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ADVANCED MICROECONOMIC THEORY

THIRD EDITION

Pearson

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To Rana and Kamran

G.A.J.

To Dianne, Lisa, and Elizabeth

P.J.R.

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Preface

In preparing this third edition of our text, we wanted to provide long-time readers with new and updated material in a familiar format, while offering first-time readers an accessible, self-contained treatment of the essential core of modern microeconomic theory.

To those ends, every chapter has been revised and updated. The more significant changes include a new introduction to general equilibrium with contingent commodities in Chapter 5, along with a simplified proof of Arrow's theorem and a new, careful development of the Gibbard-Satterthwaite theorem in Chapter 6. Chapter 7 includes many refinements and extensions, especially in our presentation on Bayesian games. The biggest change – one we hope readers find interesting and useful – is an extensive, integrated presentation in Chapter 9 of many of the central results of mechanism design in the quasi-linear utility, private-values environment.

We continue to believe that working through exercises is the surest way to master the material in this text. New exercises have been added to virtually every chapter, and others have been updated and revised. Many of the new exercises guide readers in developing for themselves extensions, refinements or alternative approaches to important material covered in the text. Hints and answers for selected exercises are provided at the end of the book, along with lists of theorems and definitions appearing in the text. We will continue to maintain a readers' forum on the web, where readers can exchange solutions to exercises in the text. It can be reached at *http://alfred.vassar.edu.*

The two full chapters of the Mathematical Appendix still provide students with a lengthy and largely self-contained development of the set theory, real analysis, topology, calculus, and modern optimisation theory which are indispensable in modern microeconomics. Readers of this edition will now find a fuller, self-contained development of Lagrangian and Kuhn-Tucker methods, along with new material on the Theorem of the Maximum and two separation theorems. The exposition is formal but presumes nothing more than a good grounding in single-variable calculus and simple linear algebra as a starting point. We suggest that even students who are very well-prepared in mathematics browse both chapters of the appendix early on. That way, if and when some review or reference is needed, the reader will have a sense of how that material is organised.

Before we begin to develop the theory itself, we ought to say a word to new readers about the role mathematics will play in this text. Often, you will notice we make certain assumptions purely for the sake of mathematical expediency. The justification for proceeding this way is simple, and it is the same in every other branch of science. These abstractions from 'reality' allow us to bring to bear powerful mathematical methods that, by the rigour of the logical discipline they impose, help extend our insights into areas beyond the reach of our intuition and experience. In the physical world, there is 'no such thing' as a frictionless plane or a perfect vacuum. In economics, as in physics, allowing ourselves to accept assumptions like these frees us to focus on more important aspects of the problem and thereby helps to establish benchmarks in theory against which to gauge experience and observation in the real world. This does not mean that you must wholeheartedly embrace every 'unrealistic' or purely formal aspect of the theory. Far from it. It is *always* worthwhile to cast a critical eye on these matters as they arise and to ask yourself what is gained, and what is sacrificed, by the abstraction at hand. Thought and insight on these points are the stuff of which advances in theory and knowledge are made. From here on, however, we will take the theory as it is and seek to understand it on its own terms, leaving much of its critical appraisal to your moments away from this book.

Finally, we wish to acknowledge the many readers and colleagues who have provided helpful comments and pointed out errors in previous editions. Your keen eyes and good judgements have helped us make this third edition better and more complete than it otherwise would be. While we cannot thank all of you personally, we must thank Eddie Dekel, Roger Myerson, Derek Neal, Motty Perry, Arthur Robson, Steve Williams, and Jörgen Weibull for their thoughtful comments.

PART I

ECONOMIC AGENTS

CHAPTER 1

CONSUMER THEORY

In the first two chapters of this volume, we will explore the essential features of modern consumer theory – a bedrock foundation on which so many theoretical structures in economics are built. Some time later in your study of economics, you will begin to notice just how central this theory is to the economist's way of thinking. Time and time again you will hear the echoes of consumer theory in virtually every branch of the discipline – how it is conceived, how it is constructed, and how it is applied.

1.1 PRIMITIVE NOTIONS

There are four building blocks in any model of consumer choice. They are the consumption set, the feasible set, the preference relation, and the behavioural assumption. Each is conceptually distinct from the others, though it is quite common sometimes to lose sight of that fact. This basic structure is extremely general, and so, very flexible. By specifying the form each of these takes in a given problem, many different situations involving choice can be formally described and analysed. Although we will tend to concentrate here on specific formalisations that have come to dominate economists' view of an individual consumer's behaviour, it is well to keep in mind that 'consumer theory' *per se* is in fact a very rich and flexible *theory of choice*.

The notion of a **consumption set** is straightforward. We let the consumption set, X, represent the set of all alternatives, or complete consumption plans, that the consumer can conceive – whether some of them will be achievable in practice or not. What we intend to capture here is the universe of alternative choices over which the consumer's mind is capable of wandering, unfettered by consideration of the realities of his present situation. The consumption set is sometimes also called the **choice set**.

Let each commodity be measured in some infinitely divisible units. Let $x_i \in \mathbb{R}$ represent the number of units of good i. We assume that only non-negative units of each good are meaningful and that it is always possible to conceive of having *no* units of any particular commodity. Further, we assume there is a finite, fixed, but arbitrary number n of different goods. We let $\mathbf{x} = (x_1, \ldots, x_n)$ be a vector containing different quantities of each of the n commodities and call $\mathbf{x}$ a **consumption bundle** or a **consumption plan**. A consumption bundle $\mathbf{x} \in X$ is thus represented by a point $\mathbf{x} \in \mathbb{R}^n_+$. Usually, we'll simplify things and just think of the consumption set as the *entire* non-negative orthant, $X = \mathbb{R}^n_+$. In this case, it is easy to see that each of the following basic requirements is satisfied.

ASSUMPTION 1.1 ***Properties of the Consumption Set,*** X

The minimal requirements on the consumption set are

1. $X \subseteq \mathbb{R}^n_+$.
2. X *is closed.*
3. X *is convex.*
4. $\mathbf{0} \in X$.

The notion of a **feasible set** is likewise very straightforward. We let B represent all those alternative consumption plans that are both conceivable and, more important, realistically obtainable given the consumer's circumstances. What we intend to capture here are precisely those alternatives that are *achievable* given the economic realities the consumer faces. The feasible set B then is that subset of the consumption set X that remains after we have accounted for any constraints on the consumer's access to commodities due to the practical, institutional, or economic realities of the world. How we specify those realities in a given situation will determine the precise configuration and additional properties that B must have. For now, we will simply say that $B \subset X$.

A **preference relation** typically specifies the limits, if any, on the consumer's ability to perceive in situations involving choice the form of consistency or inconsistency in the consumer's choices, and information about the consumer's tastes for the different objects of choice. The preference relation plays a crucial role in any theory of choice. Its special form in the theory of consumer behaviour is sufficiently subtle to warrant special examination in the next section.

Finally, the model is 'closed' by specifying some **behavioural assumption**. This expresses the guiding principle the consumer uses to make final choices and so identifies the ultimate objectives in choice. It is supposed that *the consumer seeks to identify and select an available alternative that is most preferred in the light of his personal tastes*.

1.2 PREFERENCES AND UTILITY

In this section, we examine the consumer's preference relation and explore its connection to modern usage of the term 'utility'. Before we begin, however, a brief word on the evolution of economists' thinking will help to place what follows in its proper context.

In earlier periods, the so-called 'Law of Demand' was built on some extremely strong assumptions. In the classical theory of Edgeworth, Mill, and other proponents of the utilitarian school of philosophy, 'utility' was thought to be something of substance. 'Pleasure' and 'pain' were held to be well-defined entities that could be measured and compared between individuals. In addition, the 'Principle of Diminishing Marginal Utility' was accepted as a psychological 'law', and early statements of the Law of Demand depended on it. These are awfully strong assumptions about the inner workings of human beings.

The more recent history of consumer theory has been marked by a drive to render its foundations as general as possible. Economists have sought to pare away as many of the traditional assumptions, explicit or implicit, as they could and still retain a coherent theory with predictive power. Pareto (1896) can be credited with suspecting that the idea of a measurable 'utility' was inessential to the theory of demand. Slutsky (1915) undertook the first systematic examination of demand theory without the concept of a measurable substance called utility. Hicks (1939) demonstrated that the Principle of Diminishing Marginal Utility was neither necessary, nor sufficient, for the Law of Demand to hold. Finally, Debreu (1959) completed the reduction of standard consumer theory to those bare essentials we will consider here. Today's theory bears close and important relations to its earlier ancestors, but it is leaner, more precise, and more general.

1.2.1 PREFERENCE RELATIONS

Consumer preferences are characterised *axiomatically*. In this method of modelling as few meaningful and distinct assumptions as possible are set forth to characterise the structure and properties of preferences. The rest of the theory then builds logically from these axioms, and predictions of behaviour are developed through the process of deduction.

These **axioms of consumer choice** are intended to give formal mathematical expression to fundamental aspects of consumer behaviour and attitudes towards the objects of choice. Together, they formalise the view that the consumer *can* choose and that choices are *consistent* in a particular way.

Formally, we represent the consumer's preferences by a *binary relation*, $\succsim$, defined on the consumption set, X. If $\mathbf{x}^1 \succsim \mathbf{x}^2$, we say that '$\mathbf{x}^1$ is at least as good as $\mathbf{x}^2$', for this consumer.

That we use a binary relation to characterise preferences is significant and worth a moment's reflection. It conveys the important point that, from the beginning, our theory requires relatively little of the consumer it describes. We require only that consumers make *binary* comparisons, that is, that they only examine two consumption plans at a time and make a decision regarding those two. The following axioms set forth basic criteria with which those binary comparisons must conform.

AXIOM 1: Completeness. *For all* $\mathbf{x}^1$ *and* $\mathbf{x}^2$ *in* X*, either* $\mathbf{x}^1 \succsim \mathbf{x}^2$ *or* $\mathbf{x}^2 \succsim \mathbf{x}^1$.

Axiom 1 formalises the notion that the consumer *can* make comparisons, that is, that he has the ability to discriminate and the necessary knowledge to evaluate alternatives. It says the consumer can examine *any* two distinct consumption plans $\mathbf{x}^1$ and $\mathbf{x}^2$ and decide whether $\mathbf{x}^1$ is at least as good as $\mathbf{x}^2$ or $\mathbf{x}^2$ is at least as good as $\mathbf{x}^1$.

AXIOM 2: Transitivity. *For any three elements* $\mathbf{x}^1$*,* $\mathbf{x}^2$*, and* $\mathbf{x}^3$ *in* X*, if* $\mathbf{x}^1 \succsim \mathbf{x}^2$ *and* $\mathbf{x}^2 \succsim \mathbf{x}^3$*, then* $\mathbf{x}^1 \succsim \mathbf{x}^3$.

Axiom 2 gives a very particular form to the requirement that the consumer's choices be *consistent*. Although we require only that the consumer be capable of comparing two alternatives at a time, the assumption of transitivity requires that those pairwise comparisons be linked together in a consistent way. At first brush, requiring that the evaluation of alternatives be transitive seems simple and only natural. Indeed, were they not transitive, our instincts would tell us that there was something peculiar about them. Nonetheless, this is a controversial axiom. Experiments have shown that in various situations, the choices of real human beings are not always transitive. Nonetheless, we will retain it in our description of the consumer, though not without some slight trepidation.

These two axioms together imply that the consumer can completely *rank* any finite number of elements in the consumption set, X, from best to worst, possibly with some ties. (Try to prove this.) We summarise the view that preferences enable the consumer to construct such a ranking by saying that those preferences can be represented by a *preference relation*.

DEFINITION 1.1 ***Preference Relation***

The binary relation $\succsim$ *on the consumption set X is called a preference relation if it satisfies Axioms 1 and 2.*

There are two additional relations that we will use in our discussion of consumer preferences. Each is determined by the preference relation, $\succsim$, and they formalise the notions of *strict preference* and *indifference*.

DEFINITION 1.2 ***Strict Preference Relation***

The binary relation $\succ$ *on the consumption set X is defined as follows:*

$$\mathbf{x}^1 \succ \mathbf{x}^2 \qquad \textit{if and only if} \qquad \mathbf{x}^1 \succsim \mathbf{x}^2 \quad \textit{and} \quad \mathbf{x}^2 \not\succsim \mathbf{x}^1.$$

The relation $\succ$ *is called the strict preference relation induced by* $\succsim$*, or simply the strict preference relation when* $\succsim$ *is clear. The phrase* $\mathbf{x}^1 \succ \mathbf{x}^2$ *is read, '*$\mathbf{x}^1$ *is strictly preferred to* $\mathbf{x}^2$*'.*

DEFINITION 1.3 ***Indifference Relation***

The binary relation $\sim$ *on the consumption set X is defined as follows:*

$$\mathbf{x}^1 \sim \mathbf{x}^2 \qquad \textit{if and only if} \qquad \mathbf{x}^1 \succsim \mathbf{x}^2 \quad \textit{and} \quad \mathbf{x}^2 \succsim \mathbf{x}^1.$$

The relation $\sim$ *is called the indifference relation induced by* $\succsim$*, or simply the indifference relation when* $\succsim$ *is clear. The phrase* $\mathbf{x}^1 \sim \mathbf{x}^2$ *is read, '*$\mathbf{x}^1$ *is indifferent to* $\mathbf{x}^2$*'.*

Building on the underlying definition of the preference relation, both the strict preference relation and the indifference relation capture the usual sense in which the terms 'strict preference' and 'indifference' are used in ordinary language. Because each is derived from the preference relation, each can be expected to share some of its properties. Some, yes, but not all. In general, both are transitive and neither is complete.

Using these two supplementary relations, we can establish something very concrete about the consumer's ranking of any two alternatives. For any pair $\mathbf{x}^1$ and $\mathbf{x}^2$, *exactly one* of three mutually exclusive possibilities holds: $\mathbf{x}^1 \succ \mathbf{x}^2$, or $\mathbf{x}^2 \succ \mathbf{x}^1$, or $\mathbf{x}^1 \sim \mathbf{x}^2$.

To this point, we have simply managed to formalise the requirement that preferences reflect an ability to make choices and display a certain kind of consistency. Let us consider how we might describe graphically a set of preferences satisfying just those first few axioms. To that end, and also because of their usefulness later on, we will use the preference relation to define some related sets. These sets focus on a single alternative in the consumption set and examine the ranking of all other alternatives relative to it.

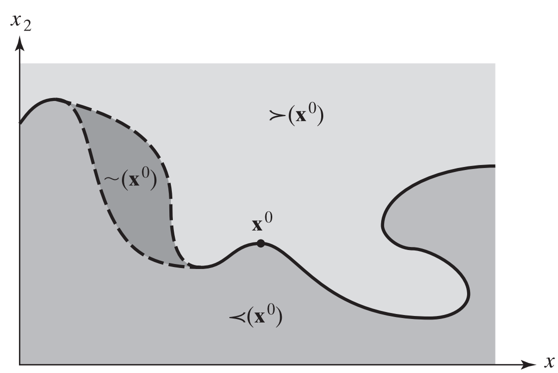
DEFINITION 1.4 ***Sets in X Derived from the Preference Relation***

Let $\mathbf{x}^0$ be any point in the consumption set, X. Relative to any such point, we can define the following subsets of X:

1. $\succsim(\mathbf{x}^0) \equiv \{\mathbf{x} \mid \mathbf{x} \in X, \mathbf{x} \succsim \mathbf{x}^0\}$, *called the 'at least as good as' set.*
2. $\precsim(\mathbf{x}^0) \equiv \{\mathbf{x} \mid \mathbf{x} \in X, \mathbf{x}^0 \succsim \mathbf{x}\}$, *called the 'no better than' set.*
3. $\prec(\mathbf{x}^0) \equiv \{\mathbf{x} \mid \mathbf{x} \in X, \mathbf{x}^0 \succ \mathbf{x}\}$, *called the 'worse than' set.*
4. $\succ(\mathbf{x}^0) \equiv \{\mathbf{x} \mid \mathbf{x} \in X, \mathbf{x} \succ \mathbf{x}^0\}$, *called the 'preferred to' set.*
5. $\sim(\mathbf{x}^0) \equiv \{\mathbf{x} \mid \mathbf{x} \in X, \mathbf{x} \sim \mathbf{x}^0\}$, *called the 'indifference' set.*

A hypothetical set of preferences satisfying Axioms 1 and 2 has been sketched in Fig. 1.1 for $X = \mathbb{R}^2_+$. Any point in the consumption set, such as $\mathbf{x}^0 = (x_1^0, x_2^0)$, represents a consumption plan consisting of a certain amount x_1^0 of commodity 1, together with a certain amount x_2^0 of commodity 2. Under Axiom 1, the consumer is able to compare $\mathbf{x}^0$ with any and every other plan in X and decide whether the other is at least as good as $\mathbf{x}^0$ or whether $\mathbf{x}^0$ is at least as good as the other. Given our definitions of the various sets relative to $\mathbf{x}^0$, Axioms 1 and 2 tell us that the consumer must place *every* point in X into

Figure 1.1. Hypothetical preferences satisfying Axioms 1 and 2.



one of three mutually exclusive categories relative to $\mathbf{x}^0$; every other point is worse than $\mathbf{x}^0$, indifferent to $\mathbf{x}^0$, or preferred to $\mathbf{x}^0$. Thus, for any bundle $\mathbf{x}^0$ the three sets $\prec(\mathbf{x}^0)$, $\sim(\mathbf{x}^0)$, and $\succ(\mathbf{x}^0)$ *partition* the consumption set.

The preferences in Fig. 1.1 may seem rather odd. They possess only the most limited structure, yet they are entirely consistent with and allowed for by the first two axioms alone. Nothing assumed so far prohibits any of the 'irregularities' depicted there, such as the 'thick' indifference zones, or the 'gaps' and 'curves' within the indifference set $\sim(\mathbf{x}^0)$. Such things can be ruled out only by imposing additional requirements on preferences.

We shall consider several new assumptions on preferences. One has very little behavioural significance and speaks almost exclusively to the purely mathematical aspects of representing preferences; the others speak directly to the issue of consumer tastes over objects in the consumption set.

The first is an axiom whose only effect is to impose a kind of topological regularity on preferences, and whose primary contribution will become clear a bit later.

From now on we explicitly set $X = \mathbb{R}^n_+$.

AXIOM 3: Continuity. *For all $\mathbf{x} \in \mathbb{R}^n_+$, the 'at least as good as' set, $\succsim(\mathbf{x})$, and the 'no better than' set, $\precsim(\mathbf{x})$, are closed in $\mathbb{R}^n_+$.*

Recall that a set is closed in a particular domain if its complement is open in that domain. Thus, to say that $\succsim(\mathbf{x})$ is closed in $\mathbb{R}^n_+$ is to say that its complement, $\prec(\mathbf{x})$, is open in $\mathbb{R}^n_+$.

The continuity axiom guarantees that sudden preference reversals do not occur. Indeed, the continuity axiom can be equivalently expressed by saying that if each element $\mathbf{y}^n$ of a sequence of bundles is at least as good as (no better than) $\mathbf{x}$, and $\mathbf{y}^n$ converges to $\mathbf{y}$, then $\mathbf{y}$ is at least as good as (no better than) $\mathbf{x}$. Note that because $\succsim(\mathbf{x})$ and $\precsim(\mathbf{x})$ are closed, so, too, is $\sim(\mathbf{x})$ because the latter is the intersection of the former two. Consequently, Axiom 3 rules out the open area in the indifference set depicted in the north-west of Fig. 1.1.

Additional assumptions on tastes lend the greater structure and regularity to preferences that you are probably familiar with from earlier economics classes. Assumptions of this sort must be selected for their appropriateness to the particular choice problem being analysed. We will consider in turn a few key assumptions on tastes that are ordinarily imposed in 'standard' consumer theory, and seek to understand the individual and collective contributions they make to the structure of preferences. Within each class of these assumptions, we will proceed from the less restrictive to the more restrictive. We will generally employ the more restrictive versions considered. Consequently, we let axioms with primed numbers indicate alternatives to the norm, which are conceptually similar but slightly less restrictive than their unprimed partners.

When representing preferences over ordinary consumption goods, we will want to express the fundamental view that 'wants' are essentially unlimited. In a very weak sense, we can express this by saying that there will always exist some adjustment in the composition of the consumer's consumption plan that he can imagine making to give himself a consumption plan he prefers. This adjustment may involve acquiring more of some commodities and less of others, or more of all commodities, or even less of all commodities.

By this assumption, we preclude the possibility that the consumer can even *imagine* having all his wants and whims for commodities completely satisfied. Formally, we state this assumption as follows, where $B_\varepsilon(\mathbf{x}^0)$ denotes the open ball of radius ε centred at $\mathbf{x}^0$:[1]

AXIOM 4′: Local Non-satiation. *For all* $\mathbf{x}^0 \in \mathbb{R}^n_+$, *and for all* $\varepsilon > 0$, *there exists some* $\mathbf{x} \in B_\varepsilon(\mathbf{x}^0) \cap \mathbb{R}^n_+$ *such that* $\mathbf{x} \succ \mathbf{x}^0$.

Axiom 4′ says that within any vicinity of a given point $\mathbf{x}^0$, no matter how small that vicinity is, there will always be at least one other point $\mathbf{x}$ that the consumer prefers to $\mathbf{x}^0$. Its effect on the structure of indifference sets is significant. It rules out the possibility of having 'zones of indifference', such as that surrounding $\mathbf{x}^1$ in Fig. 1.2. To see this, note that we can always find some $\varepsilon > 0$, and some $B_\varepsilon(\mathbf{x}^1)$, containing nothing but points indifferent to $\mathbf{x}^1$. This of course violates Axiom 4′, because it requires there *always* be at least one point strictly preferred to $\mathbf{x}^1$, regardless of the $\varepsilon > 0$ we choose. The preferences depicted in Fig. 1.3 do satisfy Axiom 4′ as well as Axioms 1 to 3.

A different and more demanding view of needs and wants is very common. According to this view, more is always better than less. Whereas local non-satiation requires

Figure 1.2. Hypothetical preferences satisfying Axioms 1, 2, and 3.

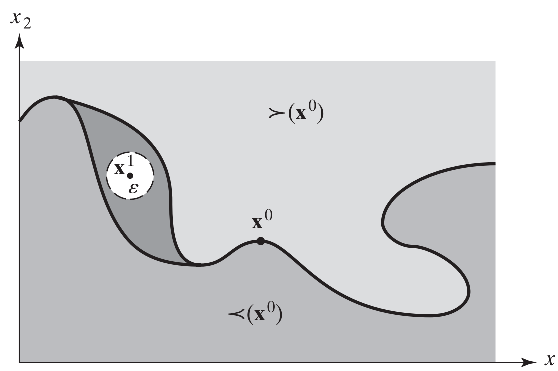
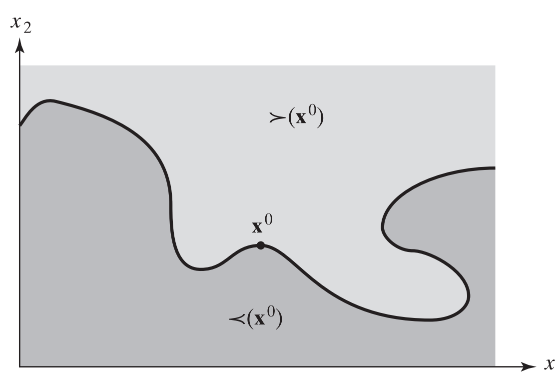


Figure 1.3. Hypothetical preferences satisfying Axioms 1, 2, 3, and 4′.



[1] See Definition A1.4 in the Mathematical Appendix.

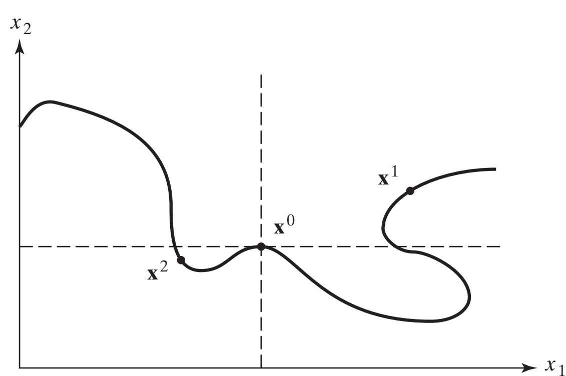
that a preferred alternative nearby always exist, it does not rule out the possibility that the preferred alternative may involve less of some or even all commodities. Specifically, it does not *imply* that giving the consumer more of everything necessarily makes that consumer better off. The alternative view takes the position that the consumer will *always* prefer a consumption plan involving more to one involving less. This is captured by the axiom of *strict monotonicity*. As a matter of notation, if the bundle $\mathbf{x}^0$ contains at least as much of every good as does $\mathbf{x}^1$ we write $\mathbf{x}^0 \geq \mathbf{x}^1$, while if $\mathbf{x}^0$ contains *strictly* more of every good than $\mathbf{x}^1$ we write $\mathbf{x}^0 \gg \mathbf{x}^1$.

AXIOM 4: Strict Monotonicity. *For all* $\mathbf{x}^0, \mathbf{x}^1 \in \mathbb{R}^n_+$, *if* $\mathbf{x}^0 \geq \mathbf{x}^1$ *then* $\mathbf{x}^0 \succsim \mathbf{x}^1$, *while if* $\mathbf{x}^0 \gg \mathbf{x}^1$, *then* $\mathbf{x}^0 \succ \mathbf{x}^1$.

Axiom 4 says that if one bundle contains at least as much of every commodity as another bundle, then the one is at least as good as the other. Moreover, it is strictly better if it contains strictly more of every good. The impact on the structure of indifference and related sets is again significant. First, it should be clear that Axiom 4 implies Axiom 4′, so if preferences satisfy Axiom 4, they automatically satisfy Axiom 4′. Thus, to require Axiom 4 will have the same effects on the structure of indifference and related sets as Axiom 4′ does, plus some additional ones. In particular, Axiom 4 eliminates the possibility that the indifference sets in $\mathbb{R}^2_+$ 'bend upward', or contain positively sloped segments. It also requires that the 'preferred to' sets be 'above' the indifference sets and that the 'worse than' sets be 'below' them.

To help see this, consider Fig. 1.4. Under Axiom 4, no points north-east of $\mathbf{x}^0$ or south-west of $\mathbf{x}^0$ may lie in the same indifference set as $\mathbf{x}^0$. Any point north-east, such as $\mathbf{x}^1$, involves more of both goods than does $\mathbf{x}^0$. All such points in the north-east quadrant must therefore be strictly preferred to $\mathbf{x}^0$. Similarly, any point in the south-west quadrant, such as $\mathbf{x}^2$, involves less of both goods. Under Axiom 4, $\mathbf{x}^0$ must be strictly preferred to $\mathbf{x}^2$ and to all other points in the south-west quadrant, so none of these can lie in the same indifference set as $\mathbf{x}^0$. For any $\mathbf{x}^0$, points north-east of the indifference set will be contained in $\succ(\mathbf{x}^0)$, and all those south-west of the indifference set will be contained in the set $\prec(\mathbf{x}^0)$. A set of preferences satisfying Axioms 1, 2, 3, and 4 is given in Fig. 1.5.

Figure 1.4. Hypothetical preferences satisfying Axioms 1, 2, 3, and 4′.



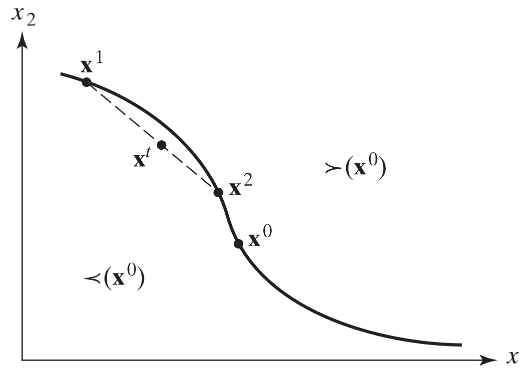


Figure 1.5. Hypothetical preferences satisfying Axioms 1, 2, 3, and 4.

The preferences in Fig. 1.5 are the closest we have seen to the kind undoubtedly familiar to you from your previous economics classes. They still differ, however, in one very important respect: typically, the kind of non-convex region in the north-west part of $\sim(\mathbf{x}^0)$ is explicitly ruled out. This is achieved by invoking one final assumption on tastes. We will state two different versions of the axiom and then consider their meaning and purpose.

AXIOM 5′: Convexity. *If* $\mathbf{x}^1 \succsim \mathbf{x}^0$, *then* $t\mathbf{x}^1 + (1-t)\mathbf{x}^0 \succsim \mathbf{x}^0$ *for all* $t \in [0, 1]$.

A slightly stronger version of this is the following:

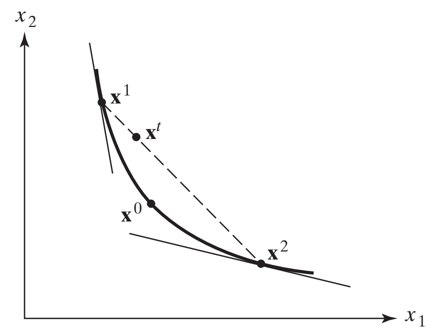
AXIOM 5: Strict Convexity. *If* $\mathbf{x}^1 \neq \mathbf{x}^0$ *and* $\mathbf{x}^1 \succsim \mathbf{x}^0$, *then* $t\mathbf{x}^1 + (1-t)\mathbf{x}^0 \succ \mathbf{x}^0$ *for all* $t \in (0, 1)$.

Notice first that either Axiom 5′ or Axiom 5 – in conjunction with Axioms 1, 2, 3, and 4 – will rule out concave-to-the-origin segments in the indifference sets, such as those in the north-west part of Fig. 1.5. To see this, choose two distinct points in the indifference set depicted there. Because $\mathbf{x}^1$ and $\mathbf{x}^2$ are both indifferent to $\mathbf{x}^0$, we clearly have $\mathbf{x}^1 \succsim \mathbf{x}^2$. Convex combinations of those two points, such as $\mathbf{x}^t$, will lie within $\prec(\mathbf{x}^0)$, violating the requirements of both Axiom 5′ and Axiom 5.

For the purposes of the consumer theory we shall develop, it turns out that Axiom 5′ can be imposed without any loss of generality. The predictive content of the theory would be the same with or without it. Although the same statement does not quite hold for the slightly stronger Axiom 5, it does greatly simplify the analysis.

There are at least two ways we can intuitively understand the implications of convexity for consumer tastes. The preferences depicted in Fig. 1.6 are consistent with both Axiom 5′ and Axiom 5. Again, suppose we choose $\mathbf{x}^1 \sim \mathbf{x}^2$. Point $\mathbf{x}^1$ represents a bundle containing a proportion of the good x_2 which is relatively 'extreme', compared to the proportion of x_2 in the other bundle $\mathbf{x}^2$. The bundle $\mathbf{x}^2$, by contrast, contains a proportion of the other good, x_1, which is relatively extreme compared to that contained in $\mathbf{x}^1$. Although each contains a relatively high proportion of one good compared to the other, the consumer is indifferent between the two bundles. Now, any convex combination of $\mathbf{x}^1$ and $\mathbf{x}^2$, such as $\mathbf{x}^t$, will be a bundle containing a more 'balanced' combination of x_1

Figure 1.6. Hypothetical preferences satisfying Axioms 1, 2, 3, 4, and 5′ or 5.



and x_2 than does either 'extreme' bundle $\mathbf{x}^1$ or $\mathbf{x}^2$. The thrust of Axiom 5′ or Axiom 5 is to forbid the consumer from preferring such extremes in consumption. Axiom 5′ requires that any such relatively balanced bundle as $\mathbf{x}^t$ be no worse than either of the two extremes between which the consumer is indifferent. Axiom 5 goes a bit further and requires that the consumer strictly prefer any such relatively balanced consumption bundle to both of the extremes between which he is indifferent. In either case, some degree of 'bias' in favour of balance in consumption is required of the consumer's tastes.

Another way to describe the implications of convexity for consumers' tastes focuses attention on the 'curvature' of the indifference sets themselves. When $X = \mathbb{R}^2_+$, the (absolute value of the) slope of an indifference curve is called the **marginal rate of substitution of good two for good one**. This slope measures, at any point, the rate at which the consumer is just willing to give up good two per unit of good one received. Thus, the consumer is indifferent after the exchange.

If preferences are strictly monotonic, any form of convexity requires the indifference curves to be at least weakly convex-shaped relative to the origin. This is equivalent to requiring that the marginal rate of substitution not increase as we move from bundles such as $\mathbf{x}^1$ towards bundles such as $\mathbf{x}^2$. Loosely, this means that the consumer is no more willing to give up x_2 in exchange for x_1 when he has relatively little x_2 and much x_1 than he is when he has relatively much x_2 and little x_1. Axiom 5′ requires the rate at which the consumer would trade x_2 for x_1 and remain indifferent to be either constant or decreasing as we move from north-west to south-east along an indifference curve. Axiom 5 goes a bit further and requires that the rate be strictly diminishing. The preferences in Fig. 1.6 display this property, sometimes called the **principle of diminishing marginal rate of substitution** in consumption.

We have taken some care to consider a number of axioms describing consumer preferences. Our goal has been to gain some appreciation of their individual and collective implications for the structure and representation of consumer preferences. We can summarise this discussion rather briefly. The axioms on consumer preferences may be roughly classified in the following way. The axioms of *completeness* and *transitivity* describe a consumer who can make consistent comparisons among alternatives. The axiom of *continuity* is intended to guarantee the existence of topologically nice 'at least as good as' and 'no better than' sets, and its purpose is primarily a mathematical one. All other axioms serve to characterise consumers' *tastes* over the objects of choice. Typically, we require that tastes display some form of non-satiation, either weak or strong, and some bias in favour of balance in consumption, either weak or strong.

1.2.2 THE UTILITY FUNCTION

In modern theory, a utility function is simply a convenient device for summarising the information contained in the consumer's preference relation – no more and no less. Sometimes it is easier to work directly with the preference relation and its associated sets. Other times, especially when one would like to employ calculus methods, it is easier to work with a utility function. In modern theory, the preference relation is taken to be the primitive, most fundamental characterisation of preferences. The utility function merely 'represents', or summarises, the information conveyed by the preference relation. A utility function is defined formally as follows.

DEFINITION 1.5 ***A Utility Function Representing the Preference Relation*** $\succsim$

A real-valued function $u\colon \mathbb{R}^n_+ \to \mathbb{R}$ *is called a utility function representing the preference relation* $\succsim$, *if for all* $\mathbf{x}^0, \mathbf{x}^1 \in \mathbb{R}^n_+$, $u(\mathbf{x}^0) \geq u(\mathbf{x}^1) \Longleftrightarrow \mathbf{x}^0 \succsim \mathbf{x}^1$.

Thus a utility function represents a consumer's preference relation if it assigns higher numbers to preferred bundles.

A question that earlier attracted a great deal of attention from theorists concerned properties that a preference relation must possess to guarantee that it can be represented by a continuous real-valued function. The question is important because the analysis of many problems in consumer theory is enormously simplified if we can work with a utility function, rather than with the preference relation itself.

Mathematically, the question is one of *existence* of a continuous utility function representing a preference relation. It turns out that a subset of the axioms we have considered so far is precisely that required to guarantee existence. It can be shown that any binary relation that is *complete*, *transitive*, and *continuous* can be represented by a continuous real-valued utility function.[2] (In the exercises, you are asked to show that these three axioms are necessary for such a representation as well.) These are simply the axioms that, together, require that the consumer be able to make basically consistent binary choices and that the preference relation possess a certain amount of topological 'regularity'. In particular, representability does *not* depend on any assumptions about consumer tastes, such as convexity or even monotonicity. We can therefore summarise preferences by a continuous utility function in an extremely broad range of problems.

Here we will take a detailed look at a slightly less general result. In addition to the three most basic axioms mentioned before, we will impose the extra requirement that preferences be strictly monotonic. Although this is not essential for representability, to

[2] See, for example, Barten and Böhm (1982). The classic reference is Debreu (1954).

require it simultaneously simplifies the purely mathematical aspects of the problem and increases the intuitive content of the proof. Notice, however, that we will not require any form of convexity.

THEOREM 1.1 ***Existence of a Real-Valued Function Representing the Preference Relation*** $\succsim$

If the binary relation $\succsim$ *is complete, transitive, continuous, and strictly monotonic, there exists a continuous real-valued function,* $u\colon \mathbb{R}^n_+ \to \mathbb{R}$, *which represents* $\succsim$.

Notice carefully that this is only an *existence* theorem. It simply claims that under the conditions stated, at least one continuous real-valued function representing the preference relation is guaranteed to exist. There may be, and in fact there always will be, more than one such function. The theorem itself, however, makes no statement on how many more there are, nor does it indicate in any way what form any of them must take. Therefore, if we can dream up just *one* function that is continuous and that represents the given preferences, we will have proved the theorem. This is the strategy we will adopt in the following proof.

Proof: Let the relation $\succsim$ be complete, transitive, continuous, and strictly monotonic. Let $\mathbf{e} \equiv (1, \ldots, 1) \in \mathbb{R}^n_+$ be a vector of ones, and consider the mapping $u\colon \mathbb{R}^n_+ \to \mathbb{R}$ defined so that the following condition is satisfied:[3]

$$u(\mathbf{x})\mathbf{e} \sim \mathbf{x}. \tag{P.1}$$

Let us first make sure we understand what this says and how it works. In words, (P.1) says, 'take any $\mathbf{x}$ in the domain $\mathbb{R}^n_+$ and assign to it the number $u(\mathbf{x})$ such that the bundle, $u(\mathbf{x})\mathbf{e}$, with $u(\mathbf{x})$ units of every commodity is ranked indifferent to $\mathbf{x}$'.

Two questions immediately arise. First, does there always exist a number $u(\mathbf{x})$ satisfying (P.1)? Second, is it uniquely determined, so that $u(\mathbf{x})$ is a well-defined function?

To settle the first question, fix $\mathbf{x} \in \mathbb{R}^n_+$ and consider the following two subsets of real numbers:

$$A \equiv \{t \geq 0 \mid t\mathbf{e} \succsim \mathbf{x}\}$$
$$B \equiv \{t \geq 0 \mid t\mathbf{e} \precsim \mathbf{x}\}.$$

Note that if $t^* \in A \cap B$, then $t^*\mathbf{e} \sim \mathbf{x}$, so that setting $u(\mathbf{x}) = t^*$ would satisfy (P.1). Thus, the first question would be answered in the affirmative if we show that $A \cap B$ is guaranteed to be non-empty. This is precisely what we shall show.

[3]For $t \geq 0$, the vector $t\mathbf{e}$ will be some point in $\mathbb{R}^n_+$ each of whose coordinates is equal to the number t, because $t\mathbf{e} = t(1, \ldots, 1) = (t, \ldots, t)$. If $t = 0$, then $t\mathbf{e} = (0, \ldots, 0)$ coincides with the origin. If $t = 1$, then $t\mathbf{e} = (1, \ldots, 1)$ coincides with $\mathbf{e}$. If $t > 1$, the point $t\mathbf{e}$ lies farther out from the origin than $\mathbf{e}$. For $0<t<1$, the point $t\mathbf{e}$ lies between the origin and $\mathbf{e}$. It should be clear that for any choice of $t \geq 0$, $t\mathbf{e}$ will be a point in $\mathbb{R}^n_+$ somewhere on the ray from the origin through $\mathbf{e}$, i.e., some point on the 45° line in Fig. 1.7.

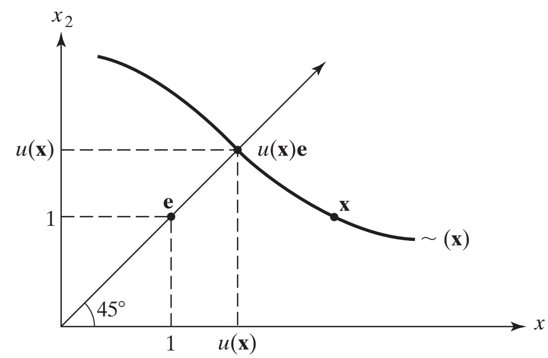


Figure 1.7. Constructing the mapping $u: \mathbb{R}^n_+ \to \mathbb{R}_+$.

According to Exercise 1.11, the continuity of $\succsim$ implies that both A and B are closed in $\mathbb{R}_+$. Also, by strict monotonicity, $t \in A$ implies $t' \in A$ for all $t' \geq t$. Consequently, A must be a closed interval of the form $[\underline{t}, \infty)$. Similarly, strict monotonicity and the closedness of B in $\mathbb{R}_+$ imply that B must be a closed interval of the form $[0, \bar{t}]$. Now for any $t \geq 0$, completeness of $\succsim$ implies that either $t\mathbf{e} \succsim \mathbf{x}$ or $t\mathbf{e} \precsim \mathbf{x}$, that is, $t \in A \cup B$. But this means that $\mathbb{R}_+ = A \cup B = [0, \bar{t}] \cup [\underline{t}, \infty]$. We conclude that $\underline{t} \leq \bar{t}$ so that $A \cap B \neq \emptyset$.

We now turn to the second question. We must show that there is *only one* number $t \geq 0$ such that $t\mathbf{e} \sim \mathbf{x}$. But this follows easily because if $t_1\mathbf{e} \sim \mathbf{x}$ and $t_2\mathbf{e} \sim \mathbf{x}$, then by the transitivity of $\sim$ (see Exercise 1.4), $t_1\mathbf{e} \sim t_2\mathbf{e}$. So, by strict monotonicity, it must be the case that $t_1 = t_2$.

We conclude that for every $\mathbf{x} \in \mathbb{R}^n_+$, there is exactly one number, $u(\mathbf{x})$, such that (P.1) is satisfied. Having constructed a utility function assigning each bundle in X a number, we show next that this utility function represents the preferences $\succsim$.

Consider two bundles $\mathbf{x}^1$ and $\mathbf{x}^2$, and their associated utility numbers $u(\mathbf{x}^1)$ and $u(\mathbf{x}^2)$, which by definition satisfy $u(\mathbf{x}^1)\mathbf{e} \sim \mathbf{x}^1$ and $u(\mathbf{x}^2)\mathbf{e} \sim \mathbf{x}^2$. Then we have the following:

$$\mathbf{x}^1 \succsim \mathbf{x}^2 \tag{P.2}$$

$$\iff u(\mathbf{x}^1)\mathbf{e} \sim \mathbf{x}^1 \succsim \mathbf{x}^2 \sim u(\mathbf{x}^2)\mathbf{e} \tag{P.3}$$

$$\iff u(\mathbf{x}^1)\mathbf{e} \succsim u(\mathbf{x}^2)\mathbf{e} \tag{P.4}$$

$$\iff u(\mathbf{x}^1) \geq u(\mathbf{x}^2). \tag{P.5}$$

Here (P.2) $\iff$ (P.3) follows by definition of u; (P.3) $\iff$ (P.4) follows from the transitivity of $\succsim$, the transitivity of $\sim$, and the definition of u; and (P.4) $\iff$ (P.5) follows from the strict monotonicity of $\succsim$. Together, (P.2) through (P.5) imply that (P.2) $\iff$ (P.5), so that $\mathbf{x}^1 \succsim \mathbf{x}^2$ if and only if $u(\mathbf{x}^1) \geq u(\mathbf{x}^2)$, as we sought to show.

It remains only to show that the utility function $u: \mathbb{R}^n_+ \to \mathbb{R}$ representing $\succsim$ is continuous. By Theorem A1.6, it suffices to show that the inverse image under u of every open ball in $\mathbb{R}$ is open in $\mathbb{R}^n_+$. Because open balls in $\mathbb{R}$ are merely open intervals, this is equivalent to showing that $u^{-1}((a, b))$ is open in $\mathbb{R}^n_+$ for every $a<b$.

Now,

$$\begin{aligned} u^{-1}((a, b)) &= \{\mathbf{x} \in \mathbb{R}^n_+ \mid a < u(\mathbf{x}) < b\} \\ &= \{\mathbf{x} \in \mathbb{R}^n_+ \mid a\mathbf{e} \prec u(\mathbf{x})\mathbf{e} \prec b\mathbf{e}\} \\ &= \{\mathbf{x} \in \mathbb{R}^n_+ \mid a\mathbf{e} \prec \mathbf{x} \prec b\mathbf{e}\}. \end{aligned}$$

The first equality follows from the definition of the inverse image; the second from the monotonicity of $\succsim$; and the third from $u(\mathbf{x})\mathbf{e} \sim \mathbf{x}$ and Exercise 1.4. Rewriting the last set on the right-hand side gives

$$u^{-1}((a, b)) = \succ (a\mathbf{e}) \bigcap \prec (b\mathbf{e}). \tag{P.6}$$

By the continuity of $\succsim$, the sets $\precsim(a\mathbf{e})$ and $\succsim(b\mathbf{e})$ are closed in $X = \mathbb{R}^n_+$. Consequently, the two sets on the right-hand side of (P.6), being the complements of these closed sets, are open in $\mathbb{R}^n_+$. Therefore, $u^{-1}((a, b))$, being the intersection of two open sets in $\mathbb{R}^n_+$, is, by Exercise A1.28, itself open in $\mathbb{R}^n_+$. ∎

Theorem 1.1 is very important. It frees us to represent preferences either in terms of the primitive set-theoretic preference relation or in terms of a numerical representation, a continuous utility function. But this utility representation is never unique. If some function u represents a consumer's preferences, then so too will the function $v = u + 5$, or the function $v = u^3$, because each of these functions ranks bundles the same way u does. This is an important point about utility functions that must be grasped. If all we require of the preference relation is that it order the bundles in the consumption set, and if all we require of a utility function representing those preferences is that it reflect that ordering of bundles by the ordering of numbers it assigns to them, then any *other* function that assigns numbers to bundles in the same order as u does will *also* represent that preference relation and will itself be just as good a utility function as u.

This is known by several different names in the literature. People sometimes say the utility function is *invariant to positive monotonic transforms* or sometimes they say that the utility function is *unique up to a positive monotonic transform*. Either way, the meaning is this: if all we require of the preference relation is that rankings between bundles be meaningful, then all any utility function representing that relation is capable of conveying to us is *ordinal* information, no more and no less. If we know that one function properly conveys the ordering of bundles, then any transform of that function that preserves that ordering of bundles will perform all the duties of a utility function just as well.

Seeing the representation issue in proper perspective thus frees us and restrains us. If we have a function u that represents some consumers' preferences, it frees us to transform u into other, perhaps more convenient or easily manipulated forms, as long as the transformation we choose is order-preserving. At the same time, we are restrained by the explicit warning here that no significance whatsoever can be attached to the actual numbers assigned by a given utility function to particular bundles – only to the ordering of those numbers.[4] This conclusion, though simple to demonstrate, is nonetheless important enough to warrant being stated formally. The proof is left as an exercise.

THEOREM 1.2 ***Invariance of the Utility Function to Positive Monotonic Transforms***

Let $\succsim$ be a preference relation on $\mathbb{R}^n_+$ and suppose $u(\mathbf{x})$ is a utility function that represents it. Then $v(\mathbf{x})$ also represents $\succsim$ if and only if $v(\mathbf{x}) = f(u(\mathbf{x}))$ for every $\mathbf{x}$, where $f: \mathbb{R} \to \mathbb{R}$ is strictly increasing on the set of values taken on by u.

Typically, we will want to make some assumptions on tastes to complete the description of consumer preferences. Naturally enough, any additional structure we impose on preferences will be reflected as additional structure on the utility function representing them. By the same token, whenever we assume the utility function to have properties beyond continuity, we will in effect be invoking some set of additional assumptions on the underlying preference relation. There is, then, an equivalence between axioms on tastes and specific mathematical properties of the utility function. We will conclude this section by briefly noting some of them. The following theorem is exceedingly simple to prove because it follows easily from the definitions involved. It is worth being convinced, however, so its proof is left as an exercise. (See Chapter A1 in the Mathematical Appendix for definitions of strictly increasing, quasiconcave, and strictly quasiconcave functions.)

THEOREM 1.3 ***Properties of Preferences and Utility Functions***

Let $\succsim$ be represented by $u: \mathbb{R}^n_+ \to \mathbb{R}$. Then:

1. *$u(\mathbf{x})$ is strictly increasing if and only if $\succsim$ is strictly monotonic.*
2. *$u(\mathbf{x})$ is quasiconcave if and only if $\succsim$ is convex.*
3. *$u(\mathbf{x})$ is strictly quasiconcave if and only if $\succsim$ is strictly convex.*

Later we will want to analyse problems using calculus tools. Until now, we have concentrated on the continuity of the utility function and properties of the preference relation that ensure it. Differentiability, of course, is a more demanding requirement than continuity. Intuitively, continuity requires there be no sudden preference reversals. It does not rule out 'kinks' or other kinds of continuous, but impolite behaviour. Differentiability specifically excludes such things and ensures indifference curves are 'smooth' as well as continuous. Differentiability of the utility function thus requires a stronger restriction on

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[4]Some theorists are so sensitive to the potential confusion between the modern usage of the term 'utility function' and the classical utilitarian notion of 'utility' as a measurable quantity of pleasure or pain that they reject the anachronistic terminology altogether and simply speak of preference relations and their 'representation functions'.

preferences than continuity. Like the axiom of continuity, what is needed is just the right mathematical condition. We shall not develop this condition here, but refer the reader to Debreu (1972) for the details. For our purposes, we are content to simply assume that the utility representation is differentiable whenever necessary.

There is a certain vocabulary we use when utility is differentiable, so we should learn it. The first-order partial derivative of $u(\mathbf{x})$ with respect to x_i is called the **marginal utility of good** i. For the case of two goods, we defined the marginal rate of substitution of good 2 for good 1 as the absolute value of the slope of an indifference curve. We can derive an expression for this in terms of the two goods' marginal utilities. To see this, consider any bundle $\mathbf{x}^1 = (x_1^1, x_2^1)$. Because the indifference curve through $\mathbf{x}^1$ is just a function in the (x_1, x_2) plane, let $x_2 = f(x_1)$ be the function describing it. Therefore, as x_1 varies, the bundle $(x_1, x_2) = (x_1, f(x_1))$ traces out the indifference curve through $\mathbf{x}^1$. Consequently, for all x_1,

$$u(x_1, f(x_1)) = \text{constant}. \tag{1.1}$$

Now the marginal rate of substitution of good two for good one at the bundle $\mathbf{x}^1 = (x_1^1, x_2^1)$, denoted $MRS_{12}(x_1^1, x_2^1)$, is the absolute value of the slope of the indifference curve through (x_1^1, x_2^1). That is,

$$MRS_{12}\big(x_1^1, x_2^1\big) \equiv \big|f'\big(x_1^1\big)\big| = -f'\big(x_1^1\big), \tag{1.2}$$

because $f' < 0$. But by (1.1), $u(x_1, f(x_1))$ is a constant function of x_1. Hence, its derivative with respect to x_1 must be zero. That is,

$$\frac{\partial u(x_1, x_2)}{\partial x_1} + \frac{\partial u(x_1, x_2)}{\partial x_2} f'(x_1) = 0. \tag{1.3}$$

But (1.2) together with (1.3) imply that

$$MRS_{12}(\mathbf{x}^1) = \frac{\partial u(\mathbf{x}^1)/\partial x_1}{\partial u(\mathbf{x}^1)/\partial x_2}.$$

Similarly, when there are more than two goods we define the marginal rate of substitution of good j for good i as the ratio of their marginal utilities,

$$MRS_{ij}(\mathbf{x}) \equiv \frac{\partial u(\mathbf{x})/\partial x_i}{\partial u(\mathbf{x})/\partial x_j}.$$

When marginal utilities are strictly positive, the $MRS_{ij}(\mathbf{x})$ is again a positive number, and it tells us the rate at which good j can be exchanged per unit of good i with no change in the consumer's utility.

When $u(\mathbf{x})$ is continuously differentiable on $\mathbb{R}^n_{++}$ and preferences are strictly monotonic, the marginal utility of every good is virtually always strictly positive. That is,

$\partial u(\mathbf{x})/\partial x_i > 0$ for 'almost all' bundles $\mathbf{x}$, and all $i = 1, \ldots, n$.[5] When preferences are strictly convex, the marginal rate of substitution between two goods is always strictly diminishing along any level surface of the utility function. More generally, for any quasiconcave utility function, its Hessian matrix $\mathbf{H}(\mathbf{x})$ of second-order partials will satisfy

$$\mathbf{y}^{\mathrm{T}}\mathbf{H}(\mathbf{x})\mathbf{y} \leq 0 \qquad \text{for all vectors } \mathbf{y} \text{ such that} \qquad \nabla u(\mathbf{x}) \cdot \mathbf{y} = 0.$$

If the inequality is strict, this says that moving from $\mathbf{x}$ in a direction $\mathbf{y}$ that is tangent to the indifference surface through $\mathbf{x}$ [i.e., $\nabla u(\mathbf{x}) \cdot \mathbf{y} = 0$] reduces utility (i.e., $\mathbf{y}^{\mathrm{T}}\mathbf{H}(\mathbf{x})\mathbf{y} < 0$).

1.3 THE CONSUMER'S PROBLEM

We have dwelt upon how to structure and represent preferences, but these are only one of four major building blocks in our theory of consumer choice. In this section, we consider the rest of them and combine them all together to construct a formal description of the central actor in much of economic theory – the humble atomistic consumer.

On the most abstract level, we view the consumer as having a consumption set, $X = \mathbb{R}^n_+$, containing all conceivable alternatives in consumption. His inclinations and attitudes toward them are described by the preference relation $\succsim$ defined on $\mathbb{R}^n_+$. The consumer's circumstances limit the alternatives he is actually able to achieve, and we collect these all together into a feasible set, $B \subset \mathbb{R}^n_+$. Finally, we assume the consumer is motivated to choose the most preferred feasible alternative according to his preference relation. Formally, the consumer seeks

$$\mathbf{x}^* \in B \text{ such that } \mathbf{x}^* \succsim \mathbf{x} \text{ for all } \mathbf{x} \in B \tag{1.4}$$

To make further progress, we make the following assumptions that will be maintained unless stated otherwise.

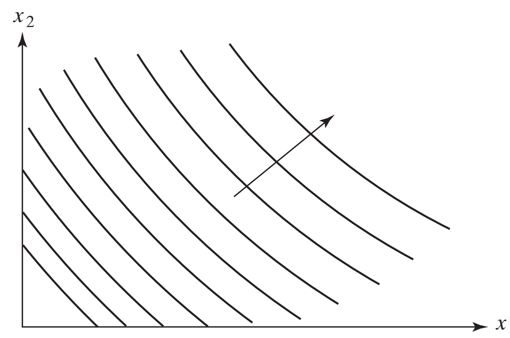
ASSUMPTION 1.2 ***Consumer Preferences***

The consumer's preference relation $\succsim$ is complete, transitive, continuous, strictly monotonic, and strictly convex on $\mathbb{R}^n_+$. Therefore, by Theorems 1.1 and 1.3 it can be represented by a real-valued utility function, u, that is continuous, strictly increasing, and strictly quasiconcave on $\mathbb{R}^n_+$.

In the two-good case, preferences like these can be represented by an indifference map whose level sets are non-intersecting, strictly convex away from the origin, and increasing north-easterly, as depicted in Fig. 1.8.

[5] In case the reader is curious, the term 'almost all' means all bundles except a set having Lebesgue measure zero. However, there is no need to be familiar with Lebesgue measure to see that some such qualifier is necessary. Consider the case of a single good, x, and the utility function $u(x) = x + \sin(x)$. Because u is strictly increasing,

Figure 1.8. Indifference map for preferences satisfying Assumption 1.2.



Next, we consider the consumer's circumstances and structure the feasible set. Our concern is with an individual consumer operating within a **market economy**. By a market economy, we mean an economic system in which transactions between agents are mediated by markets. There is a market for each commodity, and in these markets, a price p_i prevails for each commodity i. We suppose that prices are strictly positive, so $p_i > 0$, $i = 1, \ldots, n$. Moreover, we assume the individual consumer is an *insignificant force* on every market. By this we mean, specifically, that the size of each market relative to the potential purchases of the individual consumer is so large that no matter how much or how little the consumer might purchase, there will be no perceptible effect on any market price. Formally, this means we take the vector of market prices, $\mathbf{p} \gg \mathbf{0}$, as *fixed* from the consumer's point of view.

The consumer is endowed with a fixed money income $y \geq 0$. Because the purchase of x_i units of commodity i at price p_i per unit requires an expenditure of $p_i x_i$ dollars, the requirement that expenditure not exceed income can be stated as $\sum_{i=1}^{n} p_i x_i \leq y$ or, more compactly, as $\mathbf{p} \cdot \mathbf{x} \leq y$. We summarise these assumptions on the economic environment of the consumer by specifying the following structure on the feasible set, B, called the **budget set:**

$$B = \{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n_+, \mathbf{p} \cdot \mathbf{x} \leq y\}.$$

In the two-good case, B consists of all bundles lying inside or on the boundaries of the shaded region in Fig. 1.9.

If we want to, we can now recast the consumer's problem in very familiar terms. Under Assumption 1.2, preferences may be represented by a strictly increasing and strictly quasiconcave utility function $u(\mathbf{x})$ on the consumption set $\mathbb{R}^n_+$. Under our assumptions on the feasible set, total expenditure must not exceed income. The consumer's problem (1.4) can thus be cast *equivalently* as the problem of maximising the utility function subject to

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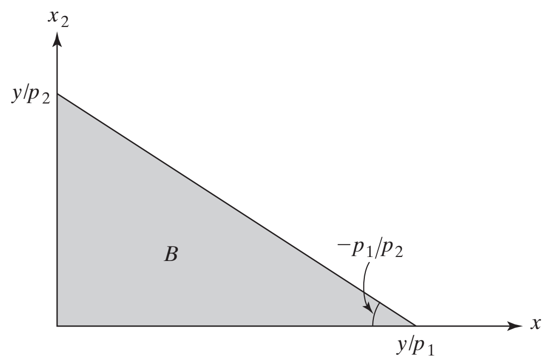
it represents strictly monotonic preferences. However, although $u'(x)$ is strictly positive for most values of x, it is zero whenever $x = \pi + 2\pi k$, $k = 0, 1, 2, \ldots$

Figure 1.9. Budget set, $B = \{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n_+,\ \mathbf{p} \cdot \mathbf{x} \leq y\}$, in the case of two commodities.



the budget constraint. Formally, the consumer's **utility-maximisation problem** is written

$$\max_{\mathbf{x} \in \mathbb{R}^n_+} u(\mathbf{x}) \qquad \text{s.t.} \qquad \mathbf{p} \cdot \mathbf{x} \leq \mathbf{y}. \tag{1.5}$$

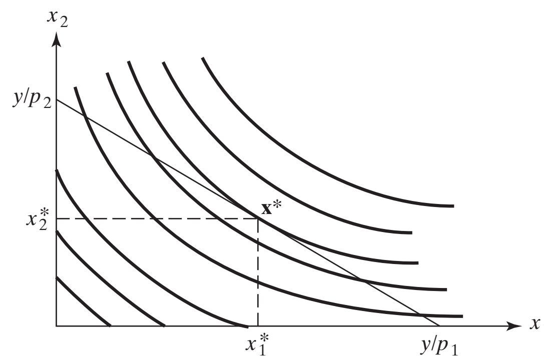
Note that if $\mathbf{x}^*$ solves this problem, then $u(\mathbf{x}^*) \geq u(\mathbf{x})$ for all $\mathbf{x} \in B$, which means that $\mathbf{x}^* \succsim \mathbf{x}$ for all $\mathbf{x} \in B$. That is, solutions to (1.5) are indeed solutions to (1.4). The converse is also true.

We should take a moment to examine the mathematical structure of this problem. As we have noted, under the assumptions on preferences, the utility function $u(\mathbf{x})$ is real-valued and continuous. The budget set B is a non-empty (it contains $\mathbf{0} \in \mathbb{R}^n_+$), closed, bounded (because all prices are strictly positive), and thus compact subset of $\mathbb{R}^n$. By the Weierstrass theorem, Theorem A1.10, we are therefore assured that a maximum of $u(\mathbf{x})$ over B exists. Moreover, because B is convex and the objective function is strictly quasiconcave, the maximiser of $u(\mathbf{x})$ over B is *unique*. Because preferences are strictly monotonic, the solution $\mathbf{x}^*$ will satisfy the budget constraint with *equality*, lying *on*, rather than inside, the boundary of the budget set. Thus, when $y > 0$ and because $\mathbf{x}^* \geq 0$, but $\mathbf{x}^* \neq 0$, we know that $x_i^* > 0$ for at least one good i. A typical solution to this problem in the two-good case is illustrated in Fig. 1.10.

Clearly, the solution vector $\mathbf{x}^*$ depends on the parameters to the consumer's problem. Because it will be unique for given values of $\mathbf{p}$ and y, we can properly view the solution to (1.5) as a *function* from the set of prices and income to the set of quantities, $X = \mathbb{R}^n_+$. We therefore will often write $x_i^* = x_i(\mathbf{p}, y)$, $i = 1, \ldots, n$, or, in vector notation, $\mathbf{x}^* = \mathbf{x}(\mathbf{p}, y)$. When viewed as functions of $\mathbf{p}$ and y, the solutions to the utility-maximisation problem are known as ordinary, or **Marshallian demand functions**. When income and all prices other than the good's own price are held fixed, the graph of the relationship between quantity demanded of x_i and its own price p_i is the standard demand curve for good i.

The relationship between the consumer's problem and consumer demand behaviour is illustrated in Fig. 1.11. In Fig. 1.11(a), the consumer faces prices p_1^0 and p_2^0 and has income y^0. Quantities $x_1(p_1^0, p_2^0, y^0)$ and $x_2(p_1^0, p_2^0, y^0)$ solve the consumer's problem and

Figure 1.10. The solution to the consumer's utility-maximisation problem.



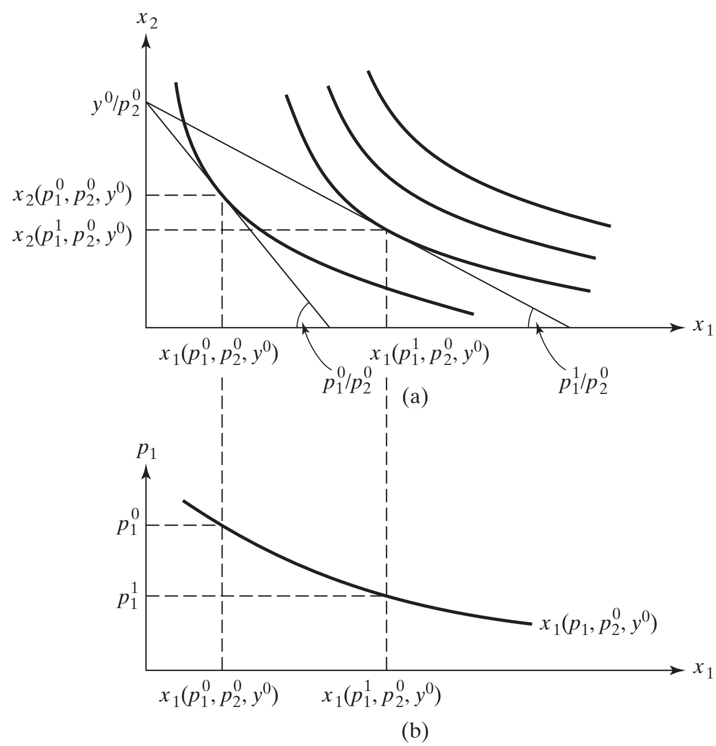


Figure 1.11. The consumer's problem and consumer demand behaviour.

maximise utility facing those prices and income. Directly below, in Fig. 1.11(b), we measure the price of good 1 on the vertical axis and the quantity demanded of good 1 on the horizontal axis. If we plot the price p_1^0 against the quantity of good 1 demanded at that price (given the price p_2^0 and income y^0), we obtain one point on the consumer's

Marshallian demand curve for good 1. At the same income and price of good 2, facing $p_1^1 < p_1^0$, the quantities $x_1(p_1^1, p_2^0, y^0)$ and $x_2(p_1^1, p_2^0, y^0)$ solve the consumer's problem and maximise utility. If we plot p_1^1 against the quantity of good 1 demanded at that price, we obtain another point on the Marshallian demand curve for good 1 in Fig. 1.11(b). By considering all possible values for p_1, we trace out the consumer's entire demand curve for good 1 in Fig. 1.11(b). As you can easily verify, different levels of income and different prices of good 2 will cause the position and shape of the demand curve for good 1 to change. That position and shape, however, will always be determined by the properties of the consumer's underlying preference relation.

If we strengthen the requirements on $u(\mathbf{x})$ to include differentiability, we can use calculus methods to further explore demand behaviour. Recall that the consumer's problem is

$$\max_{\mathbf{x}\in\mathbb{R}^n_+} u(\mathbf{x}) \qquad \text{s.t.} \qquad \mathbf{p}\cdot\mathbf{x} \leq y. \tag{1.6}$$

This is a non-linear programming problem with one inequality constraint. As we have noted, a solution $\mathbf{x}^*$ exists and is unique. If we rewrite the constraint as $\mathbf{p}\cdot\mathbf{x} - y \leq 0$ and then form the Lagrangian, we obtain

$$\mathcal{L}(\mathbf{x}, \lambda) = u(\mathbf{x}) - \lambda[\mathbf{p}\cdot\mathbf{x} - y].$$

Assuming that the solution $\mathbf{x}^*$ is strictly positive, we can apply Kuhn-Tucker methods to characterise it. If $\mathbf{x}^* \gg \mathbf{0}$ solves (1.6), then by Theorem A2.20, there exists a $\lambda^* \geq 0$ such that $(\mathbf{x}^*, \lambda^*)$ satisfy the following Kuhn-Tucker conditions:

$$\frac{\partial\mathcal{L}}{\partial x_i} = \frac{\partial u(\mathbf{x}^*)}{\partial x_i} - \lambda^* p_i = 0, \qquad i = 1, \ldots, n, \tag{1.7}$$

$$\mathbf{p}\cdot\mathbf{x}^* - y \leq 0, \tag{1.8}$$

$$\lambda^*\left[\mathbf{p}\cdot\mathbf{x}^* - y\right] = 0. \tag{1.9}$$

Now, by strict monotonicity, (1.8) must be satisfied with equality, so that (1.9) becomes redundant. Consequently, these conditions reduce to

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial x_1} &= \frac{\partial u(\mathbf{x}^*)}{\partial x_1} - \lambda^* p_1 = 0, \\
&\ \vdots \\
\frac{\partial\mathcal{L}}{\partial x_n} &= \frac{\partial u(\mathbf{x}^*)}{\partial x_n} - \lambda^* p_n = 0, \\
&\qquad \mathbf{p}\cdot\mathbf{x}^* - y = 0.
\end{aligned} \tag{1.10}$$

What do these tell us about the solution to (1.6)? There are two possibilities. Either $\nabla u(\mathbf{x}^*) = \mathbf{0}$ or $\nabla u(\mathbf{x}^*) \neq \mathbf{0}$. Under strict monotonicity, the first case is possible, but quite unlikely. We shall simply assume therefore that $\nabla u(\mathbf{x}^*) \neq \mathbf{0}$. Thus, by strict monotonicity, $\partial u(\mathbf{x}^*)/\partial x_i > 0$, for some $i = 1, \ldots, n$. Because $p_i > 0$ for all i, it is clear from (1.7) that the Lagrangian multiplier will be strictly positive at the solution, because $\lambda^* = u_i(\mathbf{x}^*)/p_i > 0$. Consequently, for all j, $\partial u(\mathbf{x}^*)/\partial x_j = \lambda^* p_j > 0$, so marginal utility is proportional to price for all goods at the optimum. Alternatively, for any two goods j and k, we can combine the conditions to conclude that

$$\frac{\partial u(\mathbf{x}^*)/\partial x_j}{\partial u(\mathbf{x}^*)/\partial x_k} = \frac{p_j}{p_k}. \tag{1.11}$$

This says that at the optimum, the marginal rate of substitution between any two goods must be equal to the ratio of the goods' prices. In the two-good case, conditions (1.10) therefore require that the slope of the indifference curve through $\mathbf{x}^*$ be equal to the slope of the budget constraint, and that $\mathbf{x}^*$ lie on, rather than inside, the budget line, as in Fig. 1.10 and Fig. 1.11(a).

In general, conditions (1.10) are merely necessary conditions for a local optimum (see the end of Section A2.3). However, for the particular problem at hand, these necessary first-order conditions are in fact *sufficient* for a global optimum. This is worthwhile stating formally.

THEOREM 1.4 ***Sufficiency of Consumer's First-Order Conditions***

Suppose that $u(\mathbf{x})$ is continuous and quasiconcave on $\mathbb{R}^n_+$, and that $(\mathbf{p}, y) \gg \mathbf{0}$. If u is differentiable at $\mathbf{x}^*$, and $(\mathbf{x}^*, \lambda^*) \gg \mathbf{0}$ solves (1.10), then $\mathbf{x}^*$ solves the consumer's maximisation problem at prices $\mathbf{p}$ and income y.

Proof: We shall employ the following fact that you are asked to prove in Exercise 1.28: For all $\mathbf{x}, \mathbf{x}^1 \geq \mathbf{0}$, because u is quasiconcave, $\nabla u(\mathbf{x})(\mathbf{x}^1 - \mathbf{x}) \geq 0$ whenever $u(\mathbf{x}^1) \geq u(\mathbf{x})$ and u is differentiable at $\mathbf{x}$.

Now, suppose that $\nabla u(\mathbf{x}^*)$ exists and $(\mathbf{x}^*, \lambda^*) \gg \mathbf{0}$ solves (1.10). Then

$$\nabla u(\mathbf{x}^*) = \lambda^* \mathbf{p}, \tag{P.1}$$

$$\mathbf{p} \cdot \mathbf{x}^* = y. \tag{P.2}$$

If $\mathbf{x}^*$ is not utility-maximising, then there must be some $\mathbf{x}^0 \geq \mathbf{0}$ such that

$$u(\mathbf{x}^0) > u(\mathbf{x}^*),$$

$$\mathbf{p} \cdot \mathbf{x}^0 \leq y.$$

Because u is continuous and $y > 0$, the preceding inequalities imply that

$$u(t\mathbf{x}^0) > u(\mathbf{x}^*), \tag{P.3}$$
$$\mathbf{p} \cdot t\mathbf{x}^0 < y. \tag{P.4}$$

for some $t \in [0, 1]$ close enough to one. Letting $\mathbf{x}^1 = t\mathbf{x}^0$, we then have

$$\begin{aligned}\nabla u(\mathbf{x}^*)(\mathbf{x}^1 - \mathbf{x}^*) &= (\lambda^*\mathbf{p}) \cdot (\mathbf{x}^1 - \mathbf{x}^*)\\ &= \lambda^*(\mathbf{p} \cdot \mathbf{x}^1 - \mathbf{p} \cdot \mathbf{x}^*)\\ &< \lambda^*(y - y)\\ &= 0,\end{aligned}$$

where the first equality follows from (P.1), and the second inequality follows from (P.2) and (P.4). However, because by (P.3) $u(\mathbf{x}^1) > u(\mathbf{x}^*)$, (P.5) contradicts the fact set forth at the beginning of the proof. ∎

With this sufficiency result in hand, it is enough to find a solution $(\mathbf{x}^*, \lambda^*) \gg \mathbf{0}$ to (1.10). Note that (1.10) is a system of $n + 1$ equations in the $n + 1$ unknowns $x_1^*, \ldots, x_n^*, \lambda^*$. These equations can typically be used to solve for the demand functions $x_i(\mathbf{p}, y)$, $i = 1, \ldots, n$, as we show in the following example.

EXAMPLE 1.1 The function, $u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{1/\rho}$, where $0 \neq \rho < 1$, is known as a **CES utility function**. You can easily verify that this utility function represents preferences that are strictly monotonic and strictly convex.

The consumer's problem is to find a non-negative consumption bundle solving

$$\max_{x_1, x_2} \left(x_1^\rho + x_2^\rho\right)^{1/\rho} \qquad \text{s.t.} \qquad p_1x_1 + p_2x_2 - y \leq 0. \tag{E.1}$$

To solve this problem, we first form the associated Lagrangian

$$\mathcal{L}(x_1, x_2, \lambda) \equiv \left(x_1^\rho + x_2^\rho\right)^{1/\rho} - \lambda(p_1x_1 + p_2x_2 - y).$$

Because preferences are monotonic, the budget constraint will hold with equality at the solution. Assuming an interior solution, the Kuhn-Tucker conditions coincide with the ordinary first-order Lagrangian conditions and the following equations must hold at the solution values x_1, x_2, and λ:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \left(x_1^\rho + x_2^\rho\right)^{(1/\rho)-1} x_1^{\rho-1} - \lambda p_1 = 0, \tag{E.2}$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = \left(x_1^\rho + x_2^\rho\right)^{(1/\rho)-1} x_2^{\rho-1} - \lambda p_2 = 0, \tag{E.3}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = p_1x_1 + p_2x_2 - y = 0. \tag{E.4}$$

Rearranging (E.2) and (E.3), then dividing the first by the second and rearranging some more, we can reduce these three equations in three unknowns to only two equations in the two unknowns of particular interest, x_1 and x_2:

$$x_1 = x_2 \left(\frac{p_1}{p_2}\right)^{1/(\rho-1)}, \tag{E.5}$$

$$y = p_1 x_1 + p_2 x_2. \tag{E.6}$$

First, substitute from (E.5) for x_1 in (E.6) to obtain the equation in x_2 alone:

$$\begin{aligned} y &= p_1 x_2 \left(\frac{p_1}{p_2}\right)^{1/(\rho-1)} + p_2 x_2 \\ &= x_2 \left(p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}\right) p_2^{-1/(\rho-1)}. \end{aligned} \tag{E.7}$$

Solving (E.7) for x_2 gives the solution value:

$$x_2 = \frac{p_2^{1/(\rho-1)} y}{p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}}. \tag{E.8}$$

To solve for x_1, substitute from (E.8) into (E.5) and obtain

$$x_1 = \frac{p_1^{1/(\rho-1)} y}{p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}}. \tag{E.9}$$

Equations (E.8) and (E.9), the solutions to the consumer's problem (E.1), are the consumer's Marshallian demand functions. If we define the parameter $r = \rho/(\rho - 1)$, we can simplify (E.8) and (E.9) and write the Marshallian demands as

$$x_1(\mathbf{p}, y) = \frac{p_1^{r-1} y}{p_1^r + p_2^r}, \tag{E.10}$$

$$x_2(\mathbf{p}, y) = \frac{p_2^{r-1} y}{p_1^r + p_2^r}. \tag{E.11}$$

Notice that the solutions to the consumer's problem depend only on its *parameters*, p_1, p_2, and y. Different prices and income, through (E.10) and (E.11), will give different quantities of each good demanded. To drive this point home, consider Fig. 1.12. There, at prices $p_1, \bar{p}_2$ and income $\bar{y}$, the solutions to the consumer's problem will be the quantities of x_1 and x_2 indicated. The pair $(p_1, x_1(p_1, \bar{p}_2, \bar{y}))$ will be a point on (one of) the consumer's demand curves for good x_1. □

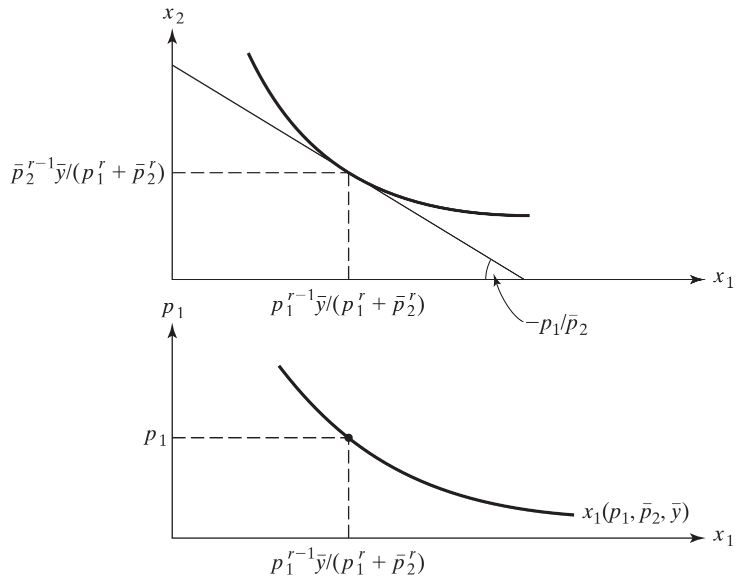


Figure 1.12. Consumer demand when preferences are represented by a CES utility function.

Finally, a word on the properties of the demand function $\mathbf{x}(\mathbf{p}, y)$ derived from the consumer's maximisation problem. We have made enough assumptions to ensure (by Theorem A2.21 (the theorem of the maximum)) that $\mathbf{x}(\mathbf{p}, y)$ will be continuous on $\mathbb{R}^n_{++}$. But we shall usually want more than this. We would like to be able to consider the slopes of demand curves and hence we would like $\mathbf{x}(\mathbf{p}, y)$ to be differentiable. From this point on, we shall simply assume that $\mathbf{x}(\mathbf{p}, y)$ is differentiable whenever we need it to be. But just to let you know what this involves, we state without proof the following result.

THEOREM 1.5 ***Differentiable Demand***

Let $\mathbf{x}^* \gg \mathbf{0}$ solve the consumer's maximisation problem at prices $\mathbf{p}^0 \gg \mathbf{0}$ and income $y^0 > 0$. If

- *u is twice continuously differentiable on $\mathbb{R}^n_{++}$,*
- *$\partial u(\mathbf{x}^*)/\partial x_i > 0$ for some $i = 1, \ldots, n$, and*
- *the bordered Hessian of u has a non-zero determinant at $\mathbf{x}^*$,*

then $\mathbf{x}(\mathbf{p}, y)$ is differentiable at $(\mathbf{p}^0, y^0)$.

1.4 INDIRECT UTILITY AND EXPENDITURE

1.4.1 THE INDIRECT UTILITY FUNCTION

The ordinary utility function, $u(\mathbf{x})$, is defined over the consumption set X and represents the consumer's preferences directly, as we have seen. It is therefore referred to as the **direct utility function**. Given prices $\mathbf{p}$ and income y, the consumer chooses a utility-maximising bundle $\mathbf{x}(\mathbf{p}, y)$. The level of utility achieved when $\mathbf{x}(\mathbf{p}, y)$ is chosen thus will be the highest level permitted by the consumer's budget constraint facing prices $\mathbf{p}$ and income y. Different prices or incomes, giving different budget constraints, will generally give rise to different choices by the consumer and so to different levels of maximised utility. The relationship among prices, income, and the maximised value of utility can be summarised by a real-valued function $v: \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R}$ defined as follows:

$$v(\mathbf{p}, y) = \max_{\mathbf{x}\in\mathbb{R}^n_+} u(\mathbf{x}) \qquad \text{s.t.} \qquad \mathbf{p}\cdot\mathbf{x} \leq y. \tag{1.12}$$

The function $v(\mathbf{p}, y)$ is called the **indirect utility function**. It is the maximum-value function corresponding to the consumer's utility maximisation problem. When $u(\mathbf{x})$ is continuous, $v(\mathbf{p}, y)$ is well-defined for all $\mathbf{p} \gg \mathbf{0}$ and $y \geq 0$ because a solution to the maximisation problem (1.12) is guaranteed to exist. If, in addition, $u(\mathbf{x})$ is strictly quasiconcave, then the solution is unique and we write it as $\mathbf{x}(\mathbf{p}, y)$, the consumer's demand function. The maximum level of utility that can be achieved when facing prices $\mathbf{p}$ and income y therefore will be that which is realised when $\mathbf{x}(\mathbf{p}, y)$ is chosen. Hence,

$$v(\mathbf{p}, y) = u(\mathbf{x}(\mathbf{p}, y)). \tag{1.13}$$

Geometrically, we can think of $v(\mathbf{p}, y)$ as giving the utility level of the highest indifference curve the consumer can reach, given prices $\mathbf{p}$ and income y, as illustrated in Fig. 1.13.

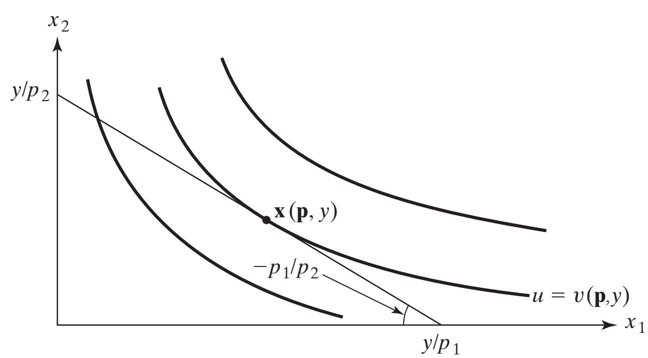


Figure 1.13. Indirect utility at prices $\mathbf{p}$ and income y.

There are several properties that the indirect utility function will possess. Continuity of the constraint function in $\mathbf{p}$ and y is sufficient to guarantee that $v(\mathbf{p}, y)$ will be continuous in $\mathbf{p}$ and y on $\mathbb{R}^n_{++} \times \mathbb{R}_+$. (See Section A2.4.) Effectively, the continuity of $v(\mathbf{p}, y)$ follows because at positive prices, 'small changes' in any of the parameters $(\mathbf{p}, y)$ fixing the location of the budget constraint will only lead to 'small changes' in the maximum level of utility the consumer can achieve. In the following theorem, we collect together a number of additional properties of $v(\mathbf{p}, y)$.

THEOREM 1.6 ***Properties of the Indirect Utility Function***

If $u(\mathbf{x})$ is continuous and strictly increasing on $\mathbb{R}^n_+$, then $v(\mathbf{p}, y)$ defined in (1.12) is

1. *Continuous on $\mathbb{R}^n_{++} \times \mathbb{R}_+$,*
2. *Homogeneous of degree zero in $(\mathbf{p}, y)$,*
3. *Strictly increasing in y,*
4. *Decreasing in $\mathbf{p}$,*
5. *Quasiconvex in $(\mathbf{p}, y)$.*

Moreover, it satisfies

6. *Roy's identity: If $v(\mathbf{p}, y)$ is differentiable at $(\mathbf{p}^0, y^0)$ and $\partial v(\mathbf{p}^0, y^0)/\partial y \neq 0$, then*

$$x_i(\mathbf{p}^0, y^0) = -\frac{\partial v(\mathbf{p}^0, y^0)/\partial p_i}{\partial v(\mathbf{p}^0, y^0)/\partial y}, \qquad i = 1, \ldots, n.$$

Proof: Property 1 follows from Theorem A2.21 (the theorem of the maximum). We shall not pursue the details.

The second property is easy to prove. We must show that $v(\mathbf{p}, y) = v(t\mathbf{p}, ty)$ for all $t > 0$. But $v(t\mathbf{p}, ty) = [\max u(\mathbf{x})$ s.t. $t\mathbf{p} \cdot \mathbf{x} \leq ty]$, which is clearly equivalent to $[\max u(\mathbf{x})$ s.t. $\mathbf{p} \cdot \mathbf{x} \leq y]$ because we may divide both sides of the constraint by $t > 0$ without affecting the set of bundles satisfying it. (See Fig. 1.14.) Consequently, $v(t\mathbf{p}, ty) = [\max u(\mathbf{x})$ s.t. $\mathbf{p} \cdot \mathbf{x} \leq y] = v(\mathbf{p}, y)$.

Intuitively, properties 3 and 4 simply say that any relaxation of the consumer's budget constraint can never cause the maximum level of achievable utility to decrease, whereas any tightening of the budget constraint can never cause that level to increase.

To prove 3 (and to practise Lagrangian methods), we shall make some additional assumptions although property 3 can be shown to hold without them. To keep things simple, we'll assume for the moment that the solution to (1.12) is strictly positive and differentiable, where $(\mathbf{p}, y) \gg \mathbf{0}$ and that $u(\cdot)$ is differentiable with $\partial u(\mathbf{x})/\partial x_i > 0$, for all $\mathbf{x} \gg \mathbf{0}$.

As we have remarked before, because $u(\cdot)$ is strictly increasing, the constraint in (1.12) must bind at the optimum. Consequently, (1.12) is equivalent to

$$v(\mathbf{p}, y) = \max_{\mathbf{x} \in \mathbb{R}^n_+} u(\mathbf{x}) \qquad \text{s.t.} \qquad \mathbf{p} \cdot \mathbf{x} = y. \tag{P.1}$$

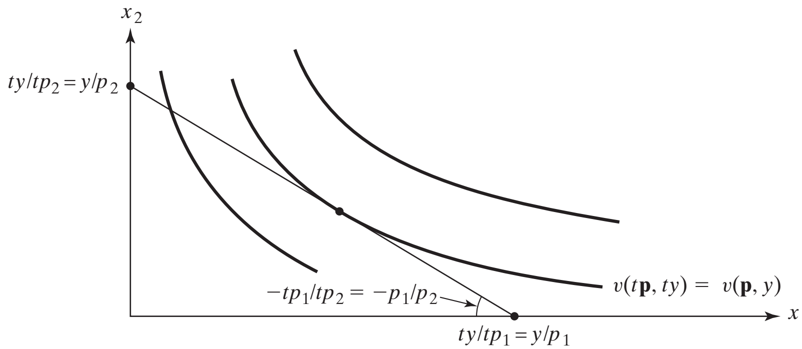


Figure 1.14. Homogeneity of the indirect utility function in prices and income.

The Lagrangian for (P.1) is

$$\mathcal{L}(\mathbf{x}, \lambda) = u(\mathbf{x}) - \lambda(\mathbf{p} \cdot \mathbf{x} - y). \tag{P.2}$$

Now, for $(\mathbf{p}, y) \gg \mathbf{0}$, let $\mathbf{x}^* = \mathbf{x}(\mathbf{p}, y)$ solve (P.1). By our additional assumption, $\mathbf{x}^* \gg \mathbf{0}$, so we may apply Lagrange's theorem to conclude that there is a $\lambda^* \in \mathbb{R}$ such that

$$\frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial x_i} = \frac{\partial u(\mathbf{x}^*)}{\partial x_i} - \lambda^* p_i = 0, \qquad i = 1, \ldots, n. \tag{P.3}$$

Note that because both p_i and $\partial u(\mathbf{x}^*)/\partial x_i$ are positive, so, too, is λ^*.

Our additional differentiability assumptions allow us to now apply Theorem A2.22, the Envelope theorem, to establish that $v(\mathbf{p}, y)$ is strictly increasing in y. According to the Envelope theorem, the partial derivative of the maximum value function $v(\mathbf{p}, y)$ with respect to y is equal to the partial derivative of the Lagrangian with respect to y evaluated at $(\mathbf{x}^*, \lambda^*)$,

$$\frac{\partial v(\mathbf{p}, y)}{\partial y} = \frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial y} = \lambda^* > 0. \tag{P.4}$$

Thus, $v(\mathbf{p}, y)$ is strictly increasing in $y > 0$. So, because v is continuous, it is then strictly increasing on $y \geq 0$.

For property 4, one can also employ the Envelope theorem. However, we shall give a more elementary proof that does not rely on any additional hypotheses. So consider $\mathbf{p}^0 \geq \mathbf{p}^1$ and let $\mathbf{x}^0$ solve (1.12) when $\mathbf{p} = \mathbf{p}^0$. Because $\mathbf{x}^0 \geq \mathbf{0}$, $(\mathbf{p}^0 - \mathbf{p}^1) \cdot \mathbf{x}^0 \geq 0$. Hence, $\mathbf{p}^1 \cdot \mathbf{x}^0 \leq \mathbf{p}^0 \cdot \mathbf{x}^0 \leq y$, so that $\mathbf{x}^0$ is feasible for (1.12) when $\mathbf{p} = \mathbf{p}^1$. We conclude that $v(\mathbf{p}^1, y) \geq u(\mathbf{x}^0) = v(\mathbf{p}^0, y)$, as desired.

Property 5 says that a consumer would prefer one of any two extreme budget sets to any average of the two. Our concern is to show that $v(\mathbf{p}, y)$ is quasiconvex in the vector of prices and income $(\mathbf{p}, y)$. The key to the proof is to concentrate on the budget sets.

Let B^1, B^2, and B^t be the budget sets available when prices and income are $(\mathbf{p}^1, y^1)$, $(\mathbf{p}^2, y^2)$, and $(\mathbf{p}^t, y^t)$, respectively, where $\mathbf{p}^t \equiv t\mathbf{p}^1 + (1-t)\mathbf{p}^2$ and $y^t \equiv y^1 + (1-t)y^2$. Then,

$$B^1 = \{\mathbf{x} \mid \mathbf{p}^1 \cdot \mathbf{x} \le y^1\},$$
$$B^2 = \{\mathbf{x} \mid \mathbf{p}^2 \cdot \mathbf{x} \le y^2\},$$
$$B^t = \{\mathbf{x} \mid \mathbf{p}^t \cdot \mathbf{x} \le y^t\}.$$

Suppose we could show that every choice the consumer can possibly make when he faces budget B^t is a choice that could have been made when he faced either budget B^1 or budget B^2. It then would be the case that every level of utility he can achieve facing B^t is a level he could have achieved either when facing B^1 or when facing B^2. Then, of course, the *maximum* level of utility that he can achieve over B^t could be no larger than *at least one* of the following: the maximum level of utility he can achieve over B^1, or the maximum level of utility he can achieve over B^2. But if this is the case, then the maximum level of utility achieved over B^t can be no greater than the *largest* of these two. If our supposition is correct, therefore, we would know that

$$v(\mathbf{p}^t, y^t) \le \max[v(\mathbf{p}^1, y^1), v(\mathbf{p}^2, y^2)] \qquad \forall\, t \in [0, 1].$$

This is equivalent to the statement that $v(\mathbf{p}, y)$ is quasiconvex in $(\mathbf{p}, y)$.

It will suffice, then, to show that our supposition on the budget sets is correct. We want to show that if $\mathbf{x} \in B^t$, then $\mathbf{x} \in B^1$ or $\mathbf{x} \in B^2$ for all $t \in [0, 1]$. If we choose either extreme value for t, B^t coincides with either B^1 or B^2, so the relations hold trivially. It remains to show that they hold for all $t \in (0, 1)$.

Suppose it were *not* true. Then we could find some $t \in (0, 1)$ and some $\mathbf{x} \in B^t$ such that $\mathbf{x} \notin B^1$ and $\mathbf{x} \notin B^2$. If $\mathbf{x} \notin B^1$ and $\mathbf{x} \notin B^2$, then

$$\mathbf{p}^1 \cdot \mathbf{x} > y^1$$

and

$$\mathbf{p}^2 \cdot \mathbf{x} > y^2,$$

respectively. Because $t \in (0, 1)$, we can multiply the first of these by t, the second by $(1-t)$, and preserve the inequalities to obtain

$$t\mathbf{p}^1 \cdot \mathbf{x} > ty^1$$

and

$$(1-t)\mathbf{p}^2 \cdot \mathbf{x} > (1-t)y^2.$$

Adding, we obtain

$$(t\mathbf{p}^1 + (1-t)\mathbf{p}^2) \cdot \mathbf{x} > ty^1 + (1-t)y^2$$

or

$$\mathbf{p}^t \cdot \mathbf{x} > y^t.$$

But this final line says that $\mathbf{x} \notin B^t$, contradicting our original assumption. We must conclude, therefore, that if $\mathbf{x} \in B^t$, then $\mathbf{x} \in B^1$ or $\mathbf{x} \in B^2$ for all $t \in [0, 1]$. By our previous argument, we can conclude that $v(\mathbf{p}, y)$ is quasiconvex in $(\mathbf{p}, y)$.

Finally, we turn to property 6, **Roy's identity**. This says that the consumer's Marshallian demand for good i is simply the ratio of the partial derivatives of indirect utility with respect to p_i and y *after a sign change*. (Note the minus sign in 6.)

We shall again invoke the additional assumptions introduced earlier in the proof because we shall again employ the Envelope theorem. (See Exercise 1.35 for a proof that does not require these additional assumptions.) Letting $\mathbf{x}^* = \mathbf{x}(\mathbf{p}, y)$ be the strictly positive solution to (1.12), as argued earlier, there must exist λ^* satisfying (P.3). Applying the Envelope theorem to evaluate $\partial v(\mathbf{p}, y)/\partial p_i$ gives

$$\frac{\partial v(\mathbf{p}, y)}{\partial p_i} = \frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial p_i} = -\lambda^* x_i^*. \tag{P.5}$$

However, according to (P.4), $\lambda^* = \partial v(\mathbf{p}, y)/\partial y > 0$. Hence, (P.5) becomes

$$-\frac{\partial v(\mathbf{p}, y)/\partial p_i}{\partial v(\mathbf{p}, y)/\partial y} = x_i^* = x_i(\mathbf{p}, y),$$

as desired. ∎

EXAMPLE 1.2 In Example 1.1, the direct utility function is the CES form, $u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{1/\rho}$, where $0 \neq \rho < 1$. There we found the Marshallian demands:

$$\begin{aligned} x_1(\mathbf{p}, y) &= \frac{p_1^{r-1} y}{p_1^r + p_2^r}, \\ x_2(\mathbf{p}, y) &= \frac{p_2^{r-1} y}{p_1^r + p_2^r}, \end{aligned} \tag{E.1}$$

for $r \equiv \rho/(\rho - 1)$. By (1.13), we can form the indirect utility function by substituting these back into the direct utility function. Doing that and rearranging, we obtain

$$\begin{aligned} v(\mathbf{p}, y) &= [(x_1(\mathbf{p}, y))^\rho + (x_2(\mathbf{p}, y))^\rho]^{1/\rho} \\ &= \left[\left(\frac{p_1^{r-1} y}{p_1^r + p_2^r} \right)^\rho + \left(\frac{p_2^{r-1} y}{p_1^r + p_2^r} \right)^\rho \right]^{1/\rho} \end{aligned} \tag{E.2}$$

$$= y\left[\frac{p_1^r + p_2^r}{\left(p_1^r + p_2^r\right)^{\rho}}\right]^{1/\rho}$$
$$= y\left(p_1^r + p_2^r\right)^{-1/r}.$$

We should verify that (E.2) satisfies all the properties of an indirect utility function detailed in Theorem 1.6. It is easy to see that $v(\mathbf{p}, y)$ is homogeneous of degree zero in prices and income, because for any $t > 0$,

$$\begin{aligned} v(t\mathbf{p}, ty) &= ty((tp_1)^r + (tp_2)^r)^{-1/r} \\ &= ty\left(t^r p_1^r + t^r p_2^r\right)^{-1/r} \\ &= tyt^{-1}\left(p_1^r + p_2^r\right)^{-1/r} \\ &= y\left(p_1^r + p_2^r\right)^{-1/r} \\ &= v(\mathbf{p}, y). \end{aligned}$$

To see that it is increasing in y and decreasing in $\mathbf{p}$, differentiate (E.2) with respect to income and any price to obtain

$$\frac{\partial v(\mathbf{p}, y)}{\partial y} = \left(p_1^r + p_2^r\right)^{-1/r} > 0, \tag{E.3}$$
$$\frac{\partial v(\mathbf{p}, y)}{\partial p_i} = -\left(p_1^r + p_2^r\right)^{(-1/r)-1} y p_i^{r-1} < 0, \qquad i = 1, 2. \tag{E.4}$$

To verify Roy's identity, form the required ratio of (E.4) to (E.3) and recall (E.1) to obtain

$$\begin{aligned} (-1)\left[\frac{\partial v(\mathbf{p}, y)/\partial p_i}{\partial v(\mathbf{p}, y)/\partial y}\right] &= (-1)\frac{-\left(p_1^r + p_2^r\right)^{(-1/r)-1} y p_i^{r-1}}{\left(p_1^r + p_2^r\right)^{-1/r}} \\ &= \frac{y p_i^{r-1}}{p_1^r + p_2^r} = x_i(\mathbf{p}, y), \qquad i = 1, 2. \end{aligned}$$

We leave as an exercise the task of verifying that (E.2) is a quasiconvex function of $(\mathbf{p}, y)$. □

1.4.2 THE EXPENDITURE FUNCTION

The indirect utility function is a neat and powerful way to summarise a great deal about the consumer's market behaviour. A companion measure, called the **expenditure function**, is equally useful. To construct the indirect utility function, we fixed market prices and

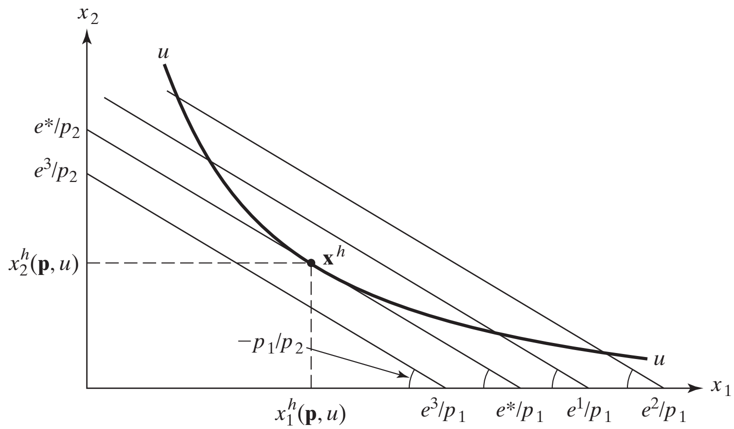


Figure 1.15. Finding the lowest level of expenditure to achieve utility level u.

income, and sought the maximum level of utility the consumer could achieve. To construct the expenditure function, we again fix prices, but we ask a different sort of question about the level of utility the consumer achieves. Specifically, we ask: what is the *minimum level of money expenditure* the consumer must make facing a given set of prices to achieve a given level of utility? In this construction, we ignore any limitations imposed by the consumer's income and simply ask what the consumer would have to spend to achieve some particular level of utility.

To better understand the type of problem we are studying, consider Fig. 1.15 and contrast it with Fig. 1.13. Each of the parallel straight lines in Fig. 1.15 depicts all bundles $\mathbf{x}$ that require the same level of total expenditure to acquire when facing prices $\mathbf{p} = (p_1, p_2)$. Each of these **isoexpenditure** curves is defined implicity by $e = p_1x_1 + p_2x_2$, for a different level of total expenditure $e > 0$. Each therefore will have the same slope, $-p_1/p_2$, but different horizontal and vertical intercepts, e/p_1 and e/p_2, respectively. Isoexpenditure curves farther out contain bundles costing more; those farther in give bundles costing less. If we fix the level of utility at u, then the indifference curve $u(\mathbf{x}) = u$ gives all bundles yielding the consumer that same level of utility.

There is no point in common with the isoexpenditure curve e^3 and the indifference curve u, indicating that e^3 dollars is insufficient at these prices to achieve utility u. However, each of the curves e^1, e^2, and e^* has at least one point in common with u, indicating that any of these levels of total expenditure is sufficient for the consumer to achieve utility u. In constructing the expenditure function, however, we seek the *minimum expenditure* the consumer requires to achieve utility u, or the lowest possible isoexpenditure curve that still has at least one point in common with indifference curve u. Clearly, that will be level e^*, and the least cost bundle that achieves utility u at prices $\mathbf{p}$ will be the bundle $\mathbf{x}^h = (x_1^h(\mathbf{p}, u), x_2^h(\mathbf{p}, u))$. If we denote the minimum expenditure necessary to achieve utility u at prices $\mathbf{p}$ by $e(\mathbf{p}, u)$, that level of expenditure will simply be equal to the cost of bundle $\mathbf{x}^h$, or $e(\mathbf{p}, u) = p_1x_1^h(\mathbf{p}, u) + p_2x_2^h(\mathbf{p}, u) = e^*$.

More generally, we define the **expenditure function** as the minimum-value function,

$$e(\mathbf{p}, u) \equiv \min_{\mathbf{x} \in \mathbb{R}^n_+} \mathbf{p} \cdot \mathbf{x} \qquad \text{s.t.} \qquad u(\mathbf{x}) \geq u \tag{1.14}$$

for all $\mathbf{p} \gg \mathbf{0}$ and all attainable utility levels u. For future reference, let $\mathcal{U} = \{u(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^n_+\}$ denote the set of attainable utility levels. Thus, the domain of $e(\cdot)$ is $\mathbb{R}^n_{++} \times \mathcal{U}$.

Note that $e(\mathbf{p}, u)$ is well-defined because for $\mathbf{p} \in \mathbb{R}^n_{++}$, $\mathbf{x} \in \mathbb{R}^n_+$, $\mathbf{p} \cdot \mathbf{x} \geq 0$. Hence, the set of numbers $\{e | e = \mathbf{p} \cdot \mathbf{x}$ for some $\mathbf{x}$ with $u(\mathbf{x}) \geq u\}$ is bounded below by zero. Moreover because $\mathbf{p} \gg \mathbf{0}$, this set can be shown to be closed. Hence, it contains a smallest number. The value $e(\mathbf{p}, u)$ is precisely this smallest number. Note that any solution vector for this minimisation problem will be non-negative and will depend on the parameters $\mathbf{p}$ and u. Notice also that if $u(\mathbf{x})$ is continuous and strictly quasiconcave, the solution will be unique, so we can denote the solution as the function $\mathbf{x}^h(\mathbf{p}, u) \geq \mathbf{0}$. As we have seen, if $\mathbf{x}^h(\mathbf{p}, u)$ solves this problem, the lowest expenditure necessary to achieve utility u at prices $\mathbf{p}$ will be exactly equal to the cost of the bundle $\mathbf{x}^h(\mathbf{p}, u)$, or

$$e(\mathbf{p}, u) = \mathbf{p} \cdot \mathbf{x}^h(\mathbf{p}, u). \tag{1.15}$$

We have seen how the consumer's utility maximisation problem is intimately related to his observable market demand behaviour. Indeed, the very solutions to that problem – the Marshallian demand functions – tell us just how much of every good we should observe the consumer buying when he faces different prices and income. We shall now interpret the solution, $\mathbf{x}^h(\mathbf{p}, u)$, of the expenditure-minimisation problem as another kind of 'demand function' – but one that is not directly observable.

Consider the following mental experiment. If we fix the level of utility the consumer is permitted to achieve at some arbitrary level u, how will his purchases of each good behave as we change the prices he faces? The kind of 'demand functions' we are imagining here are thus *utility-constant* ones. We completely ignore the level of the consumer's money income and the utility levels he actually *can* achieve. In fact, we know that when a consumer has some level of income and we change the prices he faces, there will ordinarily be some change in his purchases and some corresponding change in the level of utility he achieves. To imagine how we might then construct our hypothetical demand functions, we must imagine a process by which whenever we lower some price, and so confer a utility gain on the consumer, we compensate by reducing the consumer's income, thus conferring a corresponding utility loss sufficient to bring the consumer back to the original level of utility. Similarly, whenever we increase some price, causing a utility loss, we must imagine compensating for this by increasing the consumer's income sufficiently to give a utility gain equal to the loss. Because they reflect the net effect of this process by which we match any utility change due to a change in prices by a compensating utility change from a hypothetical adjustment in income, the hypothetical demand functions we are describing are often called **compensated demand functions**. However, because John Hicks (1939) was the first to write about them in quite this way, these hypothetical demand functions are most commonly known as **Hicksian demand functions**. As we illustrate below, the

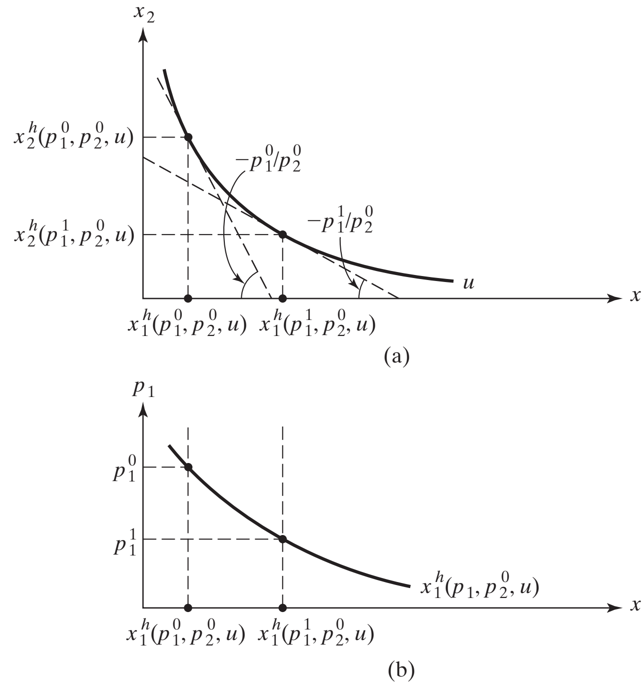


Figure 1.16. The Hicksian demand for good 1.

solution, $\mathbf{x}^h(\mathbf{p}, u)$, to the expenditure-minimisation problem is precisely the consumer's vector of Hicksian demands.

To get a clearer idea of what we have in mind, consider Fig. 1.16. If we wish to fix the level of utility the consumer can achieve at u in Fig. 1.16(a) and then confront him with prices p_1^0 and p_2^0, he must face the depicted budget constraint with slope $-p_1^0/p_2^0$. Note that his utility-maximising choices then coincide with the expenditure-minimising quantities $x_1^h(p_1^0, p_2^0, u)$ and $x_2^h(p_1^0, p_2^0, u)$. If we reduce the price of good 1 to $p_1^1 < p_1^0$, yet hold the consumer on the u-level indifference curve by an appropriate income reduction, his new budget line now has slope $-p_1^1/p_2^0$, and his utility-maximising choices change to $x_1^h(p_1^1, p_2^0, u)$ and $x_2^h(p_1^1, p_2^0, u)$. As before, if we fix price p_2^0 and we plot the own-price of good 1 in Fig. 1.16(b) against the corresponding hypothetical quantities of good 1 the consumer would 'buy' if constrained to utility level u, we would trace out a 'demand-curve-like' locus as depicted. This construction is the Hicksian demand curve for good 1, given utility level u. Clearly, there will be *different* Hicksian demand curves for different levels of utility – for different indifference curves. The shape and position of each of them, however, will always be determined by the underlying preferences.

In short, the solution, $\mathbf{x}^h(\mathbf{p}, u)$, to the expenditure-minimisation problem is precisely the vector of Hicksian demands because each of the hypothetical 'budget constraints' the consumer faces in Fig. 1.16 involves a level of expenditure exactly equal to the minimum level necessary at the given prices to achieve the utility level in question.

Thus, the expenditure function defined in (1.14) contains within it some important information on the consumer's Hicksian demands. Although the analytic importance of this construction will only become evident a bit later, we can take note here of the remarkable ease with which that information can be extracted from a knowledge of the expenditure function. The consumer's Hicksian demands can be extracted from the expenditure function by means of simple differentiation. We detail this and other important properties of the expenditure function in the following theorem.

THEOREM 1.7 ***Properties of the Expenditure Function***

If $u(\cdot)$ is continuous and strictly increasing, then $e(\mathbf{p}, u)$ defined in (1.14) is

1. *Zero when u takes on the lowest level of utility in $\mathcal{U}$,*
2. *Continuous on its domain $\mathbb{R}^n_{++} \times \mathcal{U}$,*
3. *For all $\mathbf{p} \gg \mathbf{0}$, strictly increasing and unbounded above in u,*
4. *Increasing in $\mathbf{p}$,*
5. *Homogeneous of degree 1 in $\mathbf{p}$,*
6. *Concave in $\mathbf{p}$.*

If, in addition, $u(\cdot)$ is strictly quasiconcave, we have

7. *Shephard's lemma: $e(\mathbf{p}, u)$ is differentiable in $\mathbf{p}$ at $(\mathbf{p}^0, u^0)$ with $\mathbf{p}^0 \gg \mathbf{0}$, and*

$$\frac{\partial e(\mathbf{p}^0, u^0)}{\partial p_i} = x_i^h(\mathbf{p}^0, u^0), \qquad i = 1, \ldots, n.$$

Proof: To prove property 1, note that the lowest value in $\mathcal{U}$ is $u(\mathbf{0})$ because $u(\cdot)$ is strictly increasing on $\mathbb{R}^n_+$. Consequently, $e(\mathbf{p}, u(\mathbf{0})) = 0$ because $\mathbf{x} = \mathbf{0}$ attains utility $u(\mathbf{0})$ and requires an expenditure of $\mathbf{p} \cdot \mathbf{0} = 0$.

Property 2, continuity, follows once again from Theorem A2.21 (the theorem of the maximum).

Although property 3 holds without any further assumptions, we shall be content to demonstrate it under the additional hypotheses that $\mathbf{x}^h(\mathbf{p}, u) \gg \mathbf{0}$ is differentiable $\forall\, \mathbf{p} \gg \mathbf{0}$, $u > u(\mathbf{0})$, and that $u(\cdot)$ is differentiable with $\partial u(\mathbf{x})/\partial x_i > 0$, $\forall\, i$ on $\mathbb{R}^n_{++}$.

Now, because $u(\cdot)$ is continuous and strictly increasing, and $\mathbf{p} \gg \mathbf{0}$, the constraint in (1.14) must be binding. For if $u(\mathbf{x}^1) > u$, there is a $t \in (0, 1)$ close enough to 1 such that $u(t\mathbf{x}^1) > u$. Moreover, $u \geq u(\mathbf{0})$ implies $u(\mathbf{x}^1) > u(\mathbf{0})$, so that $\mathbf{x}^1 \neq \mathbf{0}$. Therefore, $\mathbf{p} \cdot (t\mathbf{x}^1) < \mathbf{p} \cdot \mathbf{x}^1$, because $\mathbf{p} \cdot \mathbf{x}^1 > 0$. Consequently, when the constraint is not binding, there is a strictly cheaper bundle that also satisfies the constraint. Hence, at the optimum, the constraint must bind. Consequently, we may write (1.14) instead as

$$e(\mathbf{p}, u) \equiv \min_{\mathbf{x} \in \mathbb{R}^n_+} \mathbf{p} \cdot \mathbf{x} \qquad \text{s.t.} \qquad u(\mathbf{x}) = u. \tag{P.1}$$

The Lagrangian for this problem is

$$\mathcal{L}(\mathbf{x}, \lambda) = \mathbf{p} \cdot \mathbf{x} - \lambda[u(\mathbf{x}) - u]. \tag{P.2}$$

Now for $\mathbf{p} \gg \mathbf{0}$ and $u > u(\mathbf{0})$, we have that $\mathbf{x}^* = \mathbf{x}^h(\mathbf{p}, u) \gg \mathbf{0}$ solves (P.1). So, by Lagrange's theorem, there is a λ^* such that

$$\frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial x_i} = p_i - \lambda^* \frac{\partial u(\mathbf{x}^*)}{\partial x_i} = 0, \qquad i = 1, \ldots, n. \tag{P.3}$$

Note then that because p_i and $\partial u(\mathbf{x}^*)/\partial x_i$ are positive, so, too, is λ^*. Under our additional hypotheses, we can now use the Envelope theorem to show that $e(\mathbf{p}, u)$ is strictly increasing in u.

By the Envelope theorem, the partial derivative of the minimum-value function $e(\mathbf{p}, u)$ with respect to u is equal to the partial derivative of the Lagrangian with respect to u, evaluated at $(\mathbf{x}^*, \lambda^*)$. Hence,

$$\frac{\partial e(\mathbf{p}, u)}{\partial u} = \frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial u} = \lambda^* > 0.$$

Because this holds for all $u > u(\mathbf{0})$, and because $e(\cdot)$ is continuous, we may conclude that for all $\mathbf{p} \gg \mathbf{0}$, $e(\mathbf{p}, u)$ is strictly increasing in u on $\mathcal{U}$ (which includes $u(\mathbf{0})$).

That e is unbounded in u can be shown to follow from the fact that $u(\mathbf{x})$ is continuous and strictly increasing. You are asked to do so in Exercise 1.34.

Because property 4 follows from property 7, we shall defer it for the moment. Property 5 will be left as an exercise.

For property 6, we must prove that $e(\mathbf{p}, u)$ is a concave function of prices. We begin by recalling the definition of concavity. Let $\mathbf{p}^1$ and $\mathbf{p}^2$ be any two positive price vectors, let $t \in [0, 1]$, and let $\mathbf{p}^t = t\mathbf{p}^1 + (1 - t)\mathbf{p}^2$ be any convex combination of $\mathbf{p}^1$ and $\mathbf{p}^2$. Then the expenditure function will be concave in prices if

$$te(\mathbf{p}^1, u) + (1 - t)e(\mathbf{p}^2, u) \leq e(\mathbf{p}^t, u). \tag{P.4}$$

To see that this is indeed the case, simply focus on what it *means* for expenditure to be minimised at given prices. Suppose in particular that $\mathbf{x}^1$ minimises expenditure to achieve u when prices are $\mathbf{p}^1$, that $\mathbf{x}^2$ minimises expenditure to achieve u when prices are $\mathbf{p}^2$, and that $\mathbf{x}^*$ minimises expenditure to achieve u when prices are $\mathbf{p}^t$. Then the cost of $\mathbf{x}^1$ at prices $\mathbf{p}^1$ must be no more than the cost at prices $\mathbf{p}^1$ of *any other* bundle $\mathbf{x}$ that achieves utility u. Similarly, the cost of $\mathbf{x}^2$ at prices $\mathbf{p}^2$ must be no more than the cost at $\mathbf{p}^2$ of *any other* bundle $\mathbf{x}$ that achieves utility u. Now, if, as we have said,

$$\mathbf{p}^1 \cdot \mathbf{x}^1 \leq \mathbf{p}^1 \cdot \mathbf{x}$$

and

$$\mathbf{p}^2 \cdot \mathbf{x}^2 \leq \mathbf{p}^2 \cdot \mathbf{x}$$

for all $\mathbf{x}$ that achieve u, then these relations must *also* hold for $\mathbf{x}^*$, because $\mathbf{x}^*$ achieves u as well. Therefore, simply by virtue of what it means to minimise expenditure to achieve u at given prices, we know that

$$\mathbf{p}^1 \cdot \mathbf{x}^1 \leq \mathbf{p}^1 \cdot \mathbf{x}^*$$

and

$$\mathbf{p}^2 \cdot \mathbf{x}^2 \leq \mathbf{p}^2 \cdot \mathbf{x}^*.$$

But now we are home free. Because $t \geq 0$ and $(1-t) \geq 0$, we can multiply the first of these by t, the second by $(1-t)$, and add them. If we then substitute from the definition of $\mathbf{p}^t$, we obtain

$$t\mathbf{p}^1 \cdot \mathbf{x}^1 + (1-t)\mathbf{p}^2 \cdot \mathbf{x}^2 \leq \mathbf{p}^t \cdot \mathbf{x}^*.$$

The left-hand side is just the convex combination of the minimum levels of expenditure necessary at prices $\mathbf{p}^1$ and $\mathbf{p}^2$ to achieve utility u, and the right-hand side is the minimum expenditure needed to achieve utility u at the convex combination of those prices. In short, this is just the same as (P.5), and tells us that

$$te(\mathbf{p}^1, u) + (1-t)e(\mathbf{p}^2, u) \leq e(\mathbf{p}^t, u) \qquad \forall\, t \in [0, 1],$$

as we intended to show.

To prove property 7, we again appeal to the Envelope theorem but now differentiate with respect to p_i. This gives

$$\frac{\partial e(\mathbf{p}, u)}{\partial p_i} = \frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial p_i} = x_i^* \equiv x_i^h(\mathbf{p}, u),$$

as required. Because $\mathbf{x}^h(\mathbf{p}, u) \geq \mathbf{0}$, this also proves property 4. (See Exercise 1.37 for a proof of 7 that does not require any additional assumptions. Try to prove property 4 without additional assumptions as well.) ∎

EXAMPLE 1.3 Suppose the direct utility function is again the CES form, $u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{1/\rho}$, where $0 \neq \rho < 1$. We want to derive the corresponding expenditure function in this case. Because preferences are monotonic, we can formulate the expenditure minimisation problem (1.15)

$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2 \qquad \text{s.t.} \qquad \left(x_1^\rho + x_2^\rho\right)^{1/\rho} - u = 0, \qquad x_1 \geq 0,\ x_2 \geq 0,$$

and its Lagrangian,

$$\mathcal{L}(x_1, x_2, \lambda) = p_1x_1 + p_2x_2 - \lambda\big[\big(x_1^\rho + x_2^\rho\big)^{1/\rho} - u\big]. \tag{E.1}$$

Assuming an interior solution in both goods, the first-order conditions for a minimum subject to the constraint ensure that the solution values x_1, x_2, and λ satisfy the equations

$$\frac{\partial \mathcal{L}}{\partial x_1} = p_1 - \lambda\big(x_1^\rho + x_2^\rho\big)^{(1/\rho)-1}x_1^{\rho-1} = 0, \tag{E.2}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = p_2 - \lambda\big(x_1^\rho + x_2^\rho\big)^{(1/\rho)-1}x_2^{\rho-1} = 0, \tag{E.3}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \big(x_1^\rho + x_2^\rho\big)^{1/\rho} - u = 0. \tag{E.4}$$

By eliminating λ, these can be reduced to the two equations in two unknowns,

$$x_1 = x_2\left(\frac{p_1}{p_2}\right)^{1/(\rho-1)}, \tag{E.5}$$

$$u = \big(x_1^\rho + x_2^\rho\big)^{1/\rho}. \tag{E.6}$$

Substituting from (E.5) into (E.6) gives

$$u = \left[x_2^\rho\left(\frac{p_1}{p_2}\right)^{\rho/(\rho-1)} + x_2^\rho\right]^{1/\rho} = x_2\left[\left(\frac{p_1}{p_2}\right)^{\rho/(\rho-1)} \times 1\right]^{1/\rho}.$$

Solving for x_2, and letting $r \equiv \rho/(\rho - 1)$, we obtain

$$\begin{aligned} x_2 &= u\left[\left(\frac{p_1}{p_2}\right)^{\rho/(\rho-1)} + 1\right]^{-1/\rho} = u\big[p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}\big]^{-1/\rho}p_2^{1/(\rho-1)} \\ &= u\big(p_1^r + p_2^r\big)^{(1/r)-1}p_2^{r-1}. \end{aligned} \tag{E.7}$$

Substituting from (E.7) into (E.5) gives us

$$\begin{aligned} x_1 &= up_1^{1/(\rho-1)}p_2^{-1/(\rho-1)}\big(p_1^r + p_2^r\big)^{(1/r)-1}p_2^{r-1} \\ &= u\big(p_1^r + p_2^r\big)^{(1/r)-1}p_1^{r-1}. \end{aligned} \tag{E.8}$$

The solutions (E.7) and (E.8) depend on the parameters of the minimisation problem, $\mathbf{p}$ and u. These are the Hicksian demands, so we can denote (E.7) and (E.8)

$$x_1^h(\mathbf{p}, u) = u\left(p_1^r + p_2^r\right)^{(1/r)-1} p_1^{r-1}, \tag{E.9}$$

$$x_2^h(\mathbf{p}, u) = u\left(p_1^r + p_2^r\right)^{(1/r)-1} p_2^{r-1}. \tag{E.10}$$

To form the expenditure function, we invoke equation (1.15) and substitute from (E.9) and (E.10) into the objective function in (E.1) to obtain

$$\begin{aligned} e(\mathbf{p}, u) &= p_1 x_1^h(\mathbf{p}, u) + p_2 x_2^h(\mathbf{p}, u) \\ &= u p_1\left(p_1^r + p_2^r\right)^{(1/r)-1} p_1^{r-1} + u p_2\left(p_1^r + p_2^r\right)^{(1/r)-1} p_2^{r-1} \\ &= u\left(p_1^r + p_2^r\right)\left(p_1^r + p_2^r\right)^{(1/r)-1} \\ &= u\left(p_1^r + p_2^r\right)^{1/r}. \end{aligned} \tag{E.11}$$

Equation (E.11) is the expenditure function we sought. We leave as an exercise the task of verifying that it possesses the usual properties. □

1.4.3 RELATIONS BETWEEN THE TWO

Though the indirect utility function and the expenditure function are conceptually distinct, there is obviously a close relationship between them. The same can be said for the Marshallian and Hicksian demand functions.

In particular, fix $(\mathbf{p}, y)$ and let $u = v(\mathbf{p}, y)$. By the definition of v, this says that at prices $\mathbf{p}$, utility level u is the maximum that can be attained when the consumer's income is y. Consequently, at prices $\mathbf{p}$, if the consumer wished to attain a level of utility at least u, then income y would be certainly large enough to achieve this. But recall now that $e(\mathbf{p}, u)$ is the *smallest* expenditure needed to attain a level of utility at least u. Hence, we must have $e(\mathbf{p}, u) \leq y$. Consequently, the definitions of v and e lead to the following inequality:

$$e(\mathbf{p}, v(\mathbf{p}, y)) \leq y, \qquad \forall\ (\mathbf{p}, y) \gg \mathbf{0}. \tag{1.16}$$

Next, fix $(\mathbf{p}, u)$ and let $y = e(\mathbf{p}, u)$. By the definition of e, this says that at prices $\mathbf{p}$, income y is the smallest income that allows the consumer to attain at least the level of utility u. Consequently, at prices $\mathbf{p}$, if the consumer's income were in fact y, then he could attain at least the level of utility u. Because $v(\mathbf{p}, y)$ is the *largest* utility level attainable at prices $\mathbf{p}$ and with income y, this implies that $v(\mathbf{p}, y) \geq u$. Consequently, the definitions of v and e also imply that

$$v(\mathbf{p}, e(\mathbf{p}, u)) \geq u \qquad \forall\ (\mathbf{p}, u) \in \mathbb{R}^n_{++} \times \mathcal{U}. \tag{1.17}$$

The next theorem demonstrates that under certain familiar conditions on preferences, both of these inequalities, in fact, must be equalities.

THEOREM 1.8 ***Relations Between Indirect Utility and Expenditure Functions***

Let $v(\mathbf{p}, y)$ and $e(\mathbf{p}, u)$ be the indirect utility function and expenditure function for some consumer whose utility function is continuous and strictly increasing. Then for all $\mathbf{p} \gg \mathbf{0}$, $y \geq 0$, and $u \in \mathcal{U}$:

1. $e(\mathbf{p}, v(\mathbf{p}, y)) = y$.
2. $v(\mathbf{p}, e(\mathbf{p}, u)) = u$.

Proof: Because $u(\cdot)$ is strictly increasing on $\mathbb{R}^n_+$, it attains a minimum at $\mathbf{x} = \mathbf{0}$, but does not attain a maximum. Moreover, because $u(\cdot)$ is continuous, the set $\mathcal{U}$ of attainable utility numbers must be an interval. Consequently, $\mathcal{U} = [u(\mathbf{0}), \bar{u})]$ for $\bar{u} > u(\mathbf{0})$, and where $\bar{u}$ may be either finite or $+\infty$.

To prove 1, fix $(\mathbf{p}, y) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$. By (1.16), $e(\mathbf{p}, v(\mathbf{p}, y)) \leq y$. We would like to show in fact that equality must hold. So suppose not, that is, suppose $e(\mathbf{p}, u) < y$, where $u = v(\mathbf{p}, y)$. Note that by definition of $v(\cdot)$, $u \in \mathcal{U}$, so that $u < \bar{u}$. By the continuity of $e(\cdot)$ from Theorem 1.7, we may therefore choose $\varepsilon > 0$ small enough so that $u + \varepsilon < \bar{u}$, and $e(\mathbf{p}, u + \varepsilon) < y$. Letting $y_\varepsilon = e(\mathbf{p}, u + \varepsilon)$, (1.17) implies that $v(\mathbf{p}, y_\varepsilon) \geq u + \varepsilon$. Because $y_\varepsilon < y$ and v is strictly increasing in income by Theorem 1.6, $v(\mathbf{p}, y) > v(\mathbf{p}, y_\varepsilon) \geq u + \varepsilon$. But $u = v(\mathbf{p}, y)$ so this says $u \geq u + \varepsilon$, a contradiction. Hence, $e(\mathbf{p}, v(\mathbf{p}, y)) = y$.

To prove 2, fix $(\mathbf{p}, u) \in \mathbb{R}^n_{++} \times [u(\mathbf{0}), \bar{u}]$. By (1.17), $v(\mathbf{p}, e(\mathbf{p}, u)) \geq u$. Again, to show that this must be an equality, suppose to the contrary that $v(\mathbf{p}, e(\mathbf{p}, u)) > u$. There are two cases to consider: $u = u(\mathbf{0})$ and $u > u(\mathbf{0})$. We shall consider the second case only, leaving the first as an exercise. Letting $y = e(\mathbf{p}, u)$, we then have $v(\mathbf{p}, y) > u$. Now, because $e(\mathbf{p}, u(\mathbf{0})) = 0$ and because $e(\cdot)$ is strictly increasing in utility by Theorem 1.7, $y = e(\mathbf{p}, u) > 0$. Because $v(\cdot)$ is continuous by Theorem 1.6, we may choose $\varepsilon > 0$ small enough so that $y - \varepsilon > 0$ and $v(\mathbf{p}, y - \varepsilon) > u$. Thus, income $y - \varepsilon$ is sufficient, at prices $\mathbf{p}$, to achieve utility greater than u. Hence, we must have $e(\mathbf{p}, u) \leq y - \varepsilon$. But this contradicts the fact that $y = e(\mathbf{p}, u)$. ∎

Until now, if we wanted to derive a consumer's indirect utility and expenditure functions, we would have had to solve two separate constrained optimisation problems: one a maximisation problem and the other a minimisation problem. This theorem, however, points to an easy way to derive either one from knowledge of the other, thus requiring us to solve only one optimisation problem and giving us the choice of which one we care to solve.

To see how this would work, let us suppose first that we have solved the utility-maximisation problem and formed the indirect utility function. One thing we know about the indirect utility function is that it is *strictly increasing* in its income variable. But then, holding prices constant and viewing it only as a function of income, it must be possible to *invert* the indirect utility function in its income variable. From before,

$$v(\mathbf{p}, e(\mathbf{p}, u)) = u,$$

so we can apply that inverse function (call it $v^{-1}(\mathbf{p} : t)$) to both sides of this and obtain

$$e(\mathbf{p}, u) = v^{-1}(\mathbf{p} : u). \tag{1.18}$$

Whatever that expression on the right-hand side of (1.18) turns out to be, we know it will correspond exactly to the expression for the consumer's expenditure function – the expression we would eventually obtain if we solved the expenditure-minimisation problem, then substituted back into the objective function.

Suppose, instead, that we had chosen to solve the expenditure-minimisation problem and form the expenditure function, $e(\mathbf{p}, u)$. In this case, we know that $e(\mathbf{p}, u)$ is *strictly increasing* in u. Again supposing prices constant, there will be an inverse of the expenditure function in its utility variable, which we can denote $e^{-1}(\mathbf{p} : t)$. Applying this inverse to both sides of the first item in Theorem 1.8, we find that the indirect utility function can be solved for directly and will be that expression in $\mathbf{p}$ and y that results when we evaluate the utility inverse of the expenditure function at any level of income y,

$$v(\mathbf{p}, y) = e^{-1}(\mathbf{p} : y). \tag{1.19}$$

Equations (1.18) and (1.19) illustrate again the close relationship between utility maximisation and expenditure minimisation. The two are conceptually just opposite sides of the same coin. Mathematically, both the indirect utility function and the expenditure function are simply the appropriately chosen *inverses* of each other.

EXAMPLE 1.4 We can illustrate these procedures by drawing on findings from the previous examples. In Example 1.2, we found that the CES direct utility function gives the indirect utility function,

$$v(\mathbf{p}, y) = y\left(p_1^r + p_2^r\right)^{-1/r} \tag{E.1}$$

for any $\mathbf{p}$ and income level y. For an income level equal to $e(\mathbf{p}, u)$ dollars, therefore, we must have

$$v(\mathbf{p}, e(\mathbf{p}, u)) = e(\mathbf{p}, u)\left(p_1^r + p_2^r\right)^{-1/r}. \tag{E.2}$$

Next, from the second item in Theorem 1.8, we know that for any $\mathbf{p}$ and u,

$$v(\mathbf{p}, e(\mathbf{p}, u)) = u. \tag{E.3}$$

Combining (E.2) and (E.3) gives

$$e(\mathbf{p}, u)\left(p_1^r + p_2^r\right)^{-1/r} = u. \tag{E.4}$$

Solving (E.4) for $e(\mathbf{p}, u)$, we get the expression

$$e(\mathbf{p}, u) = u\left(p_1^r + p_2^r\right)^{1/r} \tag{E.5}$$

for the expenditure function. A quick look back at Example 1.3 confirms this is the same expression for the expenditure function obtained by directly solving the consumer's expenditure-minimisation problem.

Suppose, instead, we begin with knowledge of the expenditure function and want to derive the indirect utility function. For the CES direct utility function, we know from Example 1.3 that

$$e(\mathbf{p}, u) = u\left(p_1^r + p_2^r\right)^{1/r} \tag{E.6}$$

for any $\mathbf{p}$ and utility level u. Then for utility level $v(\mathbf{p}, y)$, we will have

$$e(\mathbf{p}, v(\mathbf{p}, y)) = v(\mathbf{p}, y)\left(p_1^r + p_2^r\right)^{1/r}. \tag{E.7}$$

From the first item in Theorem 1.8, for any $\mathbf{p}$ and y,

$$e(\mathbf{p}, v(\mathbf{p}, y)) = y. \tag{E.8}$$

Combining (E.7) and (E.8), we obtain

$$v(\mathbf{p}, y)\left(p_1^r + p_2^r\right)^{1/r} = y. \tag{E.9}$$

Solving (E.9) for $v(\mathbf{p}, y)$ gives the expression

$$v(\mathbf{p}, y) = y\left(p_1^r + p_2^r\right)^{-1/r} \tag{E.10}$$

for the indirect utility function. A glance at Example 1.2 confirms that (E.10) is what we obtained by directly solving the consumer's utility-maximisation problem. □

We can pursue this relationship between utility maximisation and expenditure minimisation a bit further by shifting our attention to the respective *solutions* to these two problems. The solutions to the utility-maximisation problem are the Marshallian demand functions. The solutions to the expenditure-minimisation problem are the Hicksian demand functions. In view of the close relationship between the two optimisation problems themselves, it is natural to suspect there is some equally close relationship between their respective solutions. The following theorem clarifies the links between Hicksian and Marshallian demands.

THEOREM 1.9 ***Duality Between Marshallian and Hicksian Demand Functions***

Under Assumption 1.2 we have the following relations between the Hicksian and Marshallian demand functions for $\mathbf{p} \gg \mathbf{0}$, $y \geq 0$, $u \in \mathcal{U}$, *and* $i = 1, \ldots, n$:

1. $x_i(\mathbf{p}, y) = x_i^h(\mathbf{p}, v(\mathbf{p}, y))$.
2. $x_i^h(\mathbf{p}, u) = x_i(\mathbf{p}, e(\mathbf{p}, u))$.

The first relation says that the Marshallian demand at prices $\mathbf{p}$ and income y is equal to the Hicksian demand at prices $\mathbf{p}$ and the utility level that is the maximum that can be achieved at prices $\mathbf{p}$ and income y. The second says that the Hicksian demand at any prices $\mathbf{p}$ and utility level u is the same as the Marshallian demand at those prices and an income level equal to the minimum expenditure necessary at those prices to achieve that utility level.

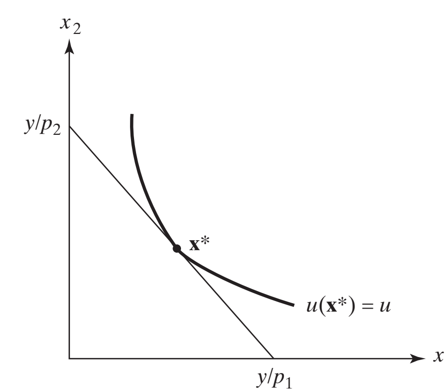
Roughly, Theorem 1.9 says that solutions to (1.12) are also solutions to (1.14), and vice versa. More precisely, if $\mathbf{x}^*$ solves (1.12) at $(\mathbf{p}, y)$, the theorem says that $\mathbf{x}^*$ solves (1.14) at $(\mathbf{p}, u)$, where $u = u(\mathbf{x}^*)$. Conversely, if $\mathbf{x}^*$ solves (1.14) at $(\mathbf{p}, u)$, then $\mathbf{x}^*$ solves (1.12) at $(\mathbf{p}, y)$, where $y = \mathbf{p} \cdot \mathbf{x}^*$. Fig. 1.17 illustrates the theorem. There, it is clear that $\mathbf{x}^*$ can be viewed either as the solution to (1.12) or the solution to (1.14). It is in this sense that $\mathbf{x}^*$ has a *dual* nature.

Proof: We will complete the proof of the first, leaving the second as an exercise.

Note that by Assumption 1.2, $u(\cdot)$ is continuous and strictly quasiconcave, so that the solutions to (1.12) and (1.14) exist and are unique. Consequently, the Marshallian and Hicksian demand fuctions are well-defined.

To prove the first relation, let $\mathbf{x}^0 = \mathbf{x}(\mathbf{p}^0, y^0)$, and let $u^0 = u(\mathbf{x}^0)$. Then $v(\mathbf{p}^0, y^0) = u^0$ by definition of $v(\cdot)$, and $\mathbf{p}^0 \cdot \mathbf{x}^0 = y^0$ because, by Assumption 1.2, $u(\cdot)$ is strictly increasing. By Theorem 1.8, $e(\mathbf{p}^0, v(\mathbf{p}^0, y^0)) = y^0$ or, equivalently, $e(\mathbf{p}^0, u^0) = y^0$. But

Figure 1.17. Expenditure minimisation and utility maximisation.



because $u(\mathbf{x}^0) = u^0$ and $\mathbf{p}^0 \cdot \mathbf{x}^0 = y^0$, this implies that $\mathbf{x}^0$ solves (1.14) when $(\mathbf{p}, u) = (\mathbf{p}^0, u^0)$. Hence, $\mathbf{x}^0 = \mathbf{x}^h(\mathbf{p}^0, u^0)$ and so $\mathbf{x}(\mathbf{p}^0, y^0) = \mathbf{x}^h(\mathbf{p}^0, v(\mathbf{p}^0, y^0))$. ∎

EXAMPLE 1.5 Let us confirm Theorem 1.9 for a CES consumer. From Example 1.3, the Hicksian demands are

$$x_i^h(\mathbf{p}, u) = u\left(p_1^r + p_2^r\right)^{(1/r)-1} p_i^{r-1}, \qquad i = 1, 2. \tag{E.1}$$

From Example 1.2, the indirect utility function is

$$v(\mathbf{p}, y) = y\left(p_1^r + p_2^r\right)^{-1/r}. \tag{E.2}$$

Substituting from (E.2) for u in (E.1) gives

$$\begin{aligned} x_i^h(\mathbf{p}, v(\mathbf{p}, y)) &= v(\mathbf{p}, y)\left(p_1^r + p_2^r\right)^{(1/r)-1} p_i^{r-1} \\ &= y\left(p_1^r + p_2^r\right)^{-1/r}\left(p_1^r + p_2^r\right)^{(1/r)-1} p_i^{r-1} \\ &= y p_i^{r-1}\left(p_1^r + p_2^r\right)^{-1} \\ &= \frac{y p_i^{r-1}}{p_1^r + p_2^r}, \qquad i = 1, 2. \end{aligned} \tag{E.3}$$

The final expression on the right-hand side of (E.3) gives the Marshallian demands we derived in Example 1.1 by solving the consumer's utility-maximisation problem. This confirms the first item in Theorem 1.9.

To confirm the second, suppose we know the Marshallian demands from Example 1.1,

$$x_i(\mathbf{p}, y) = \frac{y p_i^{r-1}}{p_1^r + p_2^r}, \qquad i = 1, 2, \tag{E.4}$$

and the expenditure function from Example 1.3,

$$e(\mathbf{p}, u) = u\left(p_1^r + p_2^r\right)^{1/r}. \tag{E.5}$$

Substituting from (E.5) into (E.4) for y yields

$$\begin{aligned} x_i(\mathbf{p}, e(\mathbf{p}, u)) &= \frac{e(\mathbf{p}, u) p_i^{r-1}}{p_1^r + p_2^r} \\ &= u\left(p_1^r + p_2^r\right)^{1/r} \frac{p_i^{r-1}}{p_1^r + p_2^r} \\ &= u p_i^{r-1}\left(p_1^r + p_2^r\right)^{(1/r)-1}, \qquad i = 1, 2. \end{aligned} \tag{E.6}$$

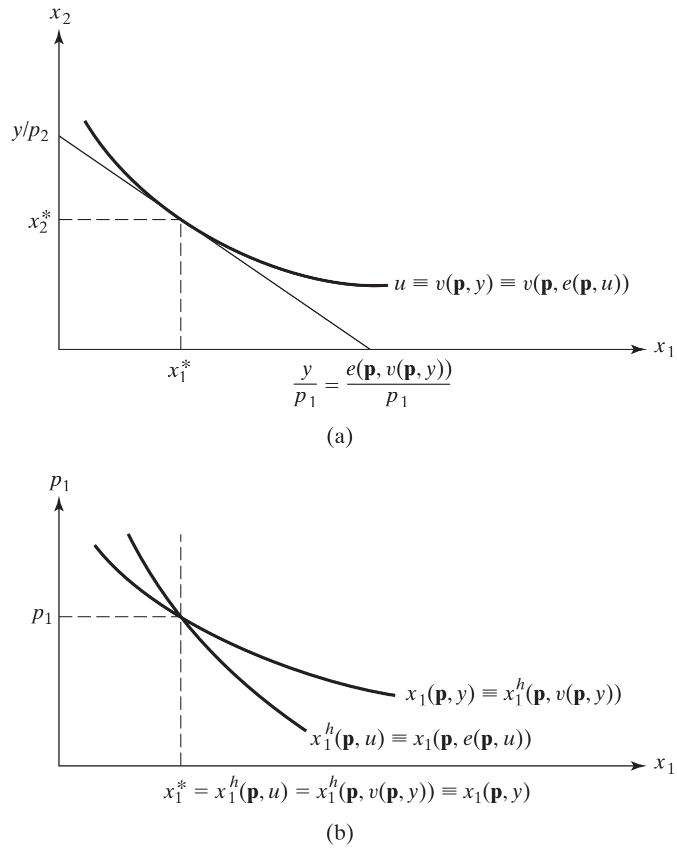


Figure 1.18. Illustration of Theorems 1.8 and 1.9.

The final expression on the right-hand side of (E.6) gives the Hicksian demands derived in Example 1.3 by directly solving the consumer's expenditure minimisation problem. □

To conclude this section, we can illustrate the four relations in Theorems 1.8 and 1.9. In Fig. 1.18(a), a consumer with income y facing prices $\mathbf{p}$ achieves maximum utility u by choosing x_1^* and x_2^*. That same u-level indifference curve therefore can be viewed as giving the level of utility $v(\mathbf{p}, y)$, and, in Fig. 1.18(b), point (p_1, x_1^*) will be a point on the Marshallian demand curve for good 1. Consider next the consumer's expenditure-minimisation problem, and suppose we seek to minimise expenditure to achieve utility u. Then, clearly, the lowest isoexpenditure curve that achieves u at prices $\mathbf{p}$ will *coincide* with the budget constraint in the previous utility-maximisation problem, and the expenditure minimising choices will again be x_1^* and x_2^*, giving the point (p_1, x_1^*) in Fig. 1.18(b) as a point on the consumer's Hicksian demand for good 1.

Considering the two problems together, we can easily see from the coincident intercepts of the budget constraint and isoexpenditure line that income y is an amount of money equal to the minimum expenditure necessary to achieve utility $v(\mathbf{p}, y)$ or that $y = e(\mathbf{p}, v(\mathbf{p}, y))$. Utility level u is both the maximum achievable at prices $\mathbf{p}$ and income y, so that $u = v(\mathbf{p}, y)$, and the maximum achievable at prices $\mathbf{p}$ and an income equal to the minimum expenditure necessary to achieve u, so that $u = v(\mathbf{p}, e(\mathbf{p}, u))$. Finally, notice that (p_1, x_1^*) must be a point on all three of the following: (1) the Hicksian demand for good 1 at prices $\mathbf{p}$ and utility level u, (2) the Hicksian demand for good 1 at prices $\mathbf{p}$ and utility level $v(\mathbf{p}, y)$, and (3) the Marshallian demand for good 1 at prices $\mathbf{p}$ and income y. Thus, $x_1(\mathbf{p}, y) = x_1^h(\mathbf{p}, v(\mathbf{p}, y))$ and $x_1^h(\mathbf{p}, u) = x_1(\mathbf{p}, e(\mathbf{p}, u))$, as we had hoped.

1.5 PROPERTIES OF CONSUMER DEMAND

The theory of consumer behaviour leads to a number of predictions about behaviour in the marketplace. We will see that *if* preferences, objectives, and circumstances are as we have modelled them to be, *then* demand behaviour must display certain observable characteristics. One then can test the theory by comparing these theoretical restrictions on demand behaviour to actual demand behaviour. Once a certain degree of confidence in the theory has been gained, it can be put to further use. For example, to statistically estimate consumer demand systems, characteristics of demand behaviour predicted by the theory can be used to provide *restrictions* on the values that estimated parameters are allowed to take. This application of the theory helps to improve the statistical precision of the estimates obtained. For both theoretical and empirical purposes, therefore, it is extremely important that we wring all the implications for observable demand behaviour we possibly can from our model of the utility-maximising consumer. This is the task of this section.

1.5.1 RELATIVE PRICES AND REAL INCOME

Economists generally prefer to measure important variables in *real*, rather than monetary, terms. This is because ‘money is a veil’, which only tends to obscure the analyst’s view of what people truly do (or should) care about: namely, real commodities. Relative prices and real income are two such real measures.

By the **relative price** of some good, we mean the number of units of some other good that must be forgone to acquire 1 unit of the good in question. If p_i is the money price of good i, it will be measured in units of dollars per unit of good i. The money price of good j will have units of dollars per unit of good j. The relative price of good i in terms of good j measures the units of good j forgone per unit of good i acquired. This will be given by the price ratio p_i/p_j because

$$\frac{p_i}{p_j} = \frac{\$/\text{unit } i}{\$/\text{unit } j} = \frac{\$}{\text{unit } i} \cdot \frac{\text{unit } j}{\$} = \frac{\text{units of } j}{\text{unit of } i}.$$

By **real income**, we mean the maximum number of units of some commodity the consumer *could* acquire if he spent his entire money income. Real income is intended to reflect the consumer's total command over all resources by measuring his potential command over a single real commodity. If y is the consumer's money income, then the ratio y/p_j is called his real income in terms of good j and will be measured in units of good j, because

$$\frac{y}{p_j} = \frac{\$}{\$/\text{unit of } j} = \text{units of } j.$$

The simplest deduction we can make from our model of the utility-maximising consumer is that only *relative prices* and *real income* affect behaviour. This is sometimes expressed by saying that the consumer's demand behaviour displays an *absence of money illusion*. To see this, simply recall the discussion of Fig. 1.14. There, equiproportionate changes in money income and the level of all prices leave the slope (relative prices) and both intercepts of the consumer's budget constraint (real income measured in terms of any good) unchanged, and so call for no change in demand behaviour. Mathematically, this amounts to saying that the consumer's demand functions are homogeneous of degree zero in prices and income. Because the only role that money has played in constructing our model is as a unit of account, it would indeed be strange if this were not the case.

For future reference, we bundle this together with the observation that consumer spending will typically exhaust income, and we give names to both results.

THEOREM 1.10 ***Homogeneity and Budget Balancedness***

Under Assumption 1.2, the consumer demand function $x_i(\mathbf{p}, y)$, $i = 1, \ldots, n$, is homogeneous of degree zero in all prices and income, and it satisfies budget balancedness, $\mathbf{p} \cdot \mathbf{x}(\mathbf{p}, y) = y$ for all $(\mathbf{p}, y)$.

Proof: We already essentially proved homogeneity in Theorem 1.6, part 2, where we showed that the indirect utility function is homogeneous of degree zero, so that

$$v(\mathbf{p}, y) = v(t\mathbf{p}, ty) \quad \text{for all } t > 0.$$

This is equivalent to the statement

$$u(\mathbf{x}(\mathbf{p}, y)) = u(\mathbf{x}(t\mathbf{p}, ty)) \quad \text{for all } t > 0.$$

Now, because the budget sets at $(\mathbf{p}, y)$ and $(t\mathbf{p}, ty)$ are the same, each of $\mathbf{x}(\mathbf{p}, y)$ and $\mathbf{x}(t\mathbf{p}, ty)$ was feasible when the other was chosen. Hence, the previous equality and the strict quasiconcavity of u imply that

$$\mathbf{x}(\mathbf{p}, y) = \mathbf{x}(t\mathbf{p}, ty) \quad \text{for all } t > 0,$$

or that the demand for every good, $x_i(\mathbf{p}, y)$, $i = 1, \ldots, n$, is homogeneous of degree zero in prices and income.

We have already mentioned on numerous occasions that because $u(\cdot)$ is strictly increasing, $\mathbf{x}(\mathbf{p}, y)$ must exhaust the consumer's income. Otherwise, he could afford to purchase strictly more of every good and strictly increase his utility. We will refer to this relationship as **budget balancedness** from now on. ∎

Homogeneity allows us to completely eliminate the yardstick of money from any analysis of demand behaviour. This is generally done by arbitrarily designating one of the n goods to serve as *numéraire* in place of money. If its money price is p_n, we can set $t = 1/p_n$ and, invoking homogeneity, conclude that

$$\mathbf{x}(\mathbf{p}, y) = \mathbf{x}(t\mathbf{p}, ty) = \mathbf{x}\left(\frac{p_1}{p_n}, \ldots, \frac{p_{n-1}}{p_n}, 1, \frac{y}{p_n}\right).$$

In words, demand for each of the n goods depends only on $n - 1$ *relative prices* and the consumer's *real income*.

1.5.2 INCOME AND SUBSTITUTION EFFECTS

An important question in our model of consumer behaviour concerns the response we should expect in quantity demanded when price changes. Ordinarily, we tend to think a consumer will buy more of a good when its price declines and less when its price increases, other things being equal. That this need not always be the case is illustrated in Fig. 1.19. In each panel, a utility-maximising consumer with strictly monotonic, convex preferences faces market-determined prices. In Fig. 1.19(a), a decrease in the price of good 1 causes the quantity of good 1 bought to increase, as we would usually expect. By contrast, in Fig. 1.19(b), a decrease in price causes no change in the amount of good 1 bought, whereas in Fig. 1.19(c), a decrease in price causes an absolute *decrease* in the amount of good 1

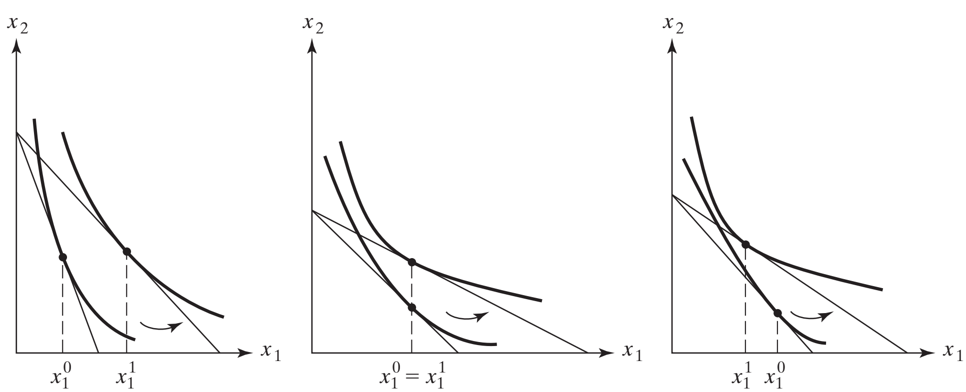


Figure 1.19. Response of quantity demanded to a change in price.

bought. Each of these cases is fully consistent with our model. What, then – if anything – does the theory predict about how someone's demand behaviour responds to changes in (relative) prices?

Let us approach it intuitively first. When the price of a good declines, there are at least two conceptually separate reasons why we expect some change in the quantity demanded. First, that good becomes relatively cheaper compared to other goods. Because all goods are desirable, even if the consumer's total command over goods were unchanged, we would expect him to substitute the relatively cheaper good for the now relatively more expensive ones. This is the **substitution effect** (*SE*). At the same time, however, whenever a price changes, the consumer's command over goods in general is *not* unchanged. When the price of any one good declines, the consumer's total command over all goods is effectively increased, allowing him to change his purchases of *all* goods in any way he sees fit. The effect on quantity demanded of this generalised increase in purchasing power is called the **income effect** (*IE*).

Although intuition tells us we can in some sense decompose the total effect (*TE*) of a price change into these two separate conceptual categories, we will have to be a great deal more precise if these ideas are to be of any analytical use. Different ways to formalise the intuition of the income and substitution effects have been proposed. We shall follow that proposed by Hicks (1939).

The Hicksian decomposition of the total effect of a price change starts with the observation that the consumer achieves some level of utility at the original prices before any change has occurred. The formalisation given to the intuitive notion of the substitution effect is the following: the substitution effect is that (hypothetical) change in consumption that *would* occur if relative prices were to change to their new levels but the maximum utility the consumer can achieve were kept the same as before the price change. The income effect is then defined as whatever is left of the total effect after the substitution effect. Notice that because the income effect is defined as a residual, the total effect is always completely explained by the sum of the substitution and the income effect. At first, this might seem a strange way to do things, but a glance at Fig. 1.20 should convince you of at least two things: its reasonable correspondence to the intuitive concepts of the income and substitution effects, and its analytical ingenuity.

Look first at Fig. 1.20(a), and suppose the consumer originally faces prices p_1^0 and p_2^0 and has income y. He originally buys quantities x_1^0 and x_2^0 and achieves utility level u^0. Suppose the price of good 1 falls to $p_1^1 < p_1^0$ and that the total effect of this price change on good 1 consumption is an increase to x_1^1, and the total effect on good 2 is a decrease to x_2^1. To apply the Hicksian decomposition, we first perform the hypothetical experiment of allowing the price of good 1 to fall to the new level p_1^1 *while holding the consumer to the original u^0 level indifference curve*. It is as if we allowed the consumer to face the new relative prices but reduced his income so that he faced the dashed hypothetical budget constraint and asked him to maximise against it. Under these circumstances, the consumer would *increase* his consumption of good 1 – the now relatively cheaper good – from x_1^0 to x_1^s, and would *decrease* his consumption of good 2 – the now relatively more expensive good – from x_2^0 to x_2^s. These hypothetical changes in consumption are the Hicksian