

A PRINCETON LIFESAVER STUDY GUIDE

$$\lim_{x \rightarrow \infty} \sin(1/x) = 0$$

$$P_N(x) = \sum_{n=0}^N \frac{f^{(n)}(a)}{n!} (x-a)^n$$
$$y = \sin(x)$$

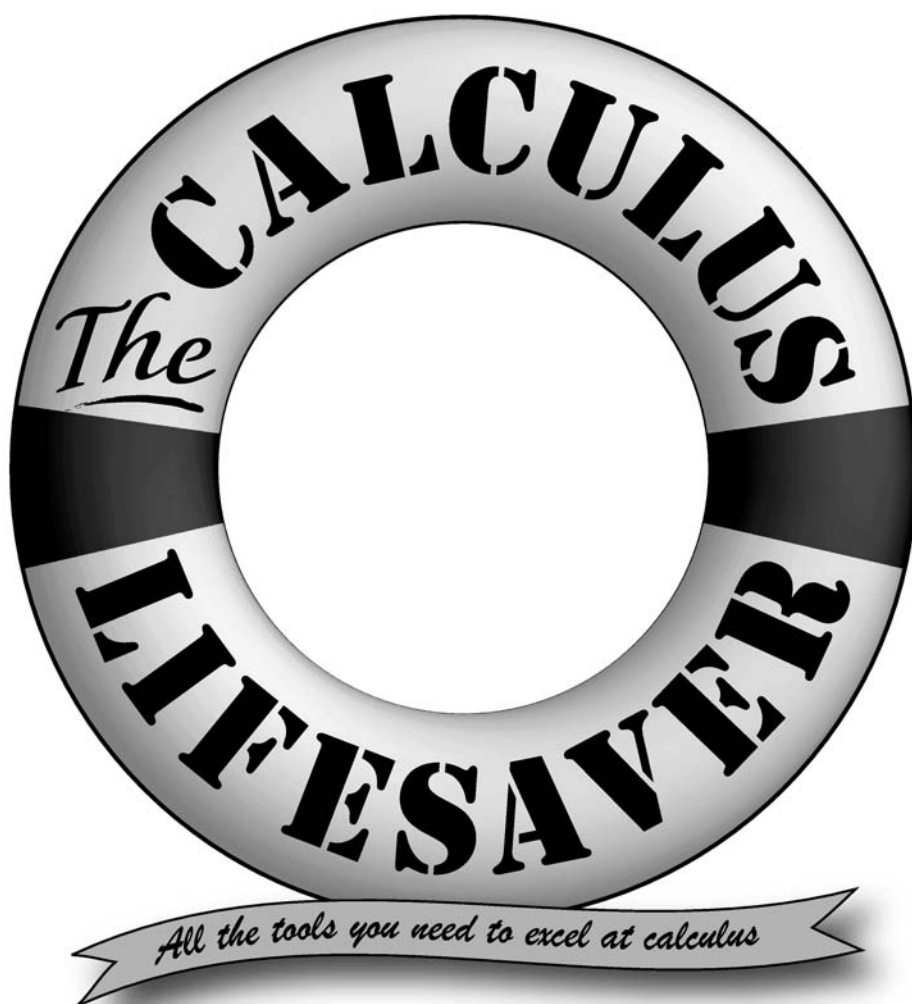
# THE CALCULUS LIFESAVER

*All the tools you need to excel at calculus*

ADRIAN BANNER

# **THE CALCULUS LIFESAVER**





**ADRIAN BANNER**

PRINCETON UNIVERSITY PRESS  
Princeton and Oxford

Copyright © 2007 by Princeton University Press  
Published by Princeton University Press, 41 William Street, Princeton,  
New Jersey 08540  
in the United Kingdom: Princeton University Press, 3 Market Place, Woodstock,  
Oxfordshire OX20 1SY

All Rights Reserved

Library of Congress Control Number: 2006939343

ISBN-13: 978-0-691-13153-5 (cloth)  
ISBN-10: 0-691-13153-8 (cloth)

ISBN-13: 978-0-691-13088-0 (paper)  
ISBN-10: 0-691-13088-4 (paper)

British Library Cataloging-in-Publication Data is available

This book has been composed in Times Roman  
The publisher would like to acknowledge the author of this volume for  
providing the camera-ready copy from which this book was printed

Printed on acid-free paper. ∞

[pup.princeton.edu](http://pup.princeton.edu)

Printed in the United States of America

1 3 5 7 9 10 8 6 4 2

To Yarry



# CONTENTS

---

<b>Welcome</b>	<b>xviii</b>
How to Use This Book to Study for an Exam	xix
Two all-purpose study tips	xx
Key sections for exam review (by topic)	xx
<b>Acknowledgments</b>	<b>xxiii</b>
<b>1 Functions, Graphs, and Lines</b>	<b>1</b>
1.1 Functions	1
1.1.1 Interval notation	3
1.1.2 Finding the domain	4
1.1.3 Finding the range using the graph	5
1.1.4 The vertical line test	6
1.2 Inverse Functions	7
1.2.1 The horizontal line test	8
1.2.2 Finding the inverse	9
1.2.3 Restricting the domain	9
1.2.4 Inverses of inverse functions	11
1.3 Composition of Functions	11
1.4 Odd and Even Functions	14
1.5 Graphs of Linear Functions	17
1.6 Common Functions and Graphs	19
<b>2 Review of Trigonometry</b>	<b>25</b>
2.1 The Basics	25
2.2 Extending the Domain of Trig Functions	28
2.2.1 The ASTC method	31
2.2.2 Trig functions outside $[0, 2\pi]$	33
2.3 The Graphs of Trig Functions	35
2.4 Trig Identities	39



<b>3</b>	<b>Introduction to Limits</b>	<b>41</b>
3.1	Limits: The Basic Idea	41
3.2	Left-Hand and Right-Hand Limits	43
3.3	When the Limit Does Not Exist	45
3.4	Limits at $\infty$ and $-\infty$	47
3.4.1	Large numbers and small numbers	48
3.5	Two Common Misconceptions about Asymptotes	50
3.6	The Sandwich Principle	51
3.7	Summary of Basic Types of Limits	54
<b>4</b>	<b>How to Solve Limit Problems Involving Polynomials</b>	<b>57</b>
4.1	Limits Involving Rational Functions as $x \rightarrow a$	57
4.2	Limits Involving Square Roots as $x \rightarrow a$	61
4.3	Limits Involving Rational Functions as $x \rightarrow \infty$	61
4.3.1	Method and examples	64
4.4	Limits Involving Poly-type Functions as $x \rightarrow \infty$	66
4.5	Limits Involving Rational Functions as $x \rightarrow -\infty$	70
4.6	Limits Involving Absolute Values	72
<b>5</b>	<b>Continuity and Differentiability</b>	<b>75</b>
5.1	Continuity	75
5.1.1	Continuity at a point	76
5.1.2	Continuity on an interval	77
5.1.3	Examples of continuous functions	77
5.1.4	The Intermediate Value Theorem	80
5.1.5	A harder IVT example	82
5.1.6	Maxima and minima of continuous functions	82
5.2	Differentiability	84
5.2.1	Average speed	84
5.2.2	Displacement and velocity	85
5.2.3	Instantaneous velocity	86
5.2.4	The graphical interpretation of velocity	87
5.2.5	Tangent lines	88
5.2.6	The derivative function	90
5.2.7	The derivative as a limiting ratio	91
5.2.8	The derivative of linear functions	93
5.2.9	Second and higher-order derivatives	94
5.2.10	When the derivative does not exist	94
5.2.11	Differentiability and continuity	96
<b>6</b>	<b>How to Solve Differentiation Problems</b>	<b>99</b>
6.1	Finding Derivatives Using the Definition	99
6.2	Finding Derivatives (the Nice Way)	102
6.2.1	Constant multiples of functions	103

6.2.2	Sums and differences of functions	103
6.2.3	Products of functions via the product rule	104
6.2.4	Quotients of functions via the quotient rule	105
6.2.5	Composition of functions via the chain rule	107
6.2.6	A nasty example	109
6.2.7	Justification of the product rule and the chain rule	111
6.3	Finding the Equation of a Tangent Line	114
6.4	Velocity and Acceleration	114
6.4.1	Constant negative acceleration	115
6.5	Limits Which Are Derivatives in Disguise	117
6.6	Derivatives of Piecewise-Defined Functions	119
6.7	Sketching Derivative Graphs Directly	123
<b>7</b>	<b>Trig Limits and Derivatives</b>	<b>127</b>
7.1	Limits Involving Trig Functions	127
7.1.1	The small case	128
7.1.2	Solving problems—the small case	129
7.1.3	The large case	134
7.1.4	The “other” case	137
7.1.5	Proof of an important limit	137
7.2	Derivatives Involving Trig Functions	141
7.2.1	Examples of differentiating trig functions	143
7.2.2	Simple harmonic motion	145
7.2.3	A curious function	146
<b>8</b>	<b>Implicit Differentiation and Related Rates</b>	<b>149</b>
8.1	Implicit Differentiation	149
8.1.1	Techniques and examples	150
8.1.2	Finding the second derivative implicitly	154
8.2	Related Rates	156
8.2.1	A simple example	157
8.2.2	A slightly harder example	159
8.2.3	A much harder example	160
8.2.4	A really hard example	162
<b>9</b>	<b>Exponentials and Logarithms</b>	<b>167</b>
9.1	The Basics	167
9.1.1	Review of exponentials	167
9.1.2	Review of logarithms	168
9.1.3	Logarithms, exponentials, and inverses	169
9.1.4	Log rules	170
9.2	Definition of $e$	173
9.2.1	A question about compound interest	173
9.2.2	The answer to our question	173
9.2.3	More about $e$ and logs	175
9.3	Differentiation of Logs and Exponentials	177

9.3.1	Examples of differentiating exponentials and logs	179
9.4	How to Solve Limit Problems Involving Exponentials or Logs	180
9.4.1	Limits involving the definition of $e$	181
9.4.2	Behavior of exponentials near 0	182
9.4.3	Behavior of logarithms near 1	183
9.4.4	Behavior of exponentials near $\infty$ or $-\infty$	184
9.4.5	Behavior of logs near $\infty$	187
9.4.6	Behavior of logs near 0	188
9.5	Logarithmic Differentiation	189
9.5.1	The derivative of $x^a$	192
9.6	Exponential Growth and Decay	193
9.6.1	Exponential growth	194
9.6.2	Exponential decay	195
9.7	Hyperbolic Functions	198
<b>10</b>	<b>Inverse Functions and Inverse Trig Functions</b>	<b>201</b>
10.1	The Derivative and Inverse Functions	201
10.1.1	Using the derivative to show that an inverse exists	201
10.1.2	Derivatives and inverse functions: what can go wrong	203
10.1.3	Finding the derivative of an inverse function	204
10.1.4	A big example	206
10.2	Inverse Trig Functions	208
10.2.1	Inverse sine	208
10.2.2	Inverse cosine	211
10.2.3	Inverse tangent	213
10.2.4	Inverse secant	216
10.2.5	Inverse cosecant and inverse cotangent	217
10.2.6	Computing inverse trig functions	218
10.3	Inverse Hyperbolic Functions	220
10.3.1	The rest of the inverse hyperbolic functions	222
<b>11</b>	<b>The Derivative and Graphs</b>	<b>225</b>
11.1	Extrema of Functions	225
11.1.1	Global and local extrema	225
11.1.2	The Extreme Value Theorem	227
11.1.3	How to find global maxima and minima	228
11.2	Rolle's Theorem	230
11.3	The Mean Value Theorem	233
11.3.1	Consequences of the Mean Value Theorem	235
11.4	The Second Derivative and Graphs	237
11.4.1	More about points of inflection	238
11.5	Classifying Points Where the Derivative Vanishes	239
11.5.1	Using the first derivative	240
11.5.2	Using the second derivative	242

<b>12 Sketching Graphs</b>	<b>245</b>
12.1 How to Construct a Table of Signs	245
12.1.1 Making a table of signs for the derivative	247
12.1.2 Making a table of signs for the second derivative	248
12.2 The Big Method	250
12.3 Examples	252
12.3.1 An example without using derivatives	252
12.3.2 The full method: example 1	254
12.3.3 The full method: example 2	256
12.3.4 The full method: example 3	259
12.3.5 The full method: example 4	262
<b>13 Optimization and Linearization</b>	<b>267</b>
13.1 Optimization	267
13.1.1 An easy optimization example	267
13.1.2 Optimization problems: the general method	269
13.1.3 An optimization example	269
13.1.4 Another optimization example	271
13.1.5 Using implicit differentiation in optimization	274
13.1.6 A difficult optimization example	275
13.2 Linearization	278
13.2.1 Linearization in general	279
13.2.2 The differential	281
13.2.3 Linearization summary and examples	283
13.2.4 The error in our approximation	285
13.3 Newton's Method	287
<b>14 L'Hôpital's Rule and Overview of Limits</b>	<b>293</b>
14.1 L'Hôpital's Rule	293
14.1.1 Type <b>A</b> : $0/0$ case	294
14.1.2 Type <b>A</b> : $\pm\infty/\pm\infty$ case	296
14.1.3 Type <b>B1</b> ( $\infty - \infty$ )	298
14.1.4 Type <b>B2</b> ( $0 \times \pm\infty$ )	299
14.1.5 Type <b>C</b> ( $1^{\pm\infty}$ , $0^0$ , or $\infty^0$ )	301
14.1.6 Summary of l'Hôpital's Rule types	302
14.2 Overview of Limits	303
<b>15 Introduction to Integration</b>	<b>307</b>
15.1 Sigma Notation	307
15.1.1 A nice sum	310
15.1.2 Telescoping series	311
15.2 Displacement and Area	314
15.2.1 Three simple cases	314
15.2.2 A more general journey	317
15.2.3 Signed area	319
15.2.4 Continuous velocity	320

15.2.5	Two special approximations	323
<b>16</b>	<b>Definite Integrals</b>	<b>325</b>
16.1	The Basic Idea	325
16.1.1	Some easy examples	327
16.2	Definition of the Definite Integral	330
16.2.1	An example of using the definition	331
16.3	Properties of Definite Integrals	334
16.4	Finding Areas	339
16.4.1	Finding the unsigned area	339
16.4.2	Finding the area between two curves	342
16.4.3	Finding the area between a curve and the $y$ -axis	344
16.5	Estimating Integrals	346
16.5.1	A simple type of estimation	347
16.6	Averages and the Mean Value Theorem for Integrals	350
16.6.1	The Mean Value Theorem for integrals	351
16.7	A Nonintegrable Function	353
<b>17</b>	<b>The Fundamental Theorems of Calculus</b>	<b>355</b>
17.1	Functions Based on Integrals of Other Functions	355
17.2	The First Fundamental Theorem	358
17.2.1	Introduction to antiderivatives	361
17.3	The Second Fundamental Theorem	362
17.4	Indefinite Integrals	364
17.5	How to Solve Problems: The First Fundamental Theorem	366
17.5.1	Variation 1: variable left-hand limit of integration	367
17.5.2	Variation 2: one tricky limit of integration	367
17.5.3	Variation 3: two tricky limits of integration	369
17.5.4	Variation 4: limit is a derivative in disguise	370
17.6	How to Solve Problems: The Second Fundamental Theorem	371
17.6.1	Finding indefinite integrals	371
17.6.2	Finding definite integrals	374
17.6.3	Unsigned areas and absolute values	376
17.7	A Technical Point	380
17.8	Proof of the First Fundamental Theorem	381
<b>18</b>	<b>Techniques of Integration, Part One</b>	<b>383</b>
18.1	Substitution	383
18.1.1	Substitution and definite integrals	386
18.1.2	How to decide what to substitute	389
18.1.3	Theoretical justification of the substitution method	392
18.2	Integration by Parts	393
18.2.1	Some variations	394
18.3	Partial Fractions	397

18.3.1	The algebra of partial fractions	398
18.3.2	Integrating the pieces	401
18.3.3	The method and a big example	404
<b>19</b>	<b>Techniques of Integration, Part Two</b>	<b>409</b>
19.1	Integrals Involving Trig Identities	409
19.2	Integrals Involving Powers of Trig Functions	413
19.2.1	Powers of sin and/or cos	413
19.2.2	Powers of tan	415
19.2.3	Powers of sec	416
19.2.4	Powers of cot	418
19.2.5	Powers of csc	418
19.2.6	Reduction formulas	419
19.3	Integrals Involving Trig Substitutions	421
19.3.1	Type 1: $\sqrt{a^2 - x^2}$	421
19.3.2	Type 2: $\sqrt{x^2 + a^2}$	423
19.3.3	Type 3: $\sqrt{x^2 - a^2}$	424
19.3.4	Completing the square and trig substitutions	426
19.3.5	Summary of trig substitutions	426
19.3.6	Technicalities of square roots and trig substitutions	427
19.4	Overview of Techniques of Integration	429
<b>20</b>	<b>Improper Integrals: Basic Concepts</b>	<b>431</b>
20.1	Convergence and Divergence	431
20.1.1	Some examples of improper integrals	433
20.1.2	Other blow-up points	435
20.2	Integrals over Unbounded Regions	437
20.3	The Comparison Test (Theory)	439
20.4	The Limit Comparison Test (Theory)	441
20.4.1	Functions asymptotic to each other	441
20.4.2	The statement of the test	443
20.5	The $p$ -test (Theory)	444
20.6	The Absolute Convergence Test	447
<b>21</b>	<b>Improper Integrals: How to Solve Problems</b>	<b>451</b>
21.1	How to Get Started	451
21.1.1	Splitting up the integral	452
21.1.2	How to deal with negative function values	453
21.2	Summary of Integral Tests	454
21.3	Behavior of Common Functions near $\infty$ and $-\infty$	456
21.3.1	Polynomials and poly-type functions near $\infty$ and $-\infty$	456
21.3.2	Trig functions near $\infty$ and $-\infty$	459
21.3.3	Exponentials near $\infty$ and $-\infty$	461
21.3.4	Logarithms near $\infty$	465
21.4	Behavior of Common Functions near 0	469

21.4.1	Polynomials and poly-type functions near 0	469
21.4.2	Trig functions near 0	470
21.4.3	Exponentials near 0	472
21.4.4	Logarithms near 0	473
21.4.5	The behavior of more general functions near 0	474
21.5	How to Deal with Problem Spots Not at 0 or $\infty$	475
<b>22</b>	<b>Sequences and Series: Basic Concepts</b>	<b>477</b>
22.1	Convergence and Divergence of Sequences	477
22.1.1	The connection between sequences and functions	478
22.1.2	Two important sequences	480
22.2	Convergence and Divergence of Series	481
22.2.1	Geometric series (theory)	484
22.3	The $n$ th Term Test (Theory)	486
22.4	Properties of Both Infinite Series and Improper Integrals	487
22.4.1	The comparison test (theory)	487
22.4.2	The limit comparison test (theory)	488
22.4.3	The $p$ -test (theory)	489
22.4.4	The absolute convergence test	490
22.5	New Tests for Series	491
22.5.1	The ratio test (theory)	492
22.5.2	The root test (theory)	493
22.5.3	The integral test (theory)	494
22.5.4	The alternating series test (theory)	497
<b>23</b>	<b>How to Solve Series Problems</b>	<b>501</b>
23.1	How to Evaluate Geometric Series	502
23.2	How to Use the $n$ th Term Test	503
23.3	How to Use the Ratio Test	504
23.4	How to Use the Root Test	508
23.5	How to Use the Integral Test	509
23.6	Comparison Test, Limit Comparison Test, and $p$ -test	510
23.7	How to Deal with Series with Negative Terms	515
<b>24</b>	<b>Taylor Polynomials, Taylor Series, and Power Series</b>	<b>519</b>
24.1	Approximations and Taylor Polynomials	519
24.1.1	Linearization revisited	520
24.1.2	Quadratic approximations	521
24.1.3	Higher-degree approximations	522
24.1.4	Taylor's Theorem	523
24.2	Power Series and Taylor Series	526
24.2.1	Power series in general	527
24.2.2	Taylor series and Maclaurin series	529
24.2.3	Convergence of Taylor series	530
24.3	A Useful Limit	534

<b>25 How to Solve Estimation Problems</b>	<b>535</b>
25.1 Summary of Taylor Polynomials and Series	535
25.2 Finding Taylor Polynomials and Series	537
25.3 Estimation Problems Using the Error Term	540
25.3.1 First example	541
25.3.2 Second example	543
25.3.3 Third example	544
25.3.4 Fourth example	546
25.3.5 Fifth example	547
25.3.6 General techniques for estimating the error term	548
25.4 Another Technique for Estimating the Error	548
<b>26 Taylor and Power Series: How to Solve Problems</b>	<b>551</b>
26.1 Convergence of Power Series	551
26.1.1 Radius of convergence	551
26.1.2 How to find the radius and region of convergence	554
26.2 Getting New Taylor Series from Old Ones	558
26.2.1 Substitution and Taylor series	560
26.2.2 Differentiating Taylor series	562
26.2.3 Integrating Taylor series	563
26.2.4 Adding and subtracting Taylor series	565
26.2.5 Multiplying Taylor series	566
26.2.6 Dividing Taylor series	567
26.3 Using Power and Taylor Series to Find Derivatives	568
26.4 Using Maclaurin Series to Find Limits	570
<b>27 Parametric Equations and Polar Coordinates</b>	<b>575</b>
27.1 Parametric Equations	575
27.1.1 Derivatives of parametric equations	578
27.2 Polar Coordinates	581
27.2.1 Converting to and from polar coordinates	582
27.2.2 Sketching curves in polar coordinates	585
27.2.3 Finding tangents to polar curves	590
27.2.4 Finding areas enclosed by polar curves	591
<b>28 Complex Numbers</b>	<b>595</b>
28.1 The Basics	595
28.1.1 Complex exponentials	598
28.2 The Complex Plane	599
28.2.1 Converting to and from polar form	601
28.3 Taking Large Powers of Complex Numbers	603
28.4 Solving $z^n = w$	604
28.4.1 Some variations	608
28.5 Solving $e^z = w$	610
28.6 Some Trigonometric Series	612



28.7 Euler's Identity and Power Series	615
<b>29 Volumes, Arc Lengths, and Surface Areas</b>	<b>617</b>
29.1 Volumes of Solids of Revolution	617
29.1.1 The disc method	619
29.1.2 The shell method	620
29.1.3 Summary ... and variations	622
29.1.4 Variation 1: regions between a curve and the $y$ -axis	623
29.1.5 Variation 2: regions between two curves	625
29.1.6 Variation 3: axes parallel to the coordinate axes	628
29.2 Volumes of General Solids	631
29.3 Arc Lengths	637
29.3.1 Parametrization and speed	639
29.4 Surface Areas of Solids of Revolution	640
<b>30 Differential Equations</b>	<b>645</b>
30.1 Introduction to Differential Equations	645
30.2 Separable First-order Differential Equations	646
30.3 First-order Linear Equations	648
30.3.1 Why the integrating factor works	652
30.4 Constant-coefficient Differential Equations	653
30.4.1 Solving first-order homogeneous equations	654
30.4.2 Solving second-order homogeneous equations	654
30.4.3 Why the characteristic quadratic method works	655
30.4.4 Nonhomogeneous equations and particular solutions	656
30.4.5 Finding a particular solution	658
30.4.6 Examples of finding particular solutions	660
30.4.7 Resolving conflicts between $y_P$ and $y_H$	662
30.4.8 Initial value problems (constant-coefficient linear)	663
30.5 Modeling Using Differential Equations	665
<b>Appendix A Limits and Proofs</b>	<b>669</b>
A.1 Formal Definition of a Limit	669
A.1.1 A little game	670
A.1.2 The actual definition	672
A.1.3 Examples of using the definition	672
A.2 Making New Limits from Old Ones	674
A.2.1 Sums and differences of limits—proofs	674
A.2.2 Products of limits—proof	675
A.2.3 Quotients of limits—proof	676
A.2.4 The sandwich principle—proof	678
A.3 Other Varieties of Limits	678
A.3.1 Infinite limits	679
A.3.2 Left-hand and right-hand limits	680
A.3.3 Limits at $\infty$ and $-\infty$	680

A.3.4	Two examples involving trig	682
A.4	Continuity and Limits	684
A.4.1	Composition of continuous functions	684
A.4.2	Proof of the Intermediate Value Theorem	686
A.4.3	Proof of the Max-Min Theorem	687
A.5	Exponentials and Logarithms Revisited	689
A.6	Differentiation and Limits	691
A.6.1	Constant multiples of functions	691
A.6.2	Sums and differences of functions	691
A.6.3	Proof of the product rule	692
A.6.4	Proof of the quotient rule	693
A.6.5	Proof of the chain rule	693
A.6.6	Proof of the Extreme Value Theorem	694
A.6.7	Proof of Rolle's Theorem	695
A.6.8	Proof of the Mean Value Theorem	695
A.6.9	The error in linearization	696
A.6.10	Derivatives of piecewise-defined functions	697
A.6.11	Proof of l'Hôpital's Rule	698
A.7	Proof of the Taylor Approximation Theorem	700
<b>Appendix B</b>	<b>Estimating Integrals</b>	<b>703</b>
B.1	Estimating Integrals Using Strips	703
B.1.1	Evenly spaced partitions	705
B.2	The Trapezoidal Rule	706
B.3	Simpson's Rule	709
B.3.1	Proof of Simpson's rule	710
B.4	The Error in Our Approximations	711
B.4.1	Examples of estimating the error	712
B.4.2	Proof of an error term inequality	714
<b>List of Symbols</b>		<b>717</b>
<b>Index</b>		<b>719</b>

# WELCOME!

---

This book is designed to help you learn the major concepts of single-variable calculus, while also concentrating on problem-solving techniques. Whether this is your first exposure to calculus, or you are studying for a test, or you've already taken calculus and want to refresh your memory, I hope that this book will be a useful resource.

The inspiration for this book came from my students at Princeton University. Over the past few years, they have found early drafts to be helpful as a study guide in conjunction with lectures, review sessions and their textbook. Here are some of the questions that they've asked along the way, which you might also be inclined to ask:

- **Why is this book so long?** I assume that you, the reader, are motivated to the extent that you'd like to master the subject. Not wanting to get by with the bare minimum, you're prepared to put in some time and effort reading—and understanding—these detailed explanations.
- **What do I need to know before I start reading?** You need to know some basic algebra and how to solve simple equations. Most of the precalculus you need is covered in the first two chapters.
- **Help! The final is in one week, and I don't know anything! Where do I start?** The next three pages describe how to use this book to study for an exam.
- **Where are all the worked solutions to examples? All I see is lots of words with a few equations.** Looking at a worked solution doesn't tell you how to think of it in the first place. So, I usually try to give a sort of "inner monologue"—what should be going through your head as you try to solve the problem. You end up with all the pieces of the solution, but you still need to write it up properly. My advice is to read the solution, then come back later and try to work it out again by yourself.
- **Where are the proofs of the theorems?** Most of the theorems in this book are justified in some way. More formal proofs can be found in Appendix A.
- **The topics are out of order! What do I do?** There's no standard order for learning calculus. The order I have chosen works, but you might have to search the table of contents to find the topics you need and ignore

the rest for now. I may also have missed out some topics too—why not try emailing me at [adrian@calclifesaver.com](mailto:adrian@calclifesaver.com) and you never know, I just might write an extra section or chapter for you (and for the next edition, if there is one!).

- **Some of the methods you use are different from the methods I learned. Who is right—my instructor or you?** Hopefully we're both right! If in doubt, ask your instructor what's acceptable.
- **Where's all the calculus history and fun facts in the margins?** Look, there's a little bit of history in this book, but let's not get too distracted here. After you get this stuff down, read a book on the history of calculus. It's interesting stuff, and deserves more attention than a couple of sentences here and there.
- **Could my school use this book as a textbook?** Paired with a good collection of exercises, this book could function as a textbook, as well as being a study guide. Your instructor might also find the book useful to help prepare lectures, particularly in regard to problem-solving techniques.
- **What's with these videos?** You can find videos of a year's supply of my review sessions, which reference a lot (but not all!) of the sections and examples from this book, at this website:

[www.calclifesaver.com](http://www.calclifesaver.com)

## How to Use This Book to Study for an Exam

There's a good chance you have a test or exam coming up soon. I am sympathetic to your plight: you don't have time to read the whole book! There's a table on the next page that identifies the main sections that will help you to review for the exam. Also, throughout the book, the following icons appear in the margin to allow you quickly to identify what's relevant:



- A worked-out example begins on this line.



- Here's something really important.



- You should try this yourself.



- Beware: this part of the text is mostly for interest. If time is limited, skip to the next section.

Also, some important formulas or theorems have boxes around them: learn these well.

## Two all-purpose study tips

- Write out your own summary of all the important points and formulas to memorize. Math isn't about memorization, but there are some key formulas and methods that you should have at your fingertips. The act of making the summary is often enough to solidify your understanding. This is the main reason why I don't summarize the important points at the end of a chapter: it's much more valuable if you do it yourself.
- Try to get your hands on similar exams—maybe your school makes previous years' finals available, for example—and take these exams under proper conditions. That means no breaks, no food, no books, no phone calls, no emails, no messaging, and so on. Then see if you can get a solution key and grade it, or ask someone (nicely!) to grade it for you.

You'll be on your way to that A if you do both of these things.

## Key sections for exam review (by topic)

Topic	Subtopic	Section(s)
Precalculus	Lines	1.5
	Other common graphs	1.6
	Trig basics	2.1
	Trig with angles outside $[0, \pi/2]$	2.2
	Trig graphs	2.3
	Trig identities	2.4
	Exponentials and logs	9.1
Limits	Sandwich principle	3.6
	Polynomial limits	all of Chapter 4
	Derivatives in disguise	6.5
	Trig limits	7.1 (skip 7.1.5)
	Exponential and log limits	9.4
	L'Hôpital's Rule	14.1
	<b>Overview of limit problems</b>	14.2
Continuity	Definition	5.1
	Intermediate Value Theorem	5.1.4
Differentiation	Definition	6.1
	Rules (e.g., product/quotient/chain rule)	6.2
	Finding tangent lines	6.3
	Derivatives of piecewise-defined functions	6.6
	Sketching the derivative	6.7
	Trig functions	7.2, 7.2.1
	Implicit differentiation	8.1
	Exponentials and logs	9.3
	Logarithmic differentiation	9.5
	Hyperbolic functions	9.7
	Inverse functions in general	10.1
	Inverse trig functions	10.2
	Inverse hyperbolic functions	10.3
	Differentiating definite integrals	17.5

Topic	Subtopic	Section(s)
Applications of differentiation	Related rates	8.2
	Exponential growth and decay	9.6
	Finding global maxima and minima	11.1.3
	Rolle's Theorem/Mean Value Theorem	11.2, 11.3
	Classifying critical points	11.5, 12.1.1
	Finding inflection points	11.4, 12.1.2
	Sketching graphs	12.2, 12.3
	Optimization	13.1
	Linearization/differentials	13.2
	Newton's method	13.3
Integration	Definition	16.2 (skip 16.2.1)
	Basic properties	16.3
	Finding areas	16.4
	Estimating definite integrals	16.5, Appendix B
	Average values/Mean Value Theorem	16.6
	Basic examples	17.4, 17.6
	Substitution	18.1
	Integration by parts	18.2
	Partial fractions	18.3
	Trig integrals	19.1, 19.2
	Trig substitutions	19.3 (skip 19.3.6)
	<b>Overview of integration techniques</b>	19.4
Motion	Velocity and acceleration	6.4
	Constant acceleration	6.4.1
	Simple harmonic motion	7.2.2
	Finding displacements	16.1.1
Improper integrals	Basics	20.1, 20.2
	Problem-solving techniques	all of Chapter 21
Infinite series	Basics	22.1.2, 22.2
	Problem-solving techniques	all of Chapter 23
Taylor series and power series	Estimation and error estimates	all of Chapter 25
	Power/Taylor series problems	all of Chapter 26
Differential equations	Separable first-order	30.2
	First-order linear	30.3
	Constant coefficients	30.4
	Modeling	30.5
Miscellaneous topics	Parametric equations	27.1
	Polar coordinates	27.2
	Complex numbers	28.1–28.5
	Volumes	29.1, 29.2
	Arc lengths	29.3
	Surface areas	29.4

Unless specified otherwise, the Section(s) column includes all subsections; for example, 6.2 includes 6.2.1 through 6.2.7.



# ACKNOWLEDGMENTS

---

There are many people I'd like to thank for supporting and helping me during the writing of this book. My students have been a source of education, entertainment, and delight; I have benefited greatly from their suggestions. I'd particularly like to thank my editor Vickie Kearn, my production editor Linny Schenck, and my designer Lorraine Doneker for all their help and support, and also Gerald Folland for his numerous excellent suggestions which have greatly improved this book. Ed Nelson, Maria Klawe, Christine Miranda, Lior Braustein, Emily Sands, Jamaal Clue, Alison Ralph, Marcher Thompson, Ioannis Avramides, Kristen Molloy, Dave Uppal, Nwanneka Onvekwusi, Ellen Zuckerman, Charles MacCluer, and Gary Slezak brought errors and omissions to my attention.

The following faculty and staff members of the Princeton University Mathematics Department have been very supportive: Eli Stein, Simon Kochen, Matthew Ferszt, and Scott Kenney. Thank you also to all of my colleagues at INTECH for their support, in particular Bob Fernholz, Camm Maguire, Marie D'Albero, and Vassilios Papathanakos, who made some excellent last-minute suggestions. I'd also like to pay tribute to my 11th- and 12th-grade math teacher, William Pender, who is surely the best calculus teacher in the world. Many of the methods in this book were inspired by his teaching. I hope he forgives me for not putting arrows on my curves, not labeling all my axes, and neglecting to write "for some constant  $C$ " after every  $+C$ .

My friends and family have been fantastic in their support, especially my parents Freda and Michael, sister Carly, grandmother Rena, and in-laws Marianna and Michael. Finally, a very special thank you to my wife Amy for putting up with me while I wrote this book and always being there for me (and also for drawing the mountain-climber!).





# CHAPTER 1

---

## Functions, Graphs, and Lines

Trying to do calculus without using functions would be one of the most pointless things you could do. If calculus had an ingredients list, functions would be first on it, and by some margin too. So, the first two chapters of this book are designed to jog your memory about the main features of functions. This chapter contains a review of the following topics:

- functions: their domain, codomain, and range, and the vertical line test;
- inverse functions and the horizontal line test;
- composition of functions;
- odd and even functions;
- graphs of linear functions and polynomials in general, as well as a brief survey of graphs of rational functions, exponentials, and logarithms; and
- how to deal with absolute values.

Trigonometric functions, or trig functions for short, are dealt with in the next chapter. So, let's kick off with a review of what a function actually is.

### 1.1 Functions

A *function* is a rule for transforming an object into another object. The object you start with is called the *input*, and comes from some set called the *domain*. What you get back is called the *output*; it comes from some set called the *codomain*.

Here are some examples of functions:

- Suppose you write  $f(x) = x^2$ . You have just defined a function  $f$  which transforms any number into its square. Since you didn't say what the domain or codomain are, it's assumed that they are both  $\mathbb{R}$ , the set of all real numbers. So you can square any real number, and get a real number back. For example,  $f$  transforms 2 into 4; it transforms  $-1/2$  into  $1/4$ ; and it transforms 1 into 1. This last one isn't much of a change at all, but that's no problem: the transformed object doesn't have to be different from the original one. When you write  $f(2) = 4$ , what you really mean

## 2 • Functions, Graphs, and Lines

is that  $f$  transforms 2 into 4. By the way,  $f$  is the transformation rule, while  $f(x)$  is the result of applying the transformation rule to the variable  $x$ . So it's technically not correct to say " $f(x)$  is a function"; it should be " $f$  is a function."

- Now, let  $g(x) = x^2$  with domain consisting only of numbers greater than or equal to 0. (Such numbers are called *nonnegative*.) This seems like the same function as  $f$ , but it's not: the domains are different. For example,  $f(-1/2) = 1/4$ , but  $g(-1/2)$  isn't defined. The function  $g$  just chokes on anything not in the domain, refusing even to touch it. Since  $g$  and  $f$  have the same rule, but the domain of  $g$  is smaller than the domain of  $f$ , we say that  $g$  is formed by *restricting the domain* of  $f$ .
- Still letting  $f(x) = x^2$ , what do you make of  $f(\text{horse})$ ? Obviously this is undefined, since you can't square a horse. On the other hand, let's set

$$h(x) = \text{number of legs } x \text{ has,}$$

where the domain of  $h$  is the set of all animals. So  $h(\text{horse}) = 4$ , while  $h(\text{ant}) = 6$  and  $h(\text{salmon}) = 0$ . The codomain could be the set of all nonnegative integers, since animals don't have negative or fractional numbers of legs. By the way, what is  $h(2)$ ? This isn't defined, of course, since 2 isn't in the domain. How many legs does a "2" have, after all? The question doesn't really make sense. You might also think that  $h(\text{chair}) = 4$ , since most chairs have four legs, but that doesn't work either, since a chair isn't an animal, and so "chair" is not in the domain of  $h$ . That is,  $h(\text{chair})$  is undefined.

- Suppose you have a dog called Junkster. Unfortunately, poor Junkster has indigestion. He eats something, then chews on it for a while and tries to digest it, fails, and hurls. Junkster has transformed the food into ... something else altogether. We could let

$$j(x) = \text{color of barf when Junkster eats } x,$$

where the domain of  $j$  is the set of foods that Junkster will eat. The codomain is the set of all colors. For this to work, we have to be confident that whenever Junkster eats a taco, his barf is always the same color (say, red). If it's sometimes red and sometimes green, that's no good: **a function must assign a unique output for each valid input.**

Now we have to look at the concept of the *range* of a function. The range is the set of all outputs that could possibly occur. You can think of the function working on transforming everything in the domain, one object at a time; the collection of transformed objects is the range. You might get duplicates, but that's OK.

So why isn't the range the same thing as the codomain? Well, the range is actually a subset of the codomain. The codomain is a set of **possible** outputs, while the range is the set of **actual** outputs. Here are the ranges of the functions we looked at above:

- If  $f(x) = x^2$  with domain  $\mathbb{R}$  and codomain  $\mathbb{R}$ , the range is the set of nonnegative numbers. After all, when you square a number, the result cannot be negative. How do you know the range is **all** the nonnegative numbers? Well, if you square every number, you definitely cover all nonnegative numbers. For example, you get 2 by squaring  $\sqrt{2}$  (or  $-\sqrt{2}$ ).
- If  $g(x) = x^2$ , where the domain of  $g$  is only the nonnegative numbers but the codomain is still all of  $\mathbb{R}$ , the range will again be the set of nonnegative numbers. When you square every nonnegative number, you still cover all the nonnegative numbers.
- If  $h(x)$  is the number of legs the animal  $x$  has, then the range is all the possible numbers of legs that **any** animal can have. I can think of animals that have 0, 2, 4, 6, and 8 legs, as well as some creepy-crawlies with more legs. If you include individual animals which have lost one or more legs, you can also include 1, 3, 5, and 7 in the mix, as well as other possibilities. In any case, the range of this function isn't so clear-cut; you probably have to be a biologist to know the real answer.
- Finally, if  $j(x)$  is the color of Junkster's barf when he eats  $x$ , then the range consists of all possible barf-colors. I dread to think what these are, but probably bright blue isn't among them.

### 1.1.1 Interval notation

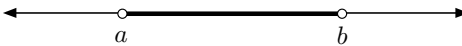
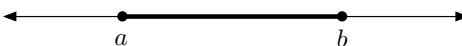
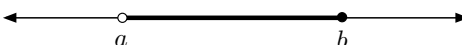
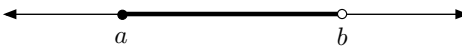
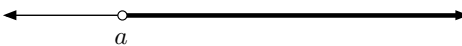
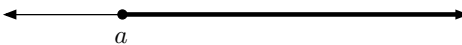
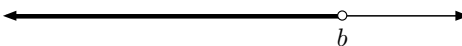


In the rest of this book, our functions will always have codomain  $\mathbb{R}$ , and the domain will always be as much of  $\mathbb{R}$  as possible (unless stated otherwise). So we'll often be dealing with subsets of the real line, especially connected intervals such as  $\{x : 2 \leq x < 5\}$ . It's a bit of a pain to write out the full set notation like this, but it sure beats having to say "all the numbers between 2 and 5, including 2 but not 5." We can do even better using interval notation.

We'll write  $[a, b]$  to mean the set of all numbers between  $a$  and  $b$ , including  $a$  and  $b$  themselves. So  $[a, b]$  means the set of all  $x$  such that  $a \leq x \leq b$ . For example,  $[2, 5]$  is the set of all real numbers between 2 and 5, including 2 and 5. (It's not just the set consisting of 2, 3, 4, and 5: don't forget that there are loads of fractions and irrational numbers between 2 and 5, such as  $5/2$ ,  $\sqrt{7}$ , and  $\pi$ .) An interval such as  $[a, b]$  is called *closed*.

If you don't want the endpoints, change the square brackets to parentheses. In particular,  $(a, b)$  is the set of all numbers between  $a$  and  $b$ , not including  $a$  or  $b$ . So if  $x$  is in the interval  $(a, b)$ , we know that  $a < x < b$ . The set  $(2, 5)$  includes all real numbers between 2 and 5, but not 2 or 5. An interval of the form  $(a, b)$  is called *open*.

You can mix and match:  $[a, b)$  consists of all numbers between  $a$  and  $b$ , including  $a$  but not  $b$ . And  $(a, b]$  includes  $b$  but not  $a$ . These intervals are closed at one end and open at the other. Sometimes such intervals are called *half-open*. An example is the set  $\{x : 2 \leq x < 5\}$  from above, which can also be written as  $[2, 5)$ .

There's also the useful notation  $(a, \infty)$  for all the numbers greater than  $a$  not including  $a$ ;  $[a, \infty)$  is the same thing but with  $a$  included. There are three other possibilities which involve  $-\infty$ ; all in all, the situation looks like this:

$(a, b)$	$\{x : a < x < b\}$	
$[a, b]$	$\{x : a \leq x \leq b\}$	
$(a, b]$	$\{x : a < x \leq b\}$	
$[a, b)$	$\{x : a \leq x < b\}$	
$(a, \infty)$	$\{x : x > a\}$	
$[a, \infty)$	$\{x : x \geq a\}$	
$(-\infty, b)$	$\{x : x < b\}$	
$(-\infty, b]$	$\{x : x \leq b\}$	
$(-\infty, \infty)$	$\mathbb{R}$	

## 1.1.2 Finding the domain

Sometimes the definition of a function will include the domain. (This was the case, for example, with our function  $g$  from Section 1.1 above.) Most of the time, however, the domain is not provided. The basic convention is that the domain consists of as much of the set of real numbers as possible. For example, if  $k(x) = \sqrt{x}$ , the domain can't be all of  $\mathbb{R}$ , since you can't take the square root of a negative number. The domain must be  $[0, \infty)$ , which is just the set of all numbers greater than or equal to 0.



OK, so square roots of negative numbers are bad. What else can cause a screw-up? Here's a list of the three most common possibilities:

1. The denominator of a fraction can't be zero.
2. You can't take the square root (or fourth root, sixth root, and so on) of a negative number.
3. You can't take the logarithm of a negative number or of 0. (Remember logs? If not, see Chapter 9!)

You might recall that  $\tan(90^\circ)$  is also a problem, but this is really a special case of the first item above. You see,

$$\tan(90^\circ) = \frac{\sin(90^\circ)}{\cos(90^\circ)} = \frac{1}{0},$$



so the reason  $\tan(90^\circ)$  is undefined is really that a hidden denominator is zero. Here's another example: if we try to define

$$f(x) = \frac{\log_{10}(x+8)\sqrt{26-2x}}{(x-2)(x+19)},$$

then what is the domain of  $f$ ? Well, for  $f(x)$  to make sense, here's what needs to happen:

- We need to take the square root of  $(26 - 2x)$ , so this quantity had better be nonnegative. That is,  $26 - 2x \geq 0$ . This can be rewritten as  $x \leq 13$ .

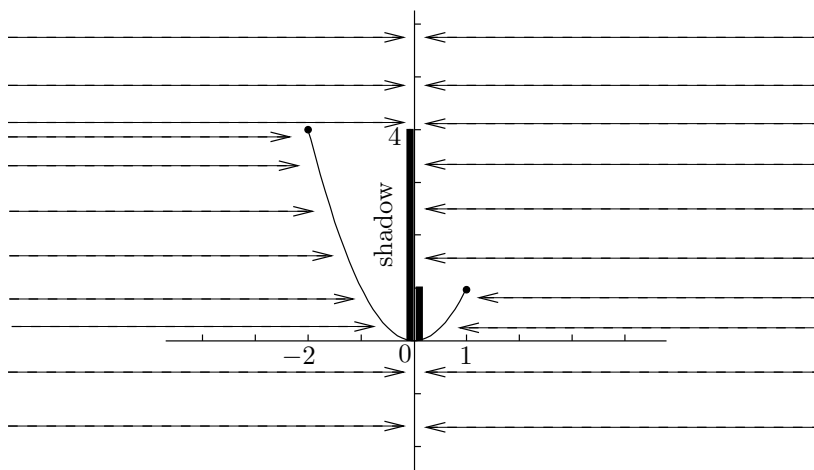
- We also need to take the logarithm of  $(x + 8)$ , so this quantity needs to be positive. (Notice the difference between logs and square roots: you can take the square root of 0, but you can't take the log of 0.) Anyway, we need  $x + 8 > 0$ , so  $x > -8$ . So far, we know that  $-8 < x \leq 13$ , so the domain is at most  $(-8, 13]$ .
- The denominator can't be 0; this means that  $(x - 2) \neq 0$  and  $(x + 19) \neq 0$ . In other words,  $x \neq 2$  and  $x \neq -19$ . This last one isn't a problem, since we already know that  $x$  lies in  $(-8, 13]$ , so  $x$  can't possibly be  $-19$ . We do have to exclude 2, though.

So we have found that the domain is the set  $(-8, 13]$  except for the number 2. This set could be written as  $(-8, 13] \setminus \{2\}$ . Here the backslash means “not including.”

### 1.1.3 Finding the range using the graph

Let's define a new function  $F$  by specifying that its domain is  $[-2, 1]$  and that  $F(x) = x^2$  on this domain. (Remember, the codomain of any function we look at will always be the set of all real numbers.) Is  $F$  the same function as  $f$ , where  $f(x) = x^2$  for all real numbers  $x$ ? The answer is no, since the two functions have different domains (even though they have the same rule). As in the case of the function  $g$  from Section 1.1 above, the function  $F$  is formed by restricting the domain of  $f$ .

Now, what is the range of  $F$ ? Well, what happens if you square every number between  $-2$  and  $1$  inclusive? You should be able to work this out directly, but this is a good opportunity to see how to use a graph to find the range of a function. The idea is to sketch the graph of the function, then imagine two rows of lights shining from the far left and far right of the graph horizontally toward the  $y$ -axis. The curve will cast two shadows, one on the left side and one on the right side of the  $y$ -axis. The range is the union of both shadows: that is, if any point on the  $y$ -axis lies in either the left-hand or the right-hand shadow, it is in the range of the function. Let's see how this works with our function  $F$ :



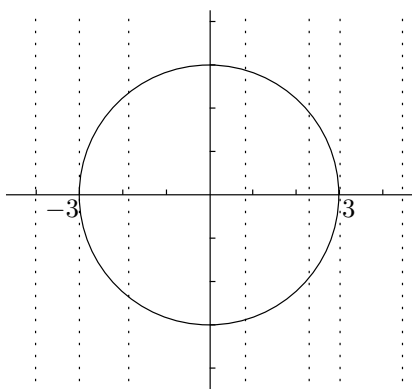
The left-hand shadow covers all the points on the  $y$ -axis between 0 and 4 inclusive, which is  $[0, 4]$ ; on the other hand, the right-hand shadow covers the points between 0 and 1 inclusive, which is  $[0, 1]$ . The right-hand shadow doesn't contribute anything extra: the total coverage is still  $[0, 4]$ . This is the range of  $F$ .

### 1.1.4 The vertical line test

In the last section, we used the graph of a function to find its range. The graph of a function is very important: it really shows you what the function “looks like.” We'll be looking at techniques for sketching graphs in Chapter 12, but for now I'd like to remind you about the vertical line test.

You can draw any figure you like on a coordinate plane, but the result may not be the graph of a function. So what's special about the graph of a function? What is the graph of a function  $f$ , anyway? Well, it's the collection of all points with coordinates  $(x, f(x))$ , where  $x$  is in the domain of  $f$ . Here's another way of looking at this: start with some number  $x$ . If  $x$  is in the domain, you plot the point  $(x, f(x))$ , which of course is at a height of  $f(x)$  units above the point  $x$  on the  $x$ -axis. If  $x$  isn't in the domain, you don't plot anything. Now repeat for every real number  $x$  to build up the graph.

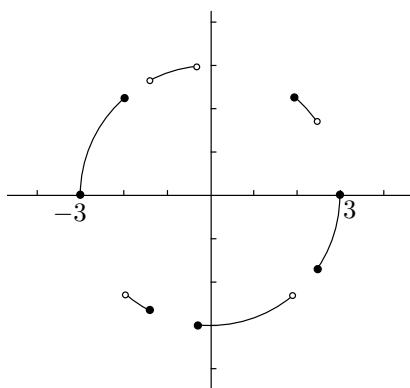
Here's the key idea: you can't have two points with the same  $x$ -coordinate. In other words, no two points on the graph can lie on the same vertical line. Otherwise, how would you know which of the two or more heights above the point  $x$  on the  $x$ -axis corresponds to the value of  $f(x)$ ? So, this leads us to the *vertical line test*: if you have some graph and you want to know whether it's the graph of a function, see whether any vertical line intersects the graph more than once. If so, it's not the graph of a function; but if no vertical line intersects the graph more than once, you are indeed dealing with the graph of a function. For example, the circle of radius 3 units centered at the origin has a graph like this:



Such a commonplace object should be a function, right? No, check the vertical lines that are shown in the diagram. Sure, to the left of  $-3$  or to the right of  $3$ , there's no problem—the vertical lines don't even hit the graph, which is fine. Even at  $-3$  or  $3$ , the vertical lines only intersect the curve in one point each, which is also fine. The problem is when  $x$  is in the interval  $(-3, 3)$ . For

any of these values of  $x$ , the vertical line through  $(x, 0)$  intersects the circle twice, which screws up the circle's potential function-status. You just don't know whether  $f(x)$  is the top point or the bottom point.

The best way to salvage the situation is to chop the circle in half horizontally and choose only the top or the bottom half. The equation for the whole circle is  $x^2 + y^2 = 9$ , whereas the equation for the top semicircle is  $y = \sqrt{9 - x^2}$ . The bottom semicircle has equation  $y = -\sqrt{9 - x^2}$ . These last two are functions, both with domain  $[-3, 3]$ . If you felt like chopping in a different way, you wouldn't actually have to take semicircles—you could chop and change between the upper and lower semicircles, as long as you don't violate the vertical line test. For example, here's the graph of a function which also has domain  $[-3, 3]$ :



The vertical line test checks out, so this is indeed the graph of a function.

## 1.2 Inverse Functions

Let's say you have a function  $f$ . You present it with an input  $x$ ; provided that  $x$  is in the domain of  $f$ , you get back an output, which we call  $f(x)$ . Now we try to do things all backward and ask this question: if you pick a number  $y$ , what input can you give to  $f$  in order to get back  $y$  as your output?

Here's how to state the problem in math-speak: given a number  $y$ , what  $x$  in the domain of  $f$  satisfies  $f(x) = y$ ? The first thing to notice is that  $y$  has to be in the range of  $f$ . Otherwise, by definition there are no values of  $x$  such that  $f(x) = y$ . There would be nothing in the domain that  $f$  would transform into  $y$ , since the range is **all** the possible outputs.

On the other hand, if  $y$  is in the range, there might be many values that work. For example, if  $f(x) = x^2$  (with domain  $\mathbb{R}$ ), and we ask what value of  $x$  transforms into 64, there are obviously two values of  $x$ : 8 and  $-8$ . On the other hand, if  $g(x) = x^3$ , and we ask the same question, there's only one value of  $x$ , which is 4. The same would be true for any number we give to  $g$  to transform, because any number has only one (real) cube root.

So, here's the situation: we're given a function  $f$ , and we pick  $y$  in the range of  $f$ . Ideally, there will be exactly one value of  $x$  which satisfies  $f(x) = y$ . If this is true for every value of  $y$  in the range, then we can define a new



function which reverses the transformation. Starting with the output  $y$ , the new function finds the one and only input  $x$  which leads to the output. The new function is called the *inverse function of  $f$* , and is written as  $f^{-1}$ . Here's a summary of the situation in mathematical language:

1. Start with a function  $f$  such that for any  $y$  in the range of  $f$ , there is exactly one number  $x$  such that  $f(x) = y$ . That is, different inputs give different outputs. Now we will define the inverse function  $f^{-1}$ .
2. The domain of  $f^{-1}$  is the same as the range of  $f$ .
3. The range of  $f^{-1}$  is the same as the domain of  $f$ .
4. The value of  $f^{-1}(y)$  is the number  $x$  such that  $f(x) = y$ . So,

$$\text{if } f(x) = y, \quad \text{then } f^{-1}(y) = x.$$

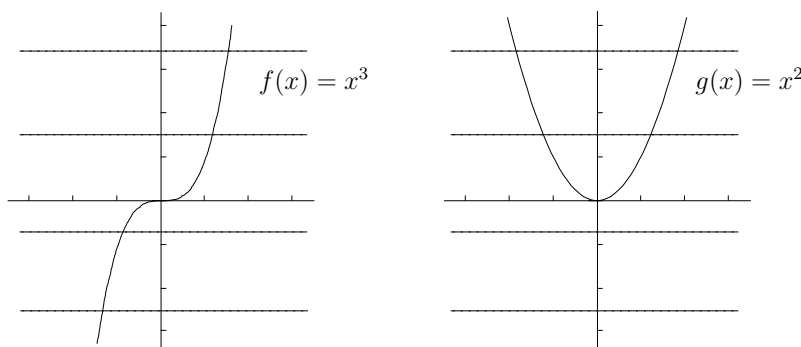
The transformation  $f^{-1}$  acts like an undo button for  $f$ : if you start with  $x$  and transform it into  $y$  using the function  $f$ , then you can undo the effect of the transformation by using the inverse function  $f^{-1}$  on  $y$  to get  $x$  back.

This raises some questions: how do you see if there's only one value of  $x$  that satisfies the equation  $f(x) = y$ ? If so, how do you find the inverse, and what does its graph look like? If not, how do you salvage the situation? We'll answer these questions in the next three sections.

### 1.2.1 The horizontal line test

For the first question—how to see that there's only one value of  $x$  that works for any  $y$  in the range—perhaps the best way is to look at the graph of your function. We want to pick  $y$  in the range of  $f$  and hopefully only have one value of  $x$  such that  $f(x) = y$ . What this means is that the horizontal line through the point  $(0, y)$  should intersect the graph exactly once, at some point  $(x, y)$ . That  $x$  is the one we want. If the horizontal line intersects the curve more than once, there would be multiple potential inverses  $x$ , which is bad. In that case, the only way to get an inverse function is to restrict the domain; we'll come back to this very shortly. What if the horizontal line doesn't intersect the curve at all? Then  $y$  isn't in the range after all, which is OK.

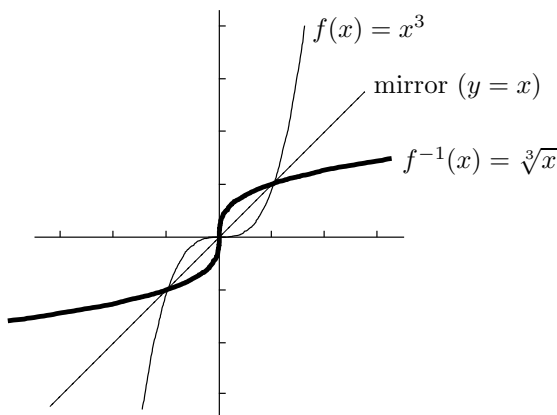
So, we have just described the *horizontal line test*: if every horizontal line intersects the graph of a function at most once, the function has an inverse. If even one horizontal line intersects the graph more than once, there isn't an inverse function. For example, look at the graphs of  $f(x) = x^3$  and  $g(x) = x^2$ :



No horizontal line hits  $y = f(x)$  more than once, so  $f$  has an inverse. On the other hand, some of the horizontal lines hit the curve  $y = g(x)$  twice, so  $g$  has no inverse. Here's the problem: if you want to solve  $y = x^2$  for  $x$ , where  $y$  is positive, then there are two solutions,  $x = \sqrt{y}$  and  $x = -\sqrt{y}$ . You don't know which one to take!

## 1.2.2 Finding the inverse

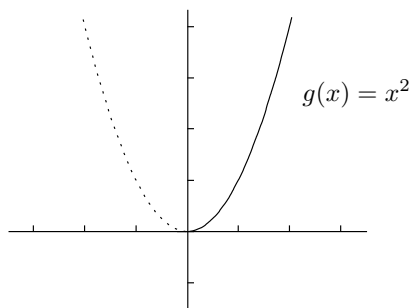
Now let's move on to the second of our questions: how do you find the inverse of a function  $f$ ? Well, you write down  $y = f(x)$  and try to solve for  $x$ . In our example of  $f(x) = x^3$ , we have  $y = x^3$ , so  $x = \sqrt[3]{y}$ . This means that  $f^{-1}(y) = \sqrt[3]{y}$ . If the variable  $y$  here offends you, by all means switch it to  $x$ : you can write  $f^{-1}(x) = \sqrt[3]{x}$  if you prefer. Of course, solving for  $x$  is not always easy and in fact is often impossible. On the other hand, if you know what the graph of your function looks like, the graph of the inverse function is easy to find. The idea is to draw the line  $y = x$  on the graph, then pretend that this line is a two-sided mirror. The inverse function is the reflection of the original function in this mirror. When  $f(x) = x^3$ , here's what  $f^{-1}$  looks like:



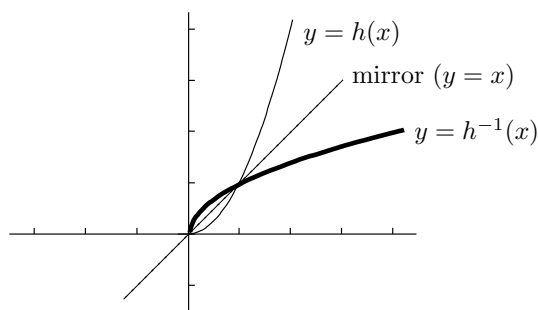
The original function  $f$  is reflected in the mirror  $y = x$  to get the inverse function. Note that the domain and range of both  $f$  and  $f^{-1}$  are the whole real line.

## 1.2.3 Restricting the domain

Finally, we'll address our third question: if the horizontal line test fails and there's no inverse, what can be done? Our problem is that there are multiple values of  $x$  that give the same  $y$ . The only way to get around the problem is to throw away all but one of these values of  $x$ . That is, we have to decide which one of our values of  $x$  we want to keep, and throw the rest away. As we saw in Section 1.1 above, this is called *restricting the domain* of our function. Effectively, we ghost out part of the curve so that what's left no longer fails the horizontal line test. For example, if  $g(x) = x^2$ , we can ghost out the left half of the graph like this:

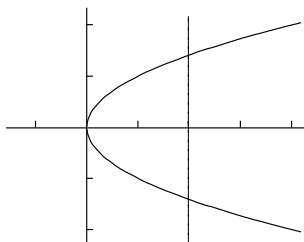


The new (unghosted) curve has the reduced domain  $[0, \infty)$  and satisfies the horizontal line test, so there is an inverse function. More precisely, the function  $h$ , which has domain  $[0, \infty)$  and is defined by  $h(x) = x^2$  on this domain, has an inverse. Let's play the reflection game to see what it looks like:



To find the equation of the inverse, we have to solve for  $x$  in the equation  $y = x^2$ . Clearly the solution is  $x = \sqrt{y}$  or  $x = -\sqrt{y}$ , but which one do we need? We know that the range of the inverse function is the same as the domain of the original function, which we have restricted to be  $[0, \infty)$ . So we need a nonnegative number as our answer, and that has to be  $x = \sqrt{y}$ . That is,  $h^{-1}(y) = \sqrt{y}$ . Of course, we could have ghosted out the right half of the original graph to restrict the domain to  $(-\infty, 0]$ . In that case, we'd get a function  $j$  which has domain  $(-\infty, 0]$  and again satisfies  $j(x) = x^2$ , but only on this domain. This function also has an inverse, but the inverse is now the negative square root:  $j^{-1}(y) = -\sqrt{y}$ .

By the way, if you take the original function  $g$  given by  $g(x) = x^2$  with domain  $(-\infty, \infty)$ , which fails the horizontal line test, and try to reflect it in the mirror  $y = x$ , you get the following picture:



Notice that the graph fails the vertical line test, so it's not the graph of a function. This illustrates the connection between the vertical and horizontal line tests—when horizontal lines are reflected in the mirror  $y = x$ , they become vertical lines.

## 1.2.4 Inverses of inverse functions

One more thing about inverse functions: if  $f$  has an inverse, it's true that  $f^{-1}(f(x)) = x$  for all  $x$  in the domain of  $f$ , and also that  $f(f^{-1}(y)) = y$  for all  $y$  in the range of  $f$ . (Remember, the range of  $f$  is the same as the domain of  $f^{-1}$ , so you can indeed take  $f^{-1}(y)$  for  $y$  in the range of  $f$  without causing any screwups.)



For example, if  $f(x) = x^3$ , then  $f$  has an inverse given by  $f^{-1}(x) = \sqrt[3]{x}$ , and so  $f^{-1}(f(x)) = \sqrt[3]{x^3} = x$  for any  $x$ . Remember, the inverse function is like an undo button. We use  $x$  as an input to  $f$ , and then give the output to  $f^{-1}$ ; this undoes the transformation and gives us back  $x$ , the original number. Similarly,  $f(f^{-1}(y)) = (\sqrt[3]{y})^3 = y$ . So  $f^{-1}$  is the inverse function of  $f$ , and  $f$  is the inverse function of  $f^{-1}$ . In other words, the inverse of the inverse is the original function.

Now, you have to be careful in the case where you restrict the domain. Let  $g(x) = x^2$ ; we've seen that you need to restrict the domain to get an inverse. Let's say we restrict the domain to  $[0, \infty)$  and carelessly continue to refer to the function as  $g$  instead of  $h$ , as in the previous section. We would then say that  $g^{-1}(x) = \sqrt{x}$ . If you calculate  $g(g^{-1}(x))$ , you find that this is  $(\sqrt{x})^2$ , which equals  $x$ , provided that  $x \geq 0$ . (Otherwise you can't take the square root in the first place.)

On the other hand, if you work out  $g^{-1}(g(x))$ , you get  $\sqrt{x^2}$ , which is not always the same thing as  $x$ . For example, if  $x = -2$ , then  $x^2 = 4$  and so  $\sqrt{x^2} = \sqrt{4} = 2$ . So it's not true in general that  $g^{-1}(g(x)) = x$ . The problem is that  $-2$  isn't in the restricted-domain version of  $g$ . Technically, you can't even compute  $g(-2)$ , since  $-2$  is no longer in the domain of  $g$ . We really should be working with  $h$ , not  $g$ , so that we remember to be more careful. Nevertheless, in practice, mathematicians will often restrict the domain without changing letters! So it will be useful to summarize the situation as follows:



If the domain of a function  $f$  can be restricted so that  $f$  has an inverse  $f^{-1}$ , then

- $f(f^{-1}(y)) = y$  for all  $y$  in the range of  $f$ ; but
- $f^{-1}(f(x))$  may not equal  $x$ ; in fact,  $f^{-1}(f(x)) = x$  only when  $x$  is in the restricted domain.

We'll be revisiting these important points in the context of inverse trig functions in Section 10.2.6 of Chapter 10.

## 1.3 Composition of Functions

Let's say we have a function  $g$  given by  $g(x) = x^2$ . You can replace  $x$  by anything you like, as long as it makes sense. For example, you can write

$g(y) = y^2$ , or  $g(x + 5) = (x + 5)^2$ . This last example shows that you need to be very careful with parentheses. It would be wrong to write  $g(x + 5) = x + 5^2$ , since this is just  $x + 25$ , which is not the same thing as  $(x + 5)^2$ . If in doubt, use parentheses. That is, if you need to write out  $f(\text{something})$ , replace every instance of  $x$  by  $(\text{something})$ , making sure to include the parentheses. Just about the only time you don't need to use parentheses is when the function is an exponential function—for example, if  $h(x) = 3^x$ , then you can just write  $h(x^2 + 6) = 3^{x^2 + 6}$ . You don't need parentheses since you're already writing the  $x^2 + 6$  as a superscript.

Now consider the function  $f$  defined by  $f(x) = \cos(x^2)$ . If I give you a number  $x$ , how do you compute  $f(x)$ ? Well, first you square it, then you take the cosine of the result. Since we can decompose the action of  $f(x)$  into these two separate actions which are performed one after the other, we might as well describe those actions as functions themselves. So, let  $g(x) = x^2$  and  $h(x) = \cos(x)$ . To simulate what  $f$  does when you use  $x$  as an input, you could first give  $x$  to  $g$  to square it, and then instead of taking the result back you could ask  $g$  to give its result to  $h$  instead. Then  $h$  spits out a number, which is the final answer. The answer will, of course, be the cosine of what came out of  $g$ , which was the square of the original  $x$ . This behavior exactly mimics  $f$ , so we can write  $f(x) = h(g(x))$ . Another way of expressing this is to write  $f = h \circ g$ ; here the circle means “composed with.” That is,  $f$  is  $h$  composed with  $g$ , or in other words,  $f$  is the *composition* of  $h$  and  $g$ . What's tricky is that you write  $h$  before  $g$  (reading from left to right as usual!) but you apply  $g$  first. I agree that it's confusing, but what can I say—you just have to deal with it.



It's useful to practice composing two or more functions together. For example, if  $g(x) = 2^x$ ,  $h(x) = 5x^4$ , and  $j(x) = 2x - 1$ , what is a formula for the function  $f = g \circ h \circ j$ ? Well, just replace one thing at a time, starting with  $j$ , then  $h$ , then  $g$ . So:

$$f(x) = g(h(j(x))) = g(h(2x - 1)) = g(5(2x - 1)^4) = 2^{5(2x - 1)^4}.$$



You should also practice reversing the process. For example, suppose you start off with

$$f(x) = \frac{1}{\tan(5 \log_2(x + 3))}.$$

How would you decompose  $f$  into simpler functions? Zoom in to where you see the quantity  $x$ . The first thing you do is add 3, so let  $g(x) = x + 3$ . Then you have to take the base 2 logarithm of the resulting quantity, so set  $h(x) = \log_2(x)$ . Next, multiply by 5, so set  $j(x) = 5x$ . Then take the tangent, so put  $k(x) = \tan(x)$ . Finally, take reciprocals, so let  $m(x) = 1/x$ . With all these definitions, you should check that

$$f(x) = m(k(j(h(g(x)))))$$

Using the composition notation, you can write

$$f = m \circ k \circ j \circ h \circ g.$$

This isn't the only way to break down  $f$ . For example, we could have combined  $h$  and  $j$  into another function  $n$ , where  $n(x) = 5 \log_2(x)$ . Then you should check that  $n = j \circ h$ , and

$$f = m \circ k \circ n \circ g.$$

Perhaps the original decomposition (involving  $j$  and  $h$ ) is better because it breaks down  $f$  into more elementary steps, but the second one (involving  $n$ ) isn't wrong. After all,  $n(x) = 5 \log_2(x)$  is still a pretty simple function of  $x$ .

Beware: composition of functions isn't the same thing as multiplying them together. For example, if  $f(x) = x^2 \sin(x)$ , then  $f$  is not the composition of two functions. To calculate  $f(x)$  for any given  $x$ , you actually have to find both  $x^2$  and  $\sin(x)$  (it doesn't matter which one you find first, unlike with composition) and then multiply these two things together. If  $g(x) = x^2$  and  $h(x) = \sin(x)$ , then we'd write  $f(x) = g(x)h(x)$ , or  $f = gh$ . Compare this to the composition of the two functions,  $j = g \circ h$ , which is given by

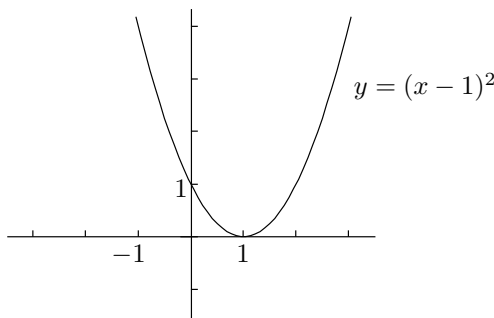
$$j(x) = g(h(x)) = g(\sin(x)) = (\sin(x))^2$$

or simply  $j(x) = \sin^2(x)$ . The function  $j$  is a completely different function from the product  $x^2 \sin(x)$ . It's also different from the function  $k = h \circ g$ , which is also a composition of  $g$  and  $h$  but in the other order:

$$k(x) = h(g(x)) = h(x^2) = \sin(x^2).$$

This is yet another completely different function. The moral of the story is that products and compositions are not the same thing, and furthermore, the order of the functions matters when you compose them, but not when you multiply them together.

One simple but important example of composition of functions occurs when you compose some function  $f$  with  $g(x) = x - a$ , where  $a$  is some constant number. You end up with a new function  $h$  given by  $h(x) = f(x - a)$ . A useful point to note is that the graph of  $y = h(x)$  is the same as the graph of  $y = f(x)$ , except that it's shifted over  $a$  units to the right. If  $a$  is negative, then the shift is to the left. (The way to think of this, for example, is that a shift of  $-3$  units to the right is the same as a shift of 3 units to the left.) So, how would you sketch the graph of  $y = (x - 1)^2$ ? This is the same as  $y = x^2$ , but with  $x$  replaced by  $x - 1$ . So the graph of  $y = x^2$  needs to be shifted to the right by 1 unit, and looks like this:



Similarly, the graph of  $y = (x + 2)^2$  is the graph of  $y = x^2$  shifted to the left by 2 units, since you can interpret  $(x + 2)$  as  $(x - (-2))$ .

## 1.4 Odd and Even Functions

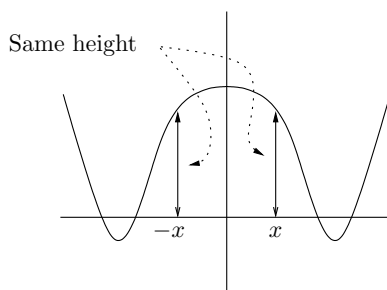
Some functions have some symmetry properties that make them easier to deal with. Consider the function  $f$  given by  $f(x) = x^2$ . Pick any positive number you like (I'll choose 3) and hit it with  $f$  (I get 9). Now take the negative of that number,  $-3$  in my case, and hit that with  $f$  (I get 9 again). You should get the same answer both times, as I did, regardless of which number you chose. You can express this phenomenon by writing  $f(-x) = f(x)$  for all  $x$ . That is, if you give  $x$  to  $f$  as an input, you get back the same answer as if you used the input  $-x$  instead. Notice that  $g(x) = x^4$  and  $h(x) = x^6$  also have this property—in fact,  $j(x) = x^n$ , where  $n$  is any even number ( $n$  could in fact be negative), has the same property. Inspired by this, we say that a function  $f$  is *even* if  $f(-x) = f(x)$  for all  $x$  in the domain of  $f$ . It's not good enough for this equation to be true for some values of  $x$ ; it has to be true for **all**  $x$  in the domain of  $f$ .

Now, let's say we play the same game with  $f(x) = x^3$ . Take your favorite positive number (I'll stick with 3) and hit that with  $f$  (I get 27). Now try again with the negative of your number,  $-3$  in my case; I get  $-27$ , and you should also get the negative of what you got before. You can express this mathematically as  $f(-x) = -f(x)$ . Once again, the same property holds for  $j(x) = x^n$  when  $n$  is any odd number (and once again,  $n$  could be negative). So, we say that a function  $f$  is *odd* if  $f(-x) = -f(x)$  for all  $x$  in the domain of  $f$ .



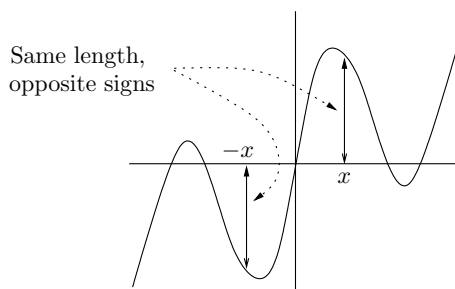
In general, a function might be odd, it might be even, or it might be neither odd nor even. Don't forget this last point! Most functions are neither odd nor even. On the other hand, there's only one function that's both odd and even, which is the rather boring function given by  $f(x) = 0$  for all  $x$  (we'll call this the "zero function"). Why is this the only odd and even function? Let's convince ourselves. If the function  $f$  is even, then  $f(-x) = f(x)$  for all  $x$ . But if it's also odd, then  $f(-x) = -f(x)$  for all  $x$ . Take the first of these equations and subtract the second from it. You should get  $0 = 2f(x)$ , which means that  $f(x) = 0$ . This is true for all  $x$ , so the function  $f$  must just be the zero function. One other nice observation is that if a function  $f$  is odd, and the number 0 is in its domain, then  $f(0) = 0$ . Why is it so? Because  $f(-x) = -f(x)$  is true for all  $x$  in the domain of  $f$ , so let's try it for  $x = 0$ . You get  $f(-0) = -f(0)$ . But  $-0$  is the same thing as 0, so we have  $f(0) = -f(0)$ . This simplifies to  $2f(0) = 0$ , or  $f(0) = 0$  as claimed.

Anyway, starting with a function  $f$ , how can you tell if it is odd, even, or neither? And so what if it is odd or even anyway? Let's look at this second question before coming back to the first one. One nice thing about knowing that a function is odd or even is that it's easier to graph the function. In fact, if you can graph the right-hand half of the function, the left-hand half is a piece of cake! Let's say that  $f$  is an even function. Then since  $f(x) = f(-x)$ , the graph of  $y = f(x)$  is at the same height above the  $x$ -coordinates  $x$  and  $-x$ . This is true for all  $x$ , so the situation looks something like this:



We can conclude that **the graph of an even function has mirror symmetry about the  $y$ -axis**. So, if you graph the right half of a function which you know is even, you can get the left half by reflecting the right half about the  $y$ -axis. Check the graph of  $y = x^2$  to make sure that it has this mirror symmetry.

On the other hand, let's say that  $f$  is an odd function. Since we have  $f(-x) = -f(x)$ , the graph of  $y = f(x)$  is at the same height **above** the  $x$ -coordinate  $x$  as it is **below** the  $x$ -coordinate  $-x$ . (Of course, if  $f(x)$  is negative, then you have to switch the words "above" and "below.") In any case, the picture looks like this:



The symmetry is now a point symmetry about the origin. That is, **the graph of an odd function has  $180^\circ$  point symmetry about the origin**. This means that if you only have the right half of a function which you know is odd, you can get the left half as follows. Pretend that the curve is sitting on top of the paper, so you can pick it up if you like but you can't change its shape. Instead of picking it up, put a pin through the curve at the origin (remember, odd functions must pass through the origin if they are defined at 0) and then spin the whole curve around half a revolution. This is what the left-hand half of the graph looks like. (This doesn't work so well if the curve isn't continuous, that is, if the curve isn't all in one piece!) Check to see that the above graph and also the graph of  $y = x^3$  have this symmetry.



Now, suppose  $f$  is defined by the equation  $f(x) = \log_5(2x^6 - 6x^2 + 3)$ . How do you tell if  $f$  is odd, even, or neither? The technique is to calculate  $f(-x)$  by replacing every instance of  $x$  with  $(-x)$ , making sure not to forget the parentheses around  $-x$ , and then simplifying the result. If you end up with the original expression  $f(x)$ , then  $f$  is even; if you end up with the negative of the original expression  $-f(x)$ , then  $f$  is odd; if you end up with a mess that isn't either  $f(x)$  or  $-f(x)$ , then  $f$  is neither (or you didn't simplify enough!).



In the example above, you'd write

$$f(-x) = \log_5(2(-x)^6 - 6(-x)^2 + 3) = \log_5(2x^6 - 6x^2 + 3),$$



which is actually equal to the original  $f(x)$ . So the function  $f$  is even. How about

$$g(x) = \frac{2x^3 + x}{3x^2 + 5} \quad \text{and} \quad h(x) = \frac{2x^3 + x - 1}{3x^2 + 5}?$$

Well, for  $g$ , we have

$$g(-x) = \frac{2(-x)^3 + (-x)}{3(-x)^2 + 5} = \frac{-2x^3 - x}{3x^2 + 5}.$$

Now you have to observe that you can take the minus sign out front and write

$$g(-x) = -\frac{2x^3 + x}{3x^2 + 5},$$

which, you notice, is equal to  $-g(x)$ . That is, apart from the minus sign, we get the original function back. So,  $g$  is an odd function. How about  $h$ ? We have

$$h(-x) = \frac{2(-x)^3 + (-x) - 1}{3(-x)^2 + 5} = \frac{-2x^3 - x - 1}{3x^2 + 5}.$$

Once again, we take out the minus sign to get

$$h(-x) = -\frac{2x^3 + x + 1}{3x^2 + 5}.$$

Hmm, this doesn't appear to be the negative of the original function, because of the  $+1$  term in the numerator. It's not the original function either, so the function  $h$  is neither odd nor even.



Let's look at one more example. Suppose you want to prove that the product of two odd functions is always an even function. How would you go about doing this? Well, it helps to have names for things, so let's say we have two odd functions  $f$  and  $g$ . We need to look at the product of these functions, so let's call the product  $h$ . That is, we define  $h(x) = f(x)g(x)$ . So, our task is to show that  $h$  is even. We'll do this by showing that  $h(-x) = h(x)$ , as usual. It will be helpful to note that  $f(-x) = -f(x)$  and  $g(-x) = -g(x)$ , since  $f$  and  $g$  are odd. Let's start with  $h(-x)$ . Since  $h$  is the product of  $f$  and  $g$ , we have  $h(-x) = f(-x)g(-x)$ . Now we use the oddness of  $f$  and  $g$  to express this last term as  $(-f(x))(-g(x))$ . The minus signs now come out front and cancel out, so this is the same thing as  $f(x)g(x)$  which of course equals  $h(x)$ . We could (and should) express all this text mathematically like this:

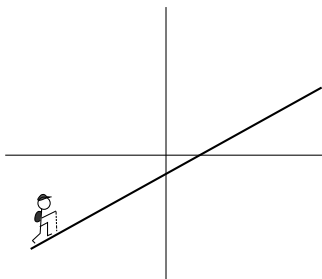
$$h(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = h(x).$$



Anyway, since  $h(-x) = h(x)$ , the function  $h$  is even. Now you should try to prove that the product of two even functions is always even, and also that the product of an odd and an even function must be odd. Go on, do it now!

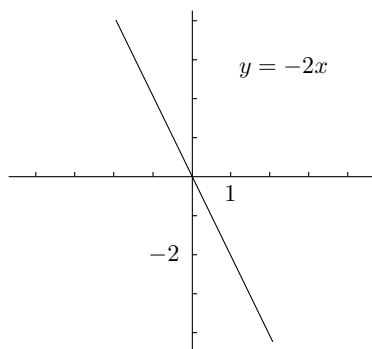
## 1.5 Graphs of Linear Functions

Functions of the form  $f(x) = mx + b$  are called *linear*. There's a good reason for this: the graphs of these functions are lines. (As far as we're concerned, the word "line" always means "straight line.") The slope of the line is given by  $m$ . Imagine for a moment that you are in the page, climbing the line as if it were a mountain. You start at the left side of the page and head to the right, like this:



If the slope  $m$  is positive, as it is in the above picture, then you are heading uphill. The bigger  $m$  is, the steeper the climb. On the other hand, if the slope is negative, then you are heading downhill. The more negative the slope, the steeper the downhill grade. If the slope is zero, then the line is flat, or horizontal—you're going neither uphill nor downhill, just trudging along a flat line.

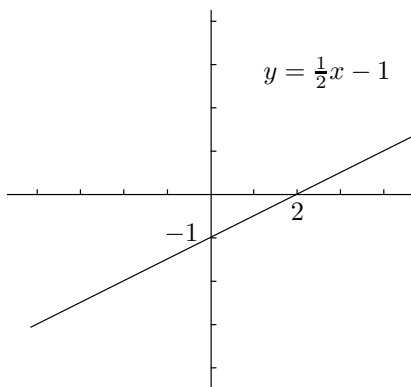
To sketch the graph of a linear function, you only need to identify two points on the graph. This is because there's only one line that goes through two different points. You just put your ruler on the points and draw the line. One point is easy to find, namely, the  $y$ -intercept. Set  $x = 0$  in the equation  $y = mx + b$ , and you see that  $y = m \times 0 + b = b$ . That is, the  $y$ -intercept is equal to  $b$ , so the line goes through  $(0, b)$ . To find another point, you could find the  $x$ -intercept by setting  $y = 0$  and finding what  $x$  is. This works pretty well except in two cases. The first case is when  $b = 0$ , in which case we are just dealing with  $y = mx$ . This goes through the origin, so the  $x$ -intercept and the  $y$ -intercept are both zero. To get another point, you'll just have to substitute in  $x = 1$  and see that  $y = m$ . So, the line  $y = mx$  goes through the origin and  $(1, m)$ . For example, the line  $y = -2x$  goes through the origin and also through  $(1, -2)$ , so it looks like this:





The other bad case is when  $m = 0$ . But then we just have  $y = b$ , which is a horizontal line through  $(0, b)$ .

For a more interesting example, consider  $y = \frac{1}{2}x - 1$ . The  $y$ -intercept is  $-1$ , and the slope is  $\frac{1}{2}$ . To sketch the line, find the  $x$ -intercept by setting  $y = 0$ . We get  $0 = \frac{1}{2}x - 1$ , which simplifies to  $x = 2$ . So, the line looks like this:



Now, let's suppose you know that you have a line in the plane, but you don't know its equation. If you know it goes through a certain point, and you know what its slope is, then you can find the equation of the line. You really, really, really need to know how to do this, since it comes up a lot. This formula, called the *point-slope* form of a linear function, is what you need to know:

If a line goes through  $(x_0, y_0)$  and has slope  $m$ , then its equation is  $y - y_0 = m(x - x_0)$ .



For example, what is the equation of the line through  $(-2, 5)$  which has slope  $-3$ ? It is  $y - 5 = -3(x - (-2))$ , which you can expand and simplify down to  $y = -3x - 1$ .

Sometimes you don't know the slope of the line, but you do know two points that it goes through. How do you find the equation? The technique is to find the slope, then use the previous idea with one of the points (your choice) to find the equation. First, you need to know this:

If a line goes through  $(x_1, y_1)$  and  $(x_2, y_2)$ , its slope is equal to  $\frac{y_2 - y_1}{x_2 - x_1}$ .



So, what is the equation of the line through  $(-3, 4)$  and  $(2, -6)$ ? Let's find the slope first:

$$\text{slope} = \frac{-6 - 4}{2 - (-3)} = \frac{-10}{5} = -2.$$

We now know that the line goes through  $(-3, 4)$  and has slope  $-2$ , so its equation is  $y - 4 = -2(x - (-3))$ , or after simplifying,  $y = -2x - 2$ . Alternatively, we could have used the other point  $(2, -6)$  with slope  $-2$  to see that the equation of the line is  $y - (-6) = -2(x - 2)$ , which simplifies to  $y = -2x - 2$ . Thankfully this is the same equation as before—it doesn't matter which point you pick, as long as you have used both points to find the slope.

## 1.6 Common Functions and Graphs

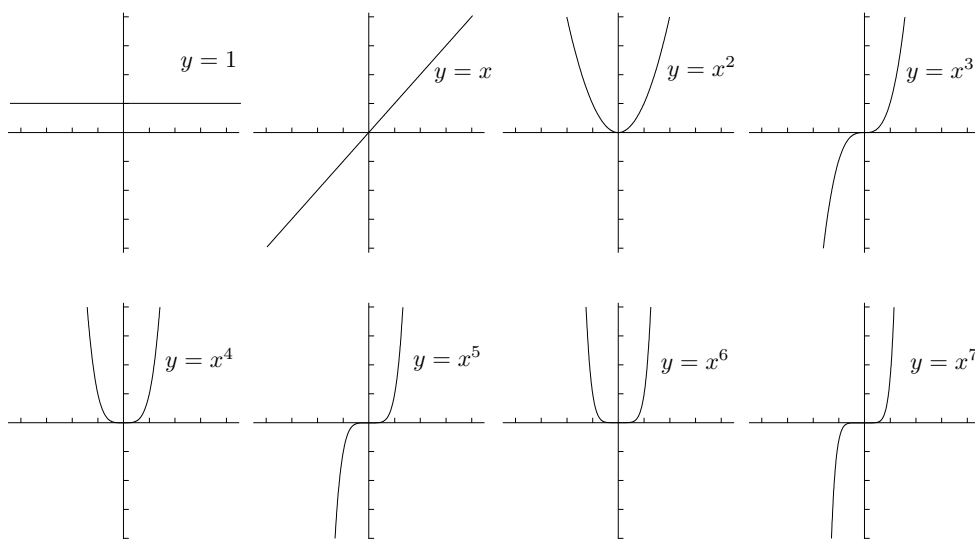
Here are the most important functions you should know about.

**1. Polynomials:** these are functions built out of nonnegative integer powers of  $x$ . You start with the building blocks  $1$ ,  $x$ ,  $x^2$ ,  $x^3$ , and so on, and you are allowed to multiply these basic functions by numbers and add a finite number of them together. For example, the polynomial  $f(x) = 5x^4 - 4x^3 + 10$  is formed by taking 5 times the building block  $x^4$ , and  $-4$  times the building block  $x^3$ , and 10 times the building block  $1$ , and adding them together. You might also want to include the intermediate building blocks  $x^2$  and  $x$ , but since they don't appear, you need to take 0 times of each. The amount that you multiply the building block  $x^n$  by is called the *coefficient* of  $x^n$ . For example, in the polynomial  $f$  above, the coefficient of  $x^4$  is 5, the coefficient of  $x^3$  is  $-4$ , the coefficients of  $x^2$  and  $x$  are both 0, and the coefficient of 1 is 10. (Why allow  $x$  and 1, by the way? They seem different from the other blocks, but they're not really:  $x = x^1$  and  $1 = x^0$ .) The highest number  $n$  such that  $x^n$  has a nonzero coefficient is called the *degree* of the polynomial. For example, the degree of the above polynomial  $f$  is 4, since no power of  $x$  greater than 4 is present. The mathematical way to write a general polynomial of degree  $n$  is

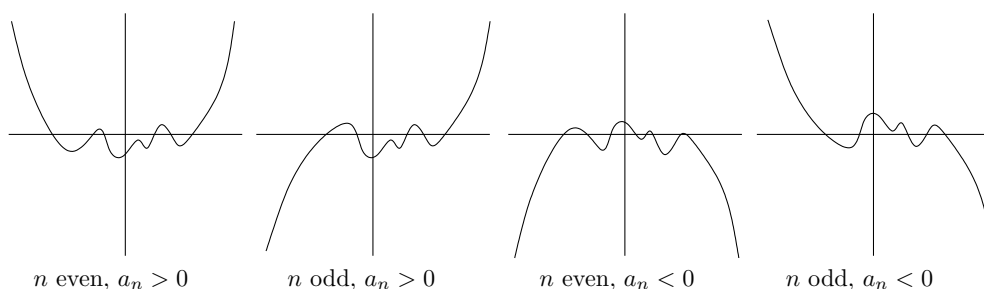
$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where  $a_n$  is the coefficient of  $x^n$ ,  $a_{n-1}$  is the coefficient of  $x^{n-1}$ , and so on down to  $a_0$ , which is the coefficient of 1.

Since the functions  $x^n$  are the building blocks of all polynomials, you should know what their graphs look like. The even powers mostly look similar to each other, and the same can be said for the odd powers. Here's what the graphs look like, from  $x^0$  up to  $x^7$ :



Sketching the graphs of more general polynomials is more difficult. Even finding the  $x$ -intercepts is often impossible unless the polynomial is very simple. There is one aspect of the graph that is fairly straightforward, which is what happens at the far left and right sides of the graph. This is determined by the so-called *leading coefficient*, which is the coefficient of the highest-degree term. This is basically the number  $a_n$  defined above. For example, in our polynomial  $f(x) = 5x^4 - 4x^3 + 10$  from above, the leading coefficient is 5. In fact, it only matters whether the leading coefficient is positive or negative. It also matters whether the degree of the polynomial is odd or even; so there are four possibilities for what the edges of the graph can look like:



The wiggles in the center of these diagrams aren't relevant—they depend on the other terms of the polynomial. The diagram is just supposed to show what the graphs look like near the left and right edges. In this sense, the graph of our polynomial  $f(x) = 5x^4 - 4x^3 + 10$  looks like the leftmost picture above, since  $n = 4$  is even and  $a_n = 5$  is positive.

Let's spend a little time on degree 2 polynomials, which are called *quadratics*. Instead of writing  $p(x) = a_2x^2 + a_1x + a_0$ , it's easier to write the coefficients as  $a$ ,  $b$ , and  $c$ , so we have  $p(x) = ax^2 + bx + c$ . Quadratics have two, one, or zero (real) roots, depending on the sign of the *discriminant*. The discriminant, which is often written as  $\Delta$ , is given by  $\Delta = b^2 - 4ac$ . There are three possibilities. If  $\Delta > 0$ , then there are two roots; if  $\Delta = 0$ , there is one root, which is called a *double root*; and if  $\Delta < 0$ , then there are no roots. In the first two cases, the roots are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$



Notice that the expression in the square root is just the discriminant. An important technique for dealing with quadratics is *completing the square*. Here's how it works. We'll use the example of the quadratic  $2x^2 - 3x + 10$ . The first step is to take out the leading coefficient as a factor. So our quadratic becomes  $2(x^2 - \frac{3}{2}x + 5)$ . This reduces the situation to dealing with a *monic* quadratic, which is a quadratic with leading coefficient equal to 1. So, let's worry about  $x^2 - \frac{3}{2}x + 5$ . The main technique now is to take the coefficient of  $x$ , which in our example is  $-\frac{3}{2}$ , divide it by 2 to get  $-\frac{3}{4}$ , and square it. We get  $\frac{9}{16}$ . We wish that the constant term were  $\frac{9}{16}$  instead of 5, so let's do some

mental gymnastics:

$$x^2 - \frac{3}{2}x + 5 = x^2 - \frac{3}{2}x + \frac{9}{16} + 5 - \frac{9}{16}.$$

Why on earth would we want to add and subtract  $\frac{9}{16}$ ? Because the first three terms combine to form  $(x - \frac{3}{4})^2$ . So, we have

$$x^2 - \frac{3}{2}x + 5 = \left(x^2 - \frac{3}{2}x + \frac{9}{16}\right) + 5 - \frac{9}{16} = \left(x - \frac{3}{4}\right)^2 + 5 - \frac{9}{16}.$$

Now we just have to work out the last little bit, which is just arithmetic:  $5 - \frac{9}{16} = \frac{71}{16}$ . Putting it all together, and restoring the factor of 2, we have

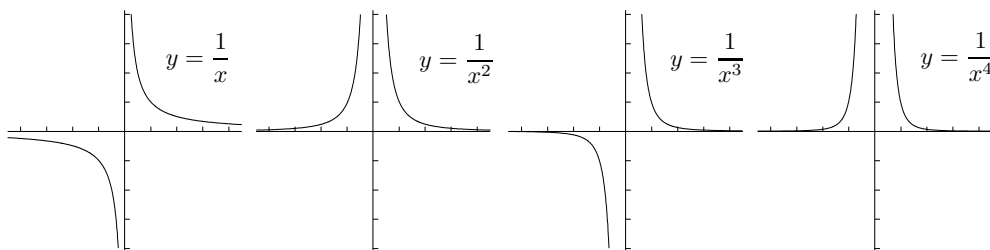
$$\begin{aligned} 2x^2 - 3x + 10 &= 2 \left(x^2 - \frac{3}{2}x + 5\right) = 2 \left(\left(x - \frac{3}{4}\right)^2 + \frac{71}{16}\right) \\ &= 2 \left(x - \frac{3}{4}\right)^2 + \frac{71}{8}. \end{aligned}$$

It turns out that this is a much nicer form to deal with in a number of situations. Make sure you know how to complete the square, since we'll be using this technique a lot in Chapters 18 and 19.

**2. Rational functions:** these are functions of the form

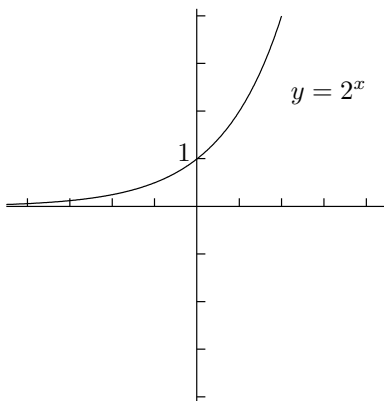
$$\frac{p(x)}{q(x)},$$

where  $p$  and  $q$  are polynomials. Rational functions will pop up in many different contexts, and the graphs can look very different depending on the polynomials  $p$  and  $q$ . The simplest examples of rational functions are polynomials themselves, which arise when  $q(x)$  is the constant polynomial 1. The next simplest examples are the functions  $1/x^n$ , where  $n$  is a positive integer. Let's look at some of the graphs of these functions:

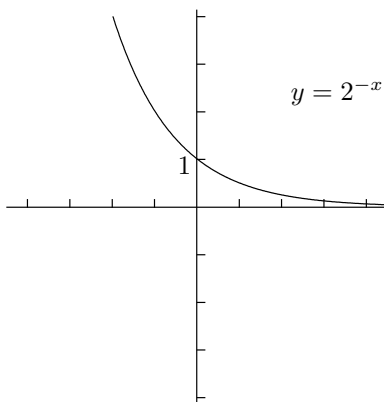


The odd powers look similar to each other, and the even powers look similar to each other. It's worth knowing what these graphs look like.

**3. Exponentials and logarithms:** you need to know what graphs of exponentials look like. For example, here is  $y = 2^x$ :

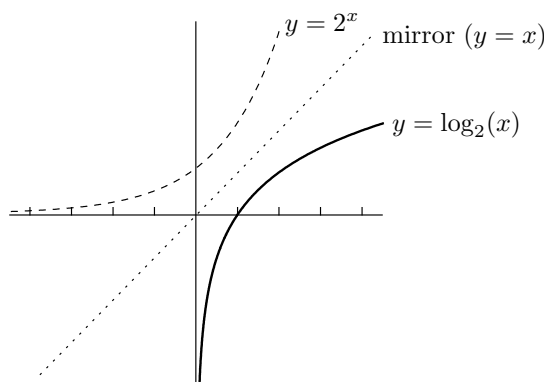


The graph of  $y = b^x$  for any other base  $b > 1$  looks similar to this. Things to notice are that the domain is the whole real line, the  $y$ -intercept is 1, the range is  $(0, \infty)$ , and there is a horizontal asymptote on the left at  $y = 0$ . In particular, the curve  $y = b^x$  does **not**, I repeat, not touch the  $x$ -axis, no matter what it looks like on your graphing calculator! (We'll be looking at asymptotes again in Chapter 3.) The graph of  $y = 2^{-x}$  is just the reflection of  $y = 2^x$  in the  $y$ -axis:



How about when the base is less than 1? For example, consider the graph of  $y = (\frac{1}{2})^x$ . Notice that  $(\frac{1}{2})^x = 1/2^x = 2^{-x}$ , so the above graph of  $y = 2^{-x}$  is also the graph of  $y = (\frac{1}{2})^x$ , since  $2^{-x}$  and  $(\frac{1}{2})^x$  are equal for any  $x$ . The same sort of thing happens for  $y = b^x$  for any  $0 < b < 1$ , not just  $b = \frac{1}{2}$ .

Now, notice that the graph of  $y = 2^x$  satisfies the horizontal line test, so there is an inverse function. This is in fact the base 2 logarithm, which is written  $y = \log_2(x)$ . Using the line  $y = x$  as a mirror, the graph of  $y = \log_2(x)$  looks like this:



The domain is  $(0, \infty)$ ; note that this backs up what I said earlier about not being able to take logarithms of a negative number or of 0. The range is all of  $(-\infty, \infty)$ , and there's a vertical asymptote at  $x = 0$ . The graphs of  $\log_{10}(x)$ , and indeed  $\log_b(x)$  for any  $b > 1$ , are very similar to this one. The log function is very important in calculus, so you should really know how to draw the above graph. We'll look at other properties of logarithms in Chapter 9.

**4. Trig functions:** these are so important that the entire next chapter is devoted to them.

**5. Functions involving absolute values:** let's take a close look at the *absolute value function*  $f$  given by  $f(x) = |x|$ . Here's the definition of  $|x|$ :

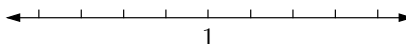
$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Another way of looking at  $|x|$  is that it is the distance between  $x$  and 0 on the number line. More generally, you should learn this nice fact:

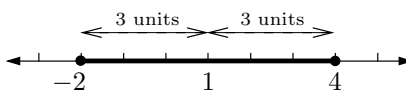
$|x - y|$  is the distance between  $x$  and  $y$  on the number line.



For example, suppose that you need to identify the region  $|x - 1| \leq 3$  on the number line. You can interpret the inequality as “the distance between  $x$  and 1 is less than or equal to 3.” That is, we are looking for all the points that are no more than 3 units away from the number 1. So, let's take a number line and mark in the number 1 as follows:



The points which are no more than 3 units away extend to  $-2$  on the left and 4 on the right, so the region we want looks like this:

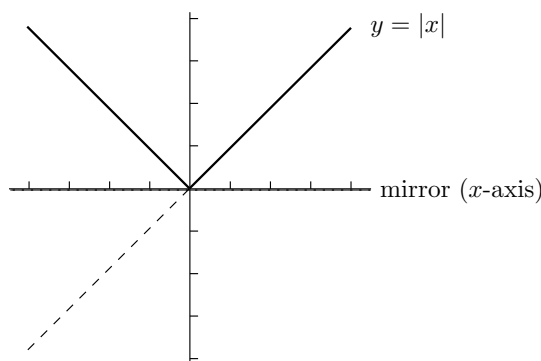


So, the region  $|x - 1| \leq 3$  can also be described as  $[-2, 4]$ .

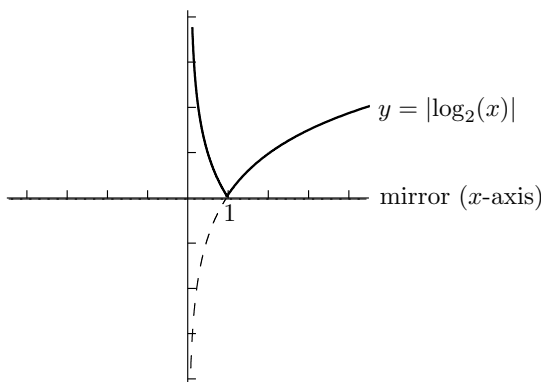


It's also true that  $|x| = \sqrt{x^2}$ . To check this, suppose that  $x \geq 0$ ; then  $\sqrt{x^2} = x$ , no problem. If instead  $x < 0$ , then it can't be true that  $\sqrt{x^2} = x$ , since the left-hand side is positive but the right-hand side is negative. The correct equation is  $\sqrt{x^2} = -x$ ; now the right-hand side is positive, since it's minus a negative number. If you look back at the definition of  $|x|$ , you'll see that we have just proved that  $|x| = \sqrt{x^2}$ . Even so, to deal with  $|x|$ , it's much better to use the piecewise definition than to write it as  $\sqrt{x^2}$ .

Finally, let's take a look at some graphs. If you know what the graph of a function looks like, you can get the graph of the absolute value of that function by reflecting everything below the  $x$ -axis up to above the  $x$ -axis, using the  $x$ -axis as your mirror. For example, here's the graph of  $y = |x|$ , which comes from reflecting the bottom portion of  $y = x$  in the  $x$ -axis:



How about the graph of  $y = |\log_2(x)|$ ? Using the reflection of the graph of  $y = \log_2(x)$  above, this is what the absolute value version looks like:



Anyway, that's all I have to say about functions, apart from trig functions which are the subject of the next chapter. Hopefully you've seen a lot of the stuff in this chapter before. Most of the material in this chapter is used over and over again in calculus, so make sure you really get on top of it all as soon as you can!

# CHAPTER 2

---

## Review of Trigonometry

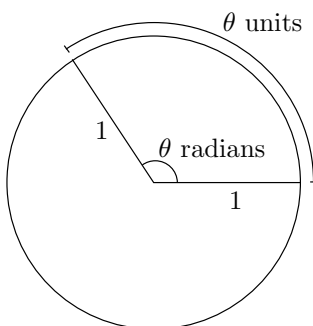
To do calculus, you really need to know trigonometry. Truth be told, we won't see much trig at first, but when it comes, it doesn't let up. So we might as well do a thorough review of the most important aspects of trig:

- angles in radians and the basics of the trig functions;
- trig functions on the real line (not just angles between  $0^\circ$  and  $90^\circ$ );
- graphs of trig functions; and
- trig identities.

Time to refresh your memory....

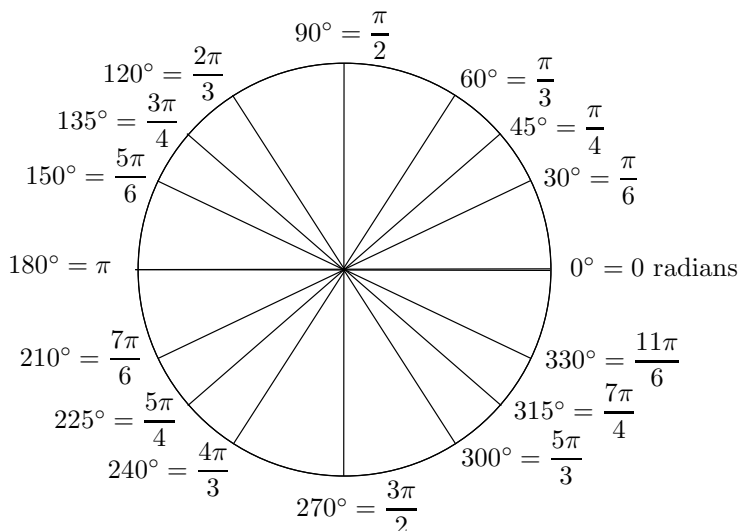
### 2.1 The Basics

The first thing I want to remind you about is the notion of radians. Instead of saying that there are 360 degrees in a full revolution, we'll say that there are  $2\pi$  *radians*. This may seem a bit wacky, but there is a reason: the circumference of a circle of radius 1 unit is  $2\pi$  units. In fact, the arc length of a wedge of this circle is exactly the angle of the wedge:



This picture is pretty and all, but the main thing is to be comfortable with the most common angles in both degree and radian form. First, you should become absolutely comfortable with the idea that  $90^\circ$  is the same as  $\pi/2$

radians, and similarly that  $180^\circ$  is the same as  $\pi$  radians and  $270^\circ$  is the same as  $3\pi/2$  radians. Once you have that in mind, try to be comfortable converting all the angles in the following picture back and forth between degrees and radians:



More generally, you can also use the formula

$$\text{angle in radians} = \frac{\pi}{180} \times \text{angle in degrees}$$

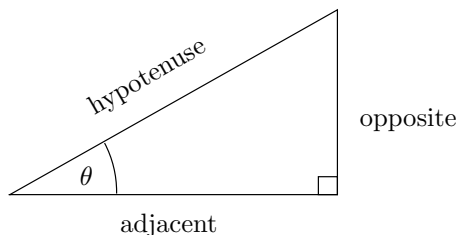


if you need to. For example, to see what  $5\pi/12$  radians is in degrees, solve

$$\frac{5\pi}{12} = \frac{\pi}{180} \times \text{angle in degrees}$$

to see that  $5\pi/12$  radians is the same as  $(180/\pi) \times (5\pi/12) = 75^\circ$ . In fact, you can think of this conversion from radians to degrees as a sort of change of units, like changing from miles to kilometers. The conversion factor is that  $\pi$  radians is the same as 180 degrees.

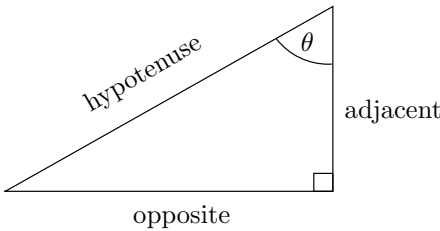
We have only looked at angles so far, so let's move on to trig functions. Obviously you have to know how the trig functions are defined in terms of triangles. Suppose you have a right-angled triangle and one of the angles, other than the right angle, is labeled  $\theta$ , like this:



Then the basic formulas are

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \text{and} \quad \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}.$$

Of course, if the angle  $\theta$  is moved, then the opposite and adjacent have to be moved as well:



The opposite is, unsurprisingly, opposite the angle  $\theta$  and the adjacent is next to it. The hypotenuse never changes, though: it is the longest side and is always across from the right angle.

We'll also be using the reciprocal functions  $\csc$ ,  $\sec$ , and  $\cot$ , which are defined as follows:

$$\csc(x) = \frac{1}{\sin(x)}, \quad \sec(x) = \frac{1}{\cos(x)}, \quad \text{and} \quad \cot(x) = \frac{1}{\tan(x)}.$$

Now, a piece of advice if you ever plan to take a calculus exam (or even if you don't!): learn the values of the trig functions at the common angles  $0$ ,  $\pi/6$ ,  $\pi/4$ ,  $\pi/3$ , and  $\pi/2$ . For example, without thinking, can you simplify  $\sin(\pi/3)$ ? How about  $\tan(\pi/4)$ ? If you can't, then at best you're wasting time trying to use a triangle to find the answer, and at worst you're throwing away easy points by not simplifying your answer all the way. The solution is to memorize the following table:



	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	*

The star means that  $\tan(\pi/2)$  is undefined. In fact, the  $\tan$  function has a vertical asymptote at  $\pi/2$  (this will be clear from the graph, which we'll look at in Section 2.3 below). Anyway, you need to be able to quote any of the entries in this table, both forward and backward! What this means is that you have to be able to answer two types of questions. Here are examples of each of these types:

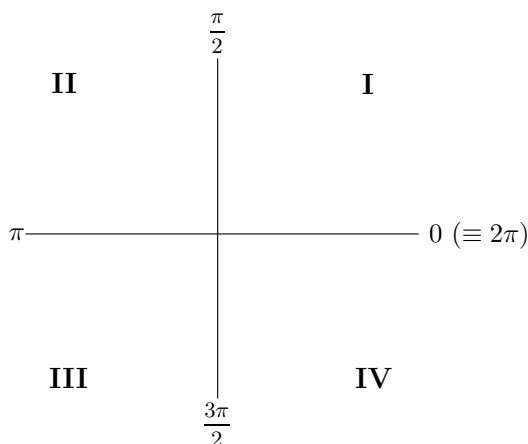
1. What is  $\sin(\pi/3)$ ? (Using the table, the answer is  $\sqrt{3}/2$ .)
2. What angle between  $0$  and  $\pi/2$  has a sine equal to  $\sqrt{3}/2$ ? (The answer is obviously  $\pi/3$ .)

Of course, you have to be able to answer these two types of questions for each entry in the table. Please, please, I beg of you, learn this table! Math isn't about memorization, but there are a few things that are worth memorizing and this table is definitely on the list. So make flash cards, get your friends to quiz you, spend one minute a day, whatever works for you, but learn the table.

## 2.2 Extending the Domain of Trig Functions

The above table (did you learn it yet?) only covers some angles ranging from 0 to  $\pi/2$ . It's possible to take  $\sin$  or  $\cos$  of any angle at all, even a negative one. For  $\tan$ , we have to be a little more careful—for example, we saw above that  $\tan(\pi/2)$  is undefined. Still, we'll be able to take  $\tan$  of just about every angle, even most negative ones.

Let's first look at angles between 0 and  $2\pi$  (remember that  $2\pi$  is the same as  $360^\circ$ ). Suppose you want to calculate  $\sin(\theta)$  (or  $\cos(\theta)$ , or  $\tan(\theta)$ ), where  $\theta$  is between 0 and  $2\pi$ . To see what this even means, start by drawing a coordinate plane with some slightly weird labels:

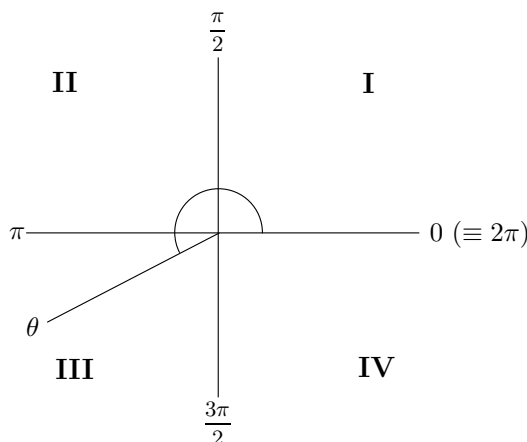


Notice that the axes divide the plane into four quadrants, which are creatively labeled from 1 to 4 (in Roman numerals), and that the labeling goes counterclockwise. These quadrants are called the first, second, third, and fourth quadrants, respectively. The next step is to draw a ray (that's half a line) starting at the origin. Which ray? It depends on  $\theta$ . Just imagine yourself standing at the origin, looking to the right along the positive  $x$ -axis. Now turn counterclockwise an angle of  $\theta$ , then march forward in a straight line. Your trail is the ray you're looking for.

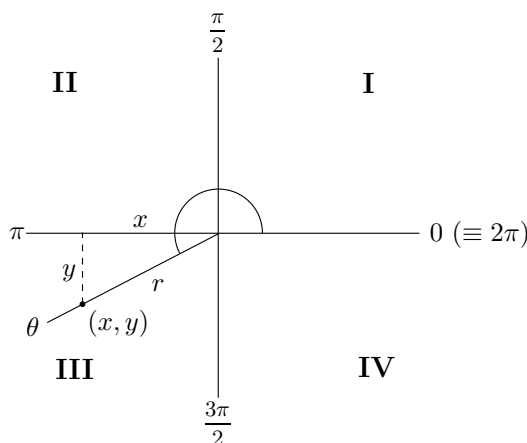
Now the other labels on the above picture (and the one on page 26) make a lot of sense. Indeed, if you turn an angle of  $\pi/2$ , you are facing up the page and you trace out the positive  $y$ -axis as you walk along. If you had instead turned an angle of  $\pi$ , you'd get the negative  $x$ -axis; and if you had turned  $3\pi/2$ , you'd get the negative  $y$ -axis. Finally, if you had turned  $2\pi$ , that would put you back to where you started, facing along the positive  $x$ -axis. It's the

same as if you hadn't turned at all! That's why the picture says  $0 \equiv 2\pi$ . As far as angles are concerned, 0 and  $2\pi$  are equivalent.

OK, let's take some angle  $\theta$  and draw in the appropriate ray. Perhaps it might be somewhere in the third quadrant, like this:



Notice that we label the **ray** as  $\theta$ , not the angle itself. Anyway, now we pick some point on the ray and drop a perpendicular from that point to the  $x$ -axis:



We're interested in three quantities: the  $x$ - and  $y$ -coordinates of the point (which are called  $x$  and  $y$ , of course!) and also the distance from the point to the origin, which is called  $r$ . Note that  $x$  and  $y$  could both potentially be negative—in fact, they both are negative in the above picture—but  $r$  is always positive, since it's a distance. In fact, by Pythagoras' Theorem, we have  $r = \sqrt{x^2 + y^2}$ , regardless of the signs of  $x$  and  $y$ . (The squares kill off any minus signs around.)

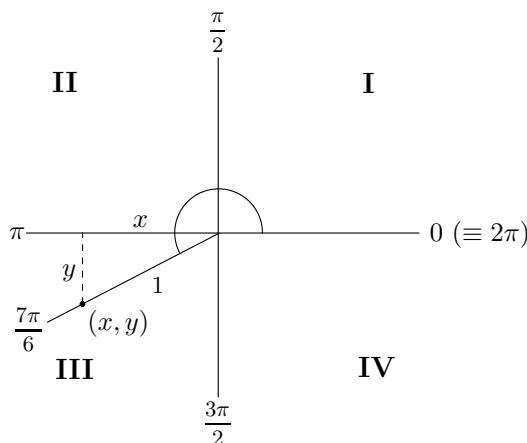
Armed with these three quantities, we can define the three trig functions as follows:

$$\sin(\theta) = \frac{y}{r}, \quad \cos(\theta) = \frac{x}{r}, \quad \text{and} \quad \tan(\theta) = \frac{y}{x}.$$

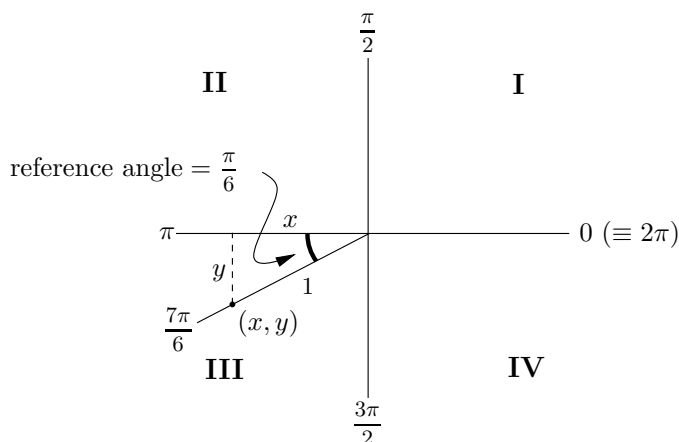
These are just the regular formulas from Section 2.1 above, with the quantities  $x$ ,  $y$ , and  $r$  interpreted as the adjacent, opposite, and hypotenuse, respectively. But wait, you say—what happens if you choose a different point on the ray? It doesn't matter, because your new triangle will be similar to the old one and the above ratios are unaffected. In fact, it is often convenient to assume that  $r = 1$ , so that the point  $(x, y)$  lies on the so-called *unit circle* (that's the circle of radius 1 centered at the origin).



Now let's look at an example. Suppose we want to find  $\sin(7\pi/6)$ . Which quadrant is  $7\pi/6$  in? We need to decide where  $7\pi/6$  fits in the list  $0, \pi/2, \pi, 3\pi/2, 2\pi$ . In fact,  $7\pi/6$  is greater than  $\pi$  but less than  $3\pi/2$ , so  $7\pi/6$  fits between  $\pi$  and  $3\pi/2$ . In fact, the picture looks pretty much like the above example:



So the angle  $7\pi/6$  is in the third quadrant. We've chosen the point on the ray which has distance  $r = 1$  from the origin, then dropped a perpendicular. We know from the above formulas that  $\sin(\theta) = y/r = y$  (since  $r = 1$ ), so we really need to find  $y$ . Well, that little angle between our ray at  $7\pi/6$  and the negative  $x$ -axis—which itself is at  $\pi$ —must be the difference between these two angles,  $\pi/6$ . The little angle is called the *reference angle*. In general, the reference angle for  $\theta$  is the smallest angle between the ray which represents  $\theta$  and the  $x$ -axis. It must be between  $0$  and  $\pi/2$ . In our example, the closest route to the  $x$ -axis is up, so the reference angle looks like this:



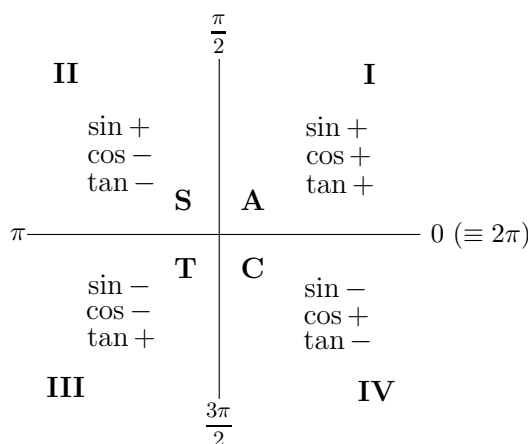
So in the little triangle, we know that  $r = 1$  and the angle is  $\pi/6$ . It looks like  $y = \sin(\pi/6) = 1/2$ , except that can't be right! Since we're below the  $x$ -axis, the quantity  $y$  must be negative. That is,  $y = -1/2$ . Since  $\sin(\theta) = y$ , we have shown that  $\sin(7\pi/6) = -1/2$ . We can also repeat this with cosine instead of sine to see that  $x = -\cos(\pi/6) = -\sqrt{3}/2$ . After all,  $x$  has to be negative, since the point  $(x, y)$  is to the left of the  $y$ -axis. This shows that  $\cos(7\pi/6) = -\sqrt{3}/2$  and we have identified our point  $(x, y)$  as  $(-\sqrt{3}/2, -1/2)$ .

## 2.2.1 The ASTC method

The key in the previous example is that  $\sin(7\pi/6)$  is related to  $\sin(\pi/6)$ , where  $\pi/6$  is the reference angle for  $7\pi/6$ . In fact, it's not hard to see that the sine of any angle is plus or minus the sine of the reference angle! This narrows it down to just two possibilities, and there's no need to mess around with  $x$ ,  $y$ , or  $r$ . So in our example, we just needed to find that the reference angle for  $7\pi/6$  is  $\pi/6$ ; this immediately told us that  $\sin(7\pi/6)$  is equal to either  $\sin(\pi/6)$  or  $-\sin(\pi/6)$  and we just had to make sure we got the correct one. We saw that it was the negative one because  $y$  was negative.

Actually, the sine of anything in the third or fourth quadrant must be negative because  $y$  is negative there. Similarly, the cosine of anything in the second or third quadrant must be negative, since  $x$  is negative there. The tangent is the ratio  $y/x$ , which is negative in the second and fourth quadrants (since one, but not both, of  $x$  and  $y$  is then negative) but positive in the first and third quadrants.

Let's summarize these findings in words as well as with a picture. First, all three functions are positive in the first quadrant (I). In the second quadrant (II), only  $\sin$  is positive; the other two functions are negative. In the third quadrant (III), only  $\tan$  is positive; the other two functions are negative. And finally, in the fourth quadrant (IV), only  $\cos$  is positive; the other two functions are negative. Here's what it all looks like:



In fact, the letters ASTC on the diagram are all you need to remember. They show you which of the functions are positive in that quadrant. “A” stands for “All,” meaning all the functions are positive in the first quadrant; the other letters obviously stand for  $\sin$ ,  $\tan$ , and  $\cos$ , respectively. In our example,



$7\pi/6$  is in the third quadrant, so only  $\tan$  is positive there. In particular,  $\sin$  is negative, so since we had narrowed the value of  $\sin(7\pi/6)$  down to  $1/2$  or  $-1/2$ , it must be the negative possibility: indeed,  $\sin(7\pi/6) = -1/2$ .

The only problem with the ASTC diagram is that it doesn't really tell you how to handle the angles  $0$ ,  $\pi/2$ ,  $\pi$ , or  $3\pi/2$ , since they lie on the axes. In this case, it's best to forget all about the ASTC stuff and draw a graph of  $y = \sin(x)$  (or  $\cos(x)$  or  $\tan(x)$ , as appropriate) and read the value off the graph. We'll discuss this in Section 2.3 below.

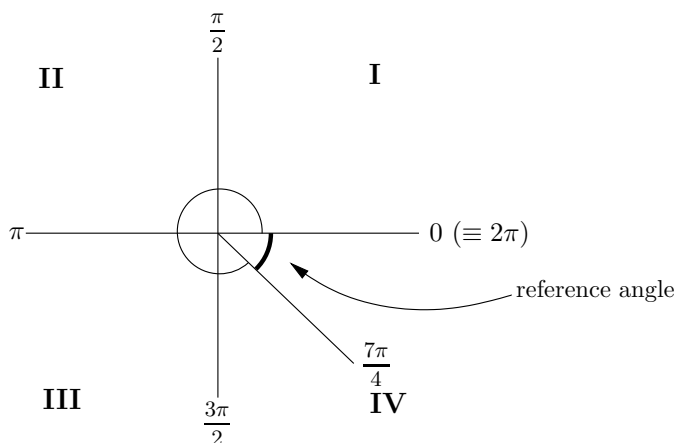


Meanwhile, here's a summary of the ASTC method for finding trig functions of angles between  $0$  and  $2\pi$ :

1. Draw the quadrant diagram, decide where in the picture the angle you care about is, and then mark that angle in the diagram.
2. If the angle you want is on the  $x$ - or  $y$ -axis (that is, not within any quadrant), draw a graph of the trig function and read the value off the graph (there are some examples in Section 2.3 below).
3. Otherwise, find the smallest angle between the one we want and the  $x$ -axis; this is called the reference angle.
4. If you can, use the important table to work out the value of the trig function of the reference angle. That's the answer you need, except that you might need a minus sign in front.
5. Use the ASTC diagram to decide whether or not you need a minus sign.



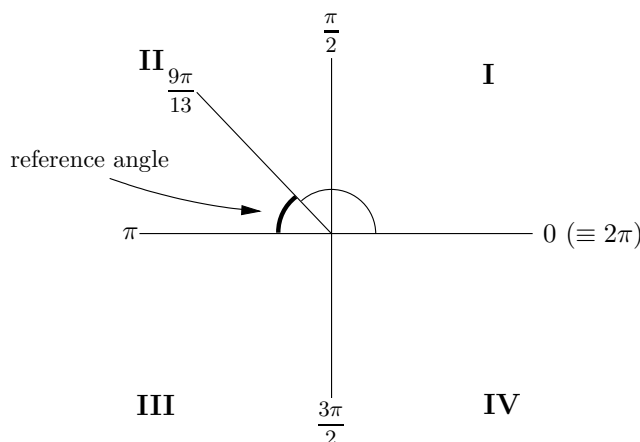
Let's look at a couple of examples. How would you find  $\cos(7\pi/4)$  and  $\tan(9\pi/13)$ ? We'll look at them one at a time. For  $\cos(7\pi/4)$ , we notice that  $7/4$  is between  $3/2$  and  $2$ , so the angle must be in the fourth quadrant:



To work out the reference angle, notice that we have to go up to  $2\pi$  (not down to  $0$ , beware!) so the reference angle is the difference between  $2\pi$  and  $7\pi/4$ , which is  $(2\pi - 7\pi/4)$  or simply  $\pi/4$ . So,  $\cos(7\pi/4)$  is plus or minus  $\cos(\pi/4)$ , which is  $1/\sqrt{2}$  according to our table. Is it plus or minus? The ASTC picture says that  $\cos$  is positive in the fourth quadrant, so it's plus:  $\cos(7\pi/4) = 1/\sqrt{2}$ .



Now let's look at  $\tan(9\pi/13)$ . We see that  $9/13$  is between  $1/2$  and  $1$ , so the angle  $9\pi/13$  is in the second quadrant:



This time we have to go up to  $\pi$  to get to the  $x$ -axis, so the reference angle is the difference between  $\pi$  and  $9\pi/13$ , which is  $\pi - 9\pi/13$  or simply  $4\pi/13$ . So, we know that  $\tan(9\pi/13)$  is plus or minus  $\tan(4\pi/13)$ . Alas, the number  $4\pi/13$  isn't in our table, so we can't simplify  $\tan(4\pi/13)$ . We also need to work out whether it's plus or minus. Well, the ASTC diagram shows that only sin is positive in the second quadrant, so tan must be negative there and we see that  $\tan(9\pi/13) = -\tan(4\pi/13)$ . That's as simplified as we can get without approximating. When solving calculus problems, I don't recommend approximating the answer unless you are explicitly asked to. A common misconception is that the number that comes out on the calculator when you calculate something like  $-\tan(4\pi/13)$  is the actual answer. On the contrary, it's just an approximation! So you shouldn't write

$$-\tan(4\pi/13) = -0.768438861,$$

since it's just not true. Instead, just leave it as  $-\tan(4\pi/13)$  unless you are specifically asked for an approximation. In that case, use the approximately-equal symbol and fewer decimal places, rounding appropriately (unless you are asked for more):

$$-\tan(4\pi/13) \cong -0.768.$$

By the way, you should rarely need to use a calculator—in fact, some colleges don't even allow them in exams! So you should try to avoid the temptation ever to use one.

## 2.2.2 Trig functions outside $[0, 2\pi]$

There's still the question of how to take trig functions of angles bigger than  $2\pi$  or less than  $0$ . In fact this isn't so bad: simply add or subtract multiples of  $2\pi$  until you get between  $0$  and  $2\pi$ . You see, it doesn't just stop at  $2\pi$ . It just keeps on wrapping around. For example, if I asked you to stand on the

spot facing due east and then turn around counterclockwise an angle of 450 degrees, it would be reasonable to assume that you'd turn a full revolution and then an extra 90 degrees. You'd be facing due north. Sure, you'd be a little dizzy than if you just did a 90-degree counterclockwise turn, but you'd be facing the same way. So 450 degrees is an equivalent angle to 90 degrees, and of course the same sort of thing is true in radians: in this case,  $5\pi/2$  radians is an equivalent angle to  $\pi/2$  radians. But why stop at one revolution? How about  $9\pi/2$  radians? That's the same as going around  $2\pi$  twice (which gets us up to  $4\pi$ ) and then an extra  $\pi/2$ , so we've done 2 useless revolutions before our final  $\pi/2$  twist. The revolutions don't matter, so once again  $9\pi/2$  is equivalent to  $\pi/2$ . This procedure can be extended indefinitely to get a whole family of angles which are equivalent to  $\pi/2$ :

$$\frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}, \frac{17\pi}{2}, \dots$$

Of course, each angle is a full revolution, or  $2\pi$ , more than the first one. Still, that's not the full story: if I'm going to insist that you do all these counterclockwise revolutions and get that dizzy, you might as well ask to be allowed to do a clockwise revolution or two to recover. This corresponds to a negative angle. In particular, if you were facing east and I asked you to turn  $-270$  degrees counterclockwise, the only sane interpretation of my bizarre request is to turn 270 degrees (or  $3\pi/2$ ) clockwise. Evidently you'll still end up facing due north, so  $-270$  degrees must be equivalent to 90 degrees. Indeed, adding 360 degrees to  $-270$  degrees just gives us 90 degrees. In radians, we see that  $-3\pi/2$  is an equivalent angle to  $\pi/2$ . In addition, we could insist on more negative (clockwise) full revolutions. In the end, here is the complete set of angles which are equivalent to  $\pi/2$ :

$$\dots, -\frac{15\pi}{2}, -\frac{11\pi}{2}, -\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}, \frac{17\pi}{2}, \dots$$

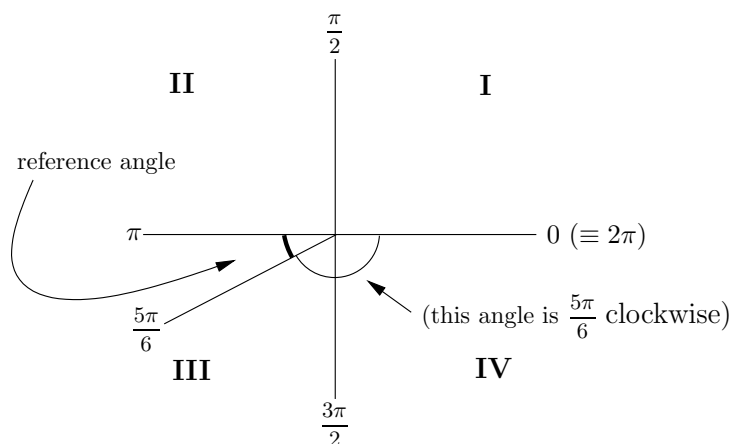
The sequence has no beginning or end; when I say it's "complete," I'm glossing over the fact that there are infinitely many angles included in the dots at the beginning and the end. We can avoid the dots by writing the collection in set notation as  $\{\pi/2 + 2\pi n\}$ , where  $n$  runs over all the integers.



Let's see if we can apply this. How would you find  $\sec(15\pi/4)$ ? The first thing to note is that if we can find  $\cos(15\pi/4)$ , all we need to do is take the reciprocal in order to get  $\sec(15\pi/4)$ . So let's find  $\cos(15\pi/4)$  first. Since  $15/4$  is more than 2, let's try lopping off 2 from it. Hmm,  $15/4 - 2 = 7/4$ , which is now between 0 and 2, so that looks promising. Restoring the  $\pi$ , we see that  $\cos(15\pi/4)$  is the same as  $\cos(7\pi/4)$  which we already saw is equal to  $1/\sqrt{2}$ . So,  $\cos(15\pi/4) = 1/\sqrt{2}$ . Taking reciprocals, we see that  $\sec(15\pi/4)$  is just  $\sqrt{2}$ .



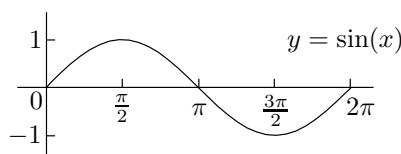
Finally, how about  $\sin(-5\pi/6)$ ? There are several ways of doing this problem, but the way suggested above is to try to add multiples of  $2\pi$  to  $-5\pi/6$  until we are between 0 and  $2\pi$ . In fact, adding  $2\pi$  to  $-5\pi/6$  gives  $7\pi/6$ , so  $\sin(-5\pi/6) = \sin(7\pi/6)$ , which we already saw is equal to  $-1/2$ . Alternatively, we could have drawn a diagram directly:



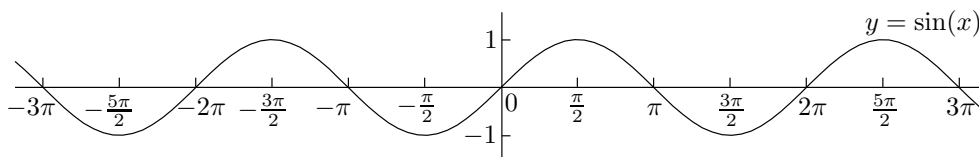
Now you have to work out the reference angle from the diagram, and it's not too hard to see that it is  $\pi/6$  and continue as before.

## 2.3 The Graphs of Trig Functions

It's really useful to remember what the graphs of the sin, cos, and tan functions look like. These functions are all *periodic*, meaning that they repeat themselves over and over again from left to right. For example, consider  $y = \sin(x)$ . The graph from 0 to  $2\pi$  looks like this:

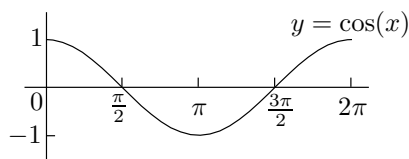


You should be able to produce this graph without thinking, including the positions of  $0$ ,  $\pi/2$ ,  $\pi$ ,  $3\pi/2$ , and  $2\pi$ . Since  $\sin(x)$  repeats every  $2\pi$  units (we say that  $\sin(x)$  is periodic in  $x$  with period  $2\pi$ ), we can extend the graph by repeating the pattern:

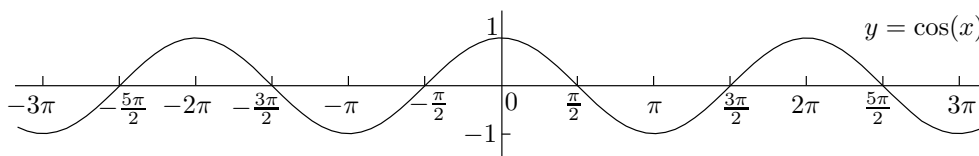


Just reading values off the graph, we can see that  $\sin(3\pi/2) = -1$  and  $\sin(-\pi) = 0$ . As noted earlier, this is how you should deal with multiples of  $\pi/2$ ; no need to mess around with reference angles. Another thing to note is that the graph has  $180^\circ$  point symmetry about the origin, which means that  $\sin(x)$  is an odd function of  $x$ . (We looked at odd and even functions in Section 1.4 of the previous chapter.)

The graph of  $y = \cos(x)$  is similar to that of  $y = \sin(x)$ . When  $x$  ranges from 0 to  $2\pi$ , it looks like this:

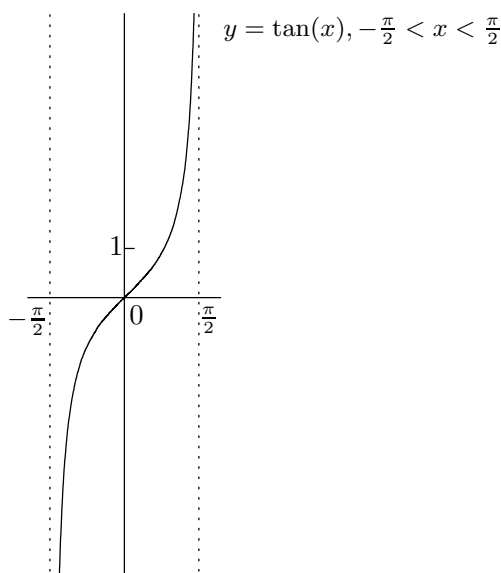


Now extend the graph using the fact that  $\cos(x)$  is periodic with period  $2\pi$ :

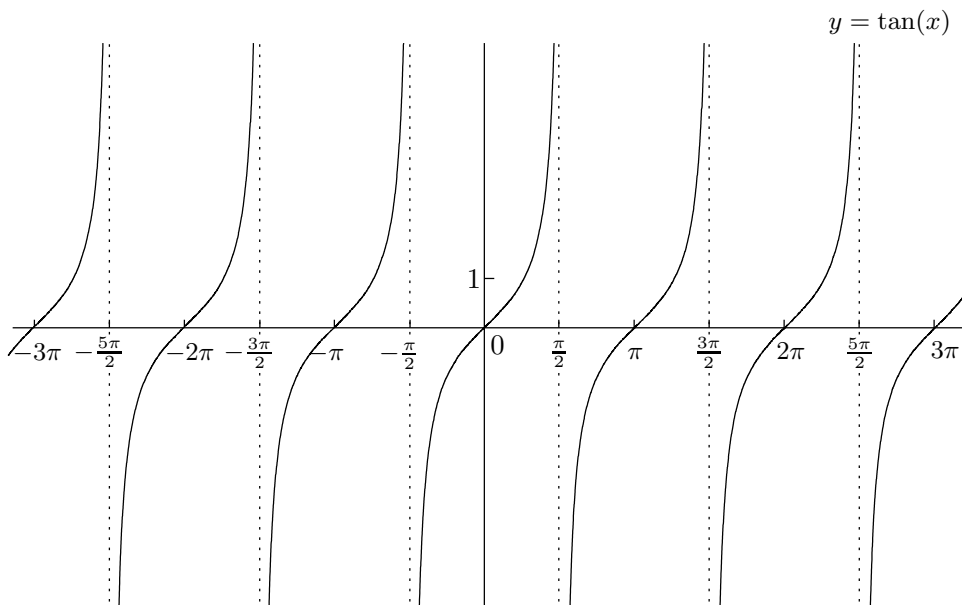


For example, if you want to find  $\cos(\pi)$ , you can see that it is  $-1$  just by reading it off the graph. Furthermore, notice that this time the graph has mirror symmetry in the  $y$ -axis. This means that  $\cos(x)$  is an even function of  $x$ .

Now  $y = \tan(x)$  is a little different. It's best to graph it for  $x$  between  $-\pi/2$  and  $\pi/2$  first:

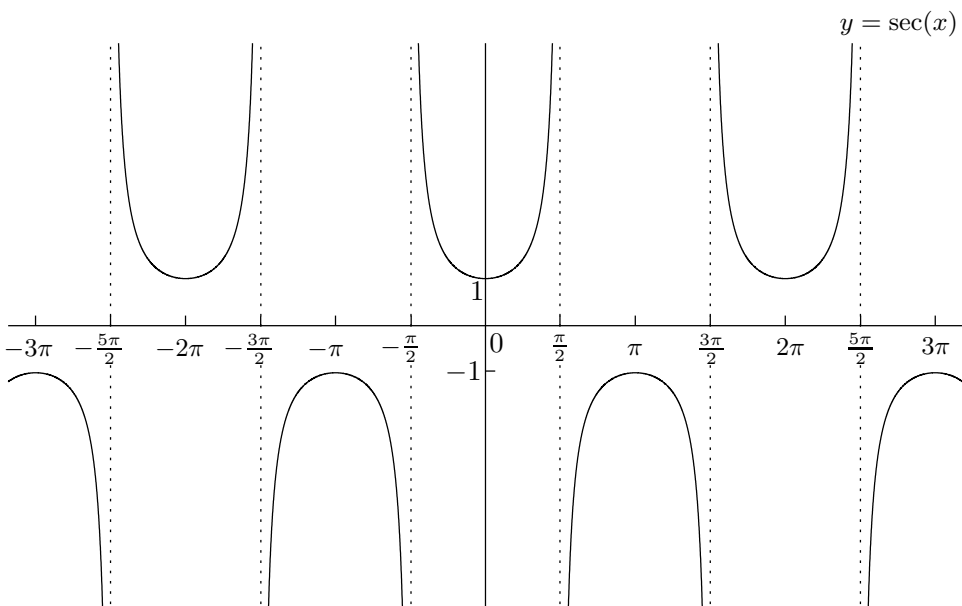


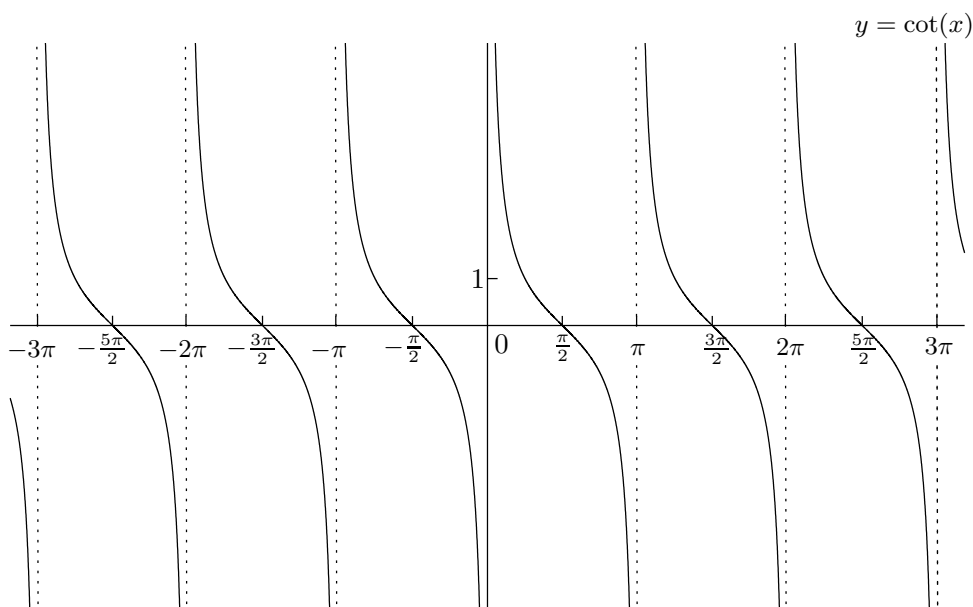
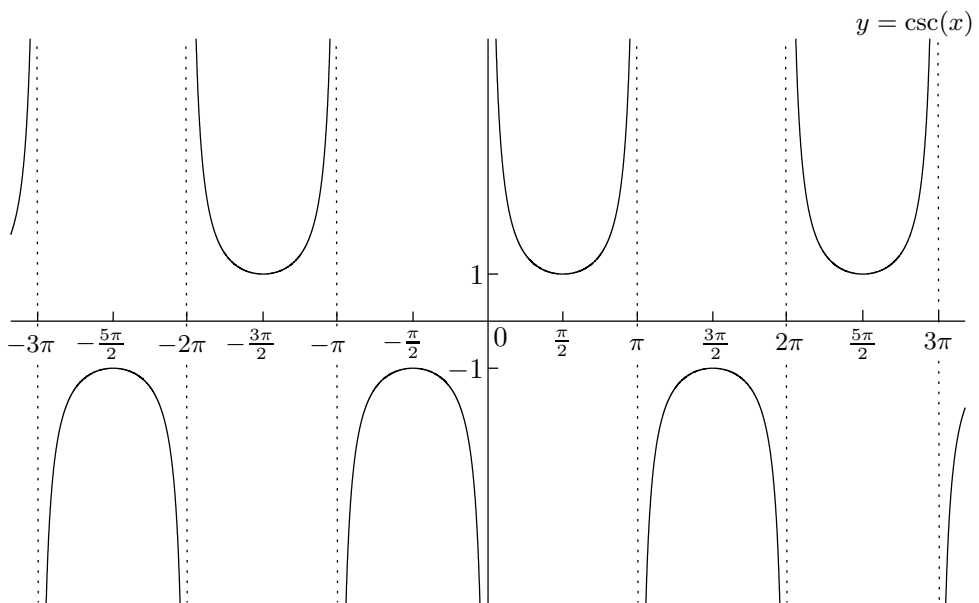
The tan function has vertical asymptotes, unlike the sin and cos functions. Also, its period is actually  $\pi$ , not  $2\pi$ , so the above pattern can be repeated to obtain the full graph of  $y = \tan(x)$ :



It's clear that  $y = \tan(x)$  has vertical asymptotes (and is consequently undefined) when  $x$  is any odd multiple of  $\pi/2$ . Also, the symmetry of the graph indicates that  $\tan(x)$  is an odd function of  $x$ .

It's also worthwhile learning the graphs of  $y = \sec(x)$ ,  $y = \csc(x)$ , and  $y = \cot(x)$ :





From their graphs, we can get the following symmetry properties of all six basic trig functions, which are worth learning:

$\sin(x)$ ,  $\tan(x)$ ,  $\cot(x)$ , and  $\csc(x)$  are odd functions of  $x$ .  
 $\cos(x)$  and  $\sec(x)$  are even functions of  $x$ .

So  $\sin(-x) = -\sin(x)$ ,  $\tan(-x) = -\tan(x)$ , and  $\cos(-x) = \cos(x)$  for all real numbers  $x$ .

## 2.4 Trig Identities

There are relations between trig functions which will come in handy. First, note that  $\tan$  and  $\cot$  may be expressed in terms of  $\sin$  and  $\cos$  as follows:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \cot(x) = \frac{\cos(x)}{\sin(x)}.$$

(Sometimes it's helpful to replace every instance of  $\tan$  or  $\cot$  by  $\sin$  and  $\cos$  using these identities, but you shouldn't really do this unless you're stuck.)

The most important of all the trig identities is Pythagoras' Theorem (written in trig form),

$$\cos^2(x) + \sin^2(x) = 1.$$

This is true for any  $x$ . (Why is this Pythagoras' Theorem? If the hypotenuse of a right-angled triangle is 1 and one of the angles is  $x$ , convince yourself that the other two sides of the triangle have lengths  $\cos(x)$  and  $\sin(x)$ .)



Now divide this equation by  $\cos^2(x)$ . I want you to check that you end up with

$$1 + \tan^2(x) = \sec^2(x).$$

This also comes up a lot in calculus. Alternatively, you could have divided the Pythagorean equation above by  $\sin^2(x)$  to get

$$\cot^2(x) + 1 = \csc^2(x).$$

This equation seems to come up less frequently than the others.

There are some more relationships between trig functions. Have you noticed that some of the names begin with the syllable “co”? This is short for the word “complementary.” To say that two angles are complementary means that they add up to  $\pi/2$  (or 90 degrees). It does not mean that they are nice to each other. All puns aside, the fact is that we have the following general relationship:

$$\text{trig function}(x) = \text{co-trig function}\left(\frac{\pi}{2} - x\right).$$

So in particular, we have

$$\sin(x) = \cos\left(\frac{\pi}{2} - x\right), \quad \tan(x) = \cot\left(\frac{\pi}{2} - x\right), \quad \text{and} \quad \sec(x) = \csc\left(\frac{\pi}{2} - x\right).$$

It even works when the trig function is already a “co”; you just have to realize that the complement of a complement is the original angle! For example, co-co-sine is really just sine, and co-co-tan is just tan. Basically this means that we can also say that

$$\cos(x) = \sin\left(\frac{\pi}{2} - x\right), \quad \cot(x) = \tan\left(\frac{\pi}{2} - x\right), \quad \text{and} \quad \csc(x) = \sec\left(\frac{\pi}{2} - x\right).$$

Finally, there's another group of identities which are worth learning. These are the identities involving sums of angles and the double-angle formulas.



Specifically, you should remember that

$$\begin{aligned}\sin(A + B) &= \sin(A) \cos(B) + \cos(A) \sin(B) \\ \cos(A + B) &= \cos(A) \cos(B) - \sin(A) \sin(B).\end{aligned}$$

It's also useful to remember that you can switch all the pluses and minuses to get some related formulas:

$$\begin{aligned}\sin(A - B) &= \sin(A) \cos(B) - \cos(A) \sin(B) \\ \cos(A - B) &= \cos(A) \cos(B) + \sin(A) \sin(B).\end{aligned}$$

A nice consequence of the  $\sin(A + B)$  and  $\cos(A + B)$  formulas in the box above is obtained by letting  $A = B = x$ . It's clear that the sine formula is  $\sin(2x) = 2 \sin(x) \cos(x)$ , but let's take a closer look at the cosine formula. This becomes  $\cos(2x) = \cos^2(x) - \sin^2(x)$ ; true as this is, it is more useful to use the Pythagorean identity  $\cos^2(x) + \sin^2(x) = 1$  to express  $\cos(2x)$  as either  $2 \cos^2(x) - 1$  or  $1 - 2 \sin^2(x)$  (convince yourself that these are both valid!). In summary, the double-angle formulas are

$$\begin{aligned}\sin(2x) &= 2 \sin(x) \cos(x) \\ \cos(2x) &= 2 \cos^2(x) - 1 = 1 - 2 \sin^2(x).\end{aligned}$$



So, how would you write  $\sin(4x)$  in terms of  $\sin(x)$  and  $\cos(x)$ ? Well, think of  $4x$  as double  $2x$  and use the sine identity to write  $\sin(4x) = 2 \sin(2x) \cos(2x)$ . Then use both identities to get

$$\sin(4x) = 2(2 \sin(x) \cos(x))(2 \cos^2(x) - 1) = 8 \sin(x) \cos^3(x) - 4 \sin(x) \cos(x).$$

Similarly,

$$\cos(4x) = 2 \cos^2(2x) - 1 = 2(2 \cos^2(x) - 1)^2 - 1 = 8 \cos^4(x) - 8 \cos^2(x) + 1.$$

You shouldn't memorize these last two formulas; instead, make sure you understand how to derive them using the double-angle formulas.

Now, if you can master all the trig in this chapter, you will be in very good shape indeed for the rest of the book. So don't leave it till too late—get cracking on a bunch of examples and make sure you learn the table and all the boxed formulas!

# CHAPTER 3

---

## Introduction to Limits

Calculus wouldn't exist without the concept of limits. This means that we are going to spend a lot of time looking at them. It turns out that it's pretty tricky to define a limit properly, but you can get an intuitive understanding of limits even without going into the gory details. This will be enough to tackle differentiation and integration. So, this chapter contains only the intuitive version; check out Appendix A for the formal version. All in all, here's what we'll look at in this chapter:

- an intuitive idea of what a limit is;
- left-hand, right-hand, and two-sided limits, and limits at  $\infty$  and  $-\infty$ ;
- when limits fail to exist; and
- the sandwich principle (also known as the “squeeze principle”).

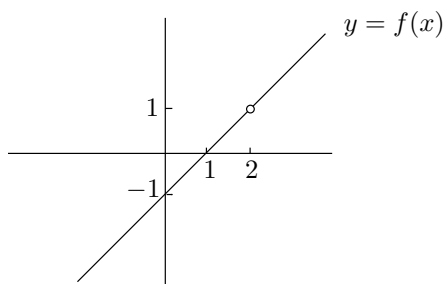
### 3.1 Limits: The Basic Idea

Let's dive in. We start with some function  $f$  and a point on the  $x$ -axis, which we'll call  $a$ . Here is what we'd like to understand: what does  $f(x)$  look like when  $x$  is really really close to  $a$ , **but not equal to  $a$** ? This is a pretty strange question to ask, which is probably why it took until relatively recently for humankind to develop calculus.

Here's an example showing why we might want to ask this question. Let  $f$  have domain  $\mathbb{R} \setminus \{2\}$  (all real numbers except for 2), and set  $f(x) = x - 1$  on this domain. Formally, you might write:

$$f(x) = x - 1 \quad \text{when } x \neq 2.$$

This seems like a weird sort of function: after all, why on earth would we want to exclude 2 from the domain? Actually, in the next chapter, we'll see that  $f$  arises quite naturally as a rational function (see the second example in Section 4.1). In the meantime, let's just take  $f$  for what it is and sketch a graph of it:



What is  $f(2)$ ? Perhaps you'd like to say that  $f(2) = 1$ , but that would be a load of bull since 2 isn't even in the domain of  $f$ . The best you can do is to say that  $f(2)$  is undefined. On the other hand, we can find the value of  $f(x)$  when  $x$  is really really close to 2 and see what happens. For example,  $f(2.01) = 1.01$ , and  $f(1.999) = 0.999$ . If you think about it, you can see that when  $x$  is really really close to 2, the value of  $f(x)$  is really really close to 1.

What's more, you can get as close as you want to 1, without actually getting to 1, by letting  $x$  be close enough to 2. For example, if you want  $f(x)$  to be within 0.0001 of 1, you could take any  $x$  between 1.9999 and 2.0001 (except of course for  $x = 2$ , which is forbidden). If you instead wanted  $f(x)$  to be within 0.000007 of 1, then you'd have to be a little more picky about your choice of  $x$ —this time you'd need to take  $x$  between 1.999993 and 2.000007 (except for  $x = 2$ , once again).

Anyway, these ideas are described in much greater detail in Section A.1 of Appendix A. Without getting bogged down, let's cut to the chase and just write

$$\lim_{x \rightarrow 2} f(x) = 1.$$

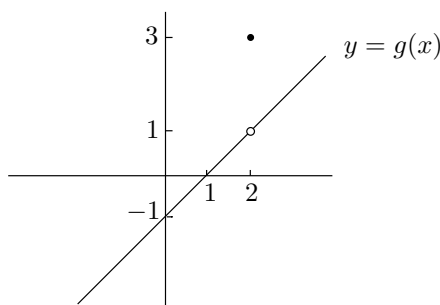
If you read this out loud, it should sound like “the limit, as  $x$  goes to 2, of  $f(x)$  is equal to 1.” Again, this means that when  $x$  is near 2 (but not equal to it), the value of  $f(x)$  is near 1. How near? As near as your heart desires. Another way of writing the above statement is

$$f(x) \rightarrow 1 \quad \text{as } x \rightarrow 2.$$

This is harder to do computations with, but its meaning is quite clear: as  $x$  journeys along the number line from the left or the right toward the number 2, the value of  $f(x)$  gets very very close to 1 (and stays close!).



Now, let's take the above function  $f$  and modify it slightly. Indeed, suppose that a new function  $g$  has the following graph:



The domain of  $g$  is all real numbers, and  $g(x)$  can be defined in piecewise fashion as

$$g(x) = \begin{cases} x - 1 & \text{if } x \neq 2, \\ 3 & \text{if } x = 2. \end{cases}$$

What is  $\lim_{x \rightarrow 2} g(x)$ ? The trick here is that the value of  $g(2)$  is irrelevant! It's only the values of  $g(x)$  where  $x$  is close to 2, not actually **at** 2, which matter. Ignoring  $x = 2$ , the function  $g$  is identical to the function  $f$  we looked at earlier. So,  $\lim_{x \rightarrow 2} g(x) = 1$  as before, even though  $g(2) = 3$ .

Here's an important point: when you write something like

$$\lim_{x \rightarrow 2} f(x) = 1,$$

the left-hand side isn't actually a function of  $x$ ! Remember, the equation means that  $f(x)$  is close to 1 when  $x$  is close to 2. We could actually replace  $x$  by any other letter and this would still be true. For example,  $f(q)$  is close to 1 when  $q$  is close to 2, so we have

$$\lim_{q \rightarrow 2} f(q) = 1.$$

We can go nuts with this and also write

$$\lim_{b \rightarrow 2} f(b) = 1, \quad \lim_{z \rightarrow 2} f(z) = 1, \quad \lim_{\alpha \rightarrow 2} f(\alpha) = 1,$$

and so on until we run out of letters and symbols! The point is that in the limit

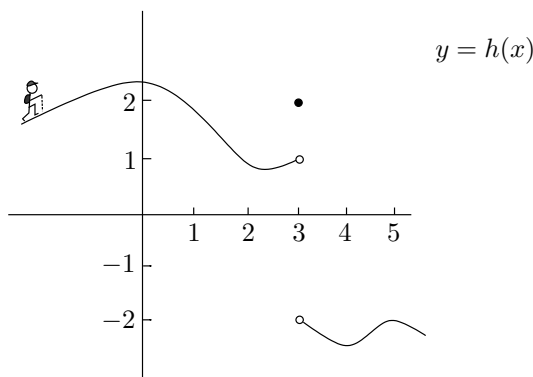
$$\lim_{x \rightarrow 2} f(x) = 1,$$

the variable  $x$  is just a *dummy variable*. It is a temporary label for some quantity that is (in this case) getting very close to 2. It can be replaced by any other letter, as long as you swap it out wherever else it appears; also, when you work out the value of the limit, the answer cannot include the dummy variable. So be smart about your dummy variables.

## 3.2 Left-Hand and Right-Hand Limits



We've seen that limits describe the behavior of a function near a certain point. Think about how you would describe the behavior of  $h(x)$  near  $x = 3$ :



Of course, the fact that  $h(3) = 2$  is irrelevant as far as the limiting behavior is concerned. Now, what happens when you approach  $x = 3$  from the left? Imagine that you're the hiker in the picture, climbing up and down the hill. The value of  $h(x)$  tells you how high up you are when your horizontal position is at  $x$ . So, if you walk rightward from the left of the picture, then when your horizontal position is close to 3, your height is close to 1. Sure, there's a sheer drop when you get to  $x = 3$  (not to mention a weird little ledge floating in space above you!), but we don't care about this for the moment. Everything to the right of  $x = 3$ , including  $x = 3$  itself, is irrelevant. So we've just seen that the *left-hand limit* of  $h(x)$  at  $x = 3$  is equal to 1.

On the other hand, if you are walking leftward from the right-hand side of the picture, your height becomes close to  $-2$  as your horizontal position gets close to  $x = 3$ . This means that the *right-hand limit* of  $h(x)$  at  $x = 3$  is equal to  $-2$ . Now everything to the left of  $x = 3$  (including  $x = 3$  itself) is irrelevant!

We can summarize our findings from above by writing

$$\lim_{x \rightarrow 3^-} h(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow 3^+} h(x) = -2.$$

The little minus sign after the 3 in the first limit above means that the limit is a left-hand limit, and the little plus sign in the second limit means that it's a right-hand limit. It's important to write the minus or plus sign **after** the 3, not before it! For example, if you write

$$\lim_{x \rightarrow -3} h(x),$$

then you are referring to the regular two-sided limit of  $h(x)$  at  $x = -3$ , not the left-hand limit at  $x = 3$ . These are two very different animals indeed. By the way, the reason that you write  $x \rightarrow 3^-$  under the limit sign for the left-hand limit is that this limit only involves values of  $x$  less than 3. That is, you need to take a little bit away from 3 to see what's going on. In a similar manner, when you write  $x \rightarrow 3^+$  for the right-hand limit, this means that you only need to consider what happens when you add a little bit onto 3.



Now, limits don't always exist, as we'll see in the next section. But here's something important: the regular two-sided limit at  $x = a$  exists **exactly when** both left-hand and right-hand limits at  $x = a$  exist **and are equal to each other!** In that case, all three limits—two-sided, left-hand, and right-hand—are the same. In math-speak, I'm saying that

$$\lim_{x \rightarrow a^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a^+} f(x) = L$$

is the same thing as

$$\lim_{x \rightarrow a} f(x) = L.$$

If the left-hand and right-hand limits are not equal, as in the case of our function  $h$  from above, then the two-sided limit does not exist. We'd just write

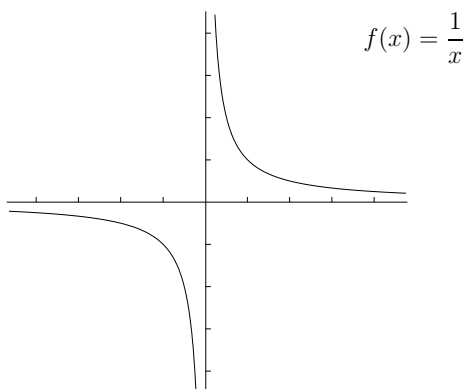
$$\lim_{x \rightarrow 3} h(x) \text{ does not exist}$$

or you could even write "DNE" instead of "does not exist."

## 3.3 When the Limit Does Not Exist



We just saw that a two-sided limit doesn't exist when the corresponding left-hand and right-hand limits are different. Here's an even more dramatic example of this. Consider the graph of  $f(x) = 1/x$ :



What is  $\lim_{x \rightarrow 0} f(x)$ ? It may be a bit much to expect the two-sided limit to exist here, so let's first try to find the right-hand limit,  $\lim_{x \rightarrow 0^+} f(x)$ . Looking at the graph, it seems as though  $f(x)$  is very large when  $x$  is positive and close to 0. It doesn't really get close to any number in particular as  $x$  slides down to 0 from the right; it just gets larger and larger. How large? Larger than anything you can imagine! We say that the limit is infinity, and write

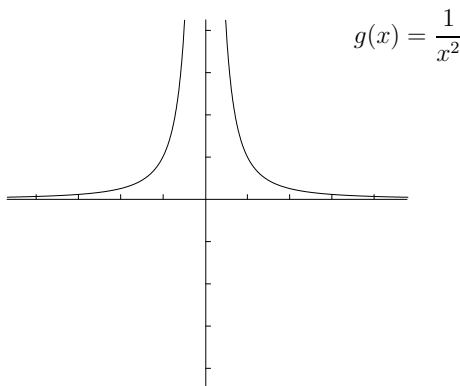
$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty.$$

Similarly, the left-hand limit here is  $-\infty$ , since  $f(x)$  gets arbitrarily more and more negative as  $x$  slides upward to 0. That is,

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty.$$



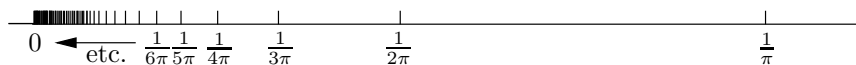
The two-sided limit certainly doesn't exist, since the left-hand and right-hand limits are different. On the other hand, consider the function  $g$  defined by  $g(x) = 1/x^2$ . The graph looks like this:



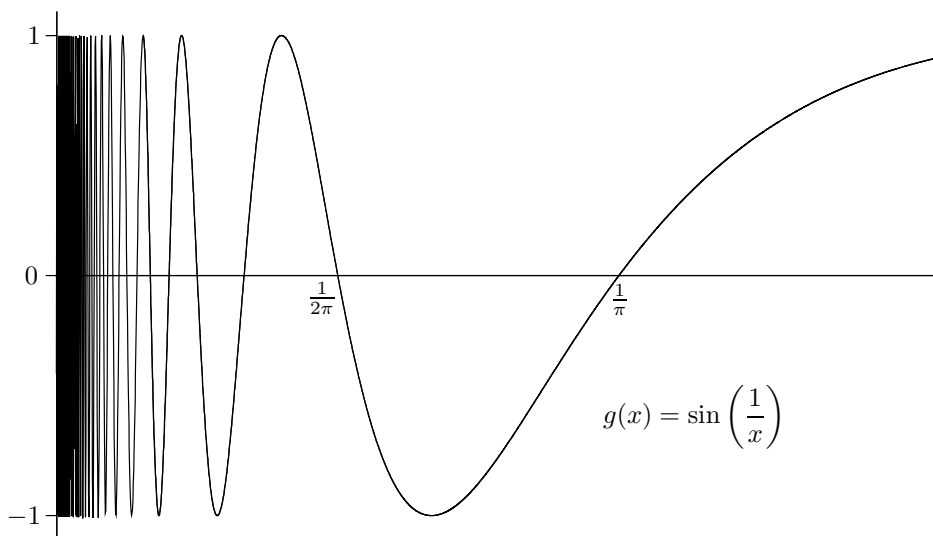
Both the left-hand and right-hand limits at  $x = 0$  are  $\infty$ , so you can say that  $\lim_{x \rightarrow 0} 1/x^2 = \infty$  as well. By the way, we now have a formal definition of the term “vertical asymptote”:

“ $f$  has a vertical asymptote at  $x = a$ ” means that at least one of  $\lim_{x \rightarrow a^+} f(x)$  and  $\lim_{x \rightarrow a^-} f(x)$  is equal to  $\infty$  or  $-\infty$ .

Now, is it possible that even a left-hand or right-hand limit fails to exist? The answer is yes! For example, let’s meet the funky function  $g$  defined by  $g(x) = \sin(1/x)$ . What does the graph of this function look like? Let’s worry about the positive values of  $x$  first. Since  $\sin(x)$  has zeroes at the values  $x = \pi, 2\pi, 3\pi, \dots$ , then  $\sin(1/x)$  will have zeroes when  $1/x = \pi, 2\pi, 3\pi, \dots$ . Taking reciprocals, we see that  $\sin(1/x)$  has zeroes when  $x = \frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \dots$ . These numbers are the  $x$ -intercepts of  $\sin(1/x)$ . On the number line, this is what they look like:



As you see, they really bunch up as you get close to 0. Now,  $\sin(x)$  goes up to 1 or down to  $-1$  between every  $x$ -intercept, so  $\sin(1/x)$  does the same. Let’s graph what we know so far:



So what is  $\lim_{x \rightarrow 0^+} \sin(1/x)$ ? The above graph is a real mess near  $x = 0$ . It oscillates infinitely often between 1 and  $-1$ , faster and faster as you move from the right toward  $x = 0$ . There is no vertical asymptote, but there’s still no limit.\* The function doesn’t tend toward any one number as  $x$  goes to 0 from the right. So, we say that  $\lim_{x \rightarrow 0^+} \sin(1/x)$  does not exist (DNE). We’ll finish the graph of  $y = \sin(1/x)$  in the next section.

---

\*See Section A.3.4 of Appendix A for a real proof of this.

## 3.4 Limits at $\infty$ and $-\infty$

There is one more type of limit that we need to investigate. We've concentrated on the behavior of a function near a point  $x = a$ . However sometimes it is important to understand how a function behaves when  $x$  gets really huge. Another way of saying this is that we are interested in the behavior of a function as its argument  $x$  goes to  $\infty$ . We'd like to write something like

$$\lim_{x \rightarrow \infty} f(x) = L$$

and mean that  $f(x)$  gets really close, and stays close, to the value  $L$  when  $x$  is large. (More details can be found in Section A.3.3 of Appendix A.) The important thing to realize is that writing " $\lim_{x \rightarrow \infty} f(x) = L$ " indicates that the graph of  $f$  has a right-hand horizontal asymptote at  $y = L$ . There is a similar notion for when  $x$  heads toward  $-\infty$ : we write

$$\lim_{x \rightarrow -\infty} f(x) = L,$$

which means that  $f(x)$  gets extremely close, and stays close, to  $L$  when  $x$  gets more and more negative (or more precisely,  $-x$  gets larger and larger). This of course corresponds to the graph of  $y = f(x)$  having a left-hand horizontal asymptote. You can turn these definitions around if you like and say:

" $f$  has a right-hand horizontal asymptote at  $y = L$ "  
means that  $\lim_{x \rightarrow \infty} f(x) = L$ .  
" $f$  has a left-hand horizontal asymptote at  $y = M$ "  
means that  $\lim_{x \rightarrow -\infty} f(x) = M$ .

Of course, something like  $y = x^2$  doesn't have any horizontal asymptotes: the values of  $y$  just go up and up as  $x$  gets larger. In symbols, we can write this as  $\lim_{x \rightarrow \infty} x^2 = \infty$ . Alternatively, the limit may not even exist. For example, what is  $\lim_{x \rightarrow \infty} \sin(x)$ ? Well, what value is  $\sin(x)$  getting closer and closer to (and staying close)? It's just oscillating back and forth between  $-1$  and  $1$ , so it never really gets anywhere. There's no horizontal asymptote, nor does the function wander off to  $\infty$  or  $-\infty$ ; the best you can do is to say that  $\lim_{x \rightarrow \infty} \sin(x)$  does not exist (DNE). Again, see Section A.3.4 of Appendix A for a proof of this.



Let's return to the function  $f$  given by  $f(x) = \sin(1/x)$  that we looked at in the previous section. What happens when  $x$  gets very large? Well, when  $x$  is large,  $1/x$  is very close to 0. Since  $\sin(0) = 0$ , it should be true that  $\sin(1/x)$  is also very close to 0. The larger  $x$  gets, the closer  $\sin(1/x)$  is to 0. My argument has been a little sketchy but hopefully you're convinced that\*

$$\lim_{x \rightarrow \infty} \sin(1/x) = 0.$$

So  $\sin(1/x)$  has a horizontal asymptote at  $y = 0$ . This allows us to extend the graph of  $y = \sin(1/x)$  that we drew above, at least to the right. We should

---

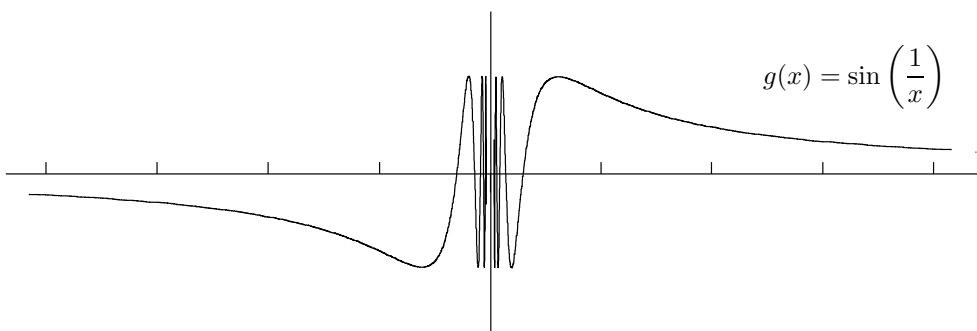
\*If not, see Section A.4.1 of Appendix A!



still worry about what happens when  $x < 0$ . This isn't too bad, since  $f$  is an odd function. Here's why:

$$f(-x) = \sin\left(\frac{1}{-x}\right) = \sin\left(-\frac{1}{x}\right) = -\sin\left(\frac{1}{x}\right) = -f(x).$$

Note that we used the fact that  $\sin(x)$  is an odd function of  $x$  to get from  $\sin(-1/x)$  to  $-\sin(1/x)$ . So, since odd functions have that nice symmetry about the origin (see Section 1.4 in Chapter 1), we can complete the graph of  $y = \sin(1/x)$  as follows:



Again, it's hard to draw what happens for  $x$  near 0. The closer  $x$  is to 0, the more wildly the function oscillates, and of course the function is undefined at  $x = 0$ . In the above picture, I chose to avoid the black smudge in the middle and just leave the oscillations up to your imagination.

### 3.4.1 Large numbers and small numbers

I hope we can all agree that 1,000,000,000,000 is a large number. So how about  $-1,000,000,000,000$ ? Perhaps controversially, I want you to think of this as a large negative number rather than a small number. An example of a small number would be 0.000000001, while  $-0.000000001$  is a small number too—more precisely, a small negative number. Funnily enough, we're not going to think of 0 itself as being small: it's just zero. So our informal definition of large numbers and small numbers looks like this:

- A number is *large* if its absolute value is a really big number.
- A number is *small* if it is really close to 0 (but not actually equal to 0).

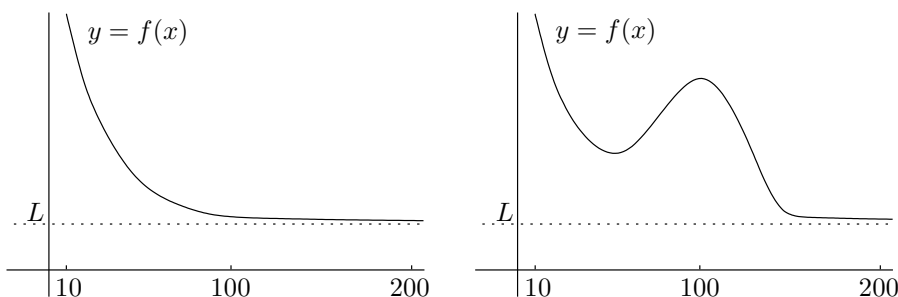


Although the above definition will serve us well in practice, it's a really lame definition. What do I mean by “really big” and “really close to 0”? Well, consider the limit equation

$$\lim_{x \rightarrow \infty} f(x) = L.$$

As we saw above, this means that when  $x$  is a large enough number, the value of  $f(x)$  is almost  $L$ . The question is, how large is “large enough”? It depends on how close to  $L$  you want  $f(x)$  to be! Still, from a practical point of view,

a number  $x$  is large enough if the graph of  $y = f(x)$  starts looking like it's getting serious about snuggling up to the horizontal asymptote at  $y = L$ . Of course, everything depends on what the function  $f$  is, as you can see from the following picture:



In both cases,  $f(10)$  is nowhere near  $L$ . In the left-hand picture, it looks like  $f(x)$  is pretty close to  $L$  when  $x$  is at least 100, so any number above 100 would be large. In the right-hand picture,  $f(100)$  is far away from  $L$ , so now 100 isn't large enough. You probably need to go up to about 200 in this case. So can't you just pick a number like 1,000,000,000,000 and say that it's always large? Nope—a function might wander around until 5,000,000,000,000 before it starts getting close to its horizontal asymptote. The point is that the term “large” has to be taken in context, relative to some function or limit. Luckily, there's plenty of room up above—even a number like 1,000,000,000,000 is pretty puny compared to  $10^{100}$  (a googol), which itself is chicken feed in comparison with  $10^{1000000}$ , and so on. By the way, we'll often use the term “near  $\infty$ ” in place of “large and positive.” (A number can't really be near  $\infty$  in the literal sense, since  $\infty$  is so far away from everything. The term “near  $\infty$ ” makes sense, though, in the context of limits as  $x \rightarrow \infty$ .)

Of course, all this also applies to limits as  $x \rightarrow -\infty$ , except that you just stick a minus sign in front of all the large positive numbers above. In this case we'll sometimes say “near  $-\infty$ ” to emphasize that we are referring to large negative numbers.

On the other hand, we'll often be looking at limit equations of the form

$$\lim_{x \rightarrow 0} f(x) = L, \quad \lim_{x \rightarrow 0^+} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow 0^-} f(x) = L.$$

In all three of these cases, we know that when  $x$  is close enough to 0, the value of  $f(x)$  is almost  $L$ . (For the right-hand limit,  $x$  also has to be positive, while for the left-hand limit,  $x$  has to be negative.) Again, how close does  $x$  have to be to 0? It depends on the function  $f$ . So, when we say a number is “small” (or “near 0”), we'll have to take this in the context of some function or limit, just as in the case of “large.”

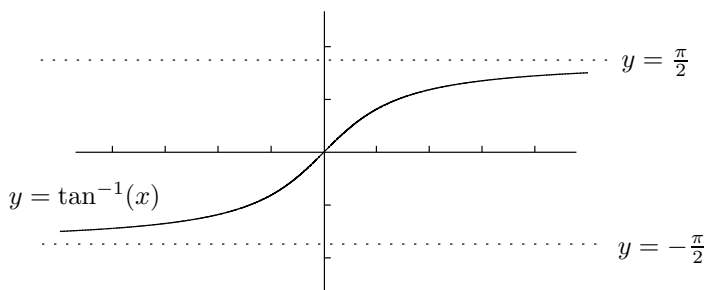
Although this discussion really tightens up the above lame definition, it's still not perfect. If you want to learn more, you should really check out Sections A.1 and A.3.3 in Appendix A.

## 3.5 Two Common Misconceptions about Asymptotes

Now seems like a good time to correct a couple of common misconceptions about horizontal asymptotes. First, a function doesn't have to have the same horizontal asymptote on the left as on the right. In the graph of  $f(x) = 1/x$  on page 45 above, there is a horizontal asymptote at  $y = 0$  on both the right-hand side and the left-hand side—which means that

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \rightarrow -\infty} -\frac{1}{x} = 0.$$

However, consider the graph of  $y = \tan^{-1}(x)$  (or if you prefer,  $y = \arctan(x)$ —this is the inverse tangent function and you can write it either way):

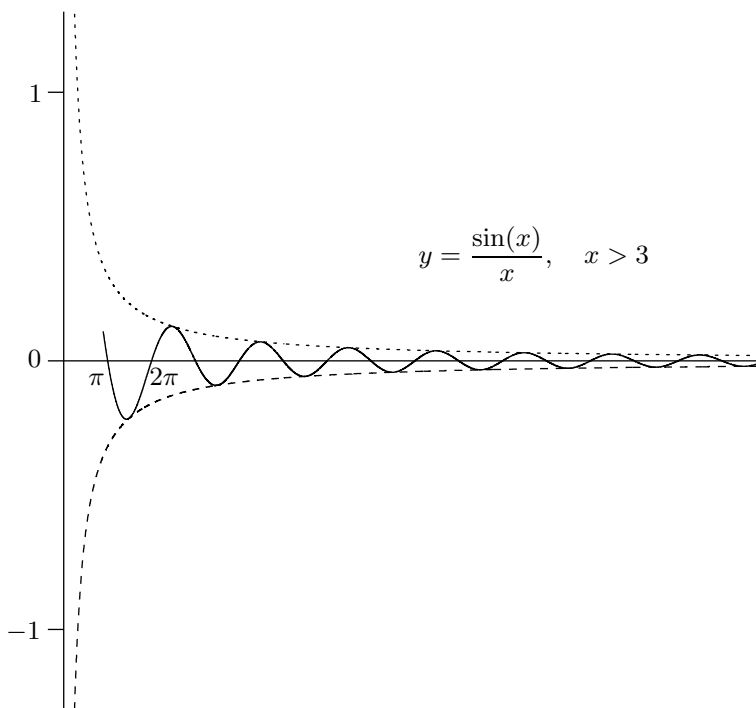


This function has a right-hand horizontal asymptote at  $y = \pi/2$  and a left-hand horizontal asymptote at  $y = -\pi/2$ ; these are not the same. We can also express this in terms of limits:

$$\boxed{\lim_{x \rightarrow \infty} \tan^{-1}(x) = \frac{\pi}{2}} \quad \text{and} \quad \boxed{\lim_{x \rightarrow -\infty} \tan^{-1}(x) = -\frac{\pi}{2}}$$

So a function can indeed have different right- and left-hand horizontal asymptotes, but there can be at most two horizontal asymptotes—one on the right and one on the left. It might have none or one: for example,  $y = 2^x$  has a left-hand horizontal asymptote but not a right-hand one (see the graph on page 22). This is in contrast to vertical asymptotes: a function can have as many of those as it feels like (for example,  $y = \tan(x)$  has infinitely many).

Another common misconception is that a function can't cross its asymptote. Perhaps you have learned that an asymptote is a line that a function gets closer and closer to without ever crossing. This just isn't true, at least when you're talking about horizontal asymptotes. For example, consider the function  $f$  given by  $f(x) = \sin(x)/x$ , where for the moment we only care about what happens when  $x$  is positive and large. The value of  $\sin(x)$  oscillates between  $-1$  and  $1$ , so the value of  $\sin(x)/x$  oscillates between the curves  $y = -1/x$  and  $y = 1/x$ . Also,  $\sin(x)/x$  has the same zeroes as  $\sin(x)$  does, namely  $\pi, 2\pi, 3\pi, \dots$ . Putting it all together, the graph looks like this:



The curves  $y = 1/x$  and  $y = -1/x$ , which are drawn as dotted curves in the graph, form what's called the envelope of the sine wave. In any event, as you can see from the graph, if there's any justice in the world, then it should be true that

$$\lim_{x \rightarrow \infty} \frac{\sin(x)}{x} = 0.$$

This means that the  $x$ -axis is a horizontal asymptote for  $f$ , even though the graph of  $y = f(x)$  crosses the axis over and over again. Now, to justify the above limit, we'll need to apply something called the sandwich principle. The justification is at the end of the next section.

## 3.6 The Sandwich Principle



The *sandwich principle*, also known as the *squeeze principle*, says that if a function  $f$  is sandwiched between two functions  $g$  and  $h$  that converge to the same limit  $L$  as  $x \rightarrow a$ , then  $f$  **also** converges to  $L$  as  $x \rightarrow a$ .

Here's a more precise statement of the principle. Suppose that for all  $x$  near  $a$ , we have  $g(x) \leq f(x) \leq h(x)$ . That is,  $f(x)$  is sandwiched (or squeezed) between  $g(x)$  and  $h(x)$ . Also, let's suppose that  $\lim_{x \rightarrow a} g(x) = L$  and  $\lim_{x \rightarrow a} h(x) = L$ . Then we can conclude that  $\lim_{x \rightarrow a} f(x) = L$ ; that is, all three functions have the same limit as  $x \rightarrow a$ . As usual, the picture tells the story: