The Structure of Atonal Music

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Preface

In 1908 a profound change in music was initiated when Arnold Schoenberg began composing his "George Lieder" Op. 15. In this work he deliberately relinquished the traditional system of tonality, which had been the basis of musical syntax for the previous two hundred and fifty years. Subsequently, Schoenberg, Anton Webern, Alban Berg, and a number of other composers created the large repertory known as atonal music.

Despite some recent and serious efforts, which stand in marked contrast to earlier simplistic formulations, the structure of this complicated music has not been well understood. Accordingly, it is the intention of the present work to provide a general theoretical framework, with reference to which the processes underlying atonal music may be systematically described. It is not claimed that all aspects of atonal music are dealt with (an improbable undertaking, in any event); instead, major emphasis has been placed upon fundamental components of structure. For instance, one can deal with pitch and disregard orchestration, but the reverse is not, in general, possible.

Although a good deal of attention has been paid to the iconoclastic nature of atonal music, there has been a tendency to overlook its significance within the art form. This circumstance is unfortunate and should be corrected. One need only remark that among the major works in this repertory are Schoenberg's Five Pieces for Orchestra Op. 16 (1909), Webern's Six Pieces for Large Orchestra Op. 6 (1910), Stravinsky's *The Rite of Spring* (1913), and Berg's *Wozzeck* (1920).

The inclusion of Stravinsky's name in the list above suggests that atonal music was not the exclusive province of Schoenberg and his circle, and that is indeed the case. Many other gifted composers contributed to the repertory: Alexander Scriabin, Charles Ives, Carl Ruggles, Ferruccio Busoni, and Karol Szymanowski–to cite only the more familiar names.

The present study draws upon the music of many of the composers mentioned above. It does not, however, deal with 12-tone music, or with what might be described as paratonal music, or with more recent music which is rooted in the atonal tradition. This is not to say that the range of applicability is narrow, however. Any composition that exhibits the structural characteristics that are discussed, and that exhibits them throughout, may be regarded as atonal.

The book is divided into two hierarchically organized parts. In Part 1 certain basic ideas are introduced and connections between them are developed. In Part 2 the concepts elaborated in Part 1 are brought within the scope of a general model of structure, the set-complex, and a number of musical excerpts are examined in detail. Thus, the exposition begins with elementary aspects of structural components and their relations and proceeds to a consideration of structures of larger scale and greater complexity.

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1 Pitch-Class Sets and Relations

1.0 Pitch combinations

The repertory of atonal music is characterized by the occurrence of pitches in novel combinations, as well as by the occurrence of familiar pitch combinations in unfamiliar environments.

1. Schoenberg, "George Lieder" Op. 15/1

As an example of a pitch combination, consider the chord at the end of the first song in Schoenberg's "George Lieder" Op. 15 (ex. 1). This pitch combination, which is reducible to one form of the all-interval tetrachord, has a very special place in atonal music. It could occur in a tonal composition only under extraordinary conditions, and even then its meaning would be determined by harmonic-contrapuntal constraints. Here, where such constraints are not operative, one is obliged to seek other explanations. Accordingly, in the sections that follow, terminology and notation will be introduced which will facilitate the discussion of certain properties of such combinations. The first task is to formulate a more general notion to replace that of pitch combination.

1.1 Pitch-Class sets

The term pitch combination introduced in section 1.0 refers to any collection of pitches represented in ordinary staff notation. The transposition of G-sharp in example 1 to the staff position an octave higher would produce a new and distinctive pitch combination. Comparison of this combination with the old one would require additional considerations. A simple and precise basis for comparing any two pitch combinations is provided by the notion *pitch-class set.**

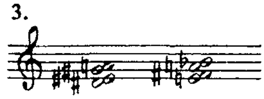
Suppose, for instance, that the chord discussed in connection with example 1 were to be compared with the chord in example 2, which occurs at the opening of Webern's Six Pieces for Orchestra Op. 6/3. By rewriting each chord

*The term and the concept were introduced by Milton Babbitt.

2. Webern, Six Pieces for Orchestra Op. 6/3

within the smallest possible range and placing the chords in adjacent positions on the same staff it is evident that the second is a transposition of the first (ex. 3).

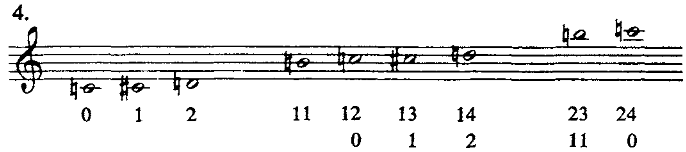
3.



In rewriting both chords it was assumed that change of register did not affect notation-class membership; E-sharp remained a member of the class E-sharp, A remained a member of the class A, and so on. Thus, the axiom of octave equivalence (which also applies to tonal music, of course) was invoked. In addition, the assertion that the second chord is a transposition of the first implies that the chords are equivalent in a very specific way: by virtue of the operation transposition. This operation will be discussed in section 1.3 and need not be pursued further here.

For the study of atonal music the assumption of octave equivalence by notation class is not sufficiently general. It is necessary to assume, further, that "enharmonic" notes are equivalent (regardless of register). Thus, E-sharp is the same as F or G-double-flat, for example. This does not imply that notation is arbitrary in an atonal composition, but merely that the notion of pitch-class set is independent of any particular notational forms.

4.



As a consequence of octave equivalence and enharmonic equivalence any notated pitch belongs to one and only one of 12 distinct pitch classes. This can be seen readily if the usual letter-names are replaced by the integers 0, 1, 2, . . . , 11. We then speak of *integer notation,* as distinct from staff notation.

The correspondence of integer notation and staff notation is shown in example 4. The integer 0 has been assigned to C (which is equivalent to B-sharp and D-double-flat, the integer 1 has been assigned to C-sharp (which is equivalent to B-double-sharp and D-flat), and so on until the integer 11 has been assigned to B, completing the octave.* If the assignment procedure were continued, as shown in the example, 12 would be assigned to the second C. But this belongs to the same class as the first C, to which the integer 0 has already been assigned. Indeed, any pitch number greater than or equal to 12 can be reduced to one of the pitch-class integers by obtaining the remainder of that number divided by 12.

A *pitch-class set,* then, is a set of distinct integers (i.e., no duplicates) representing pitch classes. Strictly speaking, one should use the term *set of pitch-class representatives,* but that is unwieldy. In fact, even the term *pitch-class set,* since it is used very often in the present volume, will usually be abbreviated to *pc set.*

A pc set is displayed in square brackets—for example, [0,1,2]. The reader is exhorted to pay attention to this and other notational conventions as they are introduced.

1.2 Normal order; the prime forms

For a number of reasons it is important to distinguish between ordered and unordered pc sets. If, for example, [0,2,3] is regarded as the same as [2,3,0] it is assumed that the difference in order does not render the sets distinct from one another; they are equivalent sets since both contain the same elements. In such case the sets are referred to as *unordered sets.* If, however, the two sets are regarded as distinct, it is evident that they are distinct on the basis of difference in order, in which case they are called *ordered sets.*

To deal with relations between two pc sets it is often necessary to take ordering into account. In particular, it is essential to be able to reduce a set to a basic ordered pattern called *normal order.*† An ordering of a set is called a permutation, and the number of distinct permutations of a set depends upon the number of elements in the set. That number is known as the *cardinal number* of the set. In general, it can be shown that for a set of n elements there are 1 X 2 X 3 X . . . X n distinct permutations. For example, for a set of 3 elements there are 6 permutations. By convention this series of multiplications is represented by the symbol n! (read "n factorial"). To determine the normal order of a set, however, it is not necessary to consider all of its permutations, but only those called *circular permutations.* Given a set in some order, the first circular permutation is formed by placing the first element

*This assignment remains fixed throughout the present volume.

†This is the same as Babbitt's normal form. See Babbitt 1961.

last. For example, the first circular permutation of the set [a,b,c] is [b,c,a]. The next circular permutation is formed in the same way and is thus [c,a,b]. The same procedure repeated once again would produce [a,b,c]. Hence, by definition, an ordered set is a circular permutation of itself, and, in general, there are n circular permutations of a set of n elements.

The normal order of a pc set can be determined as follows.* The set must be in ascending numerical order at the outset and each circular permutation must be kept in ascending numerical order. This means that 12 must be added to the first element each time it is placed in the last position to form the next circular permutation.†

Consider, as an example, the determination of normal order for the pc set [1,3,0]. The circular permutations of the set, with the addition of 12 to the shifted element each time is shown in the accompanying table.

		Difference of first and last
A_0	[0,1,3]	3
A_1	[1,3,12]	11
A_2	[3,12,13]	10

By what will be called Requirement 1, the normal order is that permutation with the least difference determined by subtracting the first integer from the last. The normal order in this instance is A_0.

In certain cases Requirement 1 is inadequate for the determination of normal order, in which case Requirement 2 must be invoked. Requirement 2 selects the best normal order as follows. If the least difference of first and last integers is the same for any two permutations, select the permutation with the least difference between first and second integers. If this is the same, select the permutation with the least difference between the first and third integers, and so on, until the difference between the first and the next to last integers has been checked. If the differences are the same each time, select one ordering arbitrarily as the normal order. Consider the following example:

A_0	[0,2,4,8]	8
A_1	[2,4,8,12]	10
A_2	[4,8,12,14]	10
A_3	[8,12,14,16]	8

Here, by Requirement 1, both A_0 and A_3 are normal orders.†† Requirement 2 selects A_0 as the best normal order.

The form of a pc set such that it is in normal order (or best normal order)

*The procedure to be described is an extension of the one used in Teitelbaum 1965.

†Since 12 is equivalent to 0 in the 12 pitch-class integer system the arithmetic value of a pitch-class integer is not changed by the addition of 12.

††In this and similar instances the two sets are inversionally related.

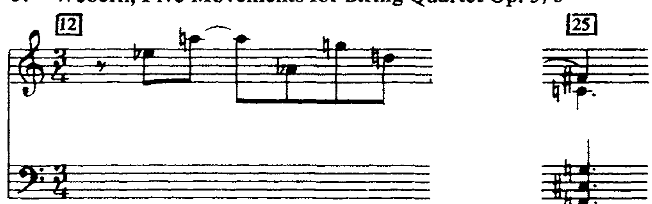
and the first integer is 0 is called a *prime form.* A complete list of prime forms for the 220 distinct pc sets is given in appendix 1. Although reference to the list will not be necessary until section 1.5, normal orders will be shown in connection with the examples in staff notation henceforth so that the reader will become accustomed to integer notation and so that certain comparisons (to be explained in the following sections) can be made easily.

1.3 Transpositionally equivalent pc sets

The notion of a pc set in normal order provides a point of departure for the development of certain fundamental procedures which will permit consequential analytical observations about structure to be made.

Let us assume that a significant kind of observation concerns the similarity or difference between two "events," such as pc sets. More precisely, given two pc sets to compare, one might ask: Are they the same, or do they differ? In order to provide an answer a definition is needed. Accordingly, two pc sets will be said to be *equivalent* if and only if* they are reducible to the same prime form by transposition or by inversion followed by transposition. This section is concerned only with transpositional equivalence.

5. Webern, Five Movements for String Quartet Op. 5/5



A : [2,3,7,8,9]　　　B : [0,1,5,6,7]

Consider, as an initial illustration, the two pitch-sets A and B in example 5.† First, we ask whether it is reasonable to compare them at all. They occur some distance apart in the composition, as indicated by the circled measure numbers. The "mode of occurrence" is different in each case: the first is a melodic line, the second a "chord." These differential surface features, however, do not preclude comparison. In fact, the only requirement necessary for comparison is that the pitch configurations be reducible to pc sets of the same cardinal number, and that requirement is met in this instance.

The pc sets (in normal order) labeled A and B correspond to the staff notation of the two configurations. For comparison, it is convenient to align

*The logical expression *if and only if* means that two pc sets are equivalent if they are reducible to the same prime form, and if they are not reducible to the same prime form they are not equivalent.

†The attentive reader will notice that a convention has been added, namely the assignment of a symbolic name to the set. This name, an upper case alphabetic character, is followed by a colon.

the integer notation as follows:

A: [2,3,7,8,9]
B: [0,1,5,6,7]

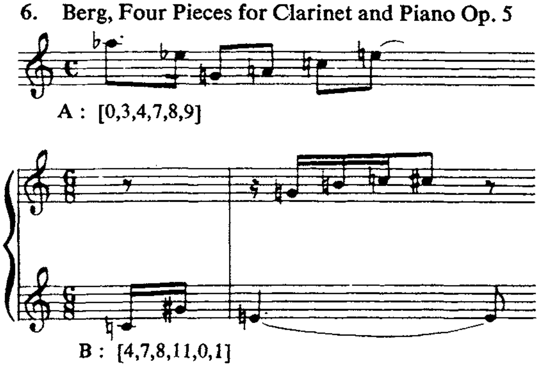
Now, the transposition of a pc integer i means that some integer t is added to i to yield a pc integer j. If j is greater than or equal to 12, j is replaced by the remainder of j divided by 12. This is called addition modulo 12, abbreviated to mod 12.

From this it is evident that if pc set A is equivalent to pc set B there must be some integer t which, added to each integer of A will yield the corresponding integer in B. By inspection it is clear that there is such a t and that its value is 10. Here and elsewhere t will be referred to as the *transposition operator.*

Since it is essential that this arithmetic interpretation of transposition be clearly understood, the additions are displayed in the accompanying table for pc sets A and B in example 5.

A		t		B
2	+	10	= 12 =	0 (mod 12)
3	+	10	= 13 =	1 (mod 12)
7	+	10	= 17 =	5 (mod 12)
8	+	10	= 18 =	6 (mod 12)
9	+	10	= 19 =	7 (mod 12)

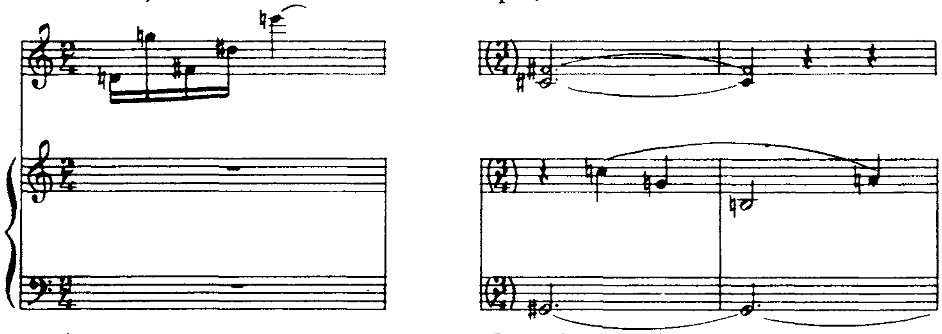
6. Berg, Four Pieces for Clarinet and Piano Op. 5



Example 6 shows the opening clarinet figure in Berg's Op. 5/1 (A) and the opening piano figure in the third piece of the same composition (B). Comparison of A and B reveals that B is transpositionally equivalent to A and that t = 4.

In example 5 (Webern Op. 5/5) two sets which occur some distance apart in the movement were found to be transpositionally equivalent, while in example 6 (Berg Op. 5) two transpositionally equivalent sets were found to occur in corresponding positions at the beginning of different movements of the same composition. These examples are intended to suggest that the operation transposition is of fundamental importance to non-tonal music and that configurations which may be dissimilar in many respects can be, in fact, equivalent at a more basic level of structure. In neither case, however, was an attempt made to further interpret the equivalence relation, since such interpretation requires additional concepts and techniques to be introduced in subsequent sections.

7. Webern, Four Pieces for Violin and Piano Op. 7/4



A : [2,3,4,6,7]   B : [7,8,9,11,0]

Example 7 provides a final illustration for this section. The reader can easily determine the value of t for himself by comparing A and B. (Note that B represents the pitch set in the piano part only.)

1.4 Inversionally equivalent pc sets

In the preceding section the process of transposing a pc set A to produce a new and equivalent pc set B was described in terms of the addition (mod 12) of some integer t, called the transposition operator, to every element of A. If we let A be [0,1,2] and t = 1, the process by which B is produced can be displayed as follows:

$$\begin{array}{ccc} & T & \\ & t=1 & \\ A & & B \\ 0 & \rightarrow & 1 \\ 1 & \rightarrow & 2 \\ 2 & \rightarrow & 3 \end{array}$$

Observe that every element of A corresponds to an element of B and that the correspondence is unique in each case–that is, some element in B does not correspond to two elements in A. Thus, transposition can be regarded as a rule of correspondence–namely, addition modulo 12–that assigns to every element of B exactly one element derived from A. We will borrow a conventional mathematical term to describe such a process and say that A is *mapped onto* B by the rule T.*

*The mapping is *into*, of course, with respect to the universal set [0,1,2,...,11]. It is not important to draw the distinction at this point.

The notion of a mapping is more than a convenience in describing relations between pc sets. It permits the development of economical and precise descriptions which could not be obtained using conventional musical terms. Its usefulness will be especially apparent as we begin to deal with the process of inversion.

Like transposition, the inversion process can also be described in terms of a rule of correspondence I which maps each element of a set A onto an element of a set B. The inversion mapping I depends upon the fixed correspondence of pc integers displayed in the following table:

$$\begin{array}{ccc} & \text{I} & \\ 0 & \leftrightarrow & 0 \\ 1 & \leftrightarrow & 11 \\ 2 & \leftrightarrow & 10 \\ 3 & \leftrightarrow & 9 \\ 4 & \leftrightarrow & 8 \\ 5 & \leftrightarrow & 7 \\ 6 & \leftrightarrow & 6 \end{array}$$

Observe that in each case the sum of the integers connected by the double arrow is 12, and recall that 12 = 0 (mod 12). We say that 0 is the *inverse** of 0, 1 is the inverse of 11 and 11 is the inverse of 1, and so on. In general, if we let a' represent the inverse of a, then

$$a' = 12 - a \pmod{12}$$

For any set A, therefore, the mapping I sends every element of A onto its inverse, producing a new and equivalent set B. For example, if A is [0,1,2] the mapping is as follows:

$$\begin{array}{ccc} & \text{I} & \\ \text{A} & & \text{B} \\ 0 & \rightarrow & 0 \\ 1 & \rightarrow & 11 \\ 2 & \rightarrow & 10 \end{array}$$

In section 1.3, in addition to equivalence by transposition, two pc sets were said to be equivalent if reducible to the same prime form by inversion *followed by* transposition. The expression *followed by* is easily understood in terms of a double mapping, as illustrated below.

*The term inverse is preferred to complement. The latter is reserved for set-theoretic complementation (sec. 1.15).

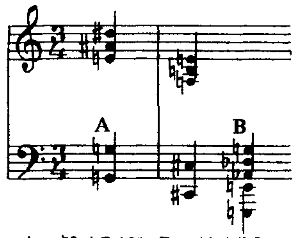
$$\begin{array}{ccccc} & \mathrm{I} & & \mathrm{T} & \\ & & & (t=1) & \\ A & & B & & C \\ 0 & \rightarrow & 0 & \rightarrow & 1 \\ 1 & \rightarrow & 11 & \rightarrow & 0 \\ 2 & \rightarrow & 10 & \rightarrow & 11 \end{array}$$

Here the first mapping I is followed by the T mapping. As a result, 0 in A is sent onto 1 in C, 1 in A is sent onto 0 in C, and 2 in A is sent onto 11 in C.

It is important to observe that whereas transposition does not imply prior inversion, inversion always implies subsequent transposition, even if it is the trivial case t = 0, as shown below.

$$\begin{array}{ccccc} & \mathrm{I} & & \mathrm{T} & \\ & & & (t=0) & \\ A & & B & & C \\ 0 & \rightarrow & 0 & \rightarrow & 0 \\ 1 & \rightarrow & 11 & \rightarrow & 11 \\ 2 & \rightarrow & 10 & \rightarrow & 10 \end{array}$$

8. Schoenberg, "George Lieder" Op. 15/6



A : [3,4,7,10] B : [1,4,7,8]

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Let us consider now some compositional examples of inversionally equivalent pc sets. In example 8, from one of Schoenberg's earliest non-tonal works, the set marked A is inversionally equivalent to B. (The second chord is not equivalent to the other two and will be disregarded for the present purpose.) The mappings by which B is derived from A are:

$$\begin{array}{ccccc} & \mathrm{I} & & \mathrm{T} & \\ & & & (t=11) & \\ A & & & & B \\ 3 & \rightarrow & 9 & \rightarrow & 8 \\ 4 & \rightarrow & 8 & \rightarrow & 7 \\ 7 & \rightarrow & 5 & \rightarrow & 4 \\ 10 & \rightarrow & 2 & \rightarrow & 1 \end{array}$$

Here, and in general, inversion of a pc set in normal order produces a pc set in descending order, as is apparent from the fixed mapping on p. 8. Therefore, to compare two pc sets for inversional equivalence, it is necessary to reverse the order of the second. The transposition operator then appears as the sum of each pair of elements, thus:

A:	[3, 4, 7, 10]
B:	[8, 7, 4, 1]
sums	11 11 11 11

Before considering another compositional example we introduce a symbolic notation for transposition and inversion which is useful for concisely describing those relations whenever they hold between two pc sets A and B.

Expression	Read as
B = T(A,t)	B is the transposition of A at level t
B = I(A)	B is the inversion of A

Since inversion always implies transposition, the latter is the same as B = T(I(A),0). For A and B in example 8 we then write B = T(I (A),11) and read "B is the inversion of A transposed at level 11" (or, "transposed with t = 11").*

9. Ives, *The Unanswered Question*

A : [7,10,11,2] B : [7,11,1,2] C : [9,10,0,4] D : [9,0,1,4]

The second illustration from the literature, example 9, is interesting from an historical standpoint since it is taken from one of the early compositions of Ives, *The Unanswered Question* (ca. 1906), and antedates Schoenberg's first atonal composition. The example shows two fragments: the first includes the

*Where it is not essential to use this notation, the form OP_t will be used, where OP is the symbol for the operation and t is the value of t. For example, IT_3 refers to the inversion of some set transposed with t = 3.

initial two notes of the first "question," played by trumpet, together with the accompanying string sonority; the second shows the corresponding music for the second "question."

The trumpet part is the same in both cases, but the accompanying string sonority is different in the second fragment. Examination in terms of equivalent sets reveals, however, that A is equivalent to D and B is equivalent to C. Specifically,

$$D = T(A,2)$$
$$C = T(I(B),11)$$

These relations effectively pair off the sets as follows:

A B C D

10. Schoenberg, Five Pieces for Orchestra Op. 16/1

A : [1,4,5,8,9]

B : [9,10,1,2,5] = T (I (A), 6)

C : [6,9,10,1,2] = T (I (B), 11)

D : [3,6,7,10,11] = T (C, 9)

Example 10 shows successive occurrences of the initial theme of Schoenberg's Five Pieces for Orchestra Op. 16/1. The second statement (B), which is rhythmically distinct from the others, is a transposed inversion of the first (A). The succession of mappings is as follows.

$$B = T(I(A),6)$$
$$C = T(I(B), 11)$$
$$D = T(C,9)$$

More will be said about this particular example in section 1.12.

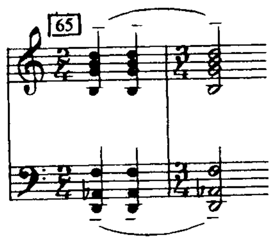
1.5 The list of prime forms; set names

It is convenient to have names for the prime forms so that a pc set can be referred to without recourse to a cumbersome description of some kind.* Accordingly, each prime form has been assigned a name consisting of numbers separated by a hyphen. The number to the left of the hyphen is the cardinal number of the set; the number to the right of the hyphen is the ordinal number of the set–that is, the position of the prime form on the list. For example, 5-31, which is the name of both pc sets in example 11 is the thirty-first set on the list of sets with cardinal number 5 (appendix 1).

11. Berg, "Altenberg Lieder" Op. 4/2

[9,0,3,5,6]

Stravinsky, Symphonies of Wind Instruments



[11,2,5,7,8]

To look up the name of a pc set on the list of prime forms it is first necessary to put the set in normal order and then to transpose the normal order so that the first integer is 0. This operation is shown below for the Berg and Stravinsky excerpts in example 11.†

	Berg	Stravinsky
normal order	[3,5,6,9,0]	[5,7,8,11,2]
transposed to level 0	[0,2,3,6,9]	[0,2,3,6,9]

Both sets have now been reduced to the same normal order. If this is the *best* normal order, as described in section 1.3, it will be found on the list of prime forms. In this case, however, the set is not in best normal order and does not appear on the list. Therefore it is necessary to take the normal order of the inversion of the set and transpose it to level 0, as follows:

given normal order	
transposed to level 0	[0,2,3,6,9]

*For example, pc set 5-22 has been described as "a diminished triad with conjunct semitone not only superimposed . . . but also subimposed" (Perle 1967, p. 228). Ad hoc descriptions of this kind usually rest upon some analytical interpretations, as in the case cited here.

†The selection of these two excerpts was made only for purposes of illustration and has no further significance.

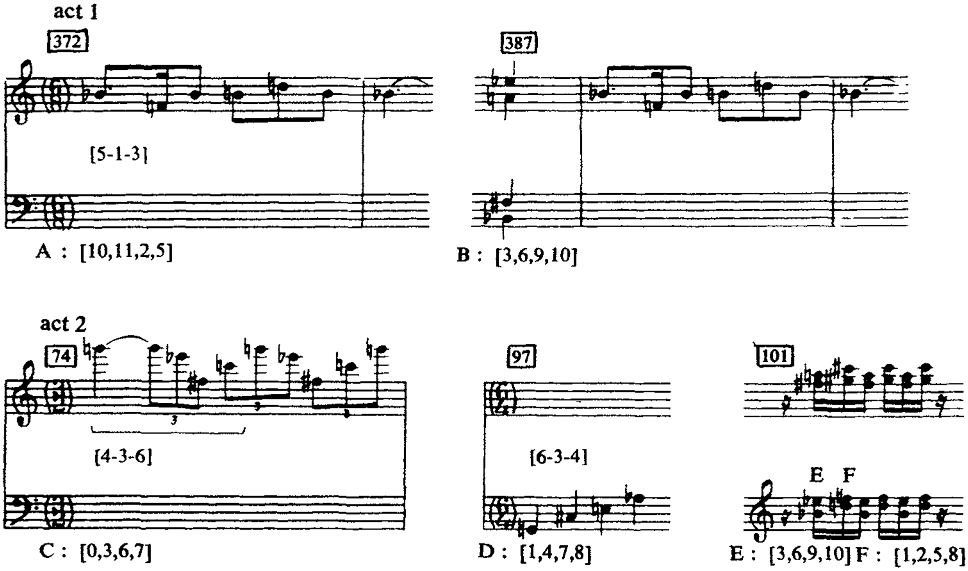
inversion	[0,10,9,6,3]
ascending order	[0,3,6,9,10]
new normal order	[9,10,0,3,6]
transposed to level 0	[0,1,3,6,9]

The last set displayed above is the best normal order, and the name associated with it on the list of prime forms is 5-31.

1.6 Intervals of a pc set; the interval vector

Thus far, pc sets have been examined from the standpoint of the elements (pc integers) they contain. In addition, equivalence relations based upon transposition and inversion have been introduced, together with some apparatus needed for identifying sets from the list of prime forms. In order to proceed to a more comprehensive study of properties of pc sets and relations between pc sets it is necessary to introduce some additional basic concepts. This section and the three which follow it are concerned with the interaction of components of a set in terms of the intervals (the term is construed in the traditional sense) which they define.

12. Berg, *Wozzeck* Op. 7
act 1



Example 12 provides a specific musical reference for the discussion that follows. Here we see six instances of pc set 4-18. Three of these, A, C, and D, are melodic statements. Observe that the intervals between interlocking note-pairs differ in each case.* This is because a particular linear ordering of a set determines a selection of intervals from the *total interval-content* of the set.† The three vertical statements of the set (B, E, and F) present all the intervals of the set. This suggests that total interval-content is the more basic and general intervallic property of a pc set. The remainder of this section is devoted to a presentation of some elementary notions which, it is hoped, will render the concept of interval both precise and useful with respect to the study of structures in non-tonal music.

The interval formed by two pc integers a and b is the arithmetic difference a − b. In order not to have to specify that a is greater than b it is assumed that the absolute value (positive value) of the difference is always taken. Thus, for example, the interval formed by 0 and 1 is $0-1 = 1$; the interval formed by 5 and 9 is $5-9 = 4$. It is not difficult to show that if the intervals between all pairs of pc integers were formed and duplicates were removed there would remain the set [0,1,2,...,11].†† Thus, there are 12 intervals, corresponding to the 12 pitch-classes.

The 12 intervals reduce to 6 *interval classes*, however, by the following defined equivalence.

If d is the difference of two pc integers
then $d \equiv d'$ mod 12.

In short, inverse-related (mod 12) intervals are defined as equivalent, and as a consequence are paired off as shown below.§

$$
\begin{array}{l}
0 \equiv 0 \\
1 \equiv 11 \\
2 \equiv 10 \\
3 \equiv 9 \\
4 \equiv 8 \\
5 \equiv 7 \\
6 \equiv 6
\end{array}
$$

Each pair thus forms an equivalence class. If we let the integers in the left column above represent the class and, further, omit for practical purposes the class 0, the reduction of 12 intervals to 6 interval classes is completed.

By analogy with the abbreviation for pitch class, pc, the abbreviation *ic* will be used for interval class. Whenever ambiguity might result from use of an

*The successive linear intervals are indicated within square brackets as numbers representing interval classes (to be explained below). See Chrisman 1969.

†Ordered sets and order relations are discussed in section 1.14.

††The binary operation-(subtraction) maps S × S *onto* S where S is the set of 12 pc integers.

§The mathematical basis of this partitioning is perhaps best explained with reference to a group-theoretic model. It does not seem necessary to undertake such an explanation here. The partitioning is the traditional one, based upon the symmetrical division of the octave.