}_2 = \frac{1}{n}$$


$$= 1\,.$$

To evaluate *casimirs*, we need to fix the overall normalization of the generators T_i of $SU(n)$. Our convention is to take

$$\delta_{ij} = \mathrm{tr}\, T_i T_j =$$


The value of the quadratic casimir for the defining rep is computed by substituting the adjoint projection operator:

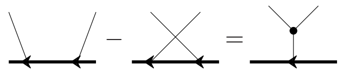
$$SU(n): \quad C_F \delta^b_a = (T_i T_i)^b_a =$$


$$= \frac{n^2-1}{n}$$

$$= \frac{n^2-1}{n}\,\delta^b_a\,. \qquad (2.6)$$

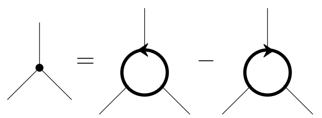
In order to evaluate the quadratic casimir for the adjoint rep, we need to replace the structure constants iC_{ijk} by their *Lie algebra* definition (see section 4.5)

$$T_iT_j - T_jT_i = iC_{ij\ell}T_\ell$$

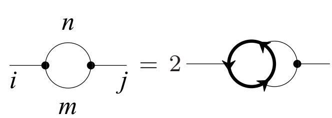


Tracing with T_k, we can express C_{ijk} in terms of the defining rep traces:

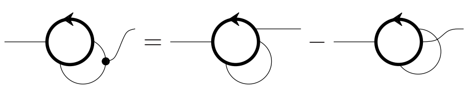
$$iC_{ijk} = \mathrm{tr}(T_iT_jT_k) - \mathrm{tr}(T_jT_iT_k)$$



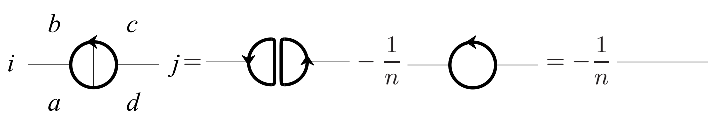
The adjoint quadratic casimir $C_{imn}C^{nmj}$ is now evaluated by first eliminating C_{ijk}'s in favor of the defining rep:

$\delta_{ij}C_A =$ 

The remaining C_{ijk} can be unwound by the Lie algebra commutator:

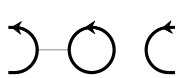
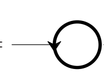


We have already evaluated the quadratic casimir (2.6) in the first term. The second term we evaluate by substituting the adjoint projection operator



$$\mathrm{tr}(T_iT_kT_jT_k) = (T_i)_a^{\ b}(P_1)_{d,\ b}^{a,\ c}(T_j)_c^{\ d} = (T_i)_a^{\ a}(T_j)_c^{\ c} - \frac{1}{n}(T_i)_a^{\ b}(T_j)_b^{\ a}\,.$$

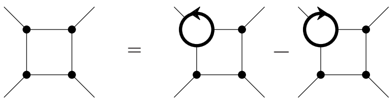
The $(T_i)_a^{\ a}(T_j)_c^{\ c}$ term vanishes by the tracelessness of T_i's. This is a consequence of the orthonormality of the two projection operators $\mathbf{P}_1$ and $\mathbf{P}_2$ in (2.5) (see (3.50)):

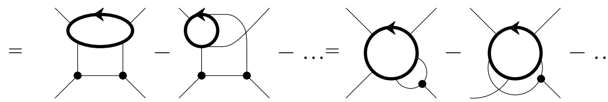
$0 = \mathbf{P}_1\mathbf{P}_2 =$  $\Rightarrow \mathrm{tr}\, T_i =$  $= 0\,.$

Combining the above expressions we finally obtain

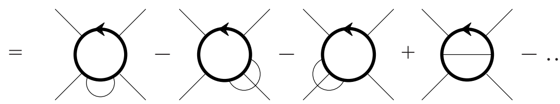
$$C_A = 2\left(\frac{n^2-1}{n} + \frac{1}{n}\right) = 2n\,.$$

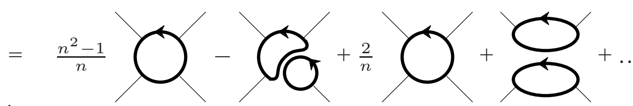
The problem (1.1) that started all this is evaluated the same way. First we relate the adjoint quartic casimir to the defining casimirs:



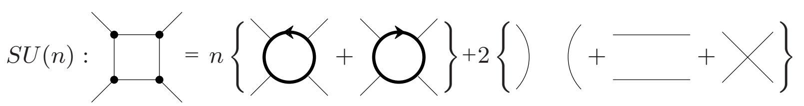


and so

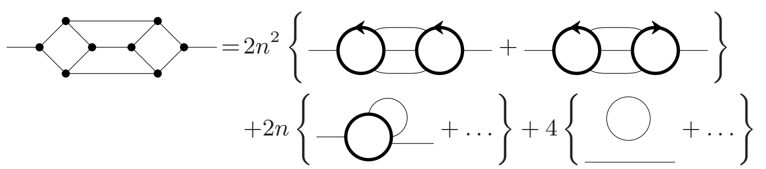




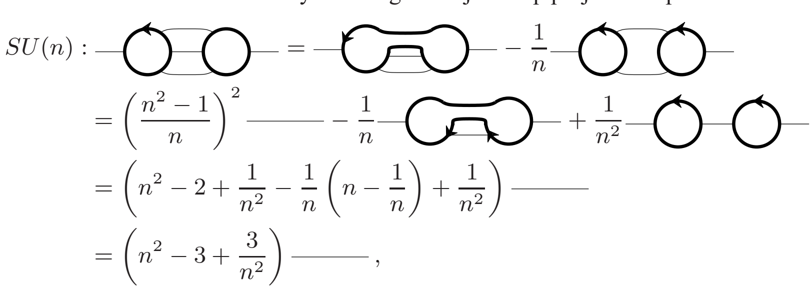
on. The result is



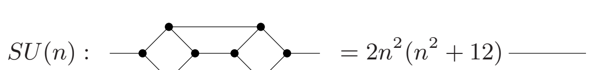
The diagram (1.1) is now reexpressed in terms of the defining rep casimirs:



The first two terms are evaluated by inserting the adjoint rep projection operators:



and the remaining terms have already been evaluated. Collecting everything together, we finally obtain



This example was unavoidably lengthy; the main point is that the evaluation is performed by a substitution algorithm and is easily automated. Any graph, no matter how complicated, is eventually reduced to a polynomial in traces of $\delta_a^a = n$, *i.e.*, the dimension of the defining rep.

2.3 SECOND EXAMPLE: E_6 FAMILY

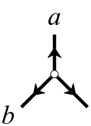
What invariance group preserves norms of complex vectors, as well as a symmetric cubic invariant,

$$D(p,q,r) = d^{abc} p_a q_b r_c = D(q,p,r) = D(p,r,q) \ ?$$

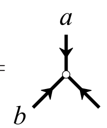
We analyze this case following the steps of the summary of section 2.1:

i) *Primitive invariant tensors*

$\delta_a^b = a \longrightarrow b\,,\quad d_{abc} =$

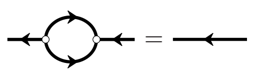


$,\quad d^{abc} = (d_{abc})^* =$



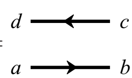
.

ii) *Primitiveness*. $d_{aef}d^{efb}$ must be proportional to δ_b^a, the only primitive 2-index tensor. We use this to fix the overall normalization of d_{abc}'s:

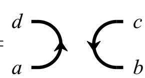


iii) *Invariant hermitian matrices*. We shall construct here the adjoint rep projection operator on the tensor product space of the defining rep and its conjugate. All invariant matrices on this space are

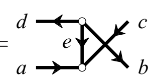
$\delta_b^a \delta_d^c =$



$,\quad \delta_d^a \delta_b^c =$



$,\quad d^{ace} d_{ebd} =$

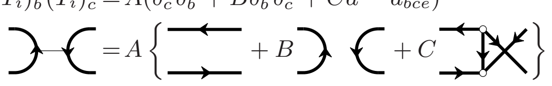


.

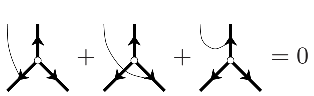
They are hermitian in the sense of being invariant under complex conjugation and transposition of indices (see (3.21)). The crucial step in constructing this basis is the primitiveness assumption: 4-leg diagrams containing loops are not primitive (see section 3.3).

The adjoint rep is always contained in the decomposition of $V \otimes \bar{V} \rightarrow V \otimes \bar{V}$ into (ir)reducible reps, so the adjoint projection operator must be expressible in terms of the 4-index invariant tensors listed above:

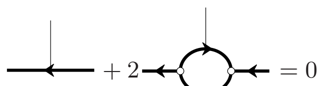
$$(T_i)^a_b (T_i)^d_c = A(\delta^a_c \delta^d_b + B \delta^a_b \delta^d_c + C d^{ade} d_{bce})$$



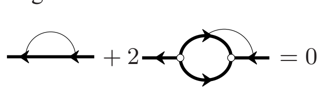
iv) *Invariance*. The cubic invariant tensor satisfies (2.4)



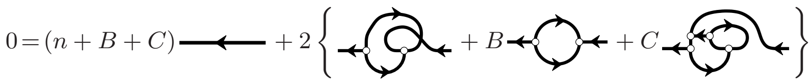
Contracting with d^{abc}, we obtain



Contracting next with $(T_i)^b_a$, we get an invariance condition on the adjoint projection operator,



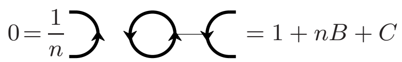
Substituting the adjoint projection operator yields the first relation between the coefficients in its expansion:



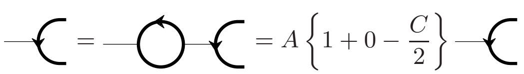
$$0 = B + C + \frac{n+2}{3} \,.$$

v) *The projection operators* should be orthonormal, $\mathbf{P}_\mu \mathbf{P}_\sigma = \mathbf{P}_\mu \delta_{\mu\sigma}$. The adjoint projection operator is orthogonal to (2.5), the singlet projection operator $\mathbf{P}_2$. This yields the second relation on the coefficients:

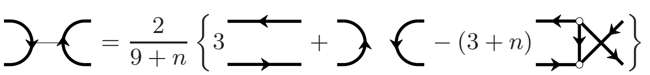
$$0 = \mathbf{P}_2 \mathbf{P}_A$$



Finally, the overall normalization factor A is fixed by $\mathbf{P}_A \mathbf{P}_A = \mathbf{P}_A$:



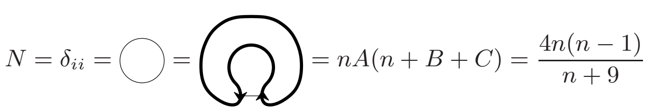
Combining the above three relations, we obtain the adjoint projection operator for the invariance group of a symmetric cubic invariant:



(2.7)

The corresponding *characteristic equation*, mentioned in the point iv) of the summary of section 2.1, is given in (18.10).

The dimension of the adjoint rep is obtained by tracing the projection operator:



This *Diophantine condition* is satisfied by a small family of invariance groups, discussed in chapter 18. The most interesting member of this family is the exceptional Lie group E_6, with $n = 27$ and $N = 78$.

The solution to problem (1.1) requires further computation, but for exceptional Lie groups the answer, given in table 7.4, turns out to be surprisingly simple. The part of the 4-loop that cannot be simplified by Lie algebra manipulations vanishes identically for all exceptional Lie groups (chapter 17.