DYNAMIC ASSET PRICING THEORY

THIRD EDITION

DARRELL DUFFIE

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For Colin

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Preface

This book is an introduction to the theory of portfolio choice and asset pricing in multiperiod settings under uncertainty. An alternate title might be *Arbitrage, Optimality, and Equilibrium,* because the book is built around the three basic constraints on asset prices: absence of arbitrage, single-agent optimality, and market equilibrium. The most important unifying principle is that any of these three conditions implies that there are "state prices," meaning positive discount factors, one for each state and date, such that the price of any security is merely the state-price weighted sum of its future payoffs. This idea can be traced to the invention by Arrow (1953) of the general equilibrium model of security markets. Identifying the state prices is the major task at hand. Technicalities are given relatively little emphasis so as to simplify these concepts and to make plain the similarities between discrete- and continuous-time models.ricing model.

To someone who came out of graduate school in the mid-eighties, the decade spanning roughly 1969–79 seems like a golden age of dynamic asset pricing theory. Robert Merton started continuous-time financial modeling with his explicit dynamic programming solution for optimal portfolio and consumption policies. This set the stage for his 1973 general equilibrium model of security prices, another milestone. His next major contribution was his arbitrage-based proof of the option pricing formula introduced by Fisher Black and Myron Scholes in 1973, and his continual development of that approach to derivative pricing. The Black-Scholes model now seems to be, by far, the most important single breakthrough of this "golden decade," and ranks alone with the Modigliani and Miller (1958) Theorem and the Capital Asset Pricing Model (CAPM) of Sharpe (1964) and Lintner (1965) in its overall importance for financial theory and practice. A tremendously influential simplification of the Black-Scholes model appeared in the "binomial" option pricing model of Cox, Ross, and Rubinstein (1979), who drew on an insight of Bill Sharpe.

Working with discrete-time models, LeRoy (1973), Rubinstein (1976), and Lucas (1978) developed multiperiod extensions of the CAPM. The "Lucas model" is the "vanilla flavor" of equilibrium asset pricing models. The simplest multiperiod representation of the CAPM finally appeared in Doug Breeden's continuous-time consumption-based CAPM, published in 1979. Although not published until 1985, the Cox-Ingersoll-Ross model of the term structure of interest rates appeared in the mid-seventies and is still the premier textbook example of a continuous-time general equilibrium asset pricing model with practical applications. It also ranks as one of the key breakthroughs of that decade. Finally, extending the ideas of Cox and Ross (1976) and Ross (1978), Harrison and Kreps (1979) gave an almost definitive conceptual structure to the whole theory of dynamic security prices.

Theoretical developments in the period since 1979, with relatively few exceptions, have been a mopping-up operation. Assumptions have been weakened, there have been noteworthy extensions and illustrative models, and the various problems have become much more unified under the umbrella of the Harrison-Kreps model of equivalent martingale measures. For example, the standard approach to optimal portfolio and consumption choice in continuous-time settings has become the martingale method of Cox and Huang (1989). An essentially final version of the relationship between the absence of arbitrage and the existence of equivalent martingale measures was finally obtained by Delbaen and Schachermayer (1999).

On the applied side, markets have experienced an explosion of new valuation techniques, hedging applications, and security innovation, much of this based on the Black-Scholes and related arbitrage models. No major investment bank, for example, lacks the experts or computer technology required to implement advanced mathematical models of the term structure. Because of the wealth of new applications, there has been a significant development of special models to treat stochastic volatility, jump behavior including default, and the term structure of interest rates, along with many econometric advances designed to take advantage of the resulting improvements in richness and tractability.

Although it is difficult to predict where the theory will go next, in order to promote faster progress by people coming into the field it seems wise to have some of the basics condensed into a textbook. This book is designed to be a streamlined course text, not a research monograph. Much generality is sacrificed for expositional reasons, and there is relatively little emphasis on mathematical rigor or on the existence of general equilibrium. As its title indicates, I am treating only the theoretical side of the story. Although it might be useful to tie the theory to the empirical side of asset pricing, we have excellent treatments of the econometric modeling of financial data, such as Campbell, Lo, and MacKinlay (1997) and Gourieroux and Jasiak (2000). I also leave out some important aspects of functioning security markets, such as asymmetric information and transactions costs. I have chosen to develop only some of the essential ideas of dynamic asset pricing, and even these are more than enough to put into one book or into a one-semester course.

Other books whose treatments overlap with some of the topics treated here include Avelleneda and Laurence (2000), Björk (1998), Dana and Jeanblanc (1998), Demange and Rochet (1992), Dewynne and Wilmott (1994), Dixit and Pindyck (1993), Dothan (1990), Duffie (1988b), Harris (1987), Huang and Litzenberger (1988), Ingersoll (1987), Jarrow (1988), Karatzas (1997), Karatzas and Shreve (1998), Lamberton and Lapeyre (1997), Magill and Quinzii (1994), Merton (1990), Musiela and Rutkowski (1997), Neftci (2000), Stokey and Lucas (1989), Willmott, Dewynne, and Howison (1993), and Wilmott, Howison, and Dewynne (1995). Each has its own aims and themes. I hope that readers will find some advantage in having yet another perspective.

A reasonable way to teach a shorter course on continuous-time asset pricing out of this book is to begin with Chapter 1 or 2 as an introduction to the basic notion of state prices and then to go directly to Chapters 5 through 11. Chapter 12, on numerical methods, could be skipped at some cost to the student's ability to implement the results. There is no direct dependence of any results in Chapters 5 through 12 on the first four chapters.

For mathematical preparation, little beyond undergraduate analysis, as in Bartle (1976), and linear algebra is assumed. Some familiarity with Royden (1968) or a similar text on functional analysis and measure theory, would also be useful. Some background in microeconomics would be useful, say Kreps (1990) or Luenberger (1995). Familiarity with probability theory at the level of Jacod and Protter (2000), for example, would also speed things along, although measure theory is not used heavily. In any case, a series of appendices supplies all of the required concepts and definitions from probability theory and stochastic calculus. Additional useful references in this regard are Brémaud (1981), Karatzas and Shreve (1988), Revuz and Yor (1991), and Protter (1990).

Students seem to learn best by doing problem exercises. Each chapter has exercises and notes to the literature. I have tried to be thorough in giving sources for results whenever possible and plead that any cases in which I have mistaken or missed sources be brought to my attention for correction. The notation and terminology throughout is fairly standard. I use $\mathbb{R}$ to denote the real line and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ for the extended real line. For any set Z and positive integer n, I use Z^n for the set of n-tuples of the form $(z_1, \ldots, z_n)$ with z_i in Z for all i. An example is $\mathbb{R}^n$. The conventions used for inequalities in any context are

- $x \geq 0$ means that x is nonnegative. For x in $\mathbb{R}^n$, this is equivalent to $x \in \mathbb{R}^n_+$;
- $x > 0$ means that x is nonnegative and not zero, but not necessarily strictly positive in all coordinates;
- $x \gg 0$ means x is strictly positive in every possible sense. The phrase "x is strictly positive" means the same thing. For x in $\mathbb{R}^n$, this is equivalent to $x \in \mathbb{R}^n_{++} \equiv \text{int}(\mathbb{R}^n_+)$.

Although warnings will be given at appropriate times, it should be kept in mind that $X = Y$ will be used to mean equality almost everywhere or almost surely, as the case may be. The same caveat applies to each of the above inequalities. A real-valued function F on an ordered set (such as $\mathbb{R}^n$) is *increasing* if $F(x) \geq F(y)$ whenever $x \geq y$ and *strictly increasing* if $F(x) > F(y)$ whenever $x > y$. When the domain and range of a function are implicitly obvious, the notation "$x \mapsto F(x)$" means the function that maps x to $F(x)$; for example, $x \mapsto x^2$ means the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = x^2$. Also, while warnings appear at appropriate places, it is worth pointing out again here that, for ease of exposition, a continuous-time "process" will be defined throughout as a jointly (product) measurable function on $\Omega \times [0, T]$, where $[0, T]$ is the given time interval and $(\Omega, \mathcal{F}, P)$ is the given underlying probability space.

The first four chapters are in a discrete-time setting with a discrete set of states. This should ease the development of intuition for the models to be found in Chapters 5 through 12. The three pillars of the theory, *arbitrage, optimality,* and *equilibrium,* are developed repeatedly in different settings. Chapter 1 is the basic single-period model. Chapter 2 extends the results of Chapter 1 to many periods. Chapter 3 specializes Chapter 2 to a Markov setting and illustrates dynamic programming as an alternate solution technique. The Ho-and-Lee and Black-Derman-Toy term-structure models are included as exercises. Chapter 4 is an infinite-horizon counterpart to Chapter 3 that has become known as the *Lucas model.*

The focus of the theory is the notion of state prices, which specify the price of any security as the state-price weighted sum or expectation of the security's state-contingent dividends. In a finite-dimensional setting, there exist state prices if and only if there is no arbitrage. The same fact is true in infinite-dimensional settings under mild technical regularity conditions. Given an agent's optimal portfolio choice, a state-price vector is given by that agent's utility gradient. In an equilibrium with Pareto optimality, a state-price vector is likewise given by a representative agent's utility gradient at the economy's aggregate consumption process.

Chapters 5 through 11 develop a continuous-time version of the theory in which uncertainty is generated by Brownian motion. In Chapter 11, there is a transition to discontinuous information, that is, settings in which the conditional probability of some events does not adjust continuously with the passage of time. An example is Poisson arrival.

Chapter 5 introduces the continuous-trading model and develops the Black-Scholes partial differential equation (PDE) for arbitrage-free prices of derivative securities. The Harrison-Kreps model of equivalent martingale measures is presented in Chapter 6 in parallel with the theory of state prices in continuous time. Chapter 7 presents models of the term structure of interest rates, including the Black-Derman-Toy, Vasicek, Cox-Ingersoll-Ross, and Heath-Jarrow-Morton models, as well as extensions. Chapter 8 presents specific classes of derivative securities, such as futures, forwards, American options, and lookback options. Chapter 8 also introduces models of option pricing with stochastic volatility. The notion of an "affine" state process is used heavily in Chapters 7 and 8 for its analytical tractability. Chapter 9 is a summary of optimal continuous-time portfolio choice, using both dynamic programming and an approach involving equivalent martingale measures or state prices. Chapter 10 is a summary of security pricing in an equilibrium setting. Included are such well-known models as Breeden's consumption-based capital asset pricing model and the general equilibrium version of the Cox-Ingersoll-Ross model of the term structure of interest rates. Chapter 11 deals with the valuation of corporate securities, such as debt and equity. The chapter moves from models based on the capital structure of the corporation, in which default is defined in terms of the sufficiency of assets, to models based on an assumed process for the default arrival intensity. Chapter 12 outlines three numerical methods for calculating derivative security prices in a continuous-time setting: binomial approximation, Monte Carlo simulation of a discrete-time approximation of security prices, and finite-difference solution of the associated PDE for the asset price or the fundamental solution.

In preparing the first edition, I relied on help from many people, in addition to those mentioned above who developed this theory. In 1982, Michael Harrison gave a class at Stanford that had a major effect on my understanding and research goals. Beside me in that class was Chi-fu Huang; we learned much of this material together, becoming close friends and collaborators. I owe him a lot. I am grateful to Niko and Vana Skiadas, who treated me with overwhelming warmth and hospitality at their home on Skiathos, where parts of the first draft were written.

I have benefited from research collaboration over the years with many co-authors, including George Constantinides, Qiang Dai, Peter DeMarzo, Larry Epstein, Nicolae Gârleanu, Mark Garman, John Geanakoplos, Pierre-Yves Geoffard, Peter Glynn, Mike Harrison, Chi-fu Huang, Ming Huang, Matt Jackson, Rui Kan, David Lando, Jun Liu, Pierre-Louis Lions, Jin Ma, Andreu Mas-Colell, Andy McLennan, Jun Pan, Lasse Pedersen, Philip Protter, Rohit Rahi, Tony Richardson, Mark Schroder, Wayne Shafer, Ken Singleton, Costis Skiadas, Richard Stanton, Jiongmin Yong, and Bill Zame. I owe a special debt to Costis Skiadas, whose generous supply of good ideas has had a big influence on the result.

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For the reader's convenience, the original preface has been revised for this third edition. Significant improvements have been made in most chapters. Chapter 11, "Corporate Securities," has been added for this edition. Errors are my own responsibility, and I hope to hear of them and any other comments from readers.

Darrell Duffie

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I
Discrete-Time Models

This first part of the book takes place in a discrete-time setting with a discrete set of states. This should ease the development of intuition for the models to be found in Part II. The three pillars of the theory, *arbitrage*, *optimality*, and *equilibrium*, are developed repeatedly in different settings. Chapter 1 is the basic single-period model. Chapter 2 extends the results of Chapter 1 to many periods. Chapter 3 specializes Chapter 2 to a Markov setting and illustrates dynamic programming as an alternate solution technique. The Ho-and-Lee and Black-Derman-Toy term-structure models are included as exercises. Chapter 4 is an infinite-horizon counterpart to Chapter 3 that has become known as the *Lucas model*.

The focus of the theory is the notion of state prices, which specify the price of any security as the state-price weighted sum or expectation of the security's state-contingent dividends. In a finite-dimensional setting, there exist state prices if and only if there is no arbitrage. The same fact is true in infinite-dimensional settings under mild technical regularity conditions. Given an agent's optimal portfolio choice, a state-price vector is given by that agent's utility gradient. In an equilibrium with Pareto optimality, a state-price vector is likewise given by a representative agent's utility gradient at the economy's aggregate consumption process.

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1
Introduction to State Pricing

THIS CHAPTER INTRODUCES the basic ideas in a finite-state one-period setting. In many basic senses, each subsequent chapter merely repeats this one from a new perspective. The objective is a characterization of security prices in terms of "state prices," one for each state of the world. The price of a given security is simply the state-price weighted sum of its payoffs in the different states. One can treat a state price as the "shadow price," or Lagrange multiplier, for wealth contingent on a given state of the world. We obtain a characterization of state prices, first based on the absence of arbitrage, then based on the first-order conditions for optimal portfolio choice of a given agent, and finally from the first-order conditions for Pareto optimality in an equilibrium with complete markets. State prices are connected with the "beta" model for excess expected returns, a special case of which is the Capital Asset Pricing Model (CAPM). Many readers will find this chapter to be a review of standard results. In most cases, here and throughout, technical conditions are imposed that give up much generality so as to simplify the exposition.

A. Arbitrage and State Prices

Uncertainty is represented here by a finite set $\{1, \ldots, S\}$ of states, one of which will be revealed as true. The N securities are given by an $N \times S$ matrix D, with D_{ij} denoting the number of units of account paid by security i in state j. The security prices are given by some q in $\mathbb{R}^N$. A *portfolio* $\theta \in \mathbb{R}^N$ has *market value* $q \cdot \theta$ and *payoff* $D^\top \theta$ in $\mathbb{R}^S$. An *arbitrage* is a portfolio θ in $\mathbb{R}^N$ with $q \cdot \theta \leq 0$ and $D^\top \theta > 0$, or $q \cdot \theta < 0$ and $D^\top \theta \geq 0$. An arbitrage is therefore, in effect, a portfolio offering "something for nothing." Not surprisingly, it will later be shown that an arbitrage is naturally ruled out, and this gives a characterization of security prices as follows. A

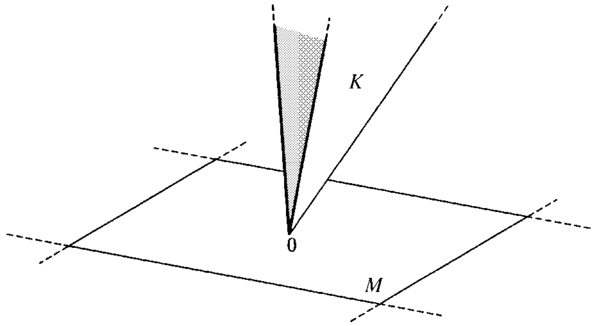

Figure 1.1. *Separating a Cone from a Linear Subspace*

state-price vector is a vector ψ in $\mathbb{R}^S_{++}$ with $q = D\psi$. We can think of ψ_j as the marginal cost of obtaining an additional unit of account in state j.

Theorem. *There is no arbitrage if and only if there is a state-price vector.*

Proof: The proof is an application of the Separating Hyperplane Theorem. Let $L = \mathbb{R} \times \mathbb{R}^S$ and $M = \{(-q \cdot \theta, D^\top \theta) : \theta \in \mathbb{R}^N\}$, a linear subspace of L. Let $K = \mathbb{R}_+ \times \mathbb{R}^S_+$, which is a cone (meaning that if x is in K, then λx is in K for each strictly positive scalar λ). Both K and M are closed and convex subsets of L. There is no arbitrage if and only if K and M intersect precisely at 0, as pictured in Figure 1.1.

Suppose $K \cap M = \{0\}$. The Separating Hyperplane Theorem (in a version for closed cones that is found in Appendix B) implies the existence of a nonzero linear functional $F : L \to \mathbb{R}$ such that $F(z) < F(x)$ for all z in M and nonzero x in K. Since M is a linear space, this implies that $F(z) = 0$ for all z in M and that $F(x) > 0$ for all nonzero x in K. The latter fact implies that there is some $\alpha > 0$ in $\mathbb{R}$ and $\psi \gg 0$ in $\mathbb{R}^S$ such that $F(v, c) = \alpha v + \psi \cdot c$, for any $(v, c) \in L$. This in turn implies that $-\alpha q \cdot \theta + \psi \cdot (D^\top \theta) = 0$ for all θ in $\mathbb{R}^N$. The vector ψ/α is therefore a state-price vector.

Conversely, if a state-price vector ψ exists, then for any θ, we have $q \cdot \theta = \psi^\top D^\top \theta$. Thus, when $D^\top \theta \geq 0$, we have $q \cdot \theta \geq 0$, and when $D^\top \theta > 0$, we have $q \cdot \theta > 0$, so there is no arbitrage. ∎

B. Risk-Neutral Probabilities

We can view any p in $\mathbb{R}^S_+$ with $p_1 + \cdots + p_S = 1$ as a vector of probabilities of the corresponding states. Given a state-price vector ψ for the dividend-price pair (D, q), let $\psi_0 = \psi_1 + \cdots + \psi_S$ and, for any state j, let $\hat{\psi}_j = \psi_j/\psi_0$.

We now have a vector $(\hat{\psi}_1, \ldots, \hat{\psi}_S)$ of probabilities and can write, for an arbitrary security i,

$$\frac{q_i}{\psi_0} = \hat{E}(D_i) \equiv \sum_{j=1}^{S} \hat{\psi}_j D_{ij},$$

viewing the normalized price of the security as its expected payoff under specially chosen "risk-neutral" probabilities. If there exists a portfolio $\bar{\theta}$ with $D^\top \bar{\theta} = (1, 1, \ldots, 1)$, then $\psi_0 = \bar{\theta} \cdot q$ is the discount on riskless borrowing and, for any security i, $q_i = \psi_0 \hat{E}(D_i)$, showing any security's price to be its discounted expected payoff in this sense of artificially constructed probabilities.

C. Optimality and Asset Pricing

Suppose the dividend-price pair (D, q) is given. An *agent* is defined by a strictly increasing *utility function* $U : \mathbb{R}^S_+ \to \mathbb{R}$ and an *endowment* e in $\mathbb{R}^S_+$. This leaves the *budget-feasible set*

$$X(q, e) = \{e + D^\top \theta \in \mathbb{R}^S_+ : \theta \in \mathbb{R}^N, q \cdot \theta \leq 0\},$$

and the problem

$$\sup_{c \in X(q,e)} U(c). \tag{1}$$

We will suppose for this section that there is some portfolio θ^0 with payoff $D^\top \theta^0 > 0$. Because U is strictly increasing, the wealth constraint $q \cdot \theta \leq 0$ is then binding at an optimum. That is, if $c^* = e + D^\top \theta^*$ solves (1), then $q \cdot \theta^* = 0$.

Proposition. *If there is a solution to* (1), *then there is no arbitrage. If U is continuous and there is no arbitrage, then there is a solution to* (1).

Proof is left as an exercise.

Theorem. *Suppose that c^* is a strictly positive solution to* (1), *that U is continuously differentiable at c^*, and that the vector $\partial U(c^*)$ of partial derivatives of U at c^* is strictly positive. Then there is some scalar $\lambda > 0$ such that $\lambda \partial U(c^*)$ is a state-price vector.*

Proof: The first-order condition for optimality is that for any θ with $q \cdot \theta = 0$, the marginal utility for buying the portfolio θ is zero. This is expressed more precisely in the following way. The strict positivity of c^* implies that for any portfolio θ, there is some scalar $k > 0$ such that $c^* + \alpha D^\top \theta \geq 0$ for all α in $[-k, k]$. Let $g_\theta : [-k, k] \to \mathbb{R}$ be defined by

$$g_\theta(\alpha) = U(c^* + \alpha D^\top \theta).$$

Suppose $q \cdot \theta = 0$. The optimality of c^* implies that g_θ is maximized at $\alpha = 0$. The first-order condition for this is that $g'_\theta(0) = \partial U(c^*)^\top D^\top \theta = 0$. We can conclude that, for any θ in $\mathbb{R}^N$, if $q \cdot \theta = 0$, then $\partial U(c^*)^\top D^\top \theta = 0$. From this, there is some scalar μ such that $\partial U(c^*)^\top D^\top = \mu q$.

By assumption, there is some portfolio θ^0 with $D^\top \theta^0 > 0$. From the existence of a solution to (1), there is no arbitrage, implying that $q \cdot \theta^0 > 0$. We have

$$\mu q \cdot \theta^0 = \partial U(c^*)^\top D^\top \theta^0 > 0.$$

Thus $\mu > 0$. We let $\lambda = 1/\mu$, obtaining

$$q = \lambda \, D \partial U(c^*), \tag{2}$$

implying that $\lambda \partial U(c^*)$ is a state-price vector. ∎

Although we have assumed that U is strictly increasing, this does not necessarily mean that $\partial U(c^*) \gg 0$. If U is concave and strictly increasing, however, it is always true that $\partial U(c^*) \gg 0$.

Corollary. *Suppose U is concave and differentiable at some $c^* = e + D^\top \theta^* \gg 0$, with $q \cdot \theta^* = 0$. Then c^* is optimal if and only if $\lambda \partial U(c^*)$ is a state-price vector for some scalar $\lambda > 0$.*

This follows from the sufficiency of the first-order optimality conditions for concave objective functions. The idea is illustrated in Figure 1.2. In that figure, there are only two states, and a state-price vector is a suitably normalized nonzero positive vector orthogonal to the set $B = \{D^\top \theta : q \cdot \theta = 0\}$ of budget-neutral consumption adjustments. The first-order condition for optimality of c^* is that movement in any feasible direction away from c^* has negative or zero marginal utility, which is equivalent to the statement that the budget-neutral set is tangent at c^* to the preferred set $\{c : U(c) \geq U(c^*)\}$, as shown in the figure. This is equivalent to the statement that $\partial U(c^*)$ is orthogonal to B, consistent with the last corollary. Figure 1.3 illustrates a strictly suboptimal consumption choice c, at which the derivative vector $\partial U(c)$ is not co-linear with the state-price vector ψ.

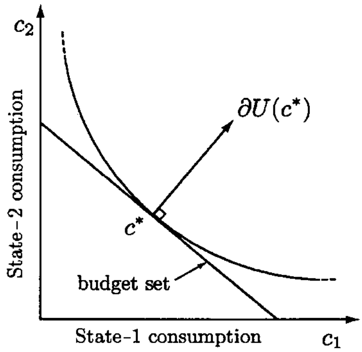

Figure 1.2. *First-Order Conditions for Optimal Consumption Choice*

We consider the special case of an *expected utility* function U, defined by a given vector p of probabilities and by some $u : \mathbb{R}_+ \to \mathbb{R}$ according to

$$U(c) = E[u(c)] \equiv \sum_{j=1}^{S} p_j u(c_j). \tag{3}$$

For $c \gg 0$, if u is differentiable, then $\partial U(c)_j = p_j u'(c_j)$. For this expected utility function, (2) therefore applies if and only if

$$q = \lambda\, E[Du'(c^*)], \tag{4}$$

with the obvious notational convention. As we saw in Section B, one can also write (2) or (4), with the "risk-neutral" probability $\hat{\psi}_j = u'(c_j^*)p_j / E[u'(c^*)]$, in the form

$$\frac{q_i}{\psi_0} = \hat{E}(D_i) \equiv \sum_{j=1}^{S} D_{ij}\hat{\psi}_j, \qquad 1 \le i \le N. \tag{5}$$

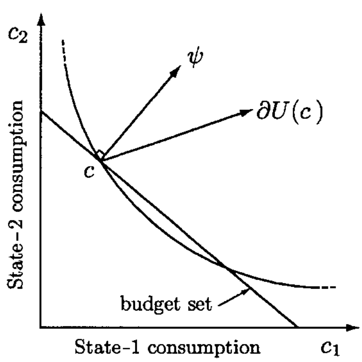

Figure 1.3. *A Strictly Suboptimal Consumption Choice*

D. Efficiency and Complete Markets

Suppose there are m agents, defined as in Section C by strictly increasing utility functions $U_1, \ldots, U_m$ and by endowments $e^1, \ldots, e^m$. An *equilibrium* for the *economy* $[(U_i, e^i), D]$ is a collection $(\theta^1, \ldots, \theta^m, q)$ such that, given the security-price vector q, for each agent i, θ^i solves $\sup_\theta U_i(e^i + D^\top \theta)$ subject to $q \cdot \theta \leq 0$, and such that $\sum_{i=1}^m \theta^i = 0$. The existence of equilibrium is treated in the exercises and in sources cited in the Notes.

With $\text{span}(D) \equiv \{D^\top \theta : \theta \in \mathbb{R}^N\}$ denoting the set of possible portfolio payoffs, markets are *complete* if $\text{span}(D) = \mathbb{R}^S$, and are otherwise *incomplete*.

Let $e = e^1 + \cdots + e^m$ denote the aggregate endowment. A consumption *allocation* $(c^1, \ldots, c^m)$ in $\left(\mathbb{R}^S_+\right)^m$ is *feasible* if $c^1 + \cdots + c^m \leq e$. A feasible allocation $(c^1, \ldots, c^m)$ is *Pareto optimal* if there is no feasible allocation $(\hat{c}^1, \ldots, \hat{c}^m)$ with $U_i(\hat{c}^i) \geq U_i(c^i)$ for all i and with $U_i(\hat{c}^i) > U_i(c^i)$ for some i. Complete markets and the Pareto optimality of equilibrium allocations are almost equivalent properties of any economy.

Proposition. *Suppose markets are complete and $(\theta^1, \ldots, \theta^m, q)$ is an equilibrium. Then the associated equilibrium allocation is Pareto optimal.*

This is sometimes known as *The First Welfare Theorem*. The proof, requiring only the strict monotonicity of utilities, is left as an exercise. We have established the sufficiency of complete markets for Pareto optimality. The necessity of complete markets for the Pareto optimality of equilibrium allocations does not always follow. For example, if the initial endowment allocation $(e^1, \ldots, e^m)$ happens by chance to be Pareto optimal, then any equilibrium allocation is also Pareto optimal, regardless of the span of securities. It would be unusual, however, for the initial endowment to be Pareto optimal. Although beyond the scope of this book, it can be shown that with incomplete markets and under natural assumptions on utility, for almost every endowment, the equilibrium allocation is not Pareto optimal.

E. Optimality and Representative Agents

Aside from its allocational implications, Pareto optimality is also a convenient property for the purpose of security pricing. In order to see this, consider, for each vector $\lambda \in \mathbb{R}^m_+$ of "agent weights," the utility function $U_\lambda : \mathbb{R}^S_+ \to \mathbb{R}$ defined by

$$U_\lambda(x) = \sup_{(c^1, \ldots, c^m)} \sum_{i=1}^m \lambda_i U_i(c^i) \qquad \text{subject to } c^1 + \cdots + c^m \leq x. \tag{6}$$

Lemma. *Suppose that, for all i, U_i is concave. An allocation $(c^1, \ldots, c^m)$ that is feasible is Pareto optimal if and only if there is some nonzero $\lambda \in \mathbb{R}^m_+$ such that $(c^1, \ldots, c^m)$ solves* (6) *at $x = e = c^1 + \cdots + c^m$.*

Proof: Suppose that $(c^1, \ldots, c^m)$ is Pareto optimal. For any allocation x, let $U(x) = (U_1(x^1), \ldots, U_m(x^m))$. Next, let

$$\mathcal{U} = \{U(x) - U(c) - z \,:\, x \in \mathcal{A},\ z \in \mathbb{R}^m_+\} \subset \mathbb{R}^m,$$

where $\mathcal{A}$ is the set of feasible allocations. Let $J = \{y \in \mathbb{R}^m_+ : y \neq 0\}$. Since $\mathcal{U}$ is convex (by the concavity of utility functions) and $J \cap \mathcal{U}$ is empty (by Pareto optimality), the Separating Hyperplane Theorem (Appendix B) implies that there is a nonzero vector λ in $\mathbb{R}^m$ such that $\lambda \cdot y \leq \lambda \cdot z$ for each y in $\mathcal{U}$ and each z in J. Since $0 \in \mathcal{U}$, we know that $\lambda \geq 0$, proving the first part of the result. The second part is easy to show as an exercise. ∎

Proposition. *Suppose that for all i, U_i is concave. Suppose that markets are complete and that $(\theta^1, \ldots, \theta^m, q)$ is an equilibrium. Then there exists some nonzero $\lambda \in \mathbb{R}^m_+$ such that $(0, q)$ is a (no-trade) equilibrium for the single-agent economy $[(U_\lambda, e), D]$ defined by* (6). *Moreover, the equilibrium consumption allocation $(c^1, \ldots, c^m)$ solves the allocation problem* (6) *at the aggregate endowment. That is, $U_\lambda(e) = \sum_i \lambda_i U_i(c^i)$.*

Proof: Since there is an equilibrium, there is no arbitrage, and therefore there is a state-price vector ψ. Since markets are complete, this implies that the problem of any agent i can be reduced to

$$\sup_{c \in \mathbb{R}^S_+} U_i(c) \quad \text{subject to } \psi \cdot c \leq \psi \cdot e^i.$$

We can assume that e^i is not zero, for otherwise $c^i = 0$ and agent i can be eliminated from the problem without loss of generality. By the Saddle Point Theorem of Appendix B, there is a Lagrange multiplier $\alpha_i \geq 0$ such that c^i solves the problem

$$\sup_{c \in \mathbb{R}^S_+} U_i(c) - \alpha_i(\psi \cdot c - \psi \cdot e^i).$$

(The Slater condition is satisfied since e^i is not zero and $\psi \gg 0$.) Since U_i is strictly increasing, $\alpha_i > 0$. Let $\lambda_i = 1/\alpha_i$. For any feasible allocation $(x^1, \ldots, x^m)$, we have

$$\sum_{i=1}^m \lambda_i U_i(c^i) = \sum_{i=1}^m [\lambda_i U_i(c^i) - \lambda_i \alpha_i(\psi \cdot c^i - \psi \cdot e^i)]$$

$$\geq \sum_{i=1}^{m} \lambda_i [U_i(x^i) - \alpha_i(\psi \cdot x^i - \psi \cdot e^i)]$$
$$= \sum_{i=1}^{m} \lambda_i U_i(x^i) - \psi \cdot \sum_{i=1}^{m}(x^i - e^i)$$
$$\geq \sum_{i=1}^{m} \lambda_i U_i(x^i).$$

This shows that $(c^1, \ldots, c^m)$ solves the allocation problem (6). We must also show that no trade is optimal for the single agent with utility function U_λ and endowment e. If not, there is some x in $\mathbb{R}_+^S$ such that $U_\lambda(x) > U_\lambda(e)$ and $\psi \cdot x \leq \psi \cdot e$. By the definition of U_λ, this would imply the existence of an allocation $(x^1, \ldots, x^m)$, not necessarily feasible, such that $\sum_i \lambda_i U_i(x^i) > \sum_i \lambda_i U_i(c^i)$ and

$$\sum_i \lambda_i \alpha_i \psi \cdot x^i = \psi \cdot x \leq \psi \cdot e = \sum_i \lambda_i \alpha_i \psi \cdot c^i.$$

Putting these two inequalities together, we have

$$\sum_{i=1}^{m} \lambda_i [U_i(x^i) - \alpha_i \psi \cdot (x^i - e^i)] > \sum_{i=1}^{m} \lambda_i [U_i(c^i) - \alpha_i \psi \cdot (c^i - e^i)],$$

which contradicts the fact that, for each agent i, (c^i, α_i) is a saddle point for that agent's problem. ∎

Corollary 1. *If, moreover, $e \gg 0$ and U_λ is continuously differentiable at e, then λ can be chosen so that $\partial U_\lambda(e)$ is a state-price vector, meaning*

$$q = D\partial U_\lambda(e). \tag{7}$$

The differentiability of U_λ at e is implied by the differentiability, for some agent i, of U_i at c^i. (See Exercise 10(C).)

Corollary 2. *Suppose there is a fixed vector p of state probabilities such that, for all i, $U_i(c) = E[u_i(c)] \equiv \sum_{j=1}^{S} p_j u_i(c_j)$, for some $u_i(\cdot)$. Then $U_\lambda(c) = E[u_\lambda(c)]$, where, for each y in $\mathbb{R}_+$,*

$$u_\lambda(y) = \max_{x \in \mathbb{R}_+^m} \sum_{i=1}^{m} \lambda_i u_i(x_i) \quad \text{subject to } x_1 + \cdots + x_m \leq y.$$

In this case, (7) *is equivalent to $q = E[Du'_\lambda(e)]$.*

Extensions of this *representative-agent* asset pricing formula will crop up frequently in later chapters.

F. State-Price Beta Models

We fix a vector $p \gg 0$ in $\mathbb{R}^S$ of probabilities for this section, and for any x in $\mathbb{R}^S$ we write $E(x) = p_1 x_1 + \cdots + p_S x_S$. For any x and π in $\mathbb{R}^S$, we take $x\pi$ to be the vector $(x_1\pi_1, \ldots, x_S\pi_S)$. The following version of the *Riesz Representation Theorem* can be shown as an exercise.

Lemma. *Suppose $F : \mathbb{R}^S \to \mathbb{R}$ is linear. Then there is a unique π in $\mathbb{R}^S$ such that, for all x in $\mathbb{R}^S$, we have $F(x) = E(\pi x)$. Moreover, F is strictly increasing if and only if $\pi \gg 0$.*

Corollary. *A dividend-price pair (D, q) admits no arbitrage if and only if there is some $\pi \gg 0$ in $\mathbb{R}^S$ such that $q = E(D\pi)$.*

Proof: Given a state-price vector ψ, let $\pi_s = \psi_s / p_s$. Conversely, if π has the assumed property, then $\psi_s = p_s \pi_s$ defines a state-price vector ψ. ∎

Given (D, q), we refer to any vector π given by this result as a *state-price deflator.* (The terms *state-price density* and *state-price kernel* are often used synonymously with state-price deflator.) For example, the representative-agent pricing model of Corollary 2 of Section E shows that we can take $\pi_s = u'_\lambda(e_s)$.

For any x and y in $\mathbb{R}^S$, the *covariance* $\text{cov}(x, y) \equiv E(xy) - E(x)E(y)$ is a measure of covariation between x and y that is useful in asset pricing applications. For any such x and y with $\text{var}(y) \equiv \text{cov}(y, y) \neq 0$, we can always represent x in the form $x = \alpha + \beta y + \epsilon$, where $\beta = \text{cov}(y, x)/\text{var}(y)$, where $\text{cov}(y, \epsilon) = E(\epsilon) = 0$, and where α is a scalar. This *linear regression* of x on y is uniquely defined. The coefficient β is called the associated *regression coefficient.*

Suppose (D, q) admits no arbitrage. For any portfolio θ with $q \cdot \theta \neq 0$, the *return* on θ is the vector R^θ in $\mathbb{R}^S$ defined by $R^\theta_s = (D^\top \theta)_s / \, q \cdot \theta$. Fixing a state-price deflator π, for any such portfolio θ, we have $E(\pi R^\theta) = 1$. Suppose there is a *riskless portfolio,* meaning some portfolio θ with constant return R^0. We then call R^0 the *riskless return.* A bit of algebra shows that for any portfolio θ with a return, we have

$$E(R^\theta) - R^0 = -\frac{\text{cov}(R^\theta, \pi)}{E(\pi)}.$$

Thus, covariation with π has a negative effect on expected return, as one might expect from the interpretation of state prices as shadow prices for wealth.

The *correlation* between any x and y in $\mathbb{R}^S$ is zero if either has zero variance, and is otherwise defined by

$$\operatorname{corr}(x, y) = \frac{\operatorname{cov}(x, y)}{\sqrt{\operatorname{var}(x)\operatorname{var}(y)}}.$$

There is always a portfolio θ^* solving the problem

$$\sup_{\theta} \operatorname{corr}(D^{\top}\theta, \pi). \tag{8}$$

If there is such a portfolio θ^* with a return R^* having nonzero variance, then it can be shown as an exercise that, for any return R^θ,

$$E(R^\theta) - R^0 = \beta_\theta[E(R^*) - R^0], \tag{9}$$

where

$$\beta_\theta = \frac{\operatorname{cov}(R^*, R^\theta)}{\operatorname{var}(R^*)}.$$

If markets are complete, then R^* is of course perfectly correlated with the state-price deflator.

Formula (9) is a *state-price beta model*, showing excess expected returns on portfolios to be proportional to the excess return on a portfolio having maximal correlation with a state-price deflator, where the constant of proportionality is the associated regression coefficient. The formula can be extended to the case in which there is no riskless return. Another exercise carries this idea, under additional assumptions, to the *Capital Asset Pricing Model*, or *CAPM*.

Exercises

1.1 The dividend-price pair (D, q) of Section A is defined to be *weakly arbitrage-free* if $q \cdot \theta \geq 0$ whenever $D^{\top}\theta \geq 0$. Show that (D, q) is weakly arbitrage-free if and only if there exist ("weak" state prices) $\psi \in \mathbb{R}^S_+$ such that $q = D\psi$. This fact is known as *Farkas's Lemma*.

1.2 Prove the assertion in Section A that (D, q) is arbitrage-free if and only if there exists some $\psi \in \mathbb{R}^S_{++}$ such that $q = D\psi$. Instead of following the proof given in Section A, use the following result, sometimes known as the *Theorem of the Alternative*.

Stiemke's Lemma. *Suppose A is an $m \times n$ matrix. Then one and only one of the following is true*:

(a) *There exists x in $\mathbb{R}^n_{++}$ with $Ax = 0$.*
(b) *There exists y in $\mathbb{R}^m$ with $y^{\top}A > 0$.*

1.3 Show, for $U(c) \equiv E[u(c)]$ as defined by (3), that (2) is equivalent to (4).

1.4 Prove the existence of an equilibrium as defined in Section D under these assumptions: There exists some portfolio θ with payoff $D^\top \theta > 0$ and, for all i, $e^i \gg 0$ and U_i is continuous, strictly concave, and strictly increasing. This is a demanding exercise, and calls for the following general result.

Kakutani's Fixed Point Theorem. *Suppose Z is a nonempty convex compact subset of $\mathbb{R}^n$, and for each x in Z, $\varphi(x)$ is a nonempty convex compact subset of Z. Suppose also that $\{(x, y) \in Z \times Z : x \in \varphi(y)\}$ is closed. Then there exists x^* in Z such that $x^* \in \varphi(x^*)$.*

1.5 Prove Proposition D. Hint: The maintained assumption of strict monotonicity of $U_i(\cdot)$ should be used.

1.6 Suppose that the endowment allocation $(e^1, \ldots, e^m)$ is Pareto optimal.

(A) Show, as claimed in Section D, that any equilibrium allocation is Pareto optimal.

(B) Suppose that there is some portfolio θ with $D^\top \theta > 0$ and, for all i, that U_i is concave and $e^i \gg 0$. Show that $(e^1, \ldots, e^m)$ is itself an equilibrium allocation.

1.7 Prove Proposition C. Hint: A continuous real-valued function on a compact set has a maximum.

1.8 Prove Corollary 1 of Proposition E.

1.9 Prove Corollary 2 of Proposition E.

1.10 Suppose, in addition to the assumptions of Proposition E, that

(a) $e = e^1 + \cdots + e^m$ is in $\mathbb{R}^S_{++}$;
(b) for all i, U_i is concave and twice continuously differentiable in $\mathbb{R}^S_{++}$;
(c) for all i, c^i is in $\mathbb{R}^S_{++}$ and the Hessian matrix $\partial^2 U(c^i)$, which is negative semi-definite by concavity, is in fact negative definite.

Property (c) can be replaced with the assumption of *regular preferences*, as defined in a source cited in the Notes.

(A) Show that the assumption that U_λ is continuously differentiable at e is justified and, moreover, that for each i there is a scalar $\gamma_i > 0$ such that $\partial U_\lambda(e) = \gamma_i \partial U_i(c^i)$. (This co-linearity is known as "equal marginal rates of substitution," a property of any Pareto optimal allocation.) Hint: Use the following:

Implicit Function Theorem. *Suppose for given m and n that $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is C^k (k times continuously differentiable) for some $k \geq 1$. Suppose also that the $n \times n$ matrix $\partial_y f(\bar{x}, \bar{y})$ of partial derivatives of f with respect to its second argument is nonsingular at some $(\bar{x}, \bar{y})$. If $f(\bar{x}, \bar{y}) = 0$, then there exist scalars $\epsilon > 0$ and $\delta > 0$ and a C^k function $Z : \mathbb{R}^m \to \mathbb{R}^n$ such that if $\|x - a\| < \epsilon$, then $f[x, Z(x)] = 0$ and $\|Z(x) - b\| < \delta$.*

(B) Show that the negative-definite part of condition (c) is satisfied if $e \gg 0$ and, for all i, U_i is an expected utility function of the form $U_i(c) = E[u_i(c)]$, where u_i is strictly concave with an unbounded derivative on $(0, \infty)$.

(C) Obtain the result of part (A) without assuming the existence of second derivatives of the utilities. (You would therefore not exploit the Hessian matrix or Implicit Function Theorem.) As the first (and main) step, show the following. Given a concave function $f : \mathbb{R}^S_+ \to \mathbb{R}$, the *superdifferential* of f at some x in $\mathbb{R}^S_+$ is

$$\partial f(x) = \{z \in \mathbb{R}^S : f(y) \leq f(x) + z \cdot (y - x), \quad y \in \mathbb{R}^S_+\}.$$

For any feasible allocation $(c^1, \ldots, c^m)$ and $\lambda \in \mathbb{R}^m_+$ satisfying $U_\lambda(e) = \sum_i \lambda_i U_i(c^i)$,

$$\partial U_\lambda(e) = \bigcap_{i=1}^{m} \lambda_i \partial U_i(c_i).$$

1.11 (Binomial Option Pricing). As an application of the results in Section A, consider the following two-state ($S = 2$) option-pricing problem. There are $N = 3$ securities:

(a) a stock, with initial price $q_1 > 0$ and dividend $D_{11} = Gq_1$ in state 1 and dividend $D_{12} = Bq_1$ in state 2, where $G > B > 0$ are the "good" and "bad" gross returns, respectively;
(b) a riskless bond, with initial price $q_2 > 0$ and dividend $D_{21} = D_{22} = Rq_2$ in both states (that is, R is the riskless return and R^{-1} is the discount);
(c) a *call option* on the stock, with initial price $q_3 = C$ and dividend $D_{3j} = (D_{1j} - K)^+ \equiv \max(D_{1j} - K, 0)$ for both states $j = 1$ and $j = 2$, where $K \geq 0$ is the *exercise price* of the option. (The call option gives its holder the right, but not the obligation, to pay K for the stock, with dividend, after the state is revealed.)

(A) Show necessary and sufficient conditions on G, B, and R for the absence of arbitrage involving only the stock and bond.

(B) Assuming no arbitrage for the three securities, calculate the call-option price C explicitly in terms of q_1, G, R, B, and K. Find the state-price probabilities $\hat{\psi}_1$ and $\hat{\psi}_2$ referred to in Section B in terms of G, B, and R, and show that $C = R^{-1}\hat{E}(D_3)$, where $\hat{E}$ denotes expectation with respect to $(\hat{\psi}_1, \hat{\psi}_2)$.

1.12 (CAPM). In the setting of Section D, suppose $(c^1, \ldots, c^m)$ is a strictly positive equilibrium consumption allocation. For any agent i, suppose utility is of the expected-utility form $U_i(c) = E[u_i(c)]$. For any agent i, suppose there are fixed positive constants $\bar{c}$ and b_i such that, for any state j, we have $c^i_j < \bar{c}$ and $u_i(x) = x - b_i x^2$ for all $x \leq \bar{c}$.

(A) In the context of Corollary 2 of Section E, show that $u'_\lambda(e) = k - Ke$ for some positive constants k and K. From this, derive the CAPM

$$q = AE(D) - B\ \mathrm{cov}(D, e), \tag{10}$$

for positive constants A and B, where $\mathrm{cov}(D, e) \in \mathbb{R}^N$ is the vector of covariances between the security dividends and the aggregate endowment.

Suppose for a given portfolio θ that each of the following is well defined:

- the return $R^\theta \equiv D^\top \theta / q \cdot \theta$;
- the return R^M on a portfolio M with payoff $D^\top M = e$;
- the return R^0 on a portfolio θ^0 with $\mathrm{cov}(D^\top \theta^0, e) = 0$;
- $\beta_\theta = \mathrm{cov}(R^\theta, R^M)/\mathrm{var}(R^M)$.

The return R^M is sometimes called the *market return*. The return R^0 is called the *zero-beta return* and is the return on a riskless bond if one exists. Prove the "beta" form of the CAPM

$$E(R^\theta - R^0) = \beta_\theta E(R^M - R^0). \tag{11}$$

(B) Part (A) relies on the completeness of markets. Without any such assumption, but assuming that the equilibrium allocation $(c^1, \ldots, c^m)$ is strictly positive, show that the same beta form (11) applies, provided we extend the definition of the market return R^M to be the return on any portfolio solving

$$\sup_{\theta \in \mathbb{R}^N} \ \mathrm{corr}(R^\theta, e). \tag{12}$$

For complete markets, $\mathrm{corr}(R^M, e) = 1$, so the result of part (A) is a special case.

(C) The CAPM applies essentially as stated without the quadratic expected-utility assumption provided that each agent i is *strictly variance-averse*, in that $U_i(x) > U_i(y)$ whenever $E(x) = E(y)$ and $\mathrm{var}(x) < \mathrm{var}(y)$. Formalize this statement by providing a reasonable set of supporting technical conditions.

We remark that a common alternative formulation of the CAPM allows security portfolios in initial endowments $\hat{\theta}^1, \ldots, \hat{\theta}^m$ with $\sum_{i=1}^m \hat{\theta}^i_j = 1$ for all j. In this case, with the total endowment e redefined by $e = \sum_{i=1}^m (e^i + D^\top \hat{\theta}^i)$, the same CAPM (11) applies. If $e^i = 0$ for all i, then even in incomplete markets, $\mathrm{corr}(R^M, e) = 1$, since (12) is solved by $\theta = (1, 1, \ldots, 1)$. The Notes provide references.

1.13 An *Arrow-Debreu equilibrium* for $[(U_i, e^i), D]$ is a nonzero vector ψ in $\mathbb{R}^S_+$ and a feasible consumption allocation $(c^1, \ldots, c^m)$ such that for each i, c^i solves $\sup_c U_i(c)$ subject to $\psi \cdot c^i \le \psi \cdot e^i$. Suppose that markets are complete, in that $\mathrm{span}(D) = \mathbb{R}^S$. Show that $(c^1, \ldots, c^m)$ is an Arrow-Debreu consumption allocation if and only if it is an equilibrium consumption allocation in the sense of Section D.

1.14 Suppose (D, q) admits no arbitrage. Show that there is a unique state-price vector if and only if markets are complete.

1.15 (Aggregation). For the "representative-agent" problem (6), suppose for all i that $U_i(c) = E[u(c)]$, where $u(c) = c^\gamma/\gamma$ for some nonzero scalar $\gamma < 1$.

(A) Show, for any nonzero agent weight vector $\lambda \in \mathbb{R}^m_+$, that $U_\lambda(c) = E[kc^\gamma/\gamma]$ for some scalar $k > 0$ and that (6) is solved by $c^i = k_i x$ for some scalar $k_i \ge 0$ that is nonzero if and only if λ_i is nonzero.

(B) With this special utility assumption, show that there exists an equilibrium with a Pareto efficient allocation, without the assumption that markets are complete, but with the assumption that $e^i \in \text{span}(D)$ for all i. Calculate the associated equilibrium allocation.

1.16 (State-Price Beta Model). This exercise is to prove and extend the state-price beta model (9) of Section F.

(A) Show problem (8) is solved by any portfolio θ such that $\pi = D^\top\theta + \epsilon$, where $\text{cov}(\epsilon, D^j) = 0$ for any security j, where $D^j \in \mathbb{R}^S$ is the payoff of security j.

(B) Given a solution θ to (8) such that R^θ is well defined with nonzero variance, prove (9).

(C) Reformulate (9) for the case in which there is no riskless return by redefining R^0 to be the expected return on any portfolio θ such that R^θ is well defined and $\text{cov}(R^\theta, \pi) = 0$, assuming such a portfolio exists.

1.17 Prove the Riesz representation lemma of Section F. The following hint is perhaps unnecessary in this simple setting but allows the result to be extended to a broad variety of spaces called *Hilbert spaces.* Given a vector space L, a function $(\cdot \mid \cdot) : L \times L \to \mathbb{R}$ is called an *inner product* for L if, for any x, y, and z in L and any scalar α, we have the five properties:

(a) $(x \mid y) = (y \mid x)$
(b) $(x + y \mid z) = (x \mid z) + (y \mid z)$
(c) $(\alpha x \mid y) = \alpha(x \mid y)$
(d) $(x \mid x) \geq 0$
(e) $(x \mid x) = 0$ if and only if $x = 0$.

Suppose a finite-dimensional vector space L has an inner product $(\cdot \mid \cdot)$. (This defines a special case of a Hilbert space.) Two vectors x and y are defined to be *orthogonal* if $(x \mid y) = 0$. For any linear subspace H of L and any x in L, it can be shown that there is a unique y in H such that $(x - y \mid z) = 0$ for all z in H. This vector y is the orthogonal projection in L of x onto H, and solves the problem $\min_{h \in H} \|x - h\|$. Let $L = \mathbb{R}^S$. For any x and y in L, let $(x \mid y) = E(xy)$. We must show that given a linear functional F, there is a unique π with $F(x) = (\pi \mid x)$ for all x. Let $J = \{x : F(x) = 0\}$. If $J = L$, then F is the zero functional, and the unique representation is $\pi = 0$. If not, there is some z such that $F(z) = 1$ and $(z \mid x) = 0$ for all x in J. Show this using the idea of orthogonal projection. Then show that $\pi = z/(z \mid z)$ represents F, using the fact that for any x, we have $x - F(x)z \in J$.

1.18 Suppose there are $m = 2$ consumers, A and B, with identical utilities for consumption c_1 and c_2 in states 1 and 2 given by $U(c_1, c_2) = 0.2\sqrt{c_1} + 0.5 \log c_2$. There is a total endowment of $e_1 = 25$ units of consumption in state 1.

(A) Suppose that markets are complete and that, in a given equilibrium, consumer A's consumption is 9 units in state 1 and 10 units in state 2. What is the total endowment e_2 in state 2?

(B) Continuing under the assumptions of part (A), suppose there are two securities. The first is a riskless bond paying 10 units of consumption in each state. The second is a risky asset paying 5 units of consumption in state 1 and 10 units in state 2. In equilibrium, what is the ratio of the price of the bond to that of the risky asset?

1.19 There are two states of the world, labeled 1 and 2, two agents, and two securities, both paying units of the consumption numeraire good. The risky security pays a total of 1 unit in state 1 and pays 3 units in state 2. The riskless security pays 1 unit in each state. Each agent is initially endowed with half of the total supply of the risky security. There are no other endowments. (The riskless security is in zero net supply.) The two agents assign equal probabilities to the two states. One of the agents is risk-neutral, with utility function $E(c)$ for state-contingent consumption c, and can consume negatively or positively in both states. The other, risk-averse, agent has utility $E(\sqrt{c})$ for nonnegative state-contingent consumption. Solve for the equilibrium allocation of the two securities in a competitive equilibrium.

1.20 Consider a setting with two assets A and B, only, both paying off the same random variable X, whose value is nonnegative in every state and nonzero with strictly positive probability. Asset A has price p, while asset B has price q. An arbitrage is then a portfolio $(\alpha, \beta) \in \mathbb{R}^2$ of the two assets whose total payoff $\alpha X + \beta X$ is nonnegative and whose initial price $\alpha p + \beta q$ is strictly negative, or whose total payoff is nonzero with strictly positive probability and always nonnegative, and whose initial price is negative or zero.

(A) Assuming no restrictions on portfolios, and no transactions costs or frictions, state the set of arbitrage-free prices (p, q). (State precisely the appropriate subset of $\mathbb{R}^2$.)

(B) Assuming no short sales ($\alpha \geq 0$ and $\beta \geq 0$), state the set of arbitrage-free prices (p, q).

(C) Now suppose that A and B can be short sold, but that asset A can be short sold only by paying an extra fee of $\phi > 0$ per unit sold short. There are no other fees of any kind. Provide the obvious new definition of "no arbitrage" in precise mathematical terms, and state the set of arbitrage-free prices.

Notes

The basic approach of this chapter follows Arrow (1953), taking a general equilibrium perspective originating with Walras (1877). Black (1995) offers a perspective on the general equilibrium approach and a critique of other approaches.

(A) The state-pricing implications of no arbitrage found in Section A originate with Ross (1978).

(B) The idea of "risk-neutral probabilities" apparently originates with Arrow (1970), a revision of Arrow (1953), and appears as well in Drèze (1971).

(C) This material is standard.

(D) Proposition D is the First Welfare Theorem of Arrow (1951) and Debreu (1954). The generic inoptimality of incomplete-markets equilibrium allocations can be gleaned from sources cited by Geanakoplos (1990). Indeed, Geanakoplos and Polemarchakis (1986) show that even a reasonable notion of constrained optimality generically fails in certain incomplete-markets settings. See, however, Kajii (1994) and references cited in the Notes of Chapter 2 for mitigating results. Mas-Colell (1987) and Werner (1991) also treat constrained optimality.

(E) The "representative-agent" approach goes back, at least, to Negishi (1960). The existence of a representative agent is no more than an illustrative simplification in this setting, and should not be confused with the more demanding notion of aggregation of Gorman (1953) found in Exercise 15. In Chapter 10, the existence of a representative agent with smooth utility, based on Exercise 1.11, is important for technical reasons.

(F) The "beta model" for pricing goes back, in the case of mean-variance preferences, to the capital asset pricing model, or CAPM, of Sharpe (1964) and Lintner (1965). The version without a riskless asset is due to Black (1972). Allingham (1991), Berk (1992), Nielsen (1990a), and Nielsen (1990b) address the existence of equilibrium in the CAPM. Characterization of the mean-variance model and two-fund separation is provided by Bottazzi, Hens, and Löffler (1994), Nielsen (1993b), and Nielsen (1993a). Löffler (1996) provides sufficient conditions for variance aversion in terms of mean-variance preferences.

Additional Topics: Ross (1976) introduced the *arbitrage pricing theory*, a multifactor model of asset returns that, in terms of expected returns, can be thought of as an extension of the CAPM. In this regard, see also Bray (1994a), Bray (1994b), and Gilles and LeRoy (1991). Balasko and Cass (1986) and Balasko, Cass, and Siconolfi (1990) treat equilibrium with constrained participation in security trading. See also Hara (1994).

Debreu (1972) provides a notion of regular preferences that substitutes for the existence of a negative-definite Hessian matrix of each agent's utility function at the equilibrium allocation. For more on regular preferences and the differential approach to general equilibrium, see Mas-Colell (1985) and Balasko (1989). Kreps (1988) reviews the theory of choice and utility representations of preferences. For Farkas's and Stiemke's Lemmas, and other forms of the Theorem of the Alternative, see Gale (1960).

Arrow and Debreu (1954) and, in a slightly different model, McKenzie (1954) are responsible for a proof of the existence of complete-markets equilibria. Debreu (1982) surveys the existence problem. Standard introductory treatments of general equilibrium theory are given by Debreu (1959) and Hildenbrand and Kirman (1989). In this setting, with incomplete markets, Polemarchakis and Siconolfi (1993) address the failure of existence unless one has a portfolio θ with payoff $D^\top \theta > 0$. Geanakoplos (1990) surveys other literature on the existence of equilibria in incomplete markets, some of which takes the alternative of defining security payoffs in nominal units of account, while allowing consumption of multiple commodities. Most of the literature allows for an initial period of consumption before the realization of the uncertain state. For a survey, see Magill and Shafer (1991). Additional results on incomplete-markets equilibrium include those of Araujo and Monteiro (1989), Berk (1997), Boyle and Wang (1999), and Weil (1992).

For related results in multiperiod settings, references are cited in the Notes of Chapter 2.

The superdifferentiability result of Exercise 10(C) is due to Skiadas (1995).

Hellwig (1996), Mas-Colell and Monteiro (1996), and Monteiro (1996) have recently shown existence of equilibrium with a continuum of states. Geanakoplos and Polemarchakis (1986) and Chae (1988) show existence in a model closely related to that studied in this chapter. Grodal and Vind (1988) and Yamazaki (1991) show existence with alternative formulations. With multiple commodities or multiple periods, existence is not guaranteed under any natural conditions, as shown by Hart (1975), who gives a counterexample. For these more delicate cases, the literature on generic existence is cited in the Notes of Chapter 2.

The binomial option-pricing formula of Exercise 1.11 is from an early edition of Sharpe (1985), and is extended in Chapter 2 to a multiperiod setting. The hint given for the demonstration of the Riesz representation exercise is condensed from the proof given by Luenberger (1969) of the *Riesz-Frechet Theorem*: For any Hilbert space H with inner product $(\cdot \mid \cdot)$, any continuous linear functional $F : H \to \mathbb{R}$ has a unique π in H such that $F(x) = (\pi \mid x)$, $x \in H$. The Fixed Point Theorem of Exercise 1.4 is from Kakutani (1941).

On the role of default and collateralization, see Geanakoplos and Zame (1999) and Sabarwal (1999). Gottardi and Kajii (1999) study the role and existence of sunspot equilibria. Pietra (1992) treats indeterminacy. Lobo, Fazel, Boyd (1999) address portfolio choice with fixed transactions costs.

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2

The Basic Multiperiod Model

THIS CHAPTER EXTENDS the results of Chapter 1 on arbitrage, optimality, and equilibrium to a multiperiod setting. A connection is drawn between state prices and martingales for the purpose of representing security prices. The exercises include the consumption-based capital asset pricing model and the multiperiod "binomial" option pricing model.

A. Uncertainty

As in Chapter 1, there is some finite set, say Ω, of states. In order to handle multiperiod issues, however, we will treat uncertainty a bit more formally as a *probability space* $(\Omega, \mathcal{F}, P)$, with $\mathcal{F}$ denoting the *tribe* of subsets of Ω that are *events* (and can therefore be assigned a probability), and with P a *probability measure* assigning to any event B in $\mathcal{F}$ its probability $P(B)$. Those not familiar with the definition of a probability space can consult Appendix A. The terms "σ-algebra" and "σ-field," among others, are often used in place of the word "tribe."

There are $T + 1$ dates: $0, 1, \ldots, T$. At each of these, a tribe $\mathcal{F}_t \subset \mathcal{F}$ denotes the set of events corresponding to the information available at time t. In effect, an event B in $\mathcal{F}_t$ is known at time t to be true or false. (A definition of tribes in terms of "partitions" of Ω is given in Exercise 2.11.) We adopt the usual convention that $\mathcal{F}_t \subset \mathcal{F}_s$ whenever $t \le s$, meaning that events are never "forgotten." For simplicity, we also take it that every event in $\mathcal{F}_0$ has probability 0 or 1, meaning roughly that there is no information at time $t = 0$. Taken altogether, the *filtration* $\mathbb{F} = \{\mathcal{F}_0, \ldots, \mathcal{F}_T\}$ represents how information is revealed through time. For any random variable Y, we let $E_t(Y) = E(Y \mid \mathcal{F}_t)$ denote the conditional expectation of Y given $\mathcal{F}_t$. (Appendix A provides definitions of random variables and of conditional expectation.) An *adapted process* is a sequence $X = \{X_0, \ldots, X_T\}$ such that for each t, X_t is a random variable with respect to $(\Omega, \mathcal{F}_t)$. Informally, this means that X_t is observable at time t. An adapted process X is a *martingale* if, for any times t and $s > t$, we have $E_t(X_s) = X_t$. As we shall see, martingales are useful in the characterization of security prices. In order to simplify things, for any two random variables Y and Z, we always write "$Y = Z$" if the probability that $Y \neq Z$ is zero.

B. Security Markets

A security is a claim to an adapted *dividend process*, say δ, with δ_t denoting the dividend paid by the security at time t. Each security has an adapted *security-price process* S, so that S_t is the price of the security, *ex dividend*, at time t. That is, at each time t, the security pays its dividend δ_t and is then available for trade at the price S_t. This convention implies that δ_0 plays no role in determining ex-dividend prices. The *cum-dividend* security price at time t is $S_t + \delta_t$.

Suppose there are N securities defined by the $\mathbb{R}^N$-valued adapted dividend process $\delta = (\delta^{(1)}, \ldots, \delta^{(N)})$. These securities have some adapted price process $S = (S^{(1)}, \ldots, S^{(N)})$. A *trading strategy* is an adapted process θ in $\mathbb{R}^N$. Here, $\theta_t = (\theta_t^{(1)}, \ldots, \theta_t^{(N)})$ represents the portfolio held after trading at time t. The dividend process δ^θ *generated* by a trading strategy θ is defined by

$$\delta_t^\theta = \theta_{t-1} \cdot (S_t + \delta_t) - \theta_t \cdot S_t, \tag{1}$$

with "θ_{-1}" taken to be zero by convention.

C. Arbitrage, State Prices, and Martingales

Given a dividend-price pair (δ, S) for N securities, a trading strategy θ is an *arbitrage* if $\delta^\theta > 0$. (The reader should become convinced that this is the same notion of arbitrage defined in Chapter 1.) Let Θ denote the space of trading strategies. For any θ and φ in Θ and scalars a and b, we have $a\delta^\theta + b\delta^\varphi = \delta^{a\theta+b\varphi}$. Thus the *marketed subspace* $M = \{\delta^\theta : \theta \in \Theta\}$ of dividend processes generated by trading strategies is a linear subspace of the space L of adapted processes.

Proposition. *There is no arbitrage if and only if there is a strictly increasing linear function $F : L \to \mathbb{R}$ such that $F(\delta^\theta) = 0$ for any trading strategy θ.*

Proof: The proof is almost identical to that of Theorem 1A. Let $L_+ = \{c \in L : c \geq 0\}$. There is no arbitrage if and only if the cone L_+ and the marketed subspace M intersect precisely at zero. Suppose there is no arbitrage. The Separating Hyperplane Theorem, in a form given in Appendix B for cones, implies the existence of a nonzero linear functional F such that $F(x) < F(y)$ for each x in M and each nonzero y in L_+. Since M is a linear subspace, this implies that $F(x) = 0$ for each x in M, and thus that $F(y) > 0$ for each nonzero y in L_+. This implies that F is strictly increasing. The converse is immediate. ∎

The following result gives a convenient *Riesz representation* of a linear function on the space of adapted processes. Proof is left as an exercise, extending the single-period Riesz representation lemma of Section 1F.

Lemma. *For each linear function $F : L \to \mathbb{R}$, there is a unique π in L, called the Riesz representation of F, such that*

$$F(x) = E\left(\sum_{t=0}^{T} \pi_t x_t\right), \qquad x \in L.$$

If F is strictly increasing, then π is strictly positive.

For convenience, we call any strictly positive adapted process a *deflator.* A deflator π is a *state-price deflator* if, for all t,

$$S_t = \frac{1}{\pi_t} E_t\left(\sum_{j=t+1}^{T} \pi_j \delta_j\right). \tag{2}$$

A state-price deflator is variously known in the literature as a *state-price density,* a *pricing kernel,* and a *marginal-rate-of-substitution process.*

For $t = T$, the right-hand side of (2) is zero, so $S_T = 0$ whenever there is a state-price deflator. The notion here of a state-price deflator is a natural extension of that of Chapter 1. It can be shown as an exercise that a deflator π is a state-price deflator if and only if, for any trading strategy θ,

$$\theta_t \cdot S_t = \frac{1}{\pi_t} E_t\left(\sum_{j=t+1}^{T} \pi_j \delta_j^{\theta}\right), \qquad t < T, \tag{3}$$

meaning roughly that the market value of a trading strategy is, at any time, the state-price discounted expected future dividends generated by the strategy. The *cum-dividend* value process V^θ of a trading strategy θ is defined by $V_t^\theta = \theta_{t-1} \cdot (S_t + \delta_t)$. If π is a state-price deflator, we have

$$V_t^\theta = \frac{1}{\pi_t} E_t\left(\sum_{j=t}^{T} \pi_j \delta_j^{\theta}\right).$$

The *gain process* G for (δ, S) is defined by $G_t = S_t + \sum_{j=1}^{t} \delta_j$, the price plus accumulated dividend. Given a deflator γ, the *deflated gain process* G^γ is defined by $G_t^\gamma = \gamma_t S_t + \sum_{j=1}^{t} \gamma_j \delta_j$. We can think of deflation as a change of numeraire.

Theorem. *The dividend-price pair* (δ, S) *admits no arbitrage if and only if there is a state-price deflator. A deflator* π *is a state-price deflator if and only if* $S_T = 0$ *and the state-price-deflated gain process* G^π *is a martingale.*

Proof: It can be shown as an easy exercise that a deflator π is a state-price deflator if and only if $S_T = 0$ and the state-price-deflated gain process G^π is a martingale.

Suppose there is no arbitrage. Then $S_T = 0$, for otherwise the strategy θ is an arbitrage when defined by $\theta_t = 0$, $t < T$, $\theta_T = -S_T$. The previous proposition implies that there is some strictly increasing linear function $F : L \to \mathbb{R}$ such that $F(\delta^\theta) = 0$ for any strategy θ. By the previous lemma, there is some deflator π such that $F(x) = E(\sum_{t=0}^{T} x_t \pi_t)$ for all x in L. This implies that $E(\sum_{t=0}^{T} \delta_t^\theta \pi_t) = 0$ for any strategy θ.

We must prove (2), or equivalently, that G^π is a martingale. From Appendix A, an adapted process X is a martingale if and only if $E(X_\tau) = X_0$ for any stopping time $\tau \leq T$. Consider, for an arbitrary security n and an arbitrary stopping time $\tau \leq T$, the trading strategy θ defined by $\theta^{(k)} = 0$ for $k \neq n$ and $\theta_t^{(n)} = 1, t < \tau$, with $\theta_t^{(n)} = 0, t \geq \tau$. Since $E(\sum_{t=0}^{T} \pi_t \delta_t^\theta) = 0$, we have

$$E\left(-S_0^{(n)} \pi_0 + \sum_{t=1}^{\tau} \pi_t \delta_t^{(n)} + \pi_\tau S_\tau^{(n)} \right) = 0,$$

implying that the deflated gain process $G^{n\pi}$ of security n satisfies $G_0^{n,\pi} = E(G_\tau^{n,\pi})$. Since τ is arbitrary, $G^{n,\pi}$ is a martingale, and since n is arbitrary, G^π is a martingale.

This shows that absence of arbitrage implies the existence of a state-price deflator. The converse is easy. ∎

D. Individual Agent Optimality

We introduce an agent, defined by a strictly increasing utility function U on the set L_+ of nonnegative adapted "consumption" processes, and by an *endowment process* e in L_+. Given a dividend-price process (δ, S), a trading strategy θ leaves the agent with the total consumption process $e + \delta^\theta$. Thus the agent has the *budget-feasible consumption set*

$$X = \{e + \delta^\theta \in L_+ : \theta \in \Theta\},$$

and the problem

$$\sup_{c \in X} U(c). \tag{4}$$

The existence of a solution to (4) implies the absence of arbitrage. Conversely, it can be shown as an exercise that if U is continuous, then the absence of arbitrage implies that there exists a solution to (4). For purposes of checking continuity or the closedness of sets in L, we will say that c_n converges to c if $E[\sum_{t=0}^{T} |c_n(t) - c(t)|] \to 0$. Then U is continuous if $U(c_n) \to U(c)$ whenever $c_n \to c$.

Suppose that (4) has a strictly positive solution c^* and that U is continuously differentiable at c^*. We can use the first-order conditions for optimality (which can be reviewed in Appendix B) to characterize security prices in terms of the derivatives of the utility function U at c^*. Specifically, for any c in L, the derivative of U at c^* in the direction c is the derivative $g'(0)$, where $g(\alpha) = U(c^* + \alpha c)$ for any scalar α sufficiently small in absolute value. That is, $g'(0)$ is the marginal rate of improvement of utility as one moves in the direction c away from c^*. This derivative is denoted $\nabla U(c^*; c)$. Because U is continuously differentiable at c^*, the function $c \mapsto \nabla U(c^*; c)$, on L into $\mathbb{R}$, is linear. Since δ^θ is a budget-feasible direction of change for any trading strategy θ, the first-order conditions for optimality of c^* imply that

$$\nabla U(c^*; \delta^\theta) = 0, \qquad \theta \in \Theta.$$

We now have a characterization of a state-price deflator.

Proposition. *Suppose that* (4) *has a strictly positive solution* c^* *and that* U *has a strictly positive continuous derivative at* c^*. *Then there is no arbitrage and a state-price deflator is given by the Riesz representation* π *of* $\nabla U(c^*)$:

$$\nabla U(c^*; x) = E\left(\sum_{t=0}^{T} \pi_t x_t\right), \qquad x \in L.$$

Despite our standing assumption that U is strictly increasing, $\nabla U(c^*; \cdot)$ need not in general be strictly increasing, but is so if U is concave.

As an example, suppose U has the *additive* form

$$U(c) = E\left[\sum_{t=0}^{T} u_t(c_t)\right], \qquad c \in L_+, \tag{5}$$

for some $u_t : \mathbb{R}_+ \to \mathbb{R}$, $t \geq 0$. It is an exercise to show that if $\nabla U(c)$ exists, then

$$\nabla U(c; x) = E\left[\sum_{t=0}^{T} u_t'(c_t)x_t\right]. \tag{6}$$

If, for all t, u_t is concave with an unbounded derivative and e is strictly positive, then any solution c^* to (4) is strictly positive.

Corollary. *Suppose U is defined by* (5). *Under the conditions of the proposition, for any times t and $\tau \geq t$,*

$$S_t = \frac{1}{u_t'(c_t^*)} E_t\left[S_\tau u_\tau'(c_\tau^*) + \sum_{j=t+1}^{\tau} \delta_j u_j'(c_j^*)\right].$$

For the case $\tau = t + 1$, this result is often called the *stochastic Euler equation*. Extending this classical result for additive utility, the exercises include other utility examples such as *habit-formation* utility and *recursive* utility. As in Chapter 1, we now turn to the multi-agent case.

E. Equilibrium and Pareto Optimality

Suppose there are m agents. Agent i is defined as above by a strictly increasing utility function $U_i : L_+ \to \mathbb{R}$ and an endowment process $e^{(i)}$ in L_+. Given a dividend process δ for N securities, an *equilibrium* is a collection $(\theta^{(1)}, \ldots, \theta^{(m)}, S)$, where S is a security-price process and, for each i, $\theta^{(i)}$ is a trading strategy solving

$$\sup_{\theta \in \Theta} U_i(c) \text{ subject to } c = e^{(i)} + \delta^\theta \in L_+, \tag{7}$$

with $\sum_{i=1}^m \theta^{(i)} = 0$.

We define markets to be *complete* if, for each process x in L, there is some trading strategy θ with $\delta_t^\theta = x_t$, $t \geq 1$. Complete markets thus means that any consumption process x can be obtained by investing some amount at time 0 in a trading strategy that generates the dividend x_t in each future period t. With the same definition of Pareto optimality, Proposition 1D carries over to this multiperiod setting. Any equilibrium $(\theta^{(1)}, \ldots, \theta^{(m)}, S)$ has an associated feasible consumption allocation $(c^{(1)}, \ldots, c^{(m)})$ defined by letting $c^{(i)} - e^{(i)}$ be the dividend process generated by $\theta^{(i)}$.

Proposition. *Suppose $(\theta^{(1)}, \ldots, \theta^{(m)}, S)$ is an equilibrium and markets are complete. Then the associated consumption allocation is Pareto optimal.*

The completeness of markets depends on the security-price process S itself. Indeed, the dependence of the marketed subspace on S makes the existence of an equilibrium a nontrivial issue. We ignore existence here and refer to the Notes for some relevant sources.

F. Equilibrium Asset Pricing

Again following the ideas in Chapter 1, we define for each λ in $\mathbb{R}^m_+$ the utility function $U_\lambda : L_+ \to \mathbb{R}$ by

$$U_\lambda(x) = \sup_{(c^{(1)},\ldots,c^{(m)})} \sum_{i=1}^{m} \lambda_i U_i(c^i) \qquad \text{subject to } c^{(1)} + \cdots + c^{(m)} \leq x. \tag{8}$$

Proposition. *Suppose for all i that U_i is concave and strictly increasing. Suppose that $(\theta^{(1)}, \ldots, \theta^{(m)}, S)$ is an equilibrium and that markets are complete. Then there exists some nonzero $\lambda \in \mathbb{R}^m_+$ such that $(0, S)$ is a (no-trade) equilibrium for the one-agent economy $[(U_\lambda, e), \delta]$, where $e = e^{(1)} + \cdots + e^{(m)}$. With this λ and with $x = e = e^{(1)} + \cdots + e^{(m)}$, problem (8) is solved by the equilibrium consumption allocation.*

Proof is assigned as an exercise. The result is essentially the same as Proposition 1E. A method of proof, as well as the intuition for this proposition, is that with complete markets, a state-price deflator π represents Lagrange multipliers for consumption in the various periods and states for all of the agents simultaneously, as well as for the *representative agent* (U_λ, e).

Corollary 1. *If, moreover, U_λ is differentiable at e, then λ can be chosen so that for any times t and $\tau \geq t$, there is a state-price deflator π equal to the Riesz representation of $\nabla U_\lambda(e)$.*

Differentiability of U_λ at e can be shown by the arguments used in Exercise 1.10.

Corollary 2. *Suppose for each i that U_i is of the additive form*

$$U_i(c) = E\left[\sum_{t=0}^{T} u_{it}(c_t)\right].$$

Then U_λ is also additive, with

$$U_\lambda(c) = E\left[\sum_{t=0}^{T} u_{\lambda t}(c_t)\right],$$

where

$$u_{\lambda t}(y) = \sup_{x \in \mathbb{R}^m_+} \sum_{i=1}^{m} \lambda_i u_{it}(x_i) \qquad \textit{subject to } x_1 + \cdots + x_m \leq y.$$

In this case, the differentiability of U_λ at e implies that for any times t and $\tau \geq t$,

$$S_t = \frac{1}{u'_{\lambda t}(e_t)} E_t \left[u'_{\lambda \tau}(e_\tau) S_\tau + \sum_{j=t+1}^{\tau} u'_{\lambda j}(e_j) \delta_j \right]. \tag{9}$$

G. Arbitrage and Martingale Measures

This section shows the equivalence between the absence of arbitrage and the existence of a probability measure Q with the property, roughly speaking, that the price of a security is the sum of Q-expected discounted dividends.

There is *short-term riskless borrowing* if, for each given time $t < T$, there is a security trading strategy θ with $\delta^\theta_{t+1} = 1$ and with $\delta^\theta_s = 0$ for $s < t$ and $s > t + 1$. The associated *discount* is $d_t = \theta_t \cdot S_t$. If there is no arbitrage, the discount d_t is uniquely defined and strictly positive, and we may define the associated *short rate* r_t by $1 + r_t = 1/d_t$. This means that at any time $t < T$, one may invest one unit of account in order to receive $1 + r_t$ units of account at time $t + 1$. We refer to $\{r_0, r_1, \ldots, r_{T-1}\}$ as the associated "short-rate process," even though r_T is not defined.

We suppose throughout this section that there is short-term riskless borrowing at some uniquely defined short-rate process r. We can define, for any times t and $\tau \leq T$,

$$R_{t,\tau} = (1 + r_t)(1 + r_{t+1}) \cdots (1 + r_{\tau-1}),$$

the payback at time τ of one unit of account borrowed risklessly at time t and "rolled over" in short-term borrowing repeatedly until date τ.

It would be a simple situation, both computationally and conceptually, if any security's price were merely the expected discounted dividends of the security. Of course, this is unlikely to be the case in a market with risk-averse investors. We can nevertheless come close to this sort of characterization of security prices by adjusting the original probability measure P. For this, we define a new probability measure Q to be *equivalent* to P if Q and P assign zero probabilities to the same events. An equivalent probability measure Q is an *equivalent martingale measure* if

$$S_t = E^Q_t \left(\sum_{j=t+1}^{T} \frac{\delta_j}{R_{t,j}} \right), \qquad t < T,$$