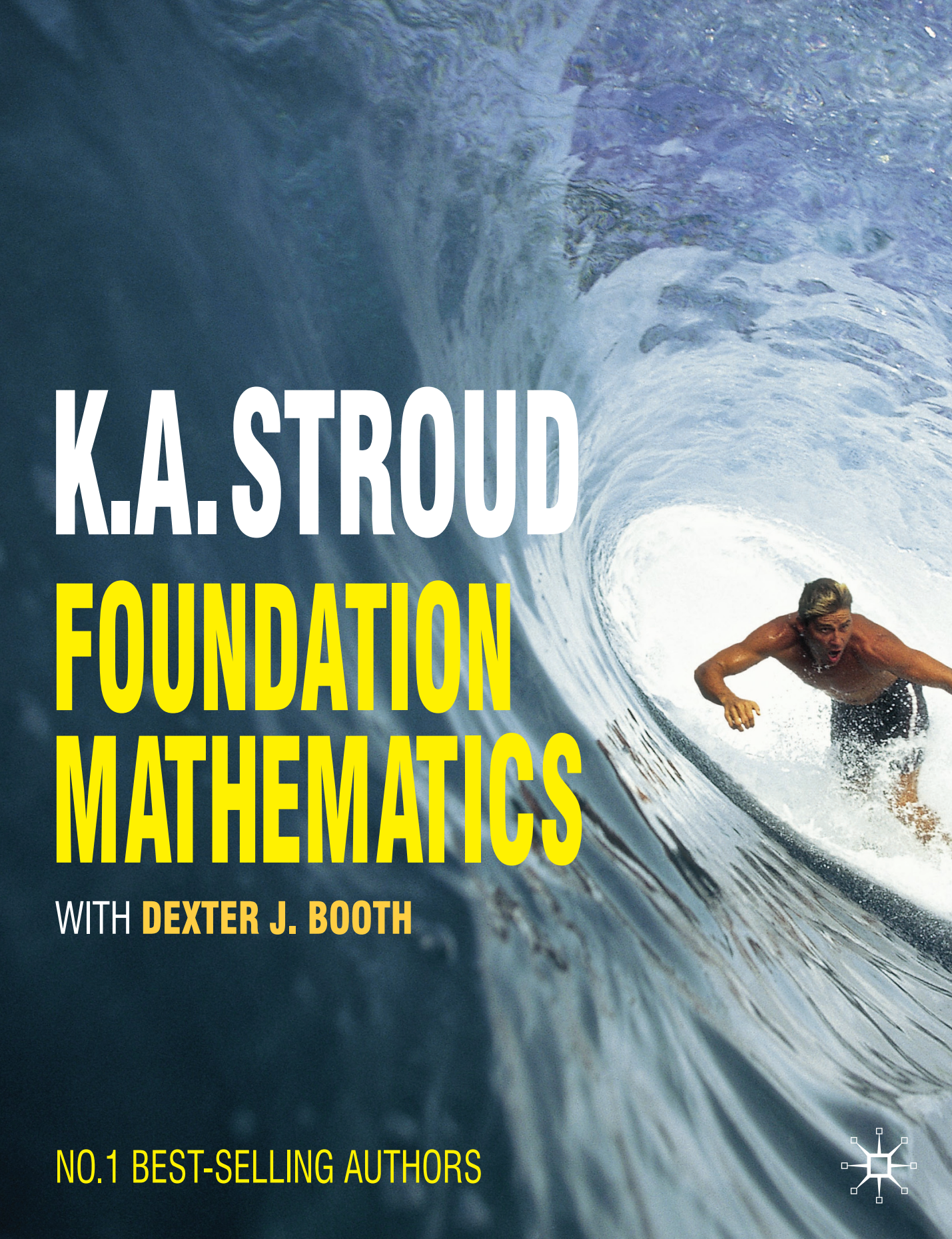


A high-action photograph of a surfer riding a massive, curling barrel wave. The surfer is positioned on the right side of the frame, leaning forward with an intense expression. The wave's face is a deep, dark blue, while the interior of the barrel is a bright, white foam. The overall scene conveys a sense of power and adventure.

K.A. STROUD

**FOUNDATION
MATHEMATICS**

WITH **DEXTER J. BOOTH**

NO.1 BEST-SELLING AUTHORS



FOUNDATION MATHEMATICS

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Summary of contents

Module 1 Arithmetic	1
Unit 1 Types of numbers	2
Unit 2 Factors and prime numbers	12
Unit 3 Fractions, ratios and percentages	17
Unit 4 Decimal numbers	27
Unit 5 Powers	35
Unit 6 Number systems	50
Module 2 Introduction to algebra	72
Unit 7 Algebraic expressions	73
Unit 8 Powers	81
Unit 9 Algebraic multiplication and division	92
Unit 10 Factorization of algebraic expressions	101
Module 3 Expressions and equations	116
Unit 11 Expressions and equations	117
Unit 12 Evaluating expressions	128
Module 4 Graphs	145
Unit 13 Graphs of equations	146
Unit 14 Using a spreadsheet	156
Unit 15 Inequalities	169
Unit 16 Absolute values	171
Module 5 Linear equations and simultaneous linear equations	184
Unit 17 Linear equations	185
Module 6 Polynomial equations	203
Unit 18 Polynomial equations	204
Module 7 Partial fractions	223
Unit 19 Partial fractions	224
Unit 20 Denominators with repeated and quadratic factors	233
Module 8 Trigonometry	245
Unit 21 Angles and trigonometric ratios	246
Unit 22 Trigonometric identities	262
Module 9 Functions	273
Unit 23 Processing numbers	274
Unit 24 Composition – ‘function of a function’	288
Unit 25 Trigonometric functions	294
Unit 26 Inverse trigonometric functions	305
Unit 27 Exponential and logarithmic functions	313
Unit 28 Odd and even functions	321
Unit 29 Limits	325

Module 10 Matrices	333
Unit 30 Matrices	334
Unit 31 Inverse matrices	344
Unit 32 Solving simultaneous linear equations	352
Module 11 Vectors	376
Unit 33 Scalar and vector quantities	377
Unit 34 Vectors in space	390
Unit 35 Products of vectors	396
Module 12 Binomial series	407
Unit 36 Factorials and combinations	408
Unit 37 Binomial series	417
Unit 38 The $\sum$ (sigma) notation	425
Module 13 Sets	436
Unit 39 Sets and subsets	437
Unit 40 Set operations	445
Unit 41 Set properties	453
Module 14 Probability	473
Unit 42 Empirical probability	474
Unit 43 Classical probability	482
Unit 44 Addition law of probability	485
Unit 45 Multiplication law and conditional probability	489
Module 15 Statistics	498
Unit 46 Data	499
Unit 47 Measures of central tendency	514
Unit 48 Dispersion	527
Unit 49 The normal distribution	533
Module 16 Regression and correlation	550
Unit 50 Regression	551
Unit 51 Correlation	561
Module 17 Introduction to differentiation	577
Unit 52 Gradients	578
Unit 53 Further differentiation	593
Module 18 Partial differentiation	620
Unit 54 Partial differentiation	621
Unit 55 Further partial differentiation	628
Unit 56 Calculating errors	636
Module 19 Integration	647
Unit 57 Integration	648
Unit 58 Integration of polynomial expressions	652
Unit 59 Integration by partial fractions	658
Unit 60 Integration by parts	663
Unit 61 Areas under curves	669
Unit 62 Integration as a summation	684

Contents

<i>Preface</i>	xx
<i>How to use this book</i>	xxi
<i>Useful background information</i>	xxii
Module 1 Arithmetic	1
Learning outcomes	1
Unit 1 Types of numbers	2
The natural numbers	2
Numerals and place value	2
Points on a line and order	2
The integers	2
Brackets	3
Addition and subtraction	3
Multiplication and division	4
Brackets and precedence rules	5
Basic laws of arithmetic	7
Estimating	8
Rounding	8
Unit 1 Review summary	10
Unit 1 Review exercise	10
Unit 1 Review test	12
Unit 2 Factors and prime numbers	12
Factors	12
Prime numbers	13
Prime factorization	13
Highest common factor (HCF)	14
Lowest common multiple (LCM)	14
Unit 2 Review summary	15
Unit 2 Review exercise	15
Unit 2 Review test	16
Unit 3 Fractions, ratios and percentages	17
Division of integers	17
Multiplying fractions	17
Of	18
Equivalent fractions	18
Dividing fractions	20
Adding and subtracting fractions	21
Fractions on a calculator	22

Ratios	23
Percentages	23
<i>Unit 3 Review summary</i>	25
<i>Unit 3 Review exercise</i>	25
<i>Unit 3 Review test</i>	27
Unit 4 Decimal numbers	27
Division of integers	27
Rounding	28
Significant figures	28
Decimal places	29
Trailing zeros	29
Fractions as decimals	30
Decimals as fractions	30
Unending decimals	30
Unending decimals as fractions	31
Rational, irrational and real numbers	32
<i>Unit 4 Review summary</i>	33
<i>Unit 4 Review exercise</i>	33
<i>Unit 4 Review test</i>	34
Unit 5 Powers	35
Raising a number to a power	35
The laws of powers	35
Powers on a calculator	39
Fractional powers and roots	40
Surds	40
Multiplication and division by integer powers of 10	41
Precedence rules	42
Standard form	42
Working in standard form	43
Using a calculator	44
Preferred standard form	45
Checking calculations	46
Accuracy	47
<i>Unit 5 Review summary</i>	48
<i>Unit 5 Review exercise</i>	48
<i>Unit 5 Review test</i>	50
Unit 6 Number systems	50
Denary (or decimal) system	50
Binary system	51
Octal system (base 8)	51
Duodecimal system (base 12)	52
Hexadecimal system (base 16)	53
An alternative method	54
Change of base from denary to a new base	57
A denary number in binary form	57
A denary number in octal form	57

A denary number in hexadecimal form	58
A denary decimal to octal form	59
Use of octals as an intermediate step	61
Reverse method	62
<i>Unit 6 Review summary</i>	63
<i>Unit 6 Review exercise</i>	64
<i>Unit 6 Review test</i>	65
Can You?	66
Test exercise 1	67
Further problems 1	68

Module 2 Introduction to algebra 72

Learning outcomes	72
--------------------------	----

Unit 7 Algebraic expressions 73

Symbols other than numerals	73
Constants	74
Variables	74
Rules of algebra	75
Rules of precedence	76
Terms and coefficients	76
Collecting like terms	76
Similar terms	77
Expanding brackets	78
Nested brackets	78
<i>Unit 7 Review summary</i>	79
<i>Unit 7 Review exercise</i>	79
<i>Unit 7 Review test</i>	81

Unit 8 Powers 81

Powers	81
Rules of indices	82
Logarithms	83
Powers	83
Logarithms	84
Rules of logarithms	85
Base 10 and base e	86
Change of base	87
Logarithmic equations	88
<i>Unit 8 Review summary</i>	90
<i>Unit 8 Review exercise</i>	91
<i>Unit 8 Review test</i>	92

Unit 9 Algebraic multiplication and division 92

Multiplication of algebraic expressions of a single variable	92
Fractions	94
Algebraic fractions	94
Addition and subtraction	94

Multiplication and division	95
Division of one expression by another	97
<i>Unit 9 Review summary</i>	98
<i>Unit 9 Review exercise</i>	99
<i>Unit 9 Review test</i>	100
Unit 10 Factorization of algebraic expressions	101
Common factors	101
Common factors by grouping	101
Useful products of two simple factors	102
Quadratic expressions as the product of two simple factors	104
Factorization of a quadratic expression $ax^2 + bx + c$ when $a = 1$	104
Factorization of a quadratic expression $ax^2 + bx + c$ when $a \neq 1$	106
Test for simple factors	108
<i>Unit 10 Review summary</i>	110
<i>Unit 10 Review exercise</i>	110
<i>Unit 10 Review test</i>	111
Can You?	112
Test exercise 2	112
Further problems 2	114
Module 3 Expressions and equations	116
Learning outcomes	116
Unit 11 Expressions and equations	117
Evaluating expressions	117
Equations	118
Evaluating independent variables	120
Transposition of formulas	122
<i>Unit 11 Review summary</i>	126
<i>Unit 11 Review exercise</i>	127
<i>Unit 11 Review test</i>	128
Unit 12 Evaluating expressions	128
Systems	128
Polynomial equations	129
Polynomial expressions	129
Evaluation of polynomials	130
Evaluation of a polynomial by nesting	130
Remainder theorem	132
Factor theorem	133
Factorization of fourth-order polynomials	136
<i>Unit 12 Review summary</i>	140
<i>Unit 12 Review exercise</i>	141
<i>Unit 12 Review test</i>	141
Can You?	142
Test exercise 3	143
Further problems 3	143

Module 4 Graphs **145**

Learning outcomes 145

Unit 13 Graphs of equations **146**

Ordered pairs of numbers 146

Cartesian axes 146

Drawing a graph 147

Unit 13 Review summary 153

Unit 13 Review exercise 154

Unit 13 Review test 155

Unit 14 Using a spreadsheet **156**

Spreadsheets 156

Rows and columns 156

Text and number entry 157

Formulas 158

Clearing entries 158

Construction of a Cartesian graph 159

Unit 14 Review summary 166

Unit 14 Review exercise 167

Unit 14 Review test 168

Unit 15 Inequalities **169**

Less than or greater than 169

Unit 15 Review summary 170

Unit 15 Review exercise 170

Unit 15 Review test 171

Unit 16 Absolute values **171**

Modulus 171

Graphs 172

Inequalities 173

Interaction 178

Unit 16 Review summary 179

Unit 16 Review exercise 180

Unit 16 Review test 181

Can You? 181

Test exercise 4 182

Further problems 4 182

Module 5 Linear equations and simultaneous linear equations **184**

Learning outcomes 184

Unit 17 Linear equations **185**

Solution of simple equations 185

Simultaneous linear equations with two unknowns 188

Solution by substitution 188

Solution by equating coefficients 189

Simultaneous linear equations with three unknowns	191
Pre-simplification	193
<i>Unit 17 Review summary</i>	196
<i>Unit 17 Review exercise</i>	197
<i>Unit 17 Review test</i>	199
Can You?	199
Test exercise 5	200
Further problems 5	200

Module 6 Polynomial equations 203

Learning outcomes	203
Unit 18 Polynomial equations	204
Quadratic equations, $ax^2 + bx + c = 0$	204
Solution by factors	204
Solution by completing the square	207
Solution by formula	209
Solution of cubic equations having at least one linear factor	210
Solution of fourth-order equations having at least two linear factors	214
<i>Unit 18 Review summary</i>	218
<i>Unit 18 Review exercise</i>	219
<i>Unit 18 Review test</i>	220
Can You?	221
Test exercise 6	221
Further problems 6	222

Module 7 Partial fractions 223

Learning outcomes	223
Unit 19 Partial fractions	224
<i>Unit 19 Review summary</i>	231
<i>Unit 19 Review exercise</i>	231
<i>Unit 19 Review test</i>	233
Unit 20 Denominators with repeated and quadratic factors	233
<i>Unit 20 Review summary</i>	241
<i>Unit 20 Review exercise</i>	241
<i>Unit 20 Review test</i>	242
Can You?	242
Test exercise 7	243
Further problems 7	243

Module 8 Trigonometry 245

Learning outcomes	245
Unit 21 Angles and trigonometric ratios	246
Rotation	246
Radians	247

Triangles	249
Trigonometric ratios	251
Reciprocal ratios	253
Pythagoras' theorem	255
Special triangles	256
Half equilateral triangles	258
<i>Unit 21 Review summary</i>	260
<i>Unit 21 Review exercise</i>	261
<i>Unit 21 Review test</i>	262
Unit 22 Trigonometric identities	262
The fundamental identity	262
Two more identities	263
Identities for compound angles	265
Trigonometric formulas	267
Sums and differences of angles	267
Double angles	267
Sums and differences of ratios	268
Products of ratios	268
<i>Unit 22 Review summary</i>	268
<i>Unit 22 Review exercise</i>	268
<i>Unit 22 Review test</i>	270
Can You?	270
Test exercise 8	271
Further problems 8	271
 Module 9 Functions	 273
Learning outcomes	273
Unit 23 Processing numbers	274
Functions are rules but not all rules are functions	275
Combining functions	278
Inverses of functions	279
Graphs of inverses	281
The graph of $y = x^3$	281
The graph of $y = x^{1/3}$	282
The graphs of $y = x^3$ and $y = x^{1/3}$ plotted together	283
<i>Unit 23 Review summary</i>	286
<i>Unit 23 Review exercise</i>	286
<i>Unit 23 Review test</i>	288
Unit 24 Composition – ‘function of a function’	288
Inverses of compositions	292
<i>Unit 24 Review summary</i>	293
<i>Unit 24 Review exercise</i>	293
<i>Unit 24 Review test</i>	294
Unit 25 Trigonometric functions	294
Rotation	294

The tangent	297
Period	298
Amplitude	300
Phase difference	302
<i>Unit 25 Review summary</i>	303
<i>Unit 25 Review exercise</i>	304
<i>Unit 25 Review test</i>	305
Unit 26 Inverse trigonometric functions	305
Trigonometric equations	307
Using inverse functions	309
<i>Unit 26 Review summary</i>	311
<i>Unit 26 Review exercise</i>	311
<i>Unit 26 Review test</i>	312
Unit 27 Exponential and logarithmic functions	313
Exponential functions	313
Logarithmic functions	314
Indicial equations	316
<i>Unit 27 Review summary</i>	319
<i>Unit 27 Review exercise</i>	319
<i>Unit 27 Review test</i>	320
Unit 28 Odd and even functions	321
Odd and even parts	321
Odd and even parts of the exponential function	323
<i>Unit 28 Review summary</i>	324
<i>Unit 28 Review exercise</i>	324
<i>Unit 28 Review test</i>	324
Unit 29 Limits	325
Limits of functions	325
The rules of limits	326
<i>Unit 29 Review summary</i>	327
<i>Unit 29 Review exercise</i>	327
<i>Unit 29 Review test</i>	328
Can You?	329
Test exercise 9	330
Further problems 9	331
 Module 10 Matrices	 333
Learning outcomes	333
Unit 30 Matrices	334
Matrix operations	334
Matrix addition	337
Matrix subtraction	337
Multiplication by a scalar	338
Multiplication of matrices	339

<i>Unit 30 Review summary</i>	342
<i>Unit 30 Review exercise</i>	342
<i>Unit 30 Review test</i>	343
Unit 31 Inverse matrices	344
Unit matrices	344
The determinant of a matrix	345
The inverse of a matrix	346
The inverse of a 2×2 matrix	347
<i>Unit 31 Review summary</i>	349
<i>Unit 31 Review exercise</i>	350
<i>Unit 31 Review test</i>	351
Unit 32 Solving simultaneous linear equations	352
Matrix solution to simultaneous linear equations	352
Determinants and Cramer's rule	357
Cramer's rule	357
Higher order determinants	362
<i>Unit 32 Review summary</i>	370
<i>Unit 32 Review exercise</i>	371
<i>Unit 32 Review test</i>	373
Can You?	373
Test exercise 10	374
Further problems 10	374

Module 11 Vectors 376

Learning outcomes	376
Unit 33 Scalar and vector quantities	377
Vector representation	378
Two equal vectors	378
Types of vectors	379
Addition of vectors	379
The sum of a number of vectors $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} + \dots$	380
Components of a given vector	382
Components of a vector in terms of unit vectors	386
<i>Unit 33 Review summary</i>	388
<i>Unit 33 Review exercise</i>	389
<i>Unit 33 Review test</i>	389
Unit 34 Vectors in space	390
Direction cosines	391
Angle between two vectors	392
<i>Unit 34 Review summary</i>	394
<i>Unit 34 Review exercise</i>	395
<i>Unit 34 Review test</i>	396
Unit 35 Products of vectors	396
Scalar product of two vectors	396
Vector product of two vectors	398

<i>Unit 35 Review summary</i>	402
<i>Unit 35 Review exercise</i>	402
<i>Unit 35 Review test</i>	403
Can You?	404
Test exercise 11	405
Further problems 11	405

Module 12 Binomial series 407

Learning outcomes	407
Unit 36 Factorials and combinations 408	408
Factorials	408
Combinations	411
Some properties of combinatorial coefficients	415
<i>Unit 36 Review summary</i>	416
<i>Unit 36 Review exercise</i>	416
<i>Unit 36 Review test</i>	417
Unit 37 Binomial series 417	417
Pascal's triangle	417
Binomial expansions	419
The general term of the binomial expansion	422
<i>Unit 37 Review summary</i>	424
<i>Unit 37 Review exercise</i>	424
<i>Unit 37 Review test</i>	425
Unit 38 The $\sum$ (sigma) notation 425	425
General terms	426
The sum of the first n natural numbers	430
Rules for manipulating sums	431
<i>Unit 38 Review summary</i>	432
<i>Unit 38 Review exercise</i>	433
<i>Unit 38 Review test</i>	433
Can You?	434
Test exercise 12	435
Further problems 12	435

Module 13 Sets 436

Learning outcomes	436
Unit 39 Sets and subsets 437	437
Definitions	437
Further set notation	438
Subsets	439
The empty set	441
The universal set	442
<i>Unit 39 Review summary</i>	443
<i>Unit 39 Review exercise</i>	443
<i>Unit 39 Review test</i>	444

Unit 40 Set operations	445
Venn diagrams	445
Intersection	446
Union	447
Complement	449
<i>Unit 40 Review summary</i>	450
<i>Unit 40 Review exercise</i>	450
<i>Unit 40 Review test</i>	452
Unit 41 Set properties	453
Union, intersection and complement	453
Distributivity rules	456
The number of elements in a set	458
Three sets	461
<i>Unit 41 Review summary</i>	466
<i>Unit 41 Review exercise</i>	466
<i>Unit 41 Review test</i>	469
Can You?	469
Test exercise 13	470
Further problems 13	471
Module 14 Probability	473
Learning outcomes	473
Unit 42 Empirical probability	474
Introduction	474
Notation	474
Sampling	474
Types of probability	474
Empirical probability	475
Expectation	475
Success or failure	476
Sample size	476
Multiple samples	477
Experiment	477
<i>Unit 42 Review summary</i>	480
<i>Unit 42 Review exercise</i>	481
<i>Unit 42 Review test</i>	481
Unit 43 Classical probability	482
Definition	482
Certain and impossible events	483
Mutually exclusive and mutually non-exclusive events	483
<i>Unit 43 Review summary</i>	484
<i>Unit 43 Review exercise</i>	484
<i>Unit 43 Review test</i>	485

Unit 44 Addition law of probability	485
<i>Unit 44 Review summary</i>	487
<i>Unit 44 Review exercise</i>	488
<i>Unit 44 Review test</i>	488
Unit 45 Multiplication law and conditional probability	489
Independent events and dependent events	489
Multiplication law of probabilities	489
Conditional probability	490
<i>Unit 45 Review summary</i>	494
<i>Unit 45 Review exercise</i>	494
<i>Unit 45 Review test</i>	495
Can You?	495
Test exercise 14	496
Further problems 14	496
Module 15 Statistics	498
Learning outcomes	498
Unit 46 Data	499
Introduction	499
Arrangement of data	499
Tally diagram	500
Grouped data	501
Grouping with continuous data	502
Relative frequency	503
Rounding off data	504
Class boundaries	505
Histograms	508
Frequency histogram	508
Relative frequency histogram	510
<i>Unit 46 Review summary</i>	511
<i>Unit 46 Review exercise</i>	512
<i>Unit 46 Review test</i>	514
Unit 47 Measures of central tendency	514
Mean	514
Coding for calculating the mean	516
Decoding	517
Coding with a grouped frequency distribution	518
Mode	519
Mode of a grouped frequency distribution	519
Median	521
Median with grouped data	521
<i>Unit 47 Review summary</i>	523
<i>Unit 47 Review exercise</i>	523
<i>Unit 47 Review test</i>	526

Unit 48 Dispersion	527
Range	527
Standard deviation	527
Unit 48 Review summary	530
Unit 48 Review exercise	531
Unit 48 Review test	532
Unit 49 The normal distribution	533
Frequency polygons	533
Frequency curves	533
Normal distribution curve	533
Standard normal curve	536
Unit 49 Review summary	542
Unit 49 Review exercise	542
Unit 49 Review test	543
Can You?	544
Test exercise 15	544
Further problems 15	546
Module 16 Regression and correlation	550
Learning outcomes	550
Unit 50 Regression	551
Explicit linear variation of two variables	551
Implicit linear variation of two variables	552
Regression – method of least squares	553
Unit 50 Review summary	558
Unit 50 Review exercise	559
Unit 50 Review test	560
Unit 51 Correlation	561
Correlation	561
Measures of correlation	561
The Pearson product-moment correlation coefficient	562
Spearman's rank correlation coefficient	566
Unit 51 Review summary	569
Unit 51 Review exercise	569
Unit 51 Review test	572
Can You?	573
Test exercise 16	573
Further problems 16	574
Module 17 Introduction to differentiation	577
Learning outcomes	577
Unit 52 Gradients	578
The gradient of a straight-line graph	578
The gradient of a curve at a given point	581

Algebraic determination of the gradient of a curve	583
Derivative of powers of x	585
Two straight lines	585
Two curves	586
Differentiation of polynomials	588
Derivatives – an alternative notation	590
<i>Unit 52 Review summary</i>	590
<i>Unit 52 Review exercise</i>	591
<i>Unit 52 Review test</i>	592
Unit 53 Further differentiation	593
Second derivatives	593
Limiting value of $\frac{\sin \theta}{\theta}$ as $\theta \rightarrow 0$	594
Standard derivatives	594
Differentiation of products of functions	597
Differentiation of a quotient of two functions	598
Function of a function	602
Differentiation of a function of a function	602
Newton–Raphson iterative method	606
Tabular display of results	608
<i>Unit 53 Review summary</i>	613
<i>Unit 53 Review exercise</i>	614
<i>Unit 53 Review test</i>	616
Can You?	617
Test exercise 17	618
Further problems 17	618
 Module 18 Partial differentiation	 620
Learning outcomes	620
Unit 54 Partial differentiation	621
Partial derivatives	621
<i>Unit 54 Review summary</i>	626
<i>Unit 54 Review exercise</i>	626
<i>Unit 54 Review test</i>	627
Unit 55 Further partial differentiation	628
Second derivatives	628
<i>Unit 55 Review summary</i>	633
<i>Unit 55 Review exercise</i>	633
<i>Unit 55 Review test</i>	635
Unit 56 Calculating errors	636
Small increments	636
<i>Unit 56 Review summary</i>	642
<i>Unit 56 Review exercise</i>	642
<i>Unit 56 Review test</i>	643

Can You?	644
Test exercise 18	644
Further problems 18	645

Module 19 Integration 647

Learning outcomes	647
-------------------	-----

Unit 57 Integration 648

Constant of integration	648
Standard integrals	649
<i>Unit 57 Review summary</i>	650
<i>Unit 57 Review exercise</i>	650
<i>Unit 57 Review test</i>	652

Unit 58 Integration of polynomial expressions 652

Function of a linear function of x	654
<i>Unit 58 Review summary</i>	655
<i>Unit 58 Review exercise</i>	655
<i>Unit 58 Review test</i>	657

Unit 59 Integration by partial fractions 658

<i>Unit 59 Review summary</i>	661
<i>Unit 59 Review exercise</i>	661
<i>Unit 59 Review test</i>	662

Unit 60 Integration by parts 663

<i>Unit 60 Review summary</i>	668
<i>Unit 60 Review exercise</i>	669
<i>Unit 60 Review test</i>	669

Unit 61 Areas under curves 669

Simpson's rule	673
<i>Unit 61 Review summary</i>	682
<i>Unit 61 Review exercise</i>	683
<i>Unit 61 Review test</i>	684

Unit 62 Integration as a summation 684

The area between a curve and an intersecting line	689
<i>Unit 62 Review summary</i>	691
<i>Unit 62 Review exercise</i>	692
<i>Unit 62 Review test</i>	695

Can You?	696
----------	-----

Test exercise 19	697
------------------	-----

Further problems 19	698
---------------------	-----

<i>Answers</i>	701
----------------	-----

<i>Index</i>	725
--------------	-----

Preface

It is now nearly 40 years since Ken Stroud first developed his approach to personalized learning with his classic text *Engineering Mathematics*, now in its sixth edition and having sold over half a million copies. That unique and hugely successful programmed learning style is exemplified in this text and I am delighted to have been asked to contribute to it. I have endeavoured to retain the very essence of his style, particularly the time-tested Stroud format with its close attention to technique development throughout. This style has, over the years, contributed significantly to the mathematical abilities of so many students all over the world.

Student readership

Over recent years there have been many developments in a wide range of university disciplines. This has led to an increase in the number of courses that require knowledge of mathematics to enable students to participate in their studies with confidence. Also, by widening access to these courses, many students arrive at university with a need to refresh and amplify the mathematical knowledge that they have previously acquired. This book was written with just those students in mind, starting as it does from the very beginnings of the subject. Indeed, the content of the book ranges from the earliest elements of arithmetic to differential and integral calculus. The material is presented in a manner that will be appreciated by those students requiring to review and extend their current mathematical abilities from a level of low confidence to one of confident proficiency.

Acknowledgements

This is a further opportunity that I have had to work on the Stroud books. It is as ever a challenge and an honour to be able to work with Ken Stroud's material. Ken had an understanding of his students and their learning and thinking processes which was second to none, and this is reflected in every page of this book. As always, my thanks go to the Stroud family for their continuing support for and encouragement of new projects and ideas which are allowing Ken's hugely successful teaching methodology to be offered to a whole new range of students. Finally, I should like to thank the entire production team at Palgrave Macmillan for all their care, and principally my editor Helen Bugler whose dedication and professionalism are an inspiration to all who work with her.

*Huddersfield
February 2009*

Dexter J Booth

How to use this book

This book contains nineteen **Modules**, each module consisting of a number of **Units**. In total there are 62 Units and each one has been written in a way that makes learning more effective and more interesting. It is like having a personal tutor because you proceed at your own rate of learning and any difficulties you may have are cleared before you have the chance to practise incorrect ideas or techniques.

You will find that each Unit is divided into numbered sections called **Frames**. When you start a Unit, begin at Frame 1. Read each frame carefully and carry out any instructions or exercise you are asked to do. In almost every frame, you are required to make a response of some kind, so have your pen and paper at the ready to test your understanding of the information in the frame. You can immediately compare your answer and how you arrived at it with the correct answer and the working given in the frame that follows. To obtain the greatest benefit, you are strongly advised to cover up the following frame until you have made your response. When a series of dots occurs, you are expected to supply the missing word, phrase, number or mathematical expression. At every stage you will be guided along the right path. There is no need to hurry: read the frames carefully and follow the directions exactly. In this way you must learn.

Each Module opens with a list of **Learning outcomes** that specify exactly what you will learn by studying the contents of the Module. The material is then presented in a number of short **Units**, each designed to be studied in a single sitting. At the end of each Unit there is a **Review summary** of the topics in the Unit. Next follows a **Review exercise** of questions that directly test your understanding of the Unit material and which comes complete with worked solutions. Finally, a **Review test** enables you to consolidate your learning of the Unit material. You are strongly recommended to study the material in each Unit in a single sitting so as to ensure that you cover a complete set of ideas without a break.

Each Module ends with a checklist of **Can You?** questions that matches the Learning outcomes at the beginning of the Module, and enables you to rate your success in having achieved them. If you feel sufficiently confident then tackle the short **Test exercise** that follows. Just like the Review tests, the Test exercise is set directly on what you have learned in the Module: the questions are straightforward and contain no tricks. To provide you with the necessary practice, a set of **Further problems** is also included: do as many of these problems as you can. Remember that in mathematics, as in many other situations, practice makes perfect – or nearly so.

Useful background information

Symbols used in the text

$=$	is equal to	$\rightarrow$	tends to
$\approx$	is approximately equal to	$\neq$	is not equal to
$>$	is greater than	$\equiv$	is identical to
$\geq$	is greater than or equal to	$<$	is less than
$n!$	factorial $n = 1 \times 2 \times 3 \times \dots \times n$	$\leq$	is less than or equal to
$ k $	modulus of k , i.e. size of k	∞	infinity
	irrespective of sign	$\lim_{n \rightarrow \infty}$	limiting value as $n \rightarrow \infty$
Σ	summation		

Useful mathematical information

1 Algebraic identities

$$\begin{aligned}
 (a+b)^2 &= a^2 + 2ab + b^2 & (a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
 (a-b)^2 &= a^2 - 2ab + b^2 & (a-b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3 \\
 (a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
 (a-b)^4 &= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \\
 a^2 - b^2 &= (a-b)(a+b) & a^3 - b^3 &= (a-b)(a^2 + ab + b^2) \\
 & & a^3 + b^3 &= (a+b)(a^2 - ab + b^2)
 \end{aligned}$$

2 Trigonometrical identities

$$(a) \sin^2 \theta + \cos^2 \theta = 1; \sec^2 \theta = 1 + \tan^2 \theta; \operatorname{cosec}^2 \theta = 1 + \cot^2 \theta$$

$$(b) \sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$(c) \text{ Let } A = B = \theta \therefore \sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 1 - 2 \sin^2 \theta = 2 \cos^2 \theta - 1$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

(d) Let $\theta = \frac{\phi}{2}$ $\therefore \sin \phi = 2 \sin \frac{\phi}{2} \cos \frac{\phi}{2}$

$$\cos \phi = \cos^2 \frac{\phi}{2} - \sin^2 \frac{\phi}{2} = 1 - 2 \sin^2 \frac{\phi}{2} = 2 \cos^2 \frac{\phi}{2} - 1$$

$$\tan \phi = \frac{2 \tan \frac{\phi}{2}}{1 - \tan^2 \frac{\phi}{2}}$$

(e) $\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$

$$\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}$$

$$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$$

$$\cos D - \cos C = 2 \sin \frac{C+D}{2} \sin \frac{C-D}{2}$$

(f) $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

$$2 \cos A \sin B = \sin(A+B) - \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

(g) Negative angles: $\sin(-\theta) = -\sin \theta$

$$\cos(-\theta) = \cos \theta$$

$$\tan(-\theta) = -\tan \theta$$

(h) Angles having the same trigonometrical ratios:

(i) Same sine: θ and $(180^\circ - \theta)$

(ii) Same cosine: θ and $(360^\circ - \theta)$, i.e. $(-\theta)$

(iii) Same tangent: θ and $(180^\circ + \theta)$

(i) $a \sin \theta + b \cos \theta = A \sin(\theta + \alpha)$

$$a \sin \theta - b \cos \theta = A \sin(\theta - \alpha)$$

$$a \cos \theta + b \sin \theta = A \cos(\theta - \alpha)$$

$$a \cos \theta - b \sin \theta = A \cos(\theta + \alpha)$$

$$\text{where } \begin{cases} A = \sqrt{a^2 + b^2} \\ \alpha = \tan^{-1} \frac{b}{a} \quad (0^\circ < \alpha < 90^\circ) \end{cases}$$

3 Standard curves

(a) *Straight line*

$$\text{Slope, } m = \frac{dy}{dx} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Angle between two lines, } \tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

For parallel lines, $m_2 = m_1$

For perpendicular lines, $m_1 m_2 = -1$

Equation of a straight line (slope = m)

- (i) Intercept c on real y -axis: $y = mx + c$
- (ii) Passing through (x_1, y_1) : $y - y_1 = m(x - x_1)$
- (iii) Joining (x_1, y_1) and (x_2, y_2) : $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$

(b) *Circle*

Centre at origin, radius r : $x^2 + y^2 = r^2$
 Centre (h, k) , radius r : $(x - h)^2 + (y - k)^2 = r^2$
 General equation: $x^2 + y^2 + 2gx + 2fy + c = 0$
 with centre $(-g, -f)$: radius = $\sqrt{g^2 + f^2 - c}$
 Parametric equations: $x = r \cos \theta$, $y = r \sin \theta$

(c) *Parabola*

Vertex at origin, focus $(a, 0)$: $y^2 = 4ax$
 Parametric equations: $x = at^2$, $y = 2at$

(d) *Ellipse*

Centre at origin, foci $(\pm\sqrt{a^2 + b^2}, 0)$: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
 where a = semi-major axis, b = semi-minor axis
 Parametric equations: $x = a \cos \theta$, $y = b \sin \theta$

(e) *Hyperbola*

Centre at origin, foci $(\pm\sqrt{a^2 + b^2}, 0)$: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
 Parametric equations: $x = a \sec \theta$, $y = b \tan \theta$
 Rectangular hyperbola:

Centre at origin, vertex $\pm\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$: $xy = \frac{a^2}{2} = c^2$

$$\text{i.e. } xy = c^2 \quad \text{where } c = \frac{a}{\sqrt{2}}$$

Parametric equations: $x = ct$, $y = c/t$

4 Laws of mathematics

(a) *Associative laws* – for addition and multiplication

$$a + (b + c) = (a + b) + c$$

$$a(bc) = (ab)c$$

(b) *Commutative laws* – for addition and multiplication

$$a + b = b + a$$

$$ab = ba$$

(c) *Distributive laws* – for multiplication and division

$$a(b + c) = ab + ac$$

$$\frac{b + c}{a} = \frac{b}{a} + \frac{c}{a} \quad (\text{provided } a \neq 0)$$

Arithmetic

Learning outcomes

When you have completed this Module you will be able to:

- Carry out the basic rules of arithmetic with integers
- Check the result of a calculation making use of rounding
- Write a natural number as a product of prime numbers
- Find the highest common factor and lowest common multiple of two natural numbers
- Manipulate fractions, ratios and percentages
- Manipulate decimal numbers
- Manipulate powers
- Use standard or preferred standard form and complete a calculation to the required level of accuracy
- Understand the construction of various number systems and convert from one number system to another.

Units

1	Types of numbers	2
2	Factors and prime numbers	12
3	Fractions, ratios and percentages	17
4	Decimal numbers	27
5	Powers	35
6	Number systems	50

Types of numbers

Unit 1

1 The natural numbers

The first numbers we ever meet are the *whole numbers*, also called the *natural numbers*, and these are written down using *numerals*.

Numerals and place value

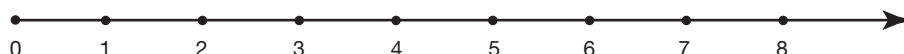
The *whole numbers* or *natural numbers* are written using the ten numerals 0, 1, ..., 9 where the position of a numeral dictates the value that it represents. For example:

246 stands for 2 hundreds and 4 tens and 6 units. That is $200 + 40 + 6$

Here the numerals 2, 4 and 6 are called the hundreds, tens and unit *coefficients* respectively. This is the place value principle.

Points on a line and order

The natural numbers can be represented by equally spaced points on a straight line where the first natural number is zero 0.

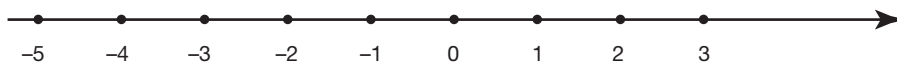


The natural numbers are ordered – they progress from small to large. As we move along the line from left to right the numbers increase as indicated by the arrow at the end of the line. On the line, numbers to the left of a given number are *less than* ($<$) the given number and numbers to the right are *greater than* ($>$) the given number. For example, $8 > 5$ because 8 is represented by a point on the line to the right of 5. Similarly, $3 < 6$ because 3 is to the left of 6.

Now move on to the next frame

2 The integers

If the straight line displaying the natural numbers is extended to the left we can plot equally spaced points to the left of zero.



These points represent *negative* numbers which are written as the natural number preceded by a minus sign, for example -4 . These positive and negative whole numbers and zero are collectively called the *integers*. The notion of order still applies. For example, $-5 < 3$ and $-2 > -4$ because the point on the line representing -5 is to the *left* of the point representing 3. Similarly, -2 is to the *right* of -4 .

The numbers -10 , 4 , 0 , -13 are of a type called

You can check your answer in the next frame

Integers

3

They are integers. The natural numbers are all positive. Now try this:

Place the appropriate symbol $<$ or $>$ between each of the following pairs of numbers:

- (a) -3 -6
 (b) 2 -4
 (c) -7 12

Complete these and check your results in the next frame

- (a) $-3 > -6$
 (b) $2 > -4$
 (c) $-7 < 12$

4

The reasons being:

- (a) $-3 > -6$ because -3 is represented on the line to the *right* of -6
 (b) $2 > -4$ because 2 is represented on the line to the *right* of -4
 (c) $-7 < 12$ because -7 is represented on the line to the *left* of 12

Now move on to the next frame

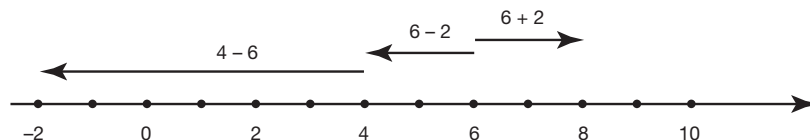
Brackets

5

Brackets should be used around negative numbers to separate the minus sign attached to the number from the arithmetic operation symbol. For example, $5 - -3$ should be written $5 - (-3)$ and 7×-2 should be written $7 \times (-2)$. *Never write two arithmetic operation symbols together without using brackets.*

Addition and subtraction

Adding two numbers gives their *sum* and subtracting two numbers gives their *difference*. For example, $6 + 2 = 8$. Adding moves to the right of the first number and subtracting moves to the left of the first number, so that $6 - 2 = 4$ and $4 - 6 = -2$:



Adding a negative number is the same as subtracting its positive counterpart. For example $7 + (-2) = 7 - 2$. The result is 5. Subtracting a negative number is the same as adding its positive counterpart. For example $7 - (-2) = 7 + 2 = 9$.

So what is the value of:

- (a) $8 + (-3)$
- (b) $9 - (-6)$
- (c) $(-4) + (-8)$
- (d) $(-14) - (-7)$?

When you have finished these check your results with the next frame

6

- | | |
|-----|-----|
| (a) | 5 |
| (b) | 15 |
| (c) | -12 |
| (d) | -7 |

Move now to Frame 7

7

Multiplication and division

Multiplying two numbers gives their *product* and dividing two numbers gives their *quotient*. Multiplying and dividing two positive or two negative numbers gives a positive number. For example:

$$12 \times 2 = 24 \text{ and } (-12) \times (-2) = 24$$

$$12 \div 2 = 6 \text{ and } (-12) \div (-2) = 6$$

Multiplying or dividing a positive number by a negative number gives a negative number. For example:

$$12 \times (-2) = -24, (-12) \div 2 = -6 \text{ and } 8 \div (-4) = -2$$

So what is the value of:

- (a) $(-5) \times 3$
- (b) $12 \div (-6)$
- (c) $(-2) \times (-8)$
- (d) $(-14) \div (-7)$?

When you have finished these check your results with the next frame

8

- | | |
|-----|-----|
| (a) | -15 |
| (b) | -2 |
| (c) | 16 |
| (d) | 2 |

Move on to Frame 9

Brackets and precedence rules

9

Brackets and the precedence rules are used to remove ambiguity in a calculation. For example, $14 - 3 \times 4$ could be either:

$$11 \times 4 = 44 \text{ or } 14 - 12 = 2$$

depending on which operation is performed first.

To remove the ambiguity we rely on the precedence rules:

In any calculation involving all four arithmetic operations we proceed as follows:

- (a) Working from the left evaluate divisions and multiplications as they are encountered;

this leaves a calculation involving just addition and subtraction.

- (b) Working from the left evaluate additions and subtractions as they are encountered.

For example, to evaluate:

$$4 + 5 \times 6 \div 2 - 12 \div 4 \times 2 - 1$$

a first sweep from left to right produces:

$$4 + 30 \div 2 - 3 \times 2 - 1$$

a second sweep from left to right produces:

$$4 + 15 - 6 - 1$$

and a final sweep produces:

$$19 - 7 = 12$$

If the calculation contains brackets then these are evaluated first, so that:

$$\begin{aligned} (4 + 5 \times 6) \div 2 - 12 \div 4 \times 2 - 1 &= 34 \div 2 - 6 - 1 \\ &= 17 - 7 \\ &= 10 \end{aligned}$$

This means that:

$$\begin{aligned} 14 - 3 \times 4 &= 14 - 12 \\ &= 2 \end{aligned}$$

because, reading from the left we multiply before we subtract. Brackets must be used to produce the alternative result:

$$\begin{aligned} (14 - 3) \times 4 &= 11 \times 4 \\ &= 44 \end{aligned}$$

because the precedence rules state that brackets are evaluated first.

So that $34 + 10 \div (2 - 3) \times 5 = \dots\dots\dots$

Result in the next frame

10**-16**

Because

$$\begin{aligned}
 34 + 10 \div (2 - 3) \times 5 &= 34 + 10 \div (-1) \times 5 && \text{we evaluate the bracket first} \\
 &= 34 + (-10) \times 5 && \text{by dividing} \\
 &= 34 + (-50) && \text{by multiplying} \\
 &= 34 - 50 && \text{finally we subtract} \\
 &= -16
 \end{aligned}$$

Notice that when brackets are used we can omit the multiplication signs and replace the division sign by a line, so that:

$$5 \times (6 - 4) \text{ becomes } 5(6 - 4)$$

and

$$(25 - 10) \div 5 \text{ becomes } (25 - 10)/5 \text{ or } \frac{25 - 10}{5}$$

When evaluating expressions containing *nested* brackets the innermost brackets are evaluated first. For example:

$$\begin{aligned}
 3(4 - 2[5 - 1]) &= 3(4 - 2 \times 4) && \text{evaluating the innermost bracket [...] first} \\
 &= 3(4 - 8) && \text{multiplication before subtraction inside the (...) bracket} \\
 &= 3(-4) && \text{subtraction completes the evaluation of the (...) bracket} \\
 &= -12 && \text{multiplication completes the calculation}
 \end{aligned}$$

so that $5 - \{8 + 7[4 - 1] - 9/3\} = \dots\dots\dots$

Work this out, the result is in the following frame

11**-21**

Because

$$\begin{aligned}
 5 - \{8 + 7[4 - 1] - 9/3\} &= 5 - \{8 + 7 \times 3 - 9 \div 3\} \\
 &= 5 - \{8 + 21 - 3\} \\
 &= 5 - \{29 - 3\} \\
 &= 5 - 26 \\
 &= -21
 \end{aligned}$$

Now move to Frame 12

Basic laws of arithmetic

12

All the work that you have done so far has been done under the assumption that you know the rules that govern the use of arithmetic operations as, indeed, you no doubt do. However, there is a difference between knowing the rules innately and being consciously aware of them, so here they are. The four basic arithmetic operations are:

addition and subtraction
multiplication and division

where each pair may be regarded as consisting of 'opposites' – in each pair one operation is the reverse operation of the other.

1 Commutativity

Two integers can be added or multiplied in either order without affecting the result. For example:

$$5 + 8 = 8 + 5 = 13 \text{ and } 5 \times 8 = 8 \times 5 = 40$$

We say that addition and multiplication are commutative operations

The order in which two integers are subtracted or divided *does* affect the result. For example:

$$4 - 2 \neq 2 - 4 \text{ because } 4 - 2 = 2 \text{ and } 2 - 4 = -2$$

Notice that $\neq$ means *is not equal to*. Also

$$4 \div 2 \neq 2 \div 4$$

We say that subtraction and division are not commutative operations

2 Associativity

The way in which three or more integers are associated under addition or multiplication does not affect the result. For example:

$$3 + (4 + 5) = (3 + 4) + 5 = 3 + 4 + 5 = 12$$

and

$$3 \times (4 \times 5) = (3 \times 4) \times 5 = 3 \times 4 \times 5 = 60$$

We say that addition and multiplication are associative operations

The way in which three or more integers are associated under subtraction or division does affect the result. For example:

$$3 - (4 - 5) \neq (3 - 4) - 5 \text{ because}$$

$$3 - (4 - 5) = 3 - (-1) = 3 + 1 = 4 \text{ and } (3 - 4) - 5 = (-1) - 5 = -6$$

Also

$$24 \div (4 \div 2) \neq (24 \div 4) \div 2 \text{ because}$$

$$24 \div (4 \div 2) = 24 \div 2 = 12 \text{ and } (24 \div 4) \div 2 = 6 \div 2 = 3$$

We say that subtraction and division are not associative operations



3 Distributivity

Multiplication is distributed over addition and subtraction from both the left and the right. For example:

$$3 \times (4 + 5) = (3 \times 4) + (3 \times 5) = 27 \text{ and } (3 + 4) \times 5 = (3 \times 5) + (4 \times 5) = 35$$

$$3 \times (4 - 5) = (3 \times 4) - (3 \times 5) = -3 \text{ and } (3 - 4) \times 5 = (3 \times 5) - (4 \times 5) = -5$$

Division is distributed over addition and subtraction from the right but not from the left. For example:

$$(60 + 15) \div 5 = (60 \div 5) + (15 \div 5) \text{ because}$$

$$(60 + 15) \div 5 = 75 \div 5 = 15 \text{ and } (60 \div 5) + (15 \div 5) = 12 + 3 = 15$$

However, $60 \div (15 + 5) \neq (60 \div 15) + (60 \div 5)$ because

$$60 \div (15 + 5) = 60 \div 20 = 3 \text{ and } (60 \div 15) + (60 \div 5) = 4 + 12 = 16$$

Also:

$$(20 - 10) \div 5 = (20 \div 5) - (10 \div 5) \text{ because}$$

$$(20 - 10) \div 5 = 10 \div 5 = 2 \text{ and } (20 \div 5) - (10 \div 5) = 4 - 2 = 2$$

but $20 \div (10 - 5) \neq (20 \div 10) - (20 \div 5)$ because

$$20 \div (10 - 5) = 20 \div 5 = 4 \text{ and } (20 \div 10) - (20 \div 5) = 2 - 4 = -2$$

On now to Frame 13

13

Estimating

Arithmetic calculations are easily performed using a calculator. However, by pressing a wrong key, wrong answers can just as easily be produced. Every calculation made using a calculator should at least be checked for the reasonableness of the final result and this can be done by *estimating* the result using *rounding*. For example, using a calculator the sum $39 + 53$ is incorrectly found to be 62 if $39 + 23$ is entered by mistake. If, now, 39 is rounded up to 40, and 53 is rounded down to 50 the reasonableness of the calculator result can be simply checked by adding 40 to 50 to give 90. This indicates that the answer 62 is wrong and that the calculation should be done again. The correct answer 92 is then seen to be close to the approximation of 90.

Rounding

An integer can be rounded to the nearest 10 as follows:

If the number is less than halfway to the next multiple of 10 then the number is rounded *down* to the previous multiple of 10. For example, 53 is rounded down to 50.

If the number is more than halfway to the next multiple of 10 then the number is rounded *up* to the next multiple of 10. For example, 39 is rounded up to 40.

If the number is exactly halfway to the next multiple of 10 then the number is rounded *up*. For example, 35 is rounded up to 40.



This principle also applies when rounding to the nearest 100, 1000, 10 000 or more. For example, 349 rounds up to 350 to the nearest 10 but rounds down to 300 to the nearest 100, and 2501 rounds up to 3000 to the nearest 1000.

Try rounding each of the following to the nearest 10, 100 and 1000 respectively:

- (a) 1846
- (b) -638
- (c) 445

Finish all three and check your results with the next frame

14

- (a) 1850, 1800, 2000
- (b) -640, -600, -1000
- (c) 450, 400, 0

Because

- (a) 1846 is nearer to 1850 than to 1840, nearer to 1800 than to 1900 and nearer to 2000 than to 1000.
- (b) -638 is nearer to -640 than to -630, nearer to -600 than to -700 and nearer to -1000 than to 0. The negative sign does not introduce any complications.
- (c) 445 rounds to 450 because it is halfway to the next multiple of 10, 445 is nearer to 400 than to 500 and nearer to 0 than 1000.

How about estimating each of the following using rounding to the nearest 10:

- (a) $18 \times 21 - 19 \div 11$
- (b) $99 \div 101 - 49 \times 8$

Check your results in Frame 15

15

- (a) 398
- (b) -499

Because

- (a) $18 \times 21 - 19 \div 11$ rounds to $20 \times 20 - 20 \div 10 = 398$
- (b) $99 \div 101 - 49 \times 8$ rounds to $100 \div 100 - 50 \times 10 = -499$

At this point let us pause and summarize the main facts so far on types of numbers



Review summary

Unit 1

16

- 1 The integers consist of the positive and negative whole numbers and zero.
- 2 The integers are ordered so that they range from large negative to small negative through zero to small positive and then large positive. They are written using the ten numerals 0 to 9 according to the principle of place value where the place of a numeral in a number dictates the value it represents.
- 3 The integers can be represented by equally spaced points on a line.
- 4 The four arithmetic operations of addition, subtraction, multiplication and division obey specific precedence rules that govern the order in which they are to be executed:

In any calculation involving all four arithmetic operations we proceed as follows:

(a) *working from the left evaluate divisions and multiplications as they are encountered.*

This leaves an expression involving just addition and subtraction:

(b) *working from the left evaluate additions and subtractions as they are encountered.*

- 5 Multiplying or dividing two positive numbers or two negative numbers produces a positive number. Multiplying or dividing a positive number and a negative number produces a negative number.
- 6 Brackets are used to group numbers and operations together. In any arithmetic expression, the contents of brackets are evaluated first.
- 7 Integers can be rounded to the nearest 10, 100 etc. and the rounded values used as estimates for the result of a calculation.



Review exercise

Unit 1

17

- 1 Place the appropriate symbol $<$ or $>$ between each of the following pairs of numbers:
 (a) -1 -6 (b) 5 -29 (c) -14 7
- 2 Find the value of each of the following:
 (a) $16 - 12 \times 4 + 8 \div 2$
 (b) $(16 - 12) \times (4 + 8) \div 2$
 (c) $9 - 3(17 + 5[5 - 7])$
 (d) $8(3[2 + 4] - 2[5 + 7])$
- 3 Show that:
 (a) $6 - (3 - 2) \neq (6 - 3) - 2$
 (b) $100 \div (10 \div 5) \neq (100 \div 10) \div 5$



(c) $24 \div (2 + 6) \neq (24 \div 2) + (24 \div 6)$

(d) $24 \div (2 - 6) \neq (24 \div 2) - (24 \div 6)$

4 Round each number to the nearest 10, 100 and 1000:

(a) 2562 (b) 1500 (c) -3451 (d) -14 525

*Complete all four questions. Take your time, there is no need to rush.**If necessary, look back at the Unit.**The answers and working are in the next frame.***18****1** (a) $-1 > -6$ because -1 is represented on the line to the right of -6 (b) $5 > -29$ because 5 is represented on the line to the right of -29 (c) $-14 < 7$ because -14 is represented on the line to the left of 7 **2** (a) $16 - 12 \times 4 + 8 \div 2 = 16 - 48 + 4 = 16 - 44 = -28$

divide and multiply before adding and subtracting

(b) $(16 - 12) \times (4 + 8) \div 2 = (4) \times (12) \div 2 = 4 \times 12 \div 2 = 4 \times 6 = 24$
brackets are evaluated first

(c) $9 - 3(17 + 5[5 - 7]) = 9 - 3(17 + 5[-2])$
 $= 9 - 3(17 - 10)$
 $= 9 - 3(7)$
 $= 9 - 21 = -12$

(d) $8(3[2 + 4] - 2[5 + 7]) = 8(3 \times 6 - 2 \times 12)$
 $= 8(18 - 24)$
 $= 8(-6) = -48$

3 (a) Left-hand side (LHS) $= 6 - (3 - 2) = 6 - (1) = 5$

Right-hand side (RHS) $= (6 - 3) - 2 = (3) - 2 = 1 \neq \text{LHS}$

(b) Left-hand side (LHS) $= 100 \div (10 \div 5) = 100 \div 2 = 50$

Right-hand side (RHS) $= (100 \div 10) \div 5 = 10 \div 5 = 2 \neq \text{LHS}$

(c) Left-hand side (LHS) $= 24 \div (2 + 6) = 24 \div 8 = 3$

Right-hand side (RHS) $= (24 \div 2) + (24 \div 6) = 12 + 4 = 16 \neq \text{LHS}$

(d) Left-hand side (LHS) $= 24 \div (2 - 6) = 24 \div (-4) = -6$

Right-hand side (RHS) $= (24 \div 2) - (24 \div 6) = 12 - 4 = 8 \neq \text{LHS}$

4 (a) 2560, 2600, 3000

(b) 1500, 1500, 2000

(c) -3450, -3500, -3000

(d) -14 530, -14 500, -15 000

Now on to the Unit test



Review test

Unit 1

19

- 1 Place the appropriate symbol $<$ or $>$ between each of the following pairs of numbers:
 (a) -3 -2 (b) 8 -13 (c) -25 0
- 2 Find the value of each of the following:
 (a) $13 + 9 \div 3 - 2 \times 5$ (b) $(13 + 9) \div (3 - 2) \times 5$
- 3 Round each number to the nearest 10, 100 and 1000:
 (a) 1354 (b) 2501 (c) -2452 (d) $-23\,625$

Factors and prime numbers

Unit 2

1 Factors

Any pair of natural numbers are called *factors* of their product. For example, the numbers 3 and 6 are factors of 18 because $3 \times 6 = 18$. These are not the only factors of 18. The complete collection of factors of 18 is 1, 2, 3, 6, 9, 18 because

$$\begin{aligned} 18 &= 1 \times 18 \\ &= 2 \times 9 \\ &= 3 \times 6 \end{aligned}$$

So the factors of:

- (a) 12
 (b) 25
 (c) 17 are

The results are in the next frame

2

- (a) 1, 2, 3, 4, 6, 12
 (b) 1, 5, 25
 (c) 1, 17

Because

- (a) $12 = 1 \times 12 = 2 \times 6 = 3 \times 4$
 (b) $25 = 1 \times 25 = 5 \times 5$
 (c) $17 = 1 \times 17$

Now move to the next frame

Prime numbers

3

If a natural number has only two factors which are itself and the number 1, the number is called a *prime number*. The first six prime numbers are 2, 3, 5, 7, 11 and 13. The number 1 is *not* a prime number because it only has one factor, namely, itself.

Prime factorization

Every natural number can be written as a product involving only prime factors. For example, the number 126 has the factors 1, 2, 3, 6, 7, 9, 14, 18, 21, 42, 63 and 126, of which 2, 3 and 7 are prime numbers and 126 can be written as:

$$126 = 2 \times 3 \times 3 \times 7$$

To obtain this *prime factorization* the number is divided by successively increasing prime numbers thus:

$$\begin{array}{r|l}
 2 & 126 \\
 \hline
 3 & 63 \\
 \hline
 3 & 21 \\
 \hline
 7 & 7 \\
 \hline
 1 &
 \end{array}
 \quad \text{so that } 126 = 2 \times 3 \times 3 \times 7$$

Notice that a prime factor may occur more than once in a prime factorization.

Now find the prime factorization of:

- (a) 84
(b) 512

Work these two out and check the working in Frame 4

4

- (a) $84 = 2 \times 2 \times 3 \times 7$
(b) $512 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

Because

$$\begin{array}{r|l}
 (a) \ 2 & 84 \\
 \hline
 2 & 42 \\
 \hline
 3 & 21 \\
 \hline
 7 & 7 \\
 \hline
 1 &
 \end{array}
 \quad \text{so that } 84 = 2 \times 2 \times 3 \times 7$$

- (b) The only prime factor of 512 is 2 which occurs 9 times. The prime factorization is:

$$512 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

Move to Frame 5

5 Highest common factor (HCF)

The *highest common factor* (HCF) of two natural numbers is the largest factor that they have in common. For example, the prime factorizations of 144 and 66 are:

$$\begin{array}{l} 144 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \\ 66 = 2 \qquad \qquad \times 3 \qquad \times 11 \end{array}$$

Only the 2 and the 3 are common to both factorizations and so the highest factor that these two numbers have in common (HCF) is $2 \times 3 = 6$.

Lowest common multiple (LCM)

The smallest natural number that each one of a pair of natural numbers divides into a whole number of times is called their *lowest common multiple* (LCM). This is also found from the prime factorization of each of the two numbers. For example:

$$\begin{array}{l} 144 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \\ 66 = 2 \qquad \qquad \times 3 \qquad \times 11 \\ \text{LCM} = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 11 = 1584 \end{array}$$

The HCF and LCM of 84 and 512 are

6

HCF: 4
LCM: 10 752

Because

84 and 512 have the prime factorizations:

$$\begin{array}{l} 84 = 2 \times 2 \qquad \qquad \qquad \times 3 \times 7 \\ 512 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \qquad \qquad \text{HCF} = 2 \times 2 = 4 \\ \text{LCM} = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 7 = 10\,752 \end{array}$$

At this point let us pause and summarize the main facts on factors and prime numbers



Review summary

Unit 2

7

- 1 A pair of natural numbers are called factors of their product.
- 2 If a natural number only has one and itself as factors it is called a prime number.
- 3 Every natural number can be written as a product of its prime factors, some of which may be repeated.
- 4 The highest common factor (HCF) is the highest factor that two natural numbers have in common.
- 5 The lowest common multiple (LCM) is the lowest natural number that two natural numbers will divide into a whole number of times.



Review exercise

Unit 2

8

- 1 Write each of the following as a product of prime factors:
(a) 429 (b) 1820 (c) 2992 (d) 3185
- 2 Find the HCF and the LCM of each pair of numbers:
(a) 63, 42 (b) 92, 34

*Complete both questions. Work through Unit 2 again if you need to.
Don't rush. Take your time.
The answers and working are in the next frame.*

9

1 (a)

3	429
11	143
13	13
1	

$429 = 3 \times 11 \times 13$

(b)

2	1820
2	910
5	455
7	91
13	13
1	

$1820 = 2 \times 2 \times 5 \times 7 \times 13$



$$\begin{array}{r}
 \text{(c)} \quad 2 \overline{) 2992} \\
 \underline{2} 1496 \\
 2 \overline{) 1496} \\
 \underline{2} 748 \\
 2 \overline{) 748} \\
 \underline{2} 374 \\
 11 \overline{) 374} \\
 \underline{11} 187 \\
 17 \overline{) 187} \\
 \underline{17} 1 \\
 \underline{1}
 \end{array}
 \quad 2992 = 2 \times 2 \times 2 \times 2 \times 11 \times 17$$

$$\begin{array}{r}
 \text{(d)} \quad 5 \overline{) 3185} \\
 \underline{5} 637 \\
 7 \overline{) 637} \\
 \underline{7} 91 \\
 7 \overline{) 91} \\
 \underline{7} 13 \\
 13 \overline{) 13} \\
 \underline{13} 1 \\
 \underline{1}
 \end{array}
 \quad 3185 = 5 \times 7 \times 7 \times 13$$

2 (a) The prime factorizations of 63 and 42 are:

$$\begin{aligned}
 63 &= 3 \times 3 \times 7 \\
 42 &= 2 \times 3 \times 7 \quad \text{HCF } 3 \times 7 = 21 \\
 \text{LCM} &= 2 \times 3 \times 3 \times 7 = 126
 \end{aligned}$$

(b) The prime factorizations of 34 and 92 are:

$$\begin{aligned}
 34 &= 2 \times 17 \\
 92 &= 2 \times 2 \times 23 \quad \text{HCF } 2 \\
 \text{LCM} &= 2 \times 2 \times 17 \times 23 = 1564
 \end{aligned}$$

Now for the Review test



Review test

Unit 2

10

1 Write each of the following as a product of prime factors:

(a) 170 (b) 455 (c) 9075 (d) 1140

2 Find the HCF and the LCM of each pair of numbers:

(a) 84, 88 (b) 105, 66

Fractions, ratios and percentages

Unit 3

Division of integers

1

A fraction is a number which is represented by one integer – the *numerator* – divided by another integer – the *denominator* (or the *divisor*). For example, $\frac{3}{5}$ is a fraction with numerator 3 and denominator 5. Because fractions are written as one integer divided by another – a *ratio* – they are called *rational* numbers. Fractions are either *proper*, *improper* or *mixed*:

- in a proper fraction the numerator is less than the denominator, for example $\frac{4}{7}$
- in an improper fraction the numerator is greater than the denominator, for example $\frac{12}{5}$
- a mixed fraction is in the form of an integer and a fraction, for example $6\frac{2}{3}$

So that $-\frac{8}{11}$ is a fraction?

The answer is in the next frame

Proper

2

Fractions can be either positive or negative.

Now to the next frame

Multiplying fractions

3

Two fractions are multiplied by multiplying their respective numerators and denominators independently. For example:

$$\frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7} = \frac{10}{21}$$

Try this one for yourself. $\frac{5}{9} \times \frac{2}{7} = \dots\dots\dots$

$\frac{10}{63}$

4

Because

$$\frac{5}{9} \times \frac{2}{7} = \frac{5 \times 2}{9 \times 7} = \frac{10}{63}$$

Correct? Then on to the next frame

5 of

The word 'of' when interposed between two fractions means multiply. For example:

Half of half a cake is one-quarter of a cake. That is

$$\frac{1}{2} \text{ of } \frac{1}{2} = \frac{1}{2} \times \frac{1}{2} = \frac{1 \times 1}{2 \times 2} = \frac{1}{4}$$

So that, for example:

$$\frac{1}{3} \text{ of } \frac{2}{5} = \frac{1}{3} \times \frac{2}{5} = \frac{1 \times 2}{3 \times 5} = \frac{2}{15}$$

So that $\frac{3}{8} \text{ of } \frac{5}{7} = \dots\dots\dots$

6

$\frac{15}{56}$

Because

$$\frac{3}{8} \text{ of } \frac{5}{7} = \frac{3}{8} \times \frac{5}{7} = \frac{3 \times 5}{8 \times 7} = \frac{15}{56}$$

On now to the next frame

7 **Equivalent fractions**

Multiplying the numerator and denominator by the same number is equivalent to multiplying the fraction by unity, that is by 1:

$$\frac{4 \times 3}{5 \times 3} = \frac{4}{5} \times \frac{3}{3} = \frac{4}{5} \times 1 = \frac{4}{5}$$

Now, $\frac{4 \times 3}{5 \times 3} = \frac{12}{15}$ so that the fraction $\frac{4}{5}$ and the fraction $\frac{12}{15}$ both represent the same number and for this reason we call $\frac{4}{5}$ and $\frac{12}{15}$ equivalent fractions.

A second fraction, equivalent to a first fraction, can be found by multiplying the numerator and the denominator of the first fraction by the same number.

So that if we multiply the numerator and denominator of the fraction $\frac{7}{5}$ by 4 we obtain the equivalent fraction $\dots\dots\dots$

Check your result in Frame 8

$$\frac{28}{20}$$

8

Because

$$\frac{7 \times 4}{5 \times 4} = \frac{28}{20}$$

We can reverse this process and find the equivalent fraction that has the smallest numerator by *cancelling out* common factors. This is known as reducing the fraction to its *lowest terms*. For example:

$\frac{16}{96}$ can be reduced to its lowest terms as follows:

$$\frac{16}{96} = \frac{4 \times 4}{24 \times 4} = \frac{4 \times \cancel{4}}{24 \times \cancel{4}} = \frac{4}{24}$$

by cancelling out the 4 in the numerator and the denominator

The fraction $\frac{4}{24}$ can also be reduced:

$$\frac{4}{24} = \frac{4}{6 \times 4} = \frac{\cancel{4}}{6 \times \cancel{4}} = \frac{1}{6}$$

Because $\frac{1}{6}$ cannot be reduced further we see that $\frac{16}{96}$ reduced to its lowest terms is $\frac{1}{6}$.

How about this one? The fraction $\frac{84}{108}$ reduced to its lowest terms is

Check with the next frame

$$\frac{7}{9}$$

9

Because

$$\frac{84}{108} = \frac{7 \times 3 \times 4}{9 \times 3 \times 4} = \frac{7 \times \cancel{3} \times \cancel{4}}{9 \times \cancel{3} \times \cancel{4}} = \frac{7}{9}$$

Now move on to the next frame

10 Dividing fractions

The expression $6 \div 3$ means the number of 3's in 6, which is 2. Similarly, the expression $1 \div \frac{1}{4}$ means the number of $\frac{1}{4}$'s in 1, which is, of course, 4. That is:

$$1 \div \frac{1}{4} = 4 = 1 \times \frac{4}{1}. \quad \text{Notice how the numerator and the denominator of the divisor are switched and the division replaced by multiplication.}$$

Two fractions are divided by switching the numerator and the denominator of the divisor and multiplying the result. For example:

$$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5} = \frac{14}{15}$$

$$\text{So that } \frac{7}{13} \div \frac{3}{4} = \dots\dots\dots$$

11

$$\frac{28}{39}$$

Because

$$\frac{7}{13} \div \frac{3}{4} = \frac{7}{13} \times \frac{4}{3} = \frac{28}{39}$$

In particular:

$$1 \div \frac{3}{5} = 1 \times \frac{5}{3} = \frac{5}{3}$$

The fraction $\frac{5}{3}$ is called the *reciprocal* of $\frac{3}{5}$

So that the reciprocal of $\frac{17}{4}$ is $\dots\dots\dots$

12

$$\frac{4}{17}$$

Because

$$1 \div \frac{17}{4} = 1 \times \frac{4}{17} = \frac{4}{17}$$

And the reciprocal of -5 is $\dots\dots\dots$

$$-\frac{1}{5}$$

13

Because

$$1 \div (-5) = 1 \div \left(-\frac{5}{1}\right) = 1 \times \left(-\frac{1}{5}\right) = -\frac{1}{5}$$

Move on to the next frame

Adding and subtracting fractions

14

Two fractions can only be added or subtracted immediately if they both possess the same denominator, in which case we add or subtract the numerators and divide by the common denominator. For example:

$$\frac{2}{7} + \frac{3}{7} = \frac{2+3}{7} = \frac{5}{7}$$

If they do not have the same denominator they must be rewritten in equivalent form so that they do have the same denominator – called the *common denominator*. For example:

$$\frac{2}{3} + \frac{1}{5} = \frac{10}{15} + \frac{3}{15} = \frac{10+3}{15} = \frac{13}{15}$$

The common denominator of the equivalent fractions is the LCM of the two original denominators. That is:

$$\frac{2}{3} + \frac{1}{5} = \frac{2 \times 5}{3 \times 5} + \frac{1 \times 3}{5 \times 3} = \frac{10}{15} + \frac{3}{15} \text{ where 15 is the LCM of 3 and 5}$$

So that $\frac{5}{9} + \frac{1}{6} = \dots\dots\dots$

The result is in Frame 15

$$\frac{13}{18}$$

15

Because

$$\begin{aligned} \text{The LCM of 9 and 6 is 18 so that } \frac{5}{9} + \frac{1}{6} &= \frac{5 \times 2}{9 \times 2} + \frac{1 \times 3}{6 \times 3} = \frac{10}{18} + \frac{3}{18} \\ &= \frac{10+3}{18} = \frac{13}{18} \end{aligned}$$

There's another one to try in the next frame

Now try $\frac{11}{15} - \frac{2}{3} = \dots\dots\dots$

16

17

$$\frac{1}{15}$$

Because

$$\begin{aligned}\frac{11}{15} - \frac{2}{3} &= \frac{11}{15} - \frac{2 \times 5}{3 \times 5} = \frac{11}{15} - \frac{10}{15} \\ &= \frac{11 - 10}{15} = \frac{1}{15} \quad (15 \text{ is the LCM of 3 and 15})\end{aligned}$$

Correct? Then on to Frame 18

18 Fractions on a calculator

The a^b/c button on a calculator enables fractions to be entered and manipulated with the results given in fractional form. For example, to evaluate $\frac{2}{3} \times 1\frac{3}{4}$ using your calculator [*note: your calculator may not produce the identical display in what follows*]:

Enter the number 2

Press the a^b/c key

Enter the number 3

The display now reads $2 \frac{_}{3}$ to represent $\frac{2}{3}$

Press the $\times$ key

Enter the number 1

Press the a^b/c key

Enter the number 3

Press the a^b/c key

Enter the number 4

The display now reads $1 \frac{_}{4}$ to represent $1\frac{3}{4}$

Press the $=$ key to display the result $1 \frac{_}{6}$, that is:

$$\frac{2}{3} \times 1\frac{3}{4} = \frac{2}{3} \times \frac{7}{4} = \frac{14}{12} = 1\frac{1}{6}$$

Now use your calculator to evaluate each of the following:

(a) $\frac{5}{7} + 3\frac{2}{3}$

(b) $\frac{8}{3} - \frac{5}{11}$

(c) $\frac{13}{5} \times \frac{4}{7} - \frac{2}{9}$

(d) $4\frac{1}{11} \div \left(-\frac{3}{5}\right) + \frac{1}{8}$

Check your answers in Frame 19

19

$$(a) 4 \text{ } \sqcup \text{ } 8 \text{ } \sqcup \text{ } 21 = 4\frac{8}{21}$$

$$(b) 2 \text{ } \sqcup \text{ } 7 \text{ } \sqcup \text{ } 33 = 2\frac{7}{33}$$

$$(c) 1 \text{ } \sqcup \text{ } 83 \text{ } \sqcup \text{ } 315 = 1\frac{83}{315}$$

$$(d) -6 \text{ } \sqcup \text{ } 61 \text{ } \sqcup \text{ } 88 = -6\frac{61}{88}$$

In (d) enter the $\frac{3}{5}$ and then press the $\frac{\text{+}}{\text{=}}$ key.

On now to the next frame

Ratios

20

If a whole number is separated into a number of fractional parts where each fraction has the same denominator, the numerators of the fractions form a *ratio*. For example, if a quantity of brine in a tank contains $\frac{1}{3}$ salt and $\frac{2}{3}$ water, the salt and water are said to be in the ratio 'one-to-two' – written 1 : 2.

What ratio do the components A, B and C form if a compound contains $\frac{3}{4}$ of A, $\frac{1}{6}$ of B and $\frac{1}{12}$ of C?

Take care here and check your results with Frame 21

9 : 2 : 1

21

Because the LCM of the denominators 4, 6 and 12 is 12, then:

$\frac{3}{4}$ of A is $\frac{9}{12}$ of A, $\frac{1}{6}$ of B is $\frac{2}{12}$ of B and the remaining $\frac{1}{12}$ is of C. This ensures that the components are in the ratio of their numerators. That is:

9 : 2 : 1

Notice that the sum of the numbers in the ratio is equal to the common denominator.

On now to the next frame

Percentages

22

A percentage is a fraction whose denominator is equal to 100. For example, if 5 out of 100 people are left-handed then the fraction of left-handers is $\frac{5}{100}$ which is written as 5%, that is 5 *per cent* (%).

So if 13 out of 100 cars on an assembly line are red, the percentage of red cars on the line is

23

13%

Because

The fraction of cars that are red is $\frac{13}{100}$ which is written as 13%.

Try this. What is the percentage of defective resistors in a batch of 25 if 12 of them are defective?

24

48%

Because

The fraction of defective resistors is $\frac{12}{25} = \frac{12 \times 4}{25 \times 4} = \frac{48}{100}$ which is written as 48%. Notice that this is the same as:

$$\left(\frac{12}{25} \times 100\right)\% = \left(\frac{12}{25} \times 25 \times 4\right)\% = (12 \times 4)\% = 48\%$$

A fraction can be converted to a percentage by multiplying the fraction by 100.

To find the percentage part of a quantity we multiply the quantity by the percentage written as a fraction. For example, 24% of 75 is:

$$\begin{aligned} 24\% \text{ of } 75 &= \frac{24}{100} \text{ of } 75 = \frac{24}{100} \times 75 = \frac{6 \times 4}{25 \times 4} \times 25 \times 3 = \frac{6 \times \cancel{4}}{\cancel{25} \times \cancel{4}} \times \cancel{25} \times 3 \\ &= 6 \times 3 = 18 \end{aligned}$$

So that 8% of 25 is

Work it through and check your results with the next frame

25

2

Because

$$\frac{8}{100} \times 25 = \frac{2 \times 4}{25 \times 4} \times 25 = \frac{2 \times \cancel{4}}{\cancel{25} \times \cancel{4}} \times \cancel{25} = 2$$

At this point let us pause and summarize the main facts on fractions, ratios and percentages



Review summary

Unit 3

26

- 1 A fraction is a number represented as one integer (the numerator) divided by another integer (the denominator or divisor).
- 2 The same number can be represented by different but equivalent fractions.
- 3 A fraction with no common factors other than unity in its numerator and denominator is said to be in its lowest terms.
- 4 Two fractions are multiplied by multiplying the numerators and denominators independently.
- 5 Two fractions can only be added or subtracted immediately when their denominators are equal.
- 6 A ratio consists of the numerators of fractions with identical denominators.
- 7 The numerator of a fraction whose denominator is 100 is called a percentage.



Review exercise

Unit 3

27

- 1 Reduce each of the following fractions to their lowest terms:

(a) $\frac{24}{30}$ (b) $\frac{72}{15}$ (c) $-\frac{52}{65}$ (d) $\frac{32}{8}$

- 2 Evaluate the following:

(a) $\frac{5}{9} \times \frac{2}{5}$ (b) $\frac{13}{25} \div \frac{2}{15}$ (c) $\frac{5}{9} + \frac{3}{14}$ (d) $\frac{3}{8} - \frac{2}{5}$

(e) $\frac{12}{7} \times \left(-\frac{3}{5}\right)$ (f) $\left(-\frac{3}{4}\right) \div \left(-\frac{12}{7}\right)$ (g) $\frac{19}{2} + \frac{7}{4}$ (h) $\frac{1}{4} - \frac{3}{8}$

- 3 Write the following proportions as ratios:

(a) $\frac{1}{2}$ of A, $\frac{2}{5}$ of B and $\frac{1}{10}$ of C

(b) $\frac{1}{3}$ of P, $\frac{1}{5}$ of Q, $\frac{1}{4}$ of R and the remainder S

- 4 Complete the following:

(a) $\frac{2}{5} = \quad \%$ (b) 58% of 25 =

(c) $\frac{7}{12} = \quad \%$ (d) 17% of 50 =

Complete the questions.

Look back at the Unit if necessary but don't rush.

The answers and working are in the next frame.

28

- 1 (a) $\frac{24}{30} = \frac{2 \times 2 \times 2 \times 3}{2 \times 3 \times 5} = \frac{2 \times 2}{5} = \frac{4}{5}$
 (b) $\frac{72}{15} = \frac{2 \times 2 \times 2 \times 3 \times 3}{3 \times 5} = \frac{2 \times 2 \times 2 \times 3}{5} = \frac{24}{5}$
 (c) $-\frac{52}{65} = -\frac{2 \times 2 \times 13}{5 \times 13} = -\frac{2 \times 2}{5} = -\frac{4}{5}$
 (d) $\frac{32}{8} = \frac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2} = 4$
- 2 (a) $\frac{5}{9} \times \frac{2}{5} = \frac{5 \times 2}{9 \times 5} = \frac{2}{9}$
 (b) $\frac{13}{25} \div \frac{2}{15} = \frac{13}{25} \times \frac{15}{2} = \frac{13 \times 15}{25 \times 2} = \frac{13 \times 3 \times 5}{5 \times 5 \times 2} = \frac{39}{10}$
 (c) $\frac{5}{9} + \frac{3}{14} = \frac{5 \times 14}{9 \times 14} + \frac{3 \times 9}{14 \times 9} = \frac{70}{126} + \frac{27}{126} = \frac{70 + 27}{126} = \frac{97}{126}$
 (d) $\frac{3}{8} - \frac{2}{5} = \frac{3 \times 5}{8 \times 5} - \frac{2 \times 8}{5 \times 8} = \frac{15}{40} - \frac{16}{40} = \frac{15 - 16}{40} = -\frac{1}{40}$
 (e) $\frac{12}{7} \times \left(-\frac{3}{5}\right) = \frac{12 \times (-3)}{7 \times 5} = \frac{-36}{35} = -\frac{36}{35}$
 (f) $\left(-\frac{3}{4}\right) \div \left(-\frac{12}{7}\right) = \left(-\frac{3}{4}\right) \times \left(-\frac{7}{12}\right) = \frac{(-3) \times (-7)}{4 \times 12} = \frac{3 \times 7}{4 \times 3 \times 4} = \frac{7}{16}$
 (g) $\frac{19}{2} + \frac{7}{4} = \frac{38}{4} + \frac{7}{4} = \frac{45}{4}$
 (h) $\frac{1}{4} - \frac{3}{8} = \frac{2}{8} - \frac{3}{8} = -\frac{1}{8}$
- 3 (a) $\frac{1}{2}, \frac{2}{5}, \frac{1}{10} = \frac{5}{10}, \frac{4}{10}, \frac{1}{10}$ so ratio is 5 : 4 : 1
 (b) $\frac{1}{3}, \frac{1}{5}, \frac{1}{4} = \frac{20}{60}, \frac{12}{60}, \frac{15}{60}$ and $\frac{20}{60} + \frac{12}{60} + \frac{15}{60} = \frac{47}{60}$
 so the fraction of S is $\frac{13}{60}$
 so P, Q, R and S are in the ratio 20 : 12 : 15 : 13
- 4 (a) $\frac{2}{5} = \frac{2 \times 20}{5 \times 20} = \frac{40}{100}$ that is 40% or $\frac{2}{5} = \frac{2}{5} \times 100\% = 40\%$
 (b) $\frac{58}{100} \times 25 = \frac{58}{4} = \frac{29}{2} = 14\frac{1}{2}$
 (c) $\frac{7}{12} = \frac{7}{12} \times 100\% = \frac{700}{12}\% = \frac{58 \times 12 + 4}{12}\% = 58\frac{4}{12}\% = 58\frac{1}{3}\%$
 (d) $\frac{17}{100} \times 50 = \frac{17}{2} = 8\frac{1}{2}$

Now for the Review test



Review test

Unit 3

29

1 Reduce each of the following fractions to their lowest terms:

(a) $\frac{12}{18}$ (b) $\frac{144}{21}$ (c) $-\frac{49}{14}$ (d) $\frac{64}{4}$

2 Evaluate the following:

(a) $\frac{3}{7} \times \frac{2}{3}$ (b) $\frac{11}{30} \div \frac{5}{6}$ (c) $\frac{3}{7} + \frac{4}{13}$ (d) $\frac{5}{16} - \frac{4}{3}$

3 Write the following proportions as ratios:

(a) $\frac{1}{2}$ of A, $\frac{1}{5}$ of B and $\frac{3}{10}$ of C

(b) $\frac{1}{4}$ of P, $\frac{1}{3}$ of Q, $\frac{1}{5}$ of R and the remainder S

4 Complete the following:

(a) $\frac{4}{5} = \quad \%$ (b) 48% of 50 =

(c) $\frac{9}{14} = \quad \%$ (d) 15% of 25 =

Decimal numbers

Unit 4

Division of integers

1

If one integer is divided by a second integer that is not one of the first integer's factors the result will not be another integer. Instead, the result will lie between two integers. For example, using a calculator it is seen that:

$$25 \div 8 = 3.125$$

which is a number greater than 3 but less than 4. As with integers, the position of a numeral within the number indicates its value. Here the number 3.125 represents

3 units + 1 tenth + 2 hundredths + 5 thousandths.

$$\text{That is } 3 + \frac{1}{10} + \frac{2}{100} + \frac{5}{1000}$$

where the decimal point shows the separation of the units from the tenths. Numbers written in this format are called *decimal numbers*.

On to the next frame

2 Rounding

All the operations of arithmetic that we have used with the integers apply to decimal numbers. However, when performing calculations involving decimal numbers it is common for the end result to be a number with a large quantity of numerals after the decimal point. For example:

$$15 \cdot 11 \div 8 \cdot 92 = 1 \cdot 6939461883 \dots$$

To make such numbers more manageable or more reasonable as the result of a calculation, they can be rounded either to a specified number of *significant figures* or to a specified number of *decimal places*.

Now to the next frame

3 Significant figures

Significant figures are counted from the first non-zero numeral encountered starting from the left of the number. When the required number of significant figures has been counted off, the remaining numerals are deleted with the following proviso:

If the first of a group of numerals to be deleted is a 5 or more, the last significant numeral is increased by 1. For example:

9·4534 to two significant figures is 9·5, to three significant figures is 9·45, and 0·001354 to two significant figures is 0·0014

Try this one for yourself. To four significant figures the number 18·7249 is

Check your result with the next frame

4

18·72

Because

The first numeral deleted is a 4 which is less than 5.

There is one further proviso. If the only numeral to be dropped is a 5 then the last numeral retained is rounded up. So that 12·235 to four significant figures (abbreviated to *sig fig*) is 12·24 and 3·465 to three sig fig is 3·47.

So 8·1265 to four sig fig is

Check with the next frame

8·127

5

Because

The only numeral deleted is a 5 and the last numeral is rounded up.

Now on to the next frame

Decimal places

6

Decimal places are counted to the right of the decimal point and the same rules as for significant figures apply for rounding to a specified number of decimal places (abbreviated to *dp*). For example:

123·4467 to one decimal place is 123·4 and to two dp is 123·45

So, 47·0235 to three dp is

47·024

7

Because

The only numeral dropped is a 5 and the last numeral retained is odd so is increased to the next even numeral.

Now move on to the next frame

Trailing zeros

8

Sometimes zeros must be inserted within a number to satisfy a condition for a specified number of either significant figures or decimal places. For example:

12 645 to two significant figures is 13 000, and 13·1 to three decimal places is 13·100.

These zeros are referred to as *trailing zeros*.

So that 1515 to two sig fig is

1500

9

And:

25·13 to four dp is

25·1300

10

On to the next frame

11 Fractions as decimals

Because a fraction is one integer divided by another it can be represented in decimal form simply by executing the division. For example:

$$\frac{7}{4} = 7 \div 4 = 1.75$$

So that the decimal form of $\frac{3}{8}$ is

12

0.375

Because

$$\frac{3}{8} = 3 \div 8 = 0.375$$

Now move on to the next frame

13 Decimals as fractions

A decimal can be represented as a fraction. For example:

$$1.224 = \frac{1224}{1000} \text{ which in lowest terms is } \frac{153}{125}$$

So that 0.52 as a fraction in lowest terms is

14

$\frac{13}{25}$

Because

$$0.52 = \frac{52}{100} = \frac{13}{25}$$

Now move on to the next frame

15 Unending decimals

Converting a fraction into its decimal form by performing the division always results in an infinite string of numerals after the decimal point. This string of numerals may contain an infinite sequence of zeros or it may contain an infinitely repeated pattern of numerals. A repeated pattern of numerals can be written in an abbreviated format. For example:

$$\frac{1}{3} = 1 \div 3 = 0.3333 \dots$$

Here the pattern after the decimal point is of an infinite number of 3's. We abbreviate this by placing a dot over the first 3 to indicate the repetition, thus:

$$0.3333 \dots = 0.\dot{3} \quad (\text{described as zero point 3 recurring})$$



For other fractions the repetition may consist of a sequence of numerals, in which case a dot is placed over the first and last numeral in the sequence. For example:

$$\frac{1}{7} = 0.142857142857142857 \dots = 0.\dot{1}4285\dot{7}$$

So that we write $\frac{2}{11} = 0.181818 \dots$ as

0. $\dot{1}8$

16

Sometimes the repeating pattern is formed by an infinite sequence of zeros, in which case we simply omit them. For example:

$$\frac{1}{5} = 0.20000 \dots \text{ is written as } 0.2$$

Next frame

Unending decimals as fractions

17

Any decimal that displays an unending repeating pattern can be converted to its fractional form. For example:

To convert $0.181818 \dots = 0.\dot{1}8$ to its fractional form we note that because there are two repeating numerals we multiply by 100 to give:

$$100 \times 0.\dot{1}8 = 18.\dot{1}8$$

Subtracting $0.\dot{1}8$ from both sides of this equation gives:

$$100 \times 0.\dot{1}8 - 0.\dot{1}8 = 18.\dot{1}8 - 0.\dot{1}8$$

That is:

$$99 \times 0.\dot{1}8 = 18.0$$

This means that:

$$0.\dot{1}8 = \frac{18}{99} = \frac{2}{11}$$



Similarly, the fractional form of $2.\dot{0}\dot{3}1\dot{5}$ is found as follows:

$2.\dot{0}\dot{3}1\dot{5} = 2.0 + 0.\dot{0}\dot{3}1\dot{5}$ and, because there are three repeating numerals:

$$1000 \times 0.\dot{0}\dot{3}1\dot{5} = 31.5\dot{3}1\dot{5}$$

Subtracting $0.\dot{0}\dot{3}1\dot{5}$ from both sides of this equation gives:

$$1000 \times 0.\dot{0}\dot{3}1\dot{5} - 0.\dot{0}\dot{3}1\dot{5} = 31.5\dot{3}1\dot{5} - 0.\dot{0}\dot{3}1\dot{5} = 31.5$$

That is:

$$999 \times 0.\dot{0}\dot{3}1\dot{5} = 31.5 \text{ so that } 0.\dot{0}\dot{3}1\dot{5} = \frac{31.5}{999} = \frac{315}{9990}$$

This means that:

$$2.\dot{0}\dot{3}1\dot{5} = 2.0 + 0.\dot{0}\dot{3}1\dot{5} = 2 + \frac{315}{9990} = 2\frac{35}{1110} = 2\frac{7}{222}$$

What are the fractional forms of $0.\dot{2}\dot{1}$ and $3.\dot{2}\dot{1}$?

The answers are in the next frame

18

$$\frac{7}{33} \text{ and } 3\frac{19}{90}$$

Because

$$100 \times 0.\dot{2}\dot{1} = 21.\dot{2}\dot{1} \text{ so that } 99 \times 0.\dot{2}\dot{1} = 21$$

$$\text{giving } 0.\dot{2}\dot{1} = \frac{21}{99} = \frac{7}{33} \text{ and}$$

$$3.\dot{2}\dot{1} = 3.2 + 0.0\dot{1} \text{ and } 10 \times 0.0\dot{1} = 0.1\dot{1} \text{ so that } 9 \times 0.0\dot{1} = 0.1 \text{ giving}$$

$$0.0\dot{1} = \frac{0.1}{9} = \frac{1}{90}, \text{ hence } 3.\dot{2}\dot{1} = \frac{32}{10} + \frac{1}{90} = \frac{289}{90} = 3\frac{19}{90}$$

19

Rational, irrational and real numbers

A number that can be expressed as a fraction is called a *rational* number. An *irrational* number is one that *cannot* be expressed as a fraction and has a decimal form consisting of an infinite string of numerals that does not display a repeating pattern. As a consequence it is not possible either to write down the complete decimal form or to devise an abbreviated decimal format. Instead, we can only round them to a specified number of significant figures or decimal places. Alternatively, we may have a numeral representation for them, such as $\sqrt{2}$, e or π . The complete collection of rational and irrational numbers is called the collection of *real* numbers.

At this point let us pause and summarize the main facts so far on decimal numbers



Review summary

Unit 4

20

- 1 Every fraction can be written as a decimal number by performing the division.
- 2 The decimal number obtained will consist of an infinitely repeating pattern of numerals to the right of one of its digits.
- 3 Other decimals, with an infinite, non-repeating sequence of numerals after the decimal point are the irrational numbers.
- 4 A decimal number can be rounded to a specified number of significant figures (sig fig) by counting from the first non-zero numeral on the left.
- 5 A decimal number can be rounded to a specified number of decimal places (dp) by counting from the decimal point.



Review exercise

Unit 4

21

- 1 Round each of the following decimal numbers, first to 3 significant figures and then to 2 decimal places:
 (a) 12.455 (b) 0.01356 (c) 0.1005 (d) 1344.555
- 2 Write each of the following in abbreviated form:
 (a) 12.110110110... (b) 0.123123123...
 (c) $-3.11111...$ (d) $-9360.936093609360...$
- 3 Convert each of the following to decimal form to 3 decimal places:
 (a) $\frac{3}{16}$ (b) $-\frac{5}{9}$
 (c) $\frac{7}{6}$ (d) $-\frac{24}{11}$
- 4 Convert each of the following to fractional form in lowest terms:
 (a) 0.6 (b) $1.\dot{4}$
 (c) $1.\dot{2}\dot{4}$ (d) $-7.\dot{3}$

Complete all four questions. Take your time, there is no need to rush.

If necessary, look back at the Unit.

The answers and working are in the next frame.

22

- 1 (a) 12.5, 12.46 (b) 0.0136, 0.01
(c) 0.101, 0.10 (d) 1340, 1344.56
- 2 (a) $12.\dot{1}1\dot{0}$ (b) $0.\dot{1}2\dot{3}$
(c) $-3.\dot{1}$ (d) $-9360.\dot{9}36\dot{0}$
- 3 (a) $\frac{3}{16} = 0.1875 = 0.188$ to 3 dp
(b) $-\frac{5}{9} = -0.555\dots = -0.556$ to 3 dp
(c) $\frac{7}{6} = 1.1666\dots = 1.167$ to 3 dp
(d) $-\frac{24}{11} = -2.1818\dots = -2.182$ to 3 dp
- 4 (a) $0.6 = \frac{6}{10} = \frac{3}{5}$
(b) $1.\dot{4} = 1 + \frac{4}{9} = \frac{13}{9}$
(c) $1.\dot{2}4 = 1 + \frac{24}{99} = \frac{123}{99} = \frac{41}{33}$
(d) $-7.\dot{3} = -\frac{73}{10}$

Now for the Review test

Review test

Unit 4

23

- 1 Round each of the following decimal numbers, first to 3 significant figures and then to 2 decimal places:
(a) 21.355 (b) 0.02456
(c) 0.3105 (d) 5134.555
- 2 Convert each of the following to decimal form to 3 decimal places:
(a) $\frac{4}{15}$ (b) $-\frac{7}{13}$ (c) $\frac{9}{5}$ (d) $-\frac{28}{13}$
- 3 Convert each of the following to fractional form in lowest terms:
(a) 0.8 (b) 2.8 (c) $3.\dot{3}\dot{2}$ (d) -5.5
- 4 Write each of the following in abbreviated form:
(a) 1.010101... (b) 9.2456456456...

Powers

Unit 5

Raising a number to a power

1

The arithmetic operation of raising a number to a *power* is devised from repetitive multiplication. For example:

$$10 \times 10 \times 10 \times 10 = 10^4 - \text{the number 10 multiplied by itself 4 times}$$

The power is also called an *index* and the number to be raised to the power is called the *base*. Here the number 4 is the power (index) and 10 is the base.

So $5 \times 5 \times 5 \times 5 \times 5 \times 5 = \dots\dots\dots$ (in the form of 5 raised to a power)

Compare your answer with the next frame

$$5^6$$

2

Because the number 5 (the base) is multiplied by itself 6 times (the power or index).

Now to the next frame

The laws of powers

3

The laws of powers are contained within the following set of rules:

- *Power unity*

Any number raised to the power 1 equals itself.

$$3^1 = 3$$

So $99^1 = \dots\dots\dots$

On to the next frame

$$99$$

4

Because any number raised to the power 1 equals itself.



- *Multiplication of numbers and the addition of powers*

If two numbers are each written as a given base raised to some power then the *product of the two numbers* is equal to the same base raised to the *sum of the powers*. For example, $16 = 2^4$ and $8 = 2^3$ so:

$$\begin{aligned} 16 \times 8 &= 2^4 \times 2^3 \\ &= (2 \times 2 \times 2 \times 2) \times (2 \times 2 \times 2) \\ &= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \\ &= 2^7 \\ &= 2^{4+3} \\ &= 128 \end{aligned}$$

Multiplication requires powers to be added.

So $8^3 \times 8^5 = \dots\dots\dots$ (in the form of 8 raised to a power)

Next frame

5

8^8

Because multiplication requires powers to be added.

Notice that we cannot combine different powers with different bases. For example:

$2^2 \times 4^3$ cannot be written as 8^5

but we can combine different bases to the same power. For example:

$3^4 \times 5^4$ can be written as 15^4 because

$$\begin{aligned} 3^4 \times 5^4 &= (3 \times 3 \times 3 \times 3) \times (5 \times 5 \times 5 \times 5) \\ &= 15 \times 15 \times 15 \times 15 \\ &= 15^4 \\ &= (3 \times 5)^4 \end{aligned}$$

So that $2^3 \times 4^3$ can be written as $\dots\dots\dots$ (in the form of a number raised to a power)

6

8^3

Next frame

7

- *Division of numbers and the subtraction of powers*

If two numbers are each written as a given base raised to some power then the *quotient of the two numbers* is equal to the same base raised to the *difference of the powers*. For example:

$$\begin{aligned}
 15\,625 \div 25 &= 5^6 \div 5^2 \\
 &= (5 \times 5 \times 5 \times 5 \times 5 \times 5) \div (5 \times 5) \\
 &= \frac{5 \times 5 \times 5 \times 5 \times 5 \times 5}{5 \times 5} \\
 &= 5 \times 5 \times 5 \times 5 \\
 &= 5^4 \\
 &= 5^{6-2} \\
 &= 625
 \end{aligned}$$

Division requires powers to be subtracted.

So $12^7 \div 12^3 = \dots\dots\dots$ (in the form of 12 raised to a power)

Check your result in the next frame

12⁴

8

Because division requires the powers to be subtracted.

- *Power zero*

Any number raised to the power 0 equals unity. For example:

$$\begin{aligned}
 1 &= 3^1 \div 3^1 \\
 &= 3^{1-1} \\
 &= 3^0
 \end{aligned}$$

So $193^0 = \dots\dots\dots$

9

1

Because any number raised to the power 0 equals unity.

- *Negative powers*

A number raised to a negative power denotes the reciprocal. For example:

$$\begin{aligned} 6^{-2} &= 6^{0-2} \\ &= 6^0 \div 6^2 && \text{subtraction of powers means division} \\ &= 1 \div 6^2 && \text{because } 6^0 = 1 \\ &= \frac{1}{6^2} \end{aligned}$$

$$\text{Also } 6^{-1} = \frac{1}{6}$$

A negative power denotes the reciprocal.

So $3^{-5} = \dots\dots\dots$

10

 $\frac{1}{3^5}$

Because

$$3^{-5} = 3^{0-5} = 3^0 \div 3^5 = \frac{1}{3^5}$$

A negative power denotes the reciprocal.

Now to the next frame

11

- *Multiplication of powers*

If a number is written as a given base raised to some power then that number *raised to a further power* is equal to the base raised to the *product of the powers*. For example:

$$\begin{aligned} (25)^3 &= (5^2)^3 \\ &= 5^2 \times 5^2 \times 5^2 \\ &= 5 \times 5 \times 5 \times 5 \times 5 \times 5 \\ &= 5^6 \\ &= 5^{2 \times 3} \\ &= 15\,625 \quad \text{Notice that } (5^2)^3 \neq 5^{2^3} \text{ because } 5^{2^3} = 5^8 = 390\,625. \end{aligned}$$

Raising to a power requires powers to be multiplied.

So $(4^2)^4 = \dots\dots\dots$ (in the form of 4 raised to a power)

$$4^8$$

12

Because raising to a power requires powers to be multiplied.

Now to the next frame.

Powers on a calculator

13

Powers on a calculator can be evaluated by using the x^y key. For example, enter the number 4, press the x^y key, enter the number 3 and press =. The result is 64 which is 4^3 .

Try this one for yourself. To two decimal places, the value of $1.3^{3.4}$ is

The result is in the following frame

$$2.44$$

14

Because

Enter the number 1.3

Press the x^y key

Enter the number 3.4

Press the = key

The number displayed is 2.44 to 2 dp.

Now try this one using the calculator:

$8^{\frac{1}{3}}$ = The $1/3$ is a problem, use the $a^{b/c}$ key.

Check your answer in the next frame

$$2$$

15

Because

Enter the number 8

Press the x^y key

Enter the number 1

Press the $a^{b/c}$ key

Enter the number 3

Press =

the number 2 is displayed.

Now move on to the next frame

16 Fractional powers and roots

We have just seen that $8^{\frac{1}{3}} = 2$. We call $8^{\frac{1}{3}}$ the *third root* or, alternatively, the *cube root* of 8 because

$$\left(8^{\frac{1}{3}}\right)^3 = 8 \quad \text{the number 8 is the result of raising the 3rd root of 8 to the power 3}$$

Roots are denoted by such fractional powers. For example, the 5th root of 6 is given as $6^{\frac{1}{5}}$ because

$$\left(6^{\frac{1}{5}}\right)^5 = 6$$

and by using a calculator $6^{\frac{1}{5}}$ can be seen to be equal to 1.431 to 3 dp. Odd roots are unique in the real number system but even roots are not. For example, there are two 2nd roots – *square roots* – of 4, namely:

$$4^{\frac{1}{2}} = 2 \text{ and } 4^{\frac{1}{2}} = -2 \text{ because } 2 \times 2 = 4 \text{ and } (-2) \times (-2) = 4$$

Similarly:

$$81^{\frac{1}{4}} = \pm 3$$

Odd roots of negative numbers are themselves negative. For example:

$$(-32)^{\frac{1}{5}} = -2 \text{ because } \left[(-32)^{\frac{1}{5}}\right]^5 = (-2)^5 = -32$$

Even roots of negative numbers, however, pose a problem. For example, because

$$\left[(-1)^{\frac{1}{2}}\right]^2 = (-1)^1 = -1$$

we conclude that the square root of -1 is $(-1)^{\frac{1}{2}}$. Unfortunately, we cannot write this as a decimal number – we cannot find its decimal value because there is no decimal number which when multiplied by itself gives -1 . We have to accept the fact that, at this stage, we cannot find the even roots of a negative number. This would be the subject matter for a book of more advanced mathematics.

Surds

An alternative notation for the square root of 4 is the surd notation $\sqrt{4}$ and, by convention, this is always taken to mean the positive square root. This notation can also be extended to other roots, for example, $\sqrt[3]{9}$ is an alternative notation for $9^{\frac{1}{3}}$.

Use your calculator to find the value of each of the following roots to 3 dp:

$$(a) 16^{\frac{1}{7}} \quad (b) \sqrt{8} \quad (c) 19^{\frac{1}{4}} \quad (d) \sqrt{-4}$$

Answers in the next frame

17

- (a) 1.486 use the x^y key
 (b) 2.828 the positive value only
 (c) ± 2.088 there are two values for even roots
 (d) We cannot find the square root of a negative number

*On now to Frame 18***Multiplication and division by integer powers of 10**

18

If a decimal number is multiplied by 10 raised to an integer power, the decimal point moves the integer number of places to the right if the integer is positive and to the left if the integer is negative. For example:

$$1.2345 \times 10^3 = 1234.5 \text{ (3 places to the right) and}$$

$$1.2345 \times 10^{-2} = 0.012345 \text{ (2 places to the left).}$$

Notice that, for example:

$$1.2345 \div 10^3 = 1.2345 \times 10^{-3} \text{ and}$$

$$1.2345 \div 10^{-2} = 1.2345 \times 10^2$$

So now try these:

- (a) 0.012045×10^4
 (b) 13.5074×10^{-3}
 (c) $144.032 \div 10^5$
 (d) $0.012045 \div 10^{-2}$

Work all four out and then check your results with the next frame

19

- (a) 120.45
 (b) 0.0135074
 (c) 0.00144032
 (d) 1.2045

Because

- (a) multiplying by 10^4 moves the decimal point 4 places to the right
 (b) multiplying by 10^{-3} moves the decimal point 3 places to the left
 (c) $144.032 \div 10^5 = 144.032 \times 10^{-5}$ move the decimal point 5 places to the left
 (d) $0.012045 \div 10^{-2} = 0.012045 \times 10^2$ move the decimal point 2 places to the right

Now move on to the next frame

20 Precedence rules

With the introduction of the arithmetic operation of raising to a power we need to amend our earlier precedence rules – *evaluating powers is performed before dividing and multiplying*. For example:

$$\begin{aligned} 5(3 \times 4^2 \div 6 - 7) &= 5(3 \times 16 \div 6 - 7) \\ &= 5(48 \div 6 - 7) \\ &= 5(8 - 7) \\ &= 5 \end{aligned}$$

So that:

$$14 \div (125 \div 5^3 \times 4 + 3) = \dots\dots\dots$$

Check your result in the next frame

21

2

Because

$$\begin{aligned} 14 \div (125 \div 5^3 \times 4 + 3) &= 14 \div (125 \div 125 \times 4 + 3) \\ &= 14 \div (4 + 3) \\ &= 2 \end{aligned}$$

22 Standard form

Any decimal number can be written as a decimal number greater than or equal to 1 and less than 10 (called the *mantissa*) multiplied by the number 10 raised to an appropriate power (the power being called the *exponent*). For example:

$$57.3 = 5.73 \times 10^1$$

$$423.8 = 4.238 \times 10^2$$

$$6042.3 = 6.0423 \times 10^3$$

and $0.267 = 2.67 \div 10 = 2.67 \times 10^{-1}$

$$0.000485 = 4.85 \div 10^4 = 4.85 \times 10^{-4} \text{ etc.}$$

So, written in standard form:

(a) $52\,674 = \dots\dots\dots$ (c) $0.0582 = \dots\dots\dots$

(b) $0.00723 = \dots\dots\dots$ (d) $1\,523\,800 = \dots\dots\dots$

23

- | | |
|---------------------------|---------------------------|
| (a) 5.2674×10^4 | (c) 5.82×10^{-2} |
| (b) 7.23×10^{-3} | (d) 1.5238×10^6 |

Working in standard form

Numbers written in standard form can be multiplied or divided by multiplying or dividing the respective mantissas and adding or subtracting the respective exponents. For example:

$$\begin{aligned}
 0.84 \times 23\,000 &= (8.4 \times 10^{-1}) \times (2.3 \times 10^4) \\
 &= (8.4 \times 2.3) \times 10^{-1} \times 10^4 \\
 &= 19.32 \times 10^3 \\
 &= 1.932 \times 10^4
 \end{aligned}$$

Another example:

$$\begin{aligned}
 175.4 \div 6340 &= (1.754 \times 10^2) \div (6.34 \times 10^3) \\
 &= (1.754 \div 6.34) \times 10^2 \div 10^3 \\
 &= 0.2767 \times 10^{-1} \\
 &= 2.767 \times 10^{-2} \text{ to 4 sig fig}
 \end{aligned}$$

Where the result obtained is not in standard form, the mantissa is written in standard number form and the necessary adjustment made to the exponent.

In the same way, then, giving the results in standard form to 4 dp:

- (a) $472.3 \times 0.000564 = \dots\dots\dots$
 (b) $752\,000 \div 0.862 = \dots\dots\dots$

24

- | |
|-----------------------------|
| (a) 2.6638×10^{-1} |
| (b) 8.7239×10^5 |

Because

$$\begin{aligned}
 \text{(a)} \quad 472.3 \times 0.000564 &= (4.723 \times 10^2) \times (5.64 \times 10^{-4}) \\
 &= (4.723 \times 5.64) \times 10^2 \times 10^{-4} \\
 &= 26.638 \times 10^{-2} = 2.6638 \times 10^{-1} \\
 \text{(b)} \quad 752\,000 \div 0.862 &= (7.52 \times 10^5) \div (8.62 \times 10^{-1}) \\
 &= (7.52 \div 8.62) \times 10^5 \times 10^1 \\
 &= 0.87239 \times 10^6 = 8.7239 \times 10^5
 \end{aligned}$$

For *addition and subtraction in standard form* the approach is slightly different.



Example 1

$$4.72 \times 10^3 + 3.648 \times 10^4$$

Before these can be added, the powers of 10 must be made the same:

$$\begin{aligned} 4.72 \times 10^3 + 3.648 \times 10^4 &= 4.72 \times 10^3 + 36.48 \times 10^3 \\ &= (4.72 + 36.48) \times 10^3 \\ &= 41.2 \times 10^3 = 4.12 \times 10^4 \text{ in standard form} \end{aligned}$$

Similarly in the next example.

Example 2

$$13.26 \times 10^{-3} - 1.13 \times 10^{-2}$$

Here again, the powers of 10 must be equalized:

$$\begin{aligned} 13.26 \times 10^{-3} - 1.13 \times 10^{-2} &= 1.326 \times 10^{-2} - 1.13 \times 10^{-2} \\ &= (1.326 - 1.13) \times 10^{-2} \\ &= 0.196 \times 10^{-2} = 1.96 \times 10^{-3} \text{ in standard form} \end{aligned}$$

Using a calculator

Numbers given in standard form can be manipulated on a calculator by making use of the EXP key. For example, to enter the number 1.234×10^3 , enter 1.234 and then press the EXP key. The display then changes to:

1.234 00

Now enter the power 3 and the display becomes:

1.234 03

Manipulating numbers in this way produces a result that is in ordinary decimal format. If the answer is required in standard form then it will have to be converted by hand. For example, using the EXP key on a calculator to evaluate $(1.234 \times 10^3) + (2.6 \times 10^2)$ results in the display 1494 which is then converted by hand to 1.494×10^3 .

Therefore, working in standard form:

- (a) $43.6 \times 10^2 + 8.12 \times 10^3 = \dots\dots\dots$
 - (b) $7.84 \times 10^5 - 12.36 \times 10^3 = \dots\dots\dots$
 - (c) $4.25 \times 10^{-3} + 1.74 \times 10^{-2} = \dots\dots\dots$
-

25

- (a) 1.248×10^4
 (b) 7.7164×10^5
 (c) 2.165×10^{-2}

Preferred standard form

In the SI system of units, it is recommended that when a number is written in standard form, the power of 10 should be restricted to powers of 10^3 , i.e. 10^3 , 10^6 , 10^{-3} , 10^{-6} , etc. Therefore in this *preferred standard form* up to three figures may appear in front of the decimal point.

In practice it is best to write the number first in standard form and to adjust the power of 10 to express this in preferred standard form.

Example 1

$$\begin{aligned} & 5.2746 \times 10^4 \text{ in standard form} \\ &= 5.2746 \times 10 \times 10^3 \\ &= 52.746 \times 10^3 \text{ in preferred standard form} \end{aligned}$$

Example 2

$$\begin{aligned} & 3.472 \times 10^8 \text{ in standard form} \\ &= 3.472 \times 10^2 \times 10^6 \\ &= 347.2 \times 10^6 \text{ in preferred standard form} \end{aligned}$$

Example 3

$$\begin{aligned} & 3.684 \times 10^{-2} \text{ in standard form} \\ &= 3.684 \times 10 \times 10^{-3} \\ &= 36.84 \times 10^{-3} \text{ in preferred standard form} \end{aligned}$$

So, rewriting the following in preferred standard form, we have

- | | |
|--|--|
| (a) $8.236 \times 10^7 = \dots\dots\dots$ | (d) $6.243 \times 10^5 = \dots\dots\dots$ |
| (b) $1.624 \times 10^{-4} = \dots\dots\dots$ | (e) $3.274 \times 10^{-2} = \dots\dots\dots$ |
| (c) $4.827 \times 10^4 = \dots\dots\dots$ | (f) $5.362 \times 10^{-7} = \dots\dots\dots$ |

26

- | | |
|----------------------------|----------------------------|
| (a) 82.36×10^6 | (d) 624.3×10^3 |
| (b) 162.4×10^{-6} | (e) 32.74×10^{-3} |
| (c) 48.27×10^3 | (f) 536.2×10^{-9} |

One final exercise on this piece of work:

Example 4

The product of (4.72×10^2) and (8.36×10^5)

- (a) in standard form =
 (b) in preferred standard form =

27

- | |
|--------------------------|
| (a) 3.9459×10^8 |
| (b) 394.59×10^6 |

Because

$$\begin{aligned} \text{(a)} \quad (4.72 \times 10^2) \times (8.36 \times 10^5) &= (4.72 \times 8.36) \times 10^2 \times 10^5 \\ &= 39.459 \times 10^7 \\ &= 3.9459 \times 10^8 \text{ in standard form} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad (4.72 \times 10^2) \times (8.36 \times 10^5) &= 3.9459 \times 10^2 \times 10^6 \\ &= 394.59 \times 10^6 \text{ in preferred standard form} \end{aligned}$$

Now move on to the next frame

28**Checking calculations**

When performing a calculation involving decimal numbers it is always a good idea to check that your result is reasonable and that an arithmetic blunder or an error in using the calculator has not been made. This can be done using standard form. For example:

$$\begin{aligned} 59.2347 \times 289.053 &= 5.92347 \times 10^1 \times 2.89053 \times 10^2 \\ &= 5.92347 \times 2.89053 \times 10^3 \end{aligned}$$

This product can then be estimated for reasonableness as:

$$6 \times 3 \times 1000 = 18\,000 \text{ (see Frames 13–15 of Unit 1)}$$

The answer using the calculator is 17 121.968 to three decimal places, which is 17 000 when rounded to the nearest 1000. This compares favourably with the estimated 18 000, indicating that the result obtained could be reasonably expected.

So, the estimated value of 800.120×0.007953 is

Check with the next frame

6.4

29

Because

$$\begin{aligned} 800.120 \times 0.007953 &= 8.00120 \times 10^2 \times 7.953 \times 10^{-3} \\ &= 8.00120 \times 7.9533 \times 10^{-1} \end{aligned}$$

This product can then be estimated for reasonableness as:

$$8 \times 8 \div 10 = 6.4$$

The exact answer is 6.36 to two decimal places.

Now move on to the next frame

Accuracy

30

Many calculations are made using numbers that have been obtained from measurements. Such numbers are only accurate to a given number of significant figures but using a calculator can produce a result that contains as many figures as its display will permit. Because any calculation involving measured values will not be accurate to *more significant figures than the least number of significant figures in any measurement*, we can justifiably round the result down to a more manageable number of significant figures.

For example:

The base length and height of a rectangle are measured as 114.8 mm and 18 mm respectively. The area of the rectangle is given as the product of these lengths. Using a calculator this product is 2066.4 mm². Because one of the lengths is only measured to 2 significant figures, the result cannot be accurate to more than 2 significant figures. It should therefore be read as 2100 mm².

Assuming the following contains numbers obtained by measurement, use a calculator to find the value to the correct level of accuracy:

$$19.1 \times 0.0053 \div 13.345$$

0.0076

31

Because

The calculator gives the result as 0.00758561 but because 0.0053 is only accurate to 2 significant figures the result cannot be accurate to more than 2 significant figures, namely 0.0076.

At this point let us pause and summarize the main facts so far on powers



Review summary

Unit 5

32

- 1 Powers are devised from repetitive multiplication of a given number.
- 2 Negative powers denote reciprocals and any number raised to the power 0 is unity.
- 3 Multiplication of a decimal number by 10 raised to an integer power moves the decimal point to the right if the power is positive and to the left if the power is negative.
- 4 A decimal number written in standard form is in the form of a mantissa (a number between 1 and 10 but excluding 10) multiplied by 10 raised to an integer power, the power being called the exponent.
- 5 Writing decimal numbers in standard form permits an estimation of the reasonableness of a calculation.
- 6 In preferred standard form the powers of 10 in the exponent are restricted to multiples of 3.
- 7 If numbers used in a calculation are obtained from measurement, the result of the calculation is a number accurate to no more than the least number of significant figures in any measurement.



Review exercise

Unit 5

33

- 1 Write each of the following as a number raised to a power:
(a) $5^8 \times 5^2$ (b) $6^4 \div 6^6$ (c) $(7^4)^3$ (d) $(19^{-8})^0$
- 2 Find the value of each of the following to 3 dp:
(a) $16^{\frac{1}{4}}$ (b) $\sqrt[3]{3}$ (c) $(-8)^{\frac{1}{5}}$ (d) $(-7)^{\frac{1}{4}}$
- 3 Write each of the following as a single decimal number:
(a) 1.0521×10^3 (b) 123.456×10^{-2}
(c) $0.0135 \div 10^{-3}$ (d) $165.21 \div 10^4$
- 4 Write each of the following in standard form:
(a) 125.87 (b) 0.0101 (c) 1.345 (d) 10.13
- 5 Write each of the following in preferred standard form:
(a) 1.3204×10^5 (b) 0.0101 (c) 1.345 (d) 9.5032×10^{-8}



- 6 In each of the following the numbers have been obtained by measurement. Evaluate each calculation to the appropriate level of accuracy:

$$(a) 13.6 \div 0.012 \times 7.63 - 9015 \quad (b) \frac{0.003 \times 194}{13.6}$$

$$(c) 19.3 \times 1.04^{2.00} \quad (d) \frac{18 \times 2.1 - 3.6 \times 0.54}{8.6 \times 2.9 + 5.7 \times 9.2}$$

Complete all of these questions.

Look back at the Unit if you need to.

You can check your answers and working in the next frame.

34

- 1 (a) $5^8 \times 5^2 = 5^{8+2} = 5^{10}$ (b) $6^4 \div 6^6 = 6^{4-6} = 6^{-2}$ (c) $(7^4)^3 = 7^{4 \times 3} = 7^{12}$
 (d) $(19^{-8})^0 = 1$ as any number raised to the power 0 equals unity
- 2 (a) $16^{\frac{1}{4}} = \pm 2.000$ (b) $\sqrt[3]{3} = 1.442$
 (c) $(-8)^{\frac{1}{5}} = -1.516$
 (d) $(-7)^{\frac{1}{4}}$ You cannot find the even root of a negative number
- 3 (a) $1.0521 \times 10^3 = 1052.1$ (b) $123.456 \times 10^{-2} = 1.23456$
 (c) $0.0135 \div 10^{-3} = 0.0135 \times 10^3 = 13.5$
 (d) $165.21 \div 10^4 = 165.21 \times 10^{-4} = 0.016521$
- 4 (a) $125.87 = 1.2587 \times 10^2$ (b) $0.0101 = 1.01 \times 10^{-2}$
 (c) $1.345 = 1.345 \times 10^0$ (d) $10.13 = 1.013 \times 10^1 = 1.013 \times 10$
- 5 (a) $1.3204 \times 10^5 = 132.04 \times 10^3$ (b) $0.0101 = 10.1 \times 10^{-3}$
 (c) $1.345 = 1.345 \times 10^0$ (d) $9.5032 \times 10^{-8} = 95.032 \times 10^{-9}$
- 6 (a) $13.6 \div 0.012 \times 7.63 - 9015 = -367.\dot{6} = -370$ to 2 sig fig
 (b) $\frac{0.003 \times 194}{13.6} = 0.042794 \dots = 0.04$ to 1 sig fig
 (c) $19.3 \times 1.04^{2.00} = 19.3 \times 1.0816 = 20.87488 = 20.9$ to 2 sig fig
 (d) $\frac{18 \times 2.1 - 3.6 \times 0.54}{8.6 \times 2.9 + 5.7 \times 9.2} = \frac{35.856}{77.38} = 0.46337554 \dots = 0.463$ to 3 sig fig

Now for the Review test



Review test

Unit 5

35

- 1 Write each of the following as a number raised to a power:
 (a) $3^6 \times 3^3$ (b) $4^3 \div 2^5$ (c) $(9^2)^3$ (d) $(7^0)^{-8}$
- 2 Find the value of each of the following to 3 dp:
 (a) $15^{\frac{1}{3}}$ (b) $\sqrt[5]{5}$ (c) $(-27)^{\frac{1}{3}}$ (d) $(-9)^{\frac{1}{2}}$
- 3 Write each of the following as a single decimal number:
 (a) 3.2044×10^3 (b) $16.1105 \div 10^{-2}$
- 4 Write each of the following in standard form:
 (a) 134.65 (b) 0.002401
- 5 Write each of the following in preferred standard form:
 (a) $16.1105 \div 10^{-2}$ (b) 9.3304
- 6 In each of the following the numbers have been obtained by measurement. Evaluate each calculation to the appropriate level of accuracy:
 (a) $11.4 \times 0.0013 \div 5.44 \times 8.810$
 (b) $\frac{1.01 \div 0.00335}{9.12 \times 6.342}$

Number systems

Unit 6

1 Denary (or decimal) system

This is our basic system in which quantities large or small can be represented by use of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 together with appropriate place values according to their positions.

For example	2	7	6	5	.	3	2	1_{10}
has place values	10^3	10^2	10^1	10^0		10^{-1}	10^{-2}	10^{-3}
	1000	100	10	1		$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$

In this case, the place values are powers of 10, which gives the name *denary* (or *decimal*) to the system. The denary system is said to have a *base* of 10. You are, of course, perfectly familiar with this system of numbers, but it is included here as it leads on to other systems which have the same type of structure but which use different place values.

So let us move on to the next system

Binary system

2

This is widely used in all forms of switching applications. The only symbols used are 0 and 1 and the place values are powers of 2, i.e. the system has a base of 2.

$$\begin{array}{rcll}
 \text{For example} & 1 & 0 & 1 & 1 & \cdot & 1 & 0 & 1_2 \\
 \text{has place values} & 2^3 & 2^2 & 2^1 & 2^0 & & 2^{-1} & 2^{-2} & 2^{-3} \\
 \text{i.e.} & 8 & 4 & 2 & 1 & & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \\
 \\
 \text{So:} & 1 & 0 & 1 & 1 & \cdot & 1 & 0 & 1 & \text{in the binary system} \\
 \\
 = & 1 \times 8 & 0 \times 4 & 1 \times 2 & 1 \times 1 & 1 \times \frac{1}{2} & 0 \times \frac{1}{4} & 1 \times \frac{1}{8} \\
 = & 8 & + & 0 & + & 2 & + & 1 & + & \frac{1}{2} & + & 0 & + & \frac{1}{8} & \text{in denary} \\
 = & 11\frac{5}{8} = 11.625 & \text{in the denary system. Therefore } 1011.101_2 = 11.625_{10}
 \end{array}$$

The small subscripts 2 and 10 indicate the bases of the two systems. In the same way, the denary equivalent of $1\ 1\ 0\ 1\cdot\ 0\ 1\ 1_2$ is to 3 dp.

$$13.375_{10}$$

3

Because

$$\begin{array}{rcll}
 & 1 & 1 & 0 & 1 & \cdot & 0 & 1 & 1_2 \\
 = & 8 & + & 4 & + & 0 & + & 1 & + & 0 & + & \frac{1}{4} & + & \frac{1}{8} \\
 = & 13\frac{3}{8} = 13.375_{10}
 \end{array}$$

Octal system (base 8)

This system uses the symbols

0, 1, 2, 3, 4, 5, 6, 7

with place values that are powers of 8.

$$\begin{array}{rcll}
 \text{For example} & 3 & 5 & 7 & \cdot & 3 & 2 & 1 & \text{in the octal system} \\
 \text{has place values} & 8^2 & 8^1 & 8^0 & & 8^{-1} & 8^{-2} & 8^{-3} \\
 \text{i.e.} & 64 & 8 & 1 & & \frac{1}{8} & \frac{1}{64} & \frac{1}{512} \\
 \\
 \text{So} & 3 & 5 & 7 & \cdot & 3 & 2 & 1 & 1_8 \\
 = & 3 \times 64 & 5 \times 8 & 7 \times 1 & 3 \times \frac{1}{8} & 2 \times \frac{1}{64} & 1 \times \frac{1}{512} \\
 = & 192 & + & 40 & + & 7 & + & \frac{3}{8} & + & \frac{1}{32} & + & \frac{1}{512} \\
 = & 239\frac{209}{512} = 239.4082_{10}
 \end{array}$$

That is

$$357.321_8 = 239.408_{10} \text{ to 3 dp}$$

