THEORY OF ORBITS

The Restricted Problem of Three Bodies

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The Restricted Problem of Three Bodies

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Foreword

The subject of this treatise, the restricted problem of three bodies, occupies a central place in analytical dynamics, celestial mechanics, and space dynamics. Entry into celestial mechanics and space dynamics can be gained by the study of the problem of two bodies. To penetrate the fundamental problems, the number of participating bodies must be increased from two to three. This step is critical. Not only is the two-body problem solved—and the meaning of "solution" may be different for astronomers, engineers, and mathematicians—but a general understanding exists regarding this dynamical system. The problem of three bodies on the other hand is neither solved nor is the behavior of the dynamical system completely understood.

The solar system provides few applications of the general problem of three bodies. This results in an unusual situation where a more general problem having considerable complexity is less useful than a comparatively simple formulation. Also it is important to realize that more is known about the restricted problem than about the general problem.

This volume is strongly influenced by the creators of modern dynamics, H. Poincaré and G. D. Birkhoff. Poincaré, in his *Méthodes Nouvelles de la Mécanique Céleste* and also in his famous *Mémoire Couronné*, "Sur le problème des trois corps et les équations de la dynamique," uses the problem of three bodies as his favorite example when presenting his work in dynamics. The same is true for G. D. Birkhoff's *Dynamical Systems*, and for C. L. Siegel's *Vorlesungen über Himmelsmechanik*. A. Wintner's *Analytical Foundations of Celestial Mechanics* was originally planned to treat the problem of three bodies, especially the restricted problem, but it actually presented more of the mathematical foundations than of the celestial mechanics. It is interesting to note that H. Happel's book is entitled *Das Dreikörperproblem*, and the subtitle reads "Vorlesungen über Himmelsmechanik," while the second volume of K. Stumpff's *Himmelsmechanik* displays the subtitle "Das Dreikörperproblem." How intimately the problem of three bodies is connected with celestial mechanics and with dynamics in general when titles, subtitles, contents, applications, and examples become interchangeable!

The applications of the restricted problem to celestial mechanics form the basis of some lunar and planetary theories. The modern applications to space mechanics are probably even more cogent if not more numerous than the classical applications. The implications of the restricted problem for cosmogony and stellar dynamics are also numerous. Finally, it can be shown that a great variety of dynamical systems can be presented by equations of motion which are formally identical with the equations of the restricted problem. One measure of the importance of a scientific endeavor is its effect on peripheral fields. While authors from Euler to Siegel recognized astronomy and dynamics as the only peripheral fields, today we know that space mechanics and stellar dynamics are fields which benefit equally.

The interest in space sciences rejuvenated celestial mechanics, and the well-established tools of the latter were immediately applied. Some of the problems were not really new and the proven methods of classical celestial mechanics—in the hands of the masters—produced immediate results. I think of several solutions of the drag-free earth-satellite problem, for instance, which today may be considered settled. It is a perturbation of the two-body problem, and the success in solving it is partly explained by the popularity of satellite problems in classical celestial mechanics. Other problems in space dynamics, closely associated with the restricted problem, are of considerable importance and interest today. Many of these problems are new, and in what follows one of them will be contrasted to a classical problem. Consider the famous classical three-body problem, the sun-earth-moon combination and the determination of the motion of the moon. We might think about two large bodies, the sun and the earth, which move around each other in approximate circles, and in their field a third body, the moon, which moves on an approximate ellipse. This configuration is stationary in a sense, since no collisions take place. This is also true for the motion of a Trojan asteroid under the continued influence of the sun and Jupiter. On the other hand, one of the central problems in space science is to create artificial bodies which may be required to move on orbits connecting the close neighborhood of two natural celestial bodies. Sometimes collision orbits are desired. Problems with close approaches and collisions were hardly ever treated in classical celestial mechanics and these problems became important in the new science of space dynamics.

The use of three essentially different approaches to dynamics, the qualitative, the quantitative, and the formalistic, is dictated by the special advantages of each and is described in the Introduction, where a number of references to the history of the restricted problem are also given.

The first chapter introduces the problem of three bodies and formulates the equations of motion in inertial and in rotating coordinate systems. The relation of the restricted problem to the general problem of three bodies is described and illustrated with examples. Several applications to cosmogony and stellar dynamics are also outlined. Chapter 2 discusses reductions of the problem and offers a comprehensive treatment of streamline analogies.

Chapter 3 is concerned with regularization and shows how the equations of motion can be written in a system free of singularities. This subject is the feature which distinguishes a work on classical celestial mechanics from one on modern applications. This chapter is probably the most important one for the reader who is working in the field of space mechanics. Chapter 4 is devoted to the principal qualitative aspect of the restricted problem—the curves of zero velocity, several uses of which are discussed. The regions of permissible motion and the location and properties of the libration points are established. Motion and nonlinear stability in the neighborhood of these equilibrium points are treated in detail in Chapter 5.

Chapter 6 contains a short introductory treatment of Hamiltonian dynamics in the extended phase space. Chapter 7 applies the principles and methods of the previous chapter to the restricted problem and to its regularization. The generating functions that are used are derived with emphasis on justification and motivation. A natural way to introduce the concept of perturbation theory is presented.

Chapter 8 discusses the problem of two bodies in a rotating coordinate system and treats periodic orbits in the restricted problem, following H. Poincaré and G. D. Birkhoff. Chapter 9 presents the quantitative aspects of the restricted problem. The results of G. Darwin, E. Strömgren, and F. R. Moulton are discussed and several of the recently established lunar and interplanetary orbits in the Soviet and American literature are compared. Chapter 10 is devoted to modifications of the restricted problem, such as the elliptic problem, the three-dimensional problem, and Hill's problem.

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Preface

This volume has been developed from my lectures and seminars on various aspects of celestial mechanics, dynamics, the restricted problem of three bodies, periodic orbits, regularization, and space dynamics. While directed primarily to the graduate student, it is intended to be sufficiently comprehensive to serve as a reference and advanced text on many applications of celestial mechanics. One purpose is to familiarize those readers who are concerned with the space applications of celestial mechanics with the next step after the problem of two bodies. The student of celestial mechanics will find both classical studies and recent developments in the restricted problem of three bodies with a survey of the pertinent literature.

This is the first book devoted to the theory of orbits in the restricted problem. My aim is to build a bridge between books written for the astronomer, mathematician, space engineer, and student of dynamics. Instead of developing the subject separately for each of these professions, it is hoped that the single subject of this volume will be useful for all its readers. Astronomers will find more references to analytical dynamics than is usual in textbooks on celestial mechanics; workers in the field of dynamics will read about astronomical applications; the needs of mathematicians and engineers will be met by the problem of establishing the totality of possible motions of our dynamical system.

Teaching experience shows that students are interested in historical reviews and remarks in the field of celestial mechanics, which is so rich in traditions and in cultural background material. Such comments are collected at the end of each chapter with the discussions of the pertinent references. Most chapters contain a generous amount of basic mathematical information. I make it a point to extend the foundations more than necessary for the building, in order to establish a more solid edifice and offer to the reader the opportunity of proceeding with his own applications.

My guiding principle has been to inform the reader of the motivation and purpose of the developments, hoping to inspire his enthusiastic interest in the subject. I try to avoid unnecessary *epsilontics* in the mathematical parts and highly specialized and undefined terms in the applications. Mathematics is a tool in dynamics, not a goal. The Wintnerian turnaround from the problem of three bodies to mathematics is avoided, and an attempt is made to emphasize the dynamics. I subject the brilliance of Poincaré and of G. D. Birkhoff to scrutiny and explanation rather than to competition. My aims are to summarize G. Darwin's eloquence, to expand Siegel's terseness, to generalize Charlier, and to particularize Moulton and E. Strömgren. Special attention is paid to the Soviet literature of the past two or three decades; it contains many significant contributions to celestial mechanics and space dynamics. Recent numerical results on earth-moon trajectories are compared with previous results, and classical orbit computations are brought up to date.

April, 1967 V. SZEBEHELY

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I extend sincere thanks to my graduate students. The computations involved in preparing the major tables were performed by Miss C. Williams. Several of the complicated figures were drawn by an automatic plotter, imputs to which were prepared by Dr. D. Pierce and Mr. M. Standish. Several of the orbits were computed by Messrs. S. Knowles and J. Lieske. In these undertakings the facilities of the Yale Computer Center were used. The burden of reading the proofs was shared mostly by Messrs. D. Bettis, J. Breedlove, P. Esposito, R. Laubscher, and P. Nacozy, graduate students. The tables were put in their final form by Dr. and Mrs. R. Duncombe of the U.S. Naval Observatory.

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Introduction

The purpose of dynamics is to characterize the totality of possible motions of a given dynamical system. Such a characterization does not necessarily mean an explicit, closed-form, general solution of the problem since this is seldom possible, and when it is possible, it is most of the time neither meaningful nor helpful in understanding the behavior of the system. An example is the problem of two bodies, which is considered solved since the properties of the totality of possible motions are known. Although the coordinates describing the motion of the bodies participating in the problem cannot be represented as explicit functions of the time in closed form, the problem is nevertheless considered solved.

Qualitative, quantitative, and formalistic dynamics are the three major approaches to the understanding of the behavior of dynamical systems. The qualitative approach is probably the most elegant and sometimes the most powerful. The formalistic method is the basis of classical celestial mechanics. The quantitative approach is often the most popular among astronomers and engineers who may want to find one particular solution of a problem rather than to study the behavior of the dynamical system. Examples are the ephemerides of the planets, representing particular solutions of the astronomers' n-body problem and Apollo trajectories, being particular solutions of the engineers' problem.

Qualitative methods in dynamics are well suited to the treatment of such questions as stability, existence problems, integrability, and reducibility. The names of H. Poincaré and G. D. Birkhoff are associated with qualitative dynamics; Hill's name is seldom thought of in this connection, in spite of his use of the zero velocity curves to establish limiting regions. His method is probably one of the most powerful and successful qualitative ideas in the restricted problem.

It is sometimes said of qualitative dynamics that its results are not helpful to "practical" men (to the "users" as opposed to the "creators"). This misconception is partly because some of the qualitative results in dynamics have not yet been interpreted and some of these results are of theoretical interest only.

Knowledge of certain qualitative properties of a dynamical system may be much more valuable than numerical solutions. An example is the existence question of periodic orbits. Solutions of nonintegrable dynamical systems are never known along the whole time axis unless they are of periodic or asymptotic nature. This is seen when we consider an attempt to establish a particular solution of the differential equations of a dynamical system with an electronic computer. Not attempting for the moment to evaluate such an undertaking, let us visualize the computer output as the time increases without limit and as various error sources contribute to the printouts. Unless some systematic behavior of the result is discovered, sooner or later the computer output becomes meaningless and no valuable information about the dynamical system will be obtained along the whole time axis. The orbit or the behavior of the system will remain unknown in spite of the numerical work.

Another example is furnished by one of the fundamental questions of dynamics: the description of the totality of possible motions of a dynamical system. For nonintegrable systems this is a major problem as no closed-form general solution is available. The practical importance of knowing all possible orbits between the earth and the moon does not need emphasis, since selection of an orbit "best" suited for a certain mission requires information regarding the possible choices. A formalistic approach to this problem is not fruitful, for even if it should furnish convergent series which give the general solution, the nature, the properties, and the totality of the solution could not in general be determined from such series. The quantitative approach to this problem is to select a region of the initial conditions which is of practical interest and to compute as many orbits as possible in this region. This set of orbits is called the "totality of orbits of interest." The deficiency in this approach is the possible omission of useful orbits or of whole families of useful orbits. When the possible range of initial conditions is considerable, the establishment of families of orbits according to six varying initial conditions is almost a hopeless task numerically. The description of the totality of possible motions should come from a combined approach (numerical and formalistic) with qualitative dynamics leading and organizing the steps. One of the most practical and most important problems in applied celestial mechanics, the selection of a suitable orbit, is therefore equivalent to one of the most advanced problems of qualitative dynamics.

Turning now to the formalistic approach we enter the stronghold of classical celestial mechanics. The formalistic methods are also called general perturbation methods, and the principal mathematical tools are series expansions. In order to have a general perturbation method the initial conditions are kept arbitrary in the solution. Justification of the method from a mathematical point of view requires scrutiny of the convergence of the series with respect to the variables. It is ironic that one of the qualitative results of dynamics, attributed to Poincaré, states that the series used in celestial mechanics are in general divergent. Nevertheless, finite parts of such series are often extremely useful in celestial mechanics since they do give results in agreement with observations. Questions connected with the behavior of the system as the time increases to infinity cannot, of course, be answered by such series solutions. The classical series of celestial mechanics become of little use when bodies approach each other closely and when they collide. Since such orbits are of central importance in modern dynamics, new formalistic approaches have had to be devised.

The hopefully expected ultimate answer of representing the totality of solutions as "simple" functions of the initial conditions and of time may come from formalistic dynamics. Such a result can probably be expected from a combined effort of the three major approaches with the formalistic approach taking the lead. Newton's approach to dynamics was to find just such explicit expressions representing the motion of dynamical systems. Advances in celestial mechanics and in other branches of science with mathematical orientation show more or less the same steps. First comes the attempt to describe the field of interest with simple analytic expressions. This leads of necessity to successive approximations and series solutions if the first attempt for simple closed-form solutions fails. Those fields, such as the "solvable" dynamical systems, in which the first step furnishes results, are considered solved and are soon abandoned. The quantitative approach to dynamics is not unlike the first step because it gives a particular solution in a simple form: a set of numbers representing the coordinates as functions of time. Those fields in which sufficient interest exists for establishing general solutions, but which at the same time are not "integrable" and therefore are not amenable to simple general solutions, are graduated to the second phase of mathematical physics: to series solutions. Some problems are solved at this stage if the series solutions furnish the properties of the general solution. This is seldom the case for systems of appreciable complexity where neither the convergence of the successive approximations nor the physical meaning of the series solution is completely clear. Those physical problems that fall in this last category are elevated to the domain of qualitative methods.

The quantitative approach to dynamics corresponds to experimentation. Its significance cannot be overestimated, especially in nonintegrable dynamical systems since, after all, this is the only method which furnishes an orbit when the convergence of the formalistic approach is in doubt. The power of properly designed experiments and the importance of proper interpretation of results are well known in physics. Dynamics' experimental tool, the computer, only recently became efficient enough to handle complex problems; therefore, experimentation in dynamics has not advanced as far as it has for example in physics. Famous classical computational results in the restricted problem were obtained without the use of digital computers by E. Strömgren and G. Darwin. Recent high-speed computational results, together with the older results, reveal several significant properties of the system that can be verified theoretically. Such a combined theoretical-experimental approach has shown great potentialities in dynamics.

The history of the restricted problem begins with Euler and Lagrange in 1772, continues with Jacobi (1836) and Hill (1878), and is followed by Poincaré (1899), Levi-Civita (1905), and Birkhoff (1915). The span of almost 200 years, from Euler until now, includes other great names and important contributions; this short historical review nevertheless will concentrate on the accomplishments of Euler, Jacobi, and Poincaré.

The first contribution was made by *Euler* in 1772 in connection with his lunar theories. His work was the first important contribution to the restricted problem and its influence on further developments of lunar theories and even on some very recent work in space dynamics is clearly evident. His principal accomplishment was the introduction of a synodic (rotating) coordinate system, the use of which led to an integral of the equations of motion, known today as the Jacobian integral. Euler himself did not discover the Jacobian integral which was first given by Jacobi (1836) who, as Wintner remarks, "rediscovered" the synodic system. The actual situation is somewhat complex since Jacobi published his integral in a sidereal (fixed) system in which its significance is definitely less than in the synodic system. The tongue-in-cheek remark of Wintner which is mentioned before, is not completely accurate, nor is his immediately following recommendation, citing Newcomb's report on lunar theory as a useful reference for the history of the restricted problem.

Prior to his lunar theory Euler (1760) gave the solution of the problem of two fixed centers of force, in which two fixed masses act on a third body according to the Newtonian law of gravitation. This dynamical system is a special and highly simplified case of the restricted problem since centrifugal and Coriolis forces do not enter. Its direct significance is limited as fixed force centers do not occur either in celestial mechanics or in its applications. In view of the fact, however, that Euler's problem of two fixed force centers can be solved in closed form its indirect applications are numerous. In the literature of space mechanics attention was called to this problem (and to its simplification by use of Bonnet's theorem) in 1959 in connection with the reliability and accuracy of digital-computer solutions. Euler's solution of the problem of two fixed centers of force can also be used in connection with the artificial satellite problem and as a reference orbit for general-perturbation calculations in the restricted problem. In fact Vinti's solution of the artificial satellite problem, established independently of Euler's result, turns out to be essentially analytically identical with it. The idea to use Euler's solution as a reference orbit in the restricted problem is not new. Unfortunately, this solution involving elliptic functions is less useful than the far simpler Encke method using conic sections as reference orbits. On the other hand, Euler's problem does include the effect of both masses while Encke's method considers only one. The third application of Euler's two fixed force centers is related to the problem of regularization. The coordinate transformation employed by Euler to treat the problem of two fixed centers of force when used for the restricted problem eliminates the singularities or in other words "regularizes" the problem. Not only did Thiele (1892) and Burrau (1906) make use of this transformation for the restricted problem, performing the regularizing process, but also the large amount of numerical work performed by the Copenhagen school under the direction of Strömgren (1935) was based on this transformation.

The *Jacobian* integral of the restricted problem is attached to the name of the second major contributor.

Implications of this integral are numerous. Since it connects the magnitude of the velocity vector (the speed) of the third body to its location, it allows us to make certain general, *qualitative* statements regarding the motion without actually solving the equations of motion. This fact gives great importance to an integral applicable to an "unsolvable" dynamical problem. It permits the establishment of a certain forbidden region from which the third body is excluded. The application of this principle to celestial mechanics was first made by Hill (1878) to show that the earth-moon distance must remain bounded from above for all time, which is to say that *if* Hill's model for the sun-earth-moon system is accepted, then the moon cannot depart from the earth's neighborhood arbitrarily far.

Poincaré's famous three volumes of *Méthodes Nouvelles* were completed in 1899. This work was so new and original that many of its implications are still not entirely clear. Probably the most significant contribution made by Poincaré was his emphasis on the qualitative aspects of celestial mechanics as opposed to the quantitative approach. Just as Euler proposed the lunar theories, which may be considered the highest computational accomplishments of mankind, Poincaré initiated analytical methods which seem to be the highest theoretical accomplishments. Just as Euler's work on the restricted problem was followed by Hill and Brown (1896), who gave the most precise lunar theory, so was Poincaré followed by Birkhoff (1915), who elevated the methods of qualitative dynamics to heights still unconquered by those who wish to apply his results.

The problem of regularization, which is so prominent in certain applications of space dynamics, is associated with the names of Thiele (1892), Painlevé (1897), Levi-Civita (1903), Burrau (1906), Sundman (1912), and Birkhoff (1915).

Interpretation and continuation of the undertaking begun by Euler and culminating in Birkhoff's work was by no means finished by the latter. In the 1920's Moulton's school published its results, while in the thirties Moiseev and again Birkhoff made their qualitative contributions, and the final quantitative results of the Strömgren school were published. In 1941 Wintner's book was published, and in the fifties came Kolmogorov's important work, and also Siegel's book. In the sixties Russian writers following Kolmogorov's work took significant steps in qualitative analysis. Also during the sixties a considerable amount of numerical experimental dynamics was performed on digital computers.

The literature of the restricted problem is closely associated with publications in celestial mechanics and with the appearance of books on dynamics. Chapters in books on celestial mechanics by Plummer (1918), by Charlier (1907), by Moulton (1914), by Brouwer–Clemence (1961), by Danby (1962), and by McCuskey (1963) offer informative descriptions of the restricted problem. Whittaker's *Analytical Dynamics* (1904) may be considered the outstanding reference text in dynamics on the general and on the restricted problem of three bodies, while chapters by Pars (1965) and Pollard (1966) give concise treatments from the points of view of dynamics and mathematics.

The justification given by Birkhoff in 1927 for the appearance of a book like this may be quoted here to close the introduction. "At a time when no physical theory can properly be termed fundamental—the known theories appear to be merely more or less fundamental in certain directions—it may be asserted with confidence that ordinary differential equations in the real domain, and particularly equations of dynamical origin, will continue to hold a position of the highest importance."

Chapter **1**

Description of the Restricted Problem

1.1 Introduction

It is often the case in physical sciences that the major difficulty in attacking a problem is the lack of clear definitions, and once the problem is stated the solution is on its way. In this spirit we shall put emphasis on describing the restricted problem in the clearest and simplest terms possible.

The offering of a clear definition is of course a necessary but not a sufficient condition for making progress. The problem of three bodies is a good example where with a little care the most precise statement of the problem can be given. This statement describes a rather simple sounding problem, the solution of which is not available.

In this chapter the simplest and most frequently occurring version of the restricted problem is described. The basic formulation seems to appear first in Euler's memoir on his second lunar theory; therefore, it is almost 200 years old.

After defining the problem, the equations of motion are derived in inertial (sidereal) and rotating (synodic) coordinate systems using physical (dimensional) and dimensionless variables. The four equations are compared and it is shown how the introduction of synodic coordinates results in the existence of the Jacobian constant and of the Jacobian integral. The derivations of these four sets of equations of motion are performed starting with basic and simple principles and using rather elementary methods in order to facilitate the understanding of the physical picture. In a later chapter (Chapter 7) the Lagrangian and Hamiltonian formulations will be given and a more sophisticated picture will be revealed.

The appearance of the restricted problem as the degenerate case of the general three-body problem is shown next to serve as the basis for two important items. As the first outcome the review of various modifications of the basic restricted problem is presented. In the last section the applicability of the restricted problem is analyzed.

1.2 Statement of the problem and equations of motion in a sidereal system

We define our problem as follows: Two bodies revolve around their center of mass in circular orbits under the influence of their mutual gravitational attraction and a third body (attracted by the previous two but not influencing their motion) moves in the plane defined by the two revolving bodies. *The restricted problem of three bodies is to describe the motion of this third body.*

The two revolving bodies are called the primaries (or the primary and the secondary, a nomenclature popular in stellar dynamics but which we will *not* follow). The masses m_1 and m_2 of these bodies are arbitrary but the bodies have such internal mass distributions that they may be considered point masses. The mass of the third body m_3 is an intricate subject which will be discussed in some detail later in this chapter. At this point the approximate statement is accepted, that m_3 is much smaller than either m_1 or m_2. This is intuitively correct since m_3 does not influence the motion of m_1 and m_2.

The circular motion of m_1 and m_2 around their mass center 0 is shown in Fig. 1.1.

Balance between the gravitational and centrifugal forces requires that

$$k^2 \frac{m_1 m_2}{l^2} = m_2 a n^2 = m_1 b n^2, \tag{1}$$

where k is the Gaussian constant of gravitation, n is the (common) angular velocity of m_1 and m_2, l is their mutual distance, and a and b are as shown in Fig. 1.1. The quantity n in celestial mechanics is called the mean motion and the angle nt^* the longitude of m_1. The symbol t^* is used for time, this way preserving t for the dimensionless time.

From this

$$k^2 m_1 = a n^2 l^2, \qquad k^2 m_2 = b n^2 l^2, \qquad k^2(m_1 + m_2) = n^2 l^3, \tag{2}$$

the last equation being Kepler's third law.

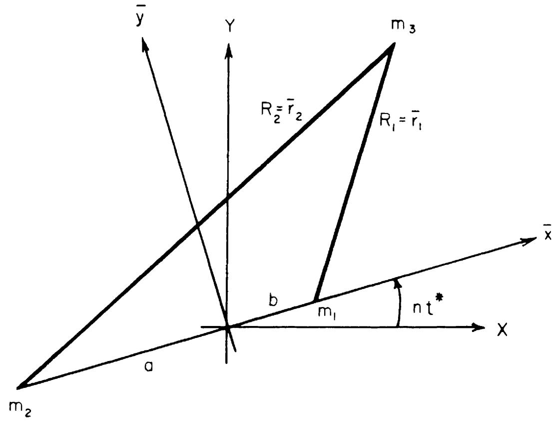

FIG. 1.1. The fixed (sidereal) and the rotating (synodic) coordinate systems ($m_1 > m_2$).

Also

$$a = \frac{m_1 l}{M} \quad \text{and} \quad b = \frac{m_2 l}{M}, \tag{3}$$

where $M = m_1 + m_2$.

The equations of motion of m_3 in an inertial (fixed) rectangular coordinate system, X being the abscissa and Y the ordinate of m_3, are

$$d^2X/dt^{*2} = \partial F/\partial X \quad \text{and} \quad d^2Y/dt^{*2} = \partial F/\partial Y. \tag{4}$$

We note that the inertial coordinate system X, Y shown in Fig. 1.1 is called the sidereal system. F is the force function or the negative potential and is given by

$$F = k^2(m_1/R_1 + m_2/R_2). \tag{5}$$

The distances R_1 and R_2 are given by

$$\begin{aligned} R_1 &= [(X - X_1)^2 + (Y - Y_1)^2]^{1/2}, \\ R_2 &= [(X - X_2)^2 + (Y - Y_2)^2]^{1/2}, \end{aligned} \tag{6}$$

where (X_1, Y_1) and (X_2, Y_2) are the time-dependent coordinates of m_1 and m_2, respectively, which are obtainable by inspecting Fig. 1.1:

$$X_1 = b \cos nt^*, \qquad X_2 = -a \cos nt^*,$$
$$Y_1 = b \sin nt^*, \qquad Y_2 = -a \sin nt^*.$$

The time dependence of the coordinates of m_1 and m_2 introduces the time explicitly in the equations of motion. This is intuitively expected since m_1 and m_2 *move* in the fixed system of coordinates. The formal proof of the explicit occurrence of the time consists of substituting the preceding time dependencies into (6), (6) into (5), and (5) into (4). Equations (4) become

$$\begin{aligned}\frac{d^2X}{dt^{*2}} &= -k^2\left[\frac{m_1(X-b\cos nt^*)}{R_1^3}+\frac{m_2(X+a\cos nt^*)}{R_2^3}\right],\\ \frac{d^2Y}{dt^{*2}} &= -k^2\left[\frac{m_1(Y-b\sin nt^*)}{R_1^3}+\frac{m_2(Y+a\sin nt^*)}{R_2^3}\right],\end{aligned} \tag{7}$$

or simply

$$\frac{d^2X}{dt^{*2}}=\frac{\partial F(X,Y,t^*)}{\partial X} \quad\text{and}\quad \frac{d^2Y}{dt^{*2}}=\frac{\partial F(X,Y,t^*)}{\partial Y}. \tag{8}$$

1.3 An invariant relation and the total energy of the system

This section discusses three questions of considerable theoretical importance. In Part (A) an invariant relation for the restricted problem in the inertial system is derived, Part (B) gives the definition of an integral of a dynamical system in general, and Part (C) discusses the conservation of energy.

(A) Inasmuch as the gravitational force field possesses a potential, an attempt might be made to derive an invariant relation corresponding to the energy conservation of the dynamical system. Multiplying Eqs. (8) by dX/dt^* and dY/dt^*, respectively, adding and integrating with respect to the time gives

$$\frac{1}{2}\left[\left(\frac{dX}{dt^*}\right)^2+\left(\frac{dY}{dt^*}\right)^2\right]=\int_{t_0^*}^{t^*}\left(\frac{\partial F}{\partial X}\frac{dX}{dt^*}+\frac{\partial F}{\partial Y}\frac{dY}{dt^*}\right)dt^*, \tag{9}$$

where, for the time being, the constant of integration is disregarded. Since

$$dF = F_X\,dX + F_Y\,dY + F_{t^*}\,dt^*,$$

where subscripts denote partial derivatives, the quadrature on the right side of (9) becomes

$$\int_{t_0^*}^{t^*} dF - F_{t^*}\,dt^* = F - \int_{t_0^*}^{t^*} F_{t^*}\,dt^*. \tag{10}$$

Equations (9) and (10) give

$$\tfrac{1}{2}V^2 = F - \int_{t_0^*}^{t^*} F_{t^*}\,dt^*, \tag{11}$$

where V is the velocity of m_3. The energy is therefore not conserved in the system since $\frac{1}{2}V^2 - F$ is a function of the time and not constant. The interpretation of the quadrature

$$\int_{t_0^*}^{t^*} \frac{\partial F(X, Y, t^*)}{\partial t^*} dt^*$$

occurring in (11) is as follows. Consider the solution given in the form $X = X(\alpha_i, t^*)$, $Y = Y(\alpha_i, t^*)$, where the constants α_i $(i = 1, 2, \ldots, 4)$ represent the initial conditions. Substituting this solution into the quadrature gives a function depending on the initial conditions and on the time. Therefore (11) becomes

$$\tfrac{1}{2}V^2 - F = C(\alpha_i, t^*).$$

The essential point is that a relation like (11) is of limited use unless additional approximations are made, since in general its interpretation requires the solution of the problem. On the other hand, if F should not depend explicitly on the time, then $F_{t^*} = 0$ and $\frac{1}{2}V^2 - F = C(\alpha_i)$; i.e., the constant of integration depends only on the initial conditions.

It is to be noticed that Eq. (11) is an invariant relation of the dynamical system in question. While it is not immediately helpful in establishing the "solution" it does serve as a useful check in numerical and formal computations. In the following we give the definition of an integral of a dynamical system to avoid any misunderstanding.

(B) Consider a dynamical system of n degrees of freedom with coordinates $q_1, q_2, \ldots, q_n$ and write the equations of motion in the form

$$d^2q_i/dt^2 = Q_i(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t), \tag{12}$$

where $i = 1, \ldots, n$ and t is used for the time. This set of n second-order equations form a system of the $2n$th order, which can be written as $2n$ equations of the first order by introducing

$$x_i = q_i \qquad \text{and} \qquad x_{n+i} = \dot{q}_i$$

or

$$x_1 = q_1, x_2 = q_2, \ldots, x_n = q_n, x_{n+1} = \dot{q}_1, x_{n+2} = \dot{q}_2, \ldots, x_{2n} = \dot{q}_n. \tag{13}$$

Equations (12) become

$$\begin{aligned} \frac{dx_i}{dt} &= x_{n+i}, \\ dx_{n+i}/dt &= Q_i(x_1, x_2, \ldots, x_{2n}, t). \end{aligned} \tag{14}$$

Explicit and detailed forms of Eqs. (14) are

$$\begin{aligned}
\dot{x}_1 &= x_{n+1}\,,\ \dot{x}_2 = x_{n+2}\,,\dots,\ \dot{x}_n = x_{2n}\,,\\
\dot{x}_{n+1} &= Q_1(x_1\,,\dots,\,x_{2n}\,,\,t),\\
\dot{x}_{n+2} &= Q_2(x_1\,,\dots,\,x_{2n}\,,\,t),\\
&\ \vdots\\
\dot{x}_{2n} &= Q_n(x_1\,,\dots,\,x_{2n}\,,\,t),
\end{aligned} \tag{15}$$

or simply

$$\dot{x}_k = P_k(x_1\,,\dots,\,x_m\,,\,t), \tag{16}$$

where $k = 1, 2,\dots, m$, $m = 2n$, and the functions P_k and their partial derivatives are defined and continuous in some domain.

Considering now the $2n$ first-order differential equations given by Eq. (16) we define an integral of this system as follows. If a function $G(x_1\,,\,x_2\,,\dots,\,x_m\,,\,t)$ with the same properties as P_k satisfies the condition

$$dG/dt = 0 \tag{17}$$

when any set of solution $x_1(t), x_2(t),\dots, x_m(t)$ is substituted, we call $G(x_1\,,\,x_2\,,\dots,\,x_m\,,\,t) =$ const an integral of the system (16). Equation (17) can be expanded:

$$\sum_{k=1}^{m} \frac{\partial G}{\partial x_k}\frac{dx_k}{dt} + \frac{\partial G}{\partial t} = 0,$$

or using (16)

$$\sum_{k=1}^{m} \frac{\partial G}{\partial x_k} P_k + \frac{\partial G}{\partial t} \equiv 0. \tag{18}$$

Equation (18) is the condition to be satisfied in order that G be an integral.

(C) We will now show that the total energy in the restricted problem is not constant. The total energy of m_3 per unit mass is

$$h_3 = \tfrac{1}{2}(dX/dt^*)^2 + \tfrac{1}{2}(dY/dt^*)^2 - F, \tag{19}$$

which, as shown in Part (A), is not constant. The question arises whether the total energy of the dynamical system formed by the three bodies is constant. The total energy of the individual particles is, of course, the sum of their separate energies, so we have

$$H = m_3 h_3 + H_{12}\,, \tag{20}$$

where H_{12} is the energy of the m_1, m_2 system, and

$$H_{12} = \tfrac{1}{2}n^2(m_1b^2 + m_2a^2) - k^2 \frac{m_1m_2}{l}. \tag{21}$$

The first term is the kinetic energy and the second the potential energy. Using the elementary relations given by Eqs. (1) and (2) one finds that

$$\tfrac{1}{2}n^2(m_1b^2 + m_2a^2) = \tfrac{1}{2}k^2 \frac{m_1m_2}{l}; \tag{22}$$

therefore, $H_{12} = -\frac{1}{2}k^2\, m_1m_2/l = \text{const}$, and the total energy of the system formed by the three bodies becomes

$$H = m_3h_3 - \tfrac{1}{2}k^2 \frac{m_1m_2}{l} \neq \text{const.} \tag{23}$$

The formal reason for the result that the total energy of the three bodies participating in the restricted problem is not constant is simply that the total energy of m_3 is not constant while that of the m_1, m_2 system is constant; therefore, the sum of the energies cannot be constant. The deeper reason for this "violation of the energy conservation" will become clear in the next chapter when the general problem of three bodies will be discussed. In the restricted problem we neglected the effect of m_3 on the motion of m_1 and m_2, creating in a way a dynamical situation which, strictly speaking, exists only when $m_3 = 0$. If this condition is satisfied then, of course, Eq. (23) gives for the total energy of the three bodies $H = -\frac{1}{2}k^2m_1m_2/l = \text{const}$.

If the mass of m_3 is different from zero it must have an effect on the motion of m_1 and m_2, which cannot move any more on their assumed circular orbits. In this case Eq. (21) is not valid and H_{12} is not constant. The total energy of the dynamical system formed by the three bodies without restrictions on the motion of m_1 and m_2 possesses a potential which does not depend explicitly on the time; therefore, the total energy of this system is conserved.

1.4 Equations of motion in a synodic coordinate system and the Jacobian integral

It was discussed in some detail that the force function F contains the time explicitly because of the motion of the primaries. Consequently the Hamiltonian depends on the time explicitly, it is not an integral, and it is not constant along an orbit as will be shown in Chapter 7.

Based on the principle that one purpose of mathematics is indeed to verify intuitive results, the question is proposed: what coordinate system would result in a force function which would show no explicit dependence on the time? The intuitive answer is that since the time dependence is a consequence of the motion of the primaries in a fixed (sidereal) system one should expect that a coordinate system in which m_1 and m_2 are fixed will show superior qualities. The following derivation shows the correctness of this intuitive guess.

The coordinate transformation is the well-known rotation which, with the notation of Fig. 1.1, becomes

$$\begin{aligned} X &= \bar{x}\cos nt^* - \bar{y}\sin nt^*, \\ Y &= \bar{x}\sin nt^* + \bar{y}\cos nt^*, \end{aligned} \tag{24}$$

or, in the notation of matrices,

$$\mathbf{R} = \mathbf{A}\bar{\mathbf{r}},$$

where the vector $\mathbf{R}$ has the components X, Y; the components of the vector $\bar{\mathbf{r}}$ are $\bar{x}$, $\bar{y}$ and the matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{pmatrix} \cos nt^* & -\sin nt^* \\ \sin nt^* & \cos nt^* \end{pmatrix}. \tag{25}$$

The transformation of Eqs. (7) is probably simplest when complex variables are introduced. Let

$$Z = ze^{int^*}, \tag{26}$$

where

$$z = \bar{x} + i\bar{y}, \qquad Z = X + iY, \qquad i = (-1)^{1/2}.$$

The distances R_1 and R_2, for instance, are given by Eqs. (6) as

$$R_1 = |Z - Z_1| \qquad \text{and} \qquad R_2 = |Z - Z_2|, \tag{27}$$

where according to Eqs. (7)

$$Z_1 = be^{int^*} \qquad \text{and} \qquad Z_2 = -ae^{int^*}. \tag{28}$$

Substituting for Z from Eq. (26) and for Z_1, Z_2 from Eqs. (28), the distances given by Eqs. (27) become

$$\begin{aligned} R_1 &= |z - b| = [(\bar{x} - b)^2 + \bar{y}^2]^{1/2}, \\ R_2 &= |z + a| = [(\bar{x} + a)^2 + \bar{y}^2]^{1/2}. \end{aligned} \tag{29}$$

The left-hand sides of Eqs. (7) in complex notation become

$$\frac{d^2Z}{dt^{*2}} = \left(\frac{d^2z}{dt^{*2}} + 2in\frac{dz}{dt^*} - n^2z\right)e^{int^*}, \tag{30}$$

and the collection and arrangement of the remaining terms of Eqs. (7) is left to the reader as a simple example. The complex form of the equations of motion in the rotating system is

$$\frac{d^2z}{dt^{*2}} + 2in\frac{dz}{dt^*} - n^2z = -k^2\left[m_1\frac{(z-b)}{|z-b|^3} + m_2\frac{(z+a)}{|z+a|^3}\right]. \tag{31}$$

The real and imaginary parts give

$$\begin{aligned} \frac{d^2\bar{x}}{dt^{*2}} - 2n\frac{d\bar{y}}{dt^*} - n^2\bar{x} &= -k^2\left[m_1\frac{(\bar{x}-b)}{\bar{r}_1^3} + m_2\frac{(\bar{x}+a)}{\bar{r}_2^3}\right], \\ \frac{d^2\bar{y}}{dt^{*2}} + 2n\frac{d\bar{x}}{dz^*} - n^2\bar{y} &= -k^2\left[\frac{m_1\bar{y}}{\bar{r}_1^3} + \frac{m_2\bar{y}}{\bar{r}_2^3}\right], \end{aligned} \tag{32}$$

where the $\bar{r}_1$, $\bar{r}_2$ notation was introduced for R_1, R_2, indicating that in the rotating coordinate system the distances show no explicit dependence on the time [cf. Eq. (29)]. Equations (32) verify the intuition that in the rotating coordinate system the force function is not expected to show explicit dependence on the time. The right-hand sides of Eqs. (7) have been simplified since Eqs. (32) do not contain t^*; the left-hand sides have become more complicated by the appearance of the first derivatives and linear terms. The terms $n^2\bar{x}$ and $n^2\bar{y}$ are of small concern since they can be combined with terms in the right-hand members, but the presence of the first derivatives raises the question whether the transformation serves any interest or whether it only complicates matters. The answer to this question is connected with the fact that the new Eqs. (32) possess a "useful" integral. In fact the only known integral of the restricted problem can be obtained directly from (32) in the same way as Eq. (9) was obtained from (8). Prior to this step, it is convenient to establish the force function belonging to Eqs. (32). For this purpose we write them as

$$\begin{aligned} \frac{d^2\bar{x}}{dt^{*2}} - 2n\frac{d\bar{y}}{dt^*} &= \frac{\partial F^*}{\partial \bar{x}}, \\ \frac{d^2\bar{y}}{dt^{*2}} + 2n\frac{d\bar{x}}{dt^*} &= \frac{\partial F^*}{\partial \bar{y}}, \end{aligned} \tag{33}$$

and find the function F^* so that

$$\begin{aligned} \frac{\partial F^*}{\partial \bar{x}} &= n^2\bar{x} - k^2\left[m_1\frac{(\bar{x}-b)}{\bar{r}_1^3} + m_2\frac{(\bar{x}+a)}{\bar{r}_2^3}\right], \\ \frac{\partial F^*}{\partial \bar{y}} &= n^2\bar{y} - k^2\left[\frac{m_1\bar{y}}{\bar{r}_1^3} + \frac{m_2\bar{y}}{\bar{r}_2^3}\right]. \end{aligned} \tag{34}$$

This problem is well known in potential theory and, since in establishing the generating functions of canonical transformations similar problems and methods will be used later, we offer a short discussion of this question in Section 1.7. At this point we give the answer to the problem proposed by Eqs. (34):

$$F^* = \frac{n^2}{2}(\bar{x}^2 + \bar{y}^2) + k^2\left(\frac{m_1}{\bar{r}_1} + \frac{m_2}{\bar{r}_2}\right). \tag{35}$$

Note that F^* may also be obtained directly from the transformation.

Equations (33) possess an integral, as may be shown by multiplying the first by $d\bar{x}/dt^*$, the second by $d\bar{y}/dt^*$, and adding and integrating with respect to the time t^*. This gives

$$\frac{1}{2}\left[\left(\frac{d\bar{x}}{dt^*}\right)^2 + \left(\frac{d\bar{y}}{dt^*}\right)^2\right] = \int_{t_0^*}^{t^*}\left(\frac{\partial F^*}{\partial \bar{x}}\,d\bar{x} + \frac{\partial F^*}{\partial \bar{y}}\,d\bar{y}\right) = F^* - \frac{C^*}{2} \tag{36}$$

since now

$$dF^* = \frac{\partial F^*}{\partial \bar{x}}\,d\bar{x} + \frac{\partial F^*}{\partial \bar{y}}\,d\bar{y}.$$

This integral and C^* are known as the Jacobian integral and the Jacobian constant after Jacobi.

Another form of Eq. (36) is obtained if we write $\bar{v}$ for the magnitude of the velocity relative to the rotating coordinate system and obtain

$$\bar{v}^2 = 2F^* - C^*. \tag{37}$$

Substituting F^* from (35) and writing $\bar{r}^2$ for $\bar{x}^2 + \bar{y}^2$ we obtain

$$\bar{v}^2 = n^2\bar{r}^2 + 2k^2\left(\frac{m_1}{\bar{r}_1} + \frac{m_2}{\bar{r}_2}\right) - C^*. \tag{38}$$

1.5 Equations of motion in dimensionless coordinates

(A) The equations of motion in the inertial system [Eqs. (7)] contain k^2, a, b, m_1, m_2, and n as physical parameters which are not all independent. It will now be shown by means of dimensionless variables that the restricted problem depends on only one parameter. For this purpose let Greek letters (excepting for the time) represent dimensionless quantities as follows:

$$\begin{gathered} \xi = X/l, \qquad \eta = Y/l, \qquad t = nt^*, \qquad \mu_1 = m_1/M = a/l, \\ \mu_2 = m_2/M = b/l, \qquad \rho_1 = R_1/l, \qquad \rho_2 = R_2/l. \end{gathered} \tag{39}$$

The equations of motion (7) become

$$d^2\xi/dt^2 = \partial\phi/\partial\xi, \qquad d^2\eta/dt^2 = \partial\phi/\partial\eta, \tag{40}$$

where

$$\phi = F/l^2n^2 = \mu_1/\rho_1 + \mu_2/\rho_2 \tag{41}$$

and

$$\begin{aligned} \rho_1{}^2 &= (\xi - \mu_2 \cos t)^2 + (\eta - \mu_2 \sin t)^2, \\ \rho_2{}^2 &= (\xi + \mu_1 \cos t)^2 + (\eta + \mu_1 \sin t)^2. \end{aligned} \tag{42}$$

The variables occurring in the equations of motion are the dimensionless coordinates (ξ, η) and the dimensionless time t. The only remaining parameters (constants) are μ_1 and μ_2, but since $m_1/M + m_2/M = 1$ we have $\mu_1 + \mu_2 = 1$, i.e., given one of the dimensionless masses, the other is determined. We are therefore left with only one parameter (either μ_1 or μ_2), the selection of which determines the problem.

The dimensionless equations corresponding to Eqs. (7) are

$$\begin{aligned} \frac{d^2\xi}{dt^2} &= -\left[\mu_1 \frac{(\xi - \mu_2 \cos t)}{\rho_1{}^3} + \mu_2 \frac{(\xi + \mu_1 \cos t)}{\rho_2{}^3}\right], \\ \frac{d^2\eta}{dt^2} &= -\left[\mu_1 \frac{(\eta - \mu_2 \sin t)}{\rho_1{}^3} + \mu_2 \frac{(\eta + \mu_1 \sin t)}{\rho_2{}^3}\right]. \end{aligned} \tag{43}$$

(B) Now we establish the equations of motion in the rotating coordinate system using dimensionless variables. We will observe that in this way we obtain the simplest form of the differential equation of motion.

Introducing

$$\begin{aligned} x &= \bar{x}/l, \qquad y = \bar{y}/l, \qquad t = nt^*, \\ r_1 &= \bar{r}_1/l, \qquad r_2 = \bar{r}_2/l, \qquad \mu_{1,2} = m_{1,2}/M, \end{aligned} \tag{44}$$

Eqs. (32) of Section 1.4 become

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \bar{\Omega}_x, \\ \ddot{y} + 2\dot{x} &= \bar{\Omega}_y, \end{aligned} \tag{45}$$

where dots denote derivatives with respect to the dimensionless time (t) and subscripts signify partial derivatives. The function $\bar{\Omega}$ corresponds to the previously introduced function ϕ for the fixed coordinate system by Eq. (41), and to the dimensionless form of F given by Eqs. (35); i.e.,

$$\bar{\Omega} = \frac{F^*}{l^2n^2} \tag{46}$$

or

$$\bar{\Omega} = \tfrac{1}{2}(x^2 + y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}, \tag{47}$$

where

$$\begin{aligned} r_1^2 &= (x - \mu_2)^2 + y^2, \\ r_2^2 &= (x + \mu_1)^2 + y^2. \end{aligned} \tag{48}$$

Equations (45) and (47), which define the problem in a synodic coordinate system, are widely used. A modification of $\bar{\Omega}$ by the addition of a constant will not affect the equations of motion and will offer a more symmetric form. Let

$$\Omega = \bar{\Omega} + \tfrac{1}{2}\mu_1\mu_2, \tag{49}$$

which results in

$$\Omega = \tfrac{1}{2}[\mu_1 r_1^2 + \mu_2 r_2^2] + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} \tag{50}$$

or

$$\Omega = \mu_1\left(\frac{r_1^2}{2} + \frac{1}{r_1}\right) + \mu_2\left(\frac{r_2^2}{2} + \frac{1}{r_2}\right). \tag{51}$$

The equations of motion are

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} + 2\dot{x} &= \Omega_y. \end{aligned} \tag{52}$$

The Jacobian integral of Eqs. (45) is

$$\dot{x}^2 + \dot{y}^2 = 2\bar{\Omega} - \bar{C} \tag{53}$$

and, using Ω instead of $\bar{\Omega}$, the integral becomes

$$\dot{x}^2 + \dot{y}^2 = 2\Omega - C, \tag{54}$$

giving

$$C = \bar{C} + \mu_1\mu_2. \tag{55}$$

The Jacobian integral (54) in the dimensionless synodic system connects the dimensionless relative velocity with the position coordinates through the Jacobian constant.

The return to the fixed system (dimensionless sidereal) is effected by the transformations

$$\begin{aligned} \xi &= x\cos t - y\sin t, \\ \eta &= x\sin t + y\cos t, \end{aligned}$$