Set Valued Mappings with Applications in Nonlinear Analysis

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Preface

This book is a collection of research articles related to the mathematical analysis of multifunctions. By a set valued map $F : X \to 2^Y$ we simply mean a map that assigns to each $x \in X$ a subset $F(x) \subseteq Y$. The theory of set valued maps is a beautiful mixture of analysis, topology and geometry. Over the last thirty years or so there has been a huge interest in this area of research. This is partly due to the rich and plentiful supply of applications in such diverse fields as for example Biology, Control theory and Optimization, Economics, Game theory and Physics. This book titled 'set valued mappings with applications in nonlinear analysis' contains 29 research articles from leading mathematicians in this area from around the world. Topological methods in the study of nonlinear phenomena is the central theme. As a result the chapters were selected accordingly and no attempt was made to cover every area in this vast field. The topics covered in this book can be grouped in the following major areas: integral inclusions, ordinary and partial differential inclusions, fixed point theorems, boundary value problems, variational inequalities, game theory, optimal control, abstract economics, and nonlinear spectra.

In particular the theory of set valued maps is used in the chapters of Agarwal, Meehan and O'Regan, Andres, Candito, Kamenski and Nistri, Kryszewski, Matzakos and Papageorgiou, and Palmucci and Papalini to present results for differential and integral inclusions in various settings. The Baire category method is used by De Blasi and Pianigiani to discuss existence problems for partial differential inclusions. Structure of solution sets is addressed by Agarwal and O'Regan, and Obukhovskii and Zecca. The chapter of Matzakos, Papageorgiou and Yannakakis contains results on optimal control for nonlinear parabolic partial differential equations. Many new fixed point theorems for set valued maps are contained in the contributions of Agarwal and O'Regan, Daffer and Kaneko, Frigon, Morales, Ricceri, and Takahashi. Nonlinear spectral theory is discussed by Appell, Conti and Santucci, random fixed point theory by Shahzad, and fuzzy mappings by Cho, Shim, Huang and Kang. In a long survey chapter Milojević presents new results in the theory of A-proper maps. Variational inequalities are discussed in the long survey article of Chowdhury and Tarafdar, and in the chapters of Isac, Tarafdar and Yuan, and Park. Maximal element principles are presented in the contributions of Ding, and Isac and Yuan. Applications of fixed point theory in abstract economies and game theory appear in the chapter of Tan and Wu. Some interesting fixed point algorithms are contained in the chapters of Reich and Zaslavski, and Verma.

We wish to express our appreciation to all the contributors. Without their cooperation this book would not have been possible.

Ravi P Agarwal
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1. Positive L^p and Continuous Solutions for Fredholm Integral Inclusions

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Abstract: In this chapter a multivalued version of Krasnoselski's fixed point theorem in a cone is used to discuss the existence of $C[0,T]$ and $L^p[0,T]$ solutions to the nonlinear integral inclusion $y(t) \in \int_0^T k(t,s)\, f(s, y(s))\, ds$. Throughout we will assume k: $[0,T] \times [0,T] \to \mathbf{R}$ and f: $[0,T] \times \mathbf{R} \to 2^{\mathbf{R}}$.

1. INTRODUCTION

In this chapter we present new results which guarantee that the Fredholm integral inclusion

$$y(t) \in \int_0^T k(t,s) f(s, y(s)) ds \tag{1.1}$$

has a positive solution $y \in L^p[0,T]$, $1 \le p < \infty$, or has a nonnegative solution $y \in C[0,T]$. Throughout this chapter $T > 0$ is fixed, k: $[0,T] \times [0,T] \to \mathbf{R}$ and f: $[0,T] \times \mathbf{R} \to 2^{\mathbf{R}}$; here $2^{\mathbf{R}}$ denotes the family of nonempty subsets of $\mathbf{R}$. It is only recently [6] that a general theory has been developed which guarantees that the operator equation, $y(t) = \int_0^T k(t,s) g(s, y(s))\, ds$ for a.e. $t \in [0,T]$, has a positive solution $y \in L^p[0,T]$ (note by a positive solution we mean $y(t) > 0$ for a.e. $t \in [0,T]$); here g: $[0,T] \times \mathbf{R} \to \mathbf{R}$ is single valued. In Section 2 using the 1991 paper of Cellina *et al.* [3] we are able to establish criteria which guarantees that (1.1) has a positive solution $y \in L^p[0,T]$. Section 3 discusses $C[0,T]$ solutions to (1.1); the results here improve those in [1].

The main idea in this chapter relies on the multivalued analogue [1] of Krasnoselski's fixed point theorem in a cone. Let $E = (E, \|\cdot\|)$ be a Banach space and $C \subseteq E$. For $\rho > 0$ let

$$\Omega_\rho = \{x \in E : \|x\| < \rho\} \quad \text{and} \quad \partial\Omega_\rho = \{x \in E : \|x\| = \rho\}.$$

Theorem 1.1: *Let $E = (E, \|\cdot\|)$ be a Banach space, $C \subseteq E$ a cone and let $\|\cdot\|$ be increasing with respect to C. Also r, R are constants with $0 < r < R$. Suppose $A: \overline{\Omega_R} \cap C \to K(C)$ (here $K(C)$ denotes the family of nonempty, convex, compact subsets of C) is an upper semicontinuous, compact map and assume* one *of the following conditions*

(*A*) $\|y\| \le \|x\|$ *for all* $y \in A(x)$ *and* $x \in \partial\Omega_R \cap C$ *and* $\|y\| > \|x\|$ *for all* $y \in A(x)$ *and* $x \in \partial\Omega_r \cap C$

or

(*B*) $\|y\| > \|x\|$ *for all* $y \in A(x)$ *and* $x \in \partial\Omega_R \cap C$ *and* $\|y\| \le \|x\|$ *for all* $y \in A(x)$ *and* $x \in \partial\Omega_r \cap C$

hold. Then A has a fixed point in $C \cap (\overline{\Omega_R} \backslash \Omega_r)$.

2. $L^p[0, T]$ SOLUTIONS

In this section we discuss the nonlinear Fredholm integral inclusion

$$y(t) \in \int_0^T k(t,s) f(s, y(s)) ds \quad \text{a.e. } t \in [0, T], \tag{2.1}$$

where k: $[0,T] \times [0,T] \to \mathbf{R}$ and f: $[0,T] \times \mathbf{R} \to K(\mathbf{R})$. We would like to know what conditions one requires on k and f in order that the inclusion (2.1) has a positive solution $y \in L^p[0,T]$, where $1 \le p < \infty$. Here by a positive solution y we mean $y(t) > 0$ for a.e. $t \in [0,T]$. Throughout this section $\|\cdot\|_q$ denotes the usual norm on L^q for $1 \le q \le \infty$.

Theorem 2.1: *Let* k: $[0,T] \times [0,T] \to \mathbf{R}$ *and* f: $[0,T] \times \mathbf{R} \to K(\mathbf{R})$ *and suppose the following conditions hold*:

$$\text{the map } u \mapsto f(t,u) \text{ is upper semicontinuous for a.e. } t \in [0,T]; \tag{2.2}$$

$$\begin{aligned} &\text{the graph of } f \text{ belongs to the } \sigma\text{-field } \mathcal{L} \otimes \mathcal{B}(\mathbf{R} \times \mathbf{R}) \\ &(\text{here } \mathcal{L} \text{ denotes the Lebesgue } \sigma\text{-field on } [0,T] \text{ and } \mathcal{B}(\mathbf{R} \times \mathbf{R}) \\ &= \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R}) \text{ is the Borel } \sigma\text{-field in } \mathbf{R} \times \mathbf{R}); \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\exists\, p_2, 1 \le p_2 < \infty, a_1 \in L^{p_2}[0,T] \text{ and } a_2 > 0 \text{ a constant, with} \\ &|f(t,y)| = \sup\{|z| : z \in f(t,y)\} \le a_1(t) + a_2 |y|^{\frac{p}{p_2}} \\ &\text{for a.e. } t \in [0,T] \text{ and all } y \in \mathbf{R}; \end{aligned} \tag{2.4}$$

$$(t,s) \mapsto k(t,s) \text{ is measurable}; \tag{2.5}$$

$$\begin{aligned} &\exists\, 0 < M \le 1, k_1 \in L^p[0,T], k_2 \in L^{p_1}[0,T], \text{ here } \tfrac{1}{p_1} + \tfrac{1}{p_2} = 1, \text{ such that} \\ &0 < k_1(t), k_2(t) \text{ a.e. } t \in [0,T] \text{ and } M k_1(t) k_2(s) \le k(t,s) \le k_1(t) k_2(s) \\ &\text{a.e. } t \in [0,T], \text{ a.e. } s \in [0,T]; \end{aligned} \tag{2.6}$$

$$\text{for a.e. } t \in [0,T] \text{ and all } y \in (0,\infty),\ u > 0 \text{ for all } u \in f(t,y); \tag{2.7}$$

$$\begin{array}{l}\exists\, q \in L^{p_2}[0,T] \text{ and } \psi\colon [0,\infty) \to [0,\infty), \psi(u) > 0 \text{ for } u > 0, \\ \text{continuous and nondecreasing with for a.e. } t \in [0,T] \text{ and } y > 0, \\ u \geq q(t)\psi(y) \text{ for all } u \in f(t,y);\end{array} \tag{2.8}$$

$$\exists\, \alpha > 0 \quad \textit{with} \quad 1 < \frac{\alpha}{2^{\frac{p_2-1}{p_2}} \|k_1\|_p \|k_2\|_{p1} \left(\|a_1\|_{p_2}^{p_2} + [a_2]^{p_2} \ \alpha^p \right)^{\frac{1}{p_2}}} \tag{2.9}$$

and

$$\exists\, \beta > 0,\ \beta \neq \alpha \quad \textit{with} \quad 1 > \frac{\beta}{M\|k_1\|_p \int_0^T k_2(s)q(s)\psi(a(s)\beta)ds}, \tag{2.10}$$

where

$$a(t) = M\frac{k_1(t)}{\|k_1\|_p}. \tag{2.11}$$

Then (2.1) has at least one positive solution $y \in L^p[0,T]$ and either

(*A*) $\quad 0 < \alpha < \|y\|_p < \beta$ *and* $y(t) \geq a(t)\alpha$ *a.e.* $t \in [0,T]$ *if* $\alpha < \beta$

or

(*B*) $\quad 0 < \beta < \|y\|_p < \alpha$ *and* $y(t) \geq a(t)\beta$ *a.e.* $t \in [0,T]$ *if* $\beta < \alpha$

holds.

Proof: Let $E = (L^p[0,T], \|\cdot\|_p)$ and

$$C = \{y \in L^p[0,T] : y(t) \geq a(t)\|y\|_p \text{ a.e. } t \in [0,T]\}.$$

It is easy to see that $C \subseteq E$ is a cone. Next let $A = K \circ N_f\colon C \to 2^E$, where the linear integral (single valued) operator K is given by

$$Ky(t) = \int_0^T k(t,s)y(s)\,ds,$$

and the multivalued Nemytskij operator N_f is given by

$$N_f u = \{y \in L^{p_2}[0,T] : y(t) \in f(t,u(t)) \text{ a.e. } t \in [0,T]\}.$$

Remark 2.1: Note A is well defined since if $x \in C$ then (2.2)–(2.4) and [3] guarantee that $N_f x \neq \emptyset$.

We first show $A\colon C \to 2^C$. To see this let $x \in C$ and $y \in Ax$. Then there exists a $v \in N_f x$ with

$$y(t) = \int_0^T k(t,s)v(s)ds \quad \text{for a.e. } t \in [0,T].$$

Now

$$|y(t)|^p \le [k_1(t)]^p \left(\int_0^T k_2(s)v(s)ds \right)^p \quad \text{for a.e. } t \in [0,T]$$

so

$$\|y\|_p \le \|k_1\|_p \int_0^T k_2(s)v(s)ds. \tag{2.12}$$

Combining this with (2.6) gives

$$y(t) \ge M \int_0^T k_1(t)k_2(s)v(s)ds \ge M \, \frac{k_1(t)}{\|k_1\|_p} \, \|y\|_p = a(t) \, \|y\|_p \quad \text{for a.e. } t \in [0,T].$$

Thus $y \in C$ so A: $C \to 2^C$. Also notice [3,6] guarantees that

$$A\colon C \to 2^C \text{ is upper semicontinuous.} \tag{2.13}$$

In addition note [8,9,10:pp. 47–49] implies K: $L^{p_2}[0,T] \to L^p[0,T]$ is completely continuous, and N_f: $L^p[0,T] \to 2^{L^{p_2}[0,T]}$ maps bounded sets into bounded sets. Consequently

$$A\colon C \to K(C) \text{ is completely continuous.} \tag{2.14}$$

Let

$$\Omega_\alpha = \{y \in L^p[0,T] : \|y\|_p < \alpha\} \quad \text{and} \quad \Omega_\beta = \{y \in L^p[0,T] : \|y\|_p < \beta\}.$$

Assume that $\beta < \alpha$ (a similar argument holds if $\alpha < \beta$). It is immediate from (2.13) and (2.14) that

$$A\colon C \cap \overline{\Omega_\alpha} \to K(C) \text{ is upper semicontinuous and compact.}$$

If we show

$$\|y\|_p < \|x\|_p \quad \text{for all } y \in Ax \text{ and } x \in C \cap \partial\Omega_\alpha \tag{2.15}$$

and

$$\|y\|_p > \|x\|_p \quad \text{for all } y \in Ax \text{ and } x \in C \cap \partial\Omega_\beta \tag{2.16}$$

are true, then Theorem 1.1 guarantees that the operator A has a fixed point in $C \cap (\overline{\Omega_\alpha} \backslash \Omega_\beta)$. This in turn implies that (2.1) has at least one solution $y \in L^p[0,T]$ with $\beta \le \|y\|_p \le \alpha$ and $y(t) \ge a(t)\beta$ for a.e. $t \in [0,T]$.

Suppose $x \in C \cap \partial\Omega_\alpha$, so $\|x\|_p = \alpha$, and $y \in Ax$. Then there exists a $v \in N_f x$ with

$$y(t) = \int_0^T k(t,s)v(s)ds \quad \text{for a.e. } t \in [0,T].$$

Now (2.4) and (2.6) guarantee that

$$|y(t)| \le k_1(t)\int_0^T k_2(s)\Big[|a_1(s)| + a_2|x(s)|^{\frac{p}{p_2}}\Big]ds \quad \text{for a.e. } t\in[0,T].$$

This together with (2.9) yields

$$\begin{aligned}
\|y\|_p &\le \|k_1\|_p\|k_2\|_{p_1}\left(\int_0^T \Big[|a_1(s)| + a_2\,|x(s)|^{\frac{p}{p_2}}\Big]^{p_2} ds\right)^{\frac{1}{p_2}}\\
&\le \|k_1\|_p\|k_2\|_{p_1}\left(2^{p_2-1}\int_0^T [|a_1(s)|^{p_2} + [a_2]^{p_2}|x(s)|^p]ds\right)^{\frac{1}{p_2}}\\
&= 2^{\frac{p_2-1}{p_2}}\|k_1\|_p\,\|k_2\|_{p_1}\left(\|a_1\|_{p_2}^{p_2} + [a_2]^{p_2}\,\|x\|_p^p\right)^{\frac{1}{p_2}}\\
&= 2^{\frac{p_2-1}{p_2}}\,\|k_1\|_p\,\|k_2\|_{p_1}\left(\|a_1\|_{p_2}^{p_2} + [a_2]^{p_2}\,\alpha^p\right)^{\frac{1}{p_2}}\\
&< \alpha = \|x\|_p
\end{aligned}$$

and so (2.15) is satisfied.

Now suppose $x\in C\cap\partial\Omega_\beta$, so $\|x\|_p=\beta$ and $x(t)\ge a(t)\beta$ for a.e. $t\in[0,T]$, and $y\in Ax$. Then there exists a $v\in N_f x$ with

$$y(t) = \int_0^T k(t,s)v(s)ds \quad \text{for a.e. } t\in[0,T].$$

Notice (2.8) guarantees that $v(s)\ge q(s)\psi(x(s))$ for a.e. $s\in[0,T]$ and this together with (2.6) yields

$$y(t) \ge Mk_1(t)\int_0^T k_2(s)q(s)\psi(x(s))ds \quad \text{for a.e. } t\in[0,T].$$

Combining with (2.10) gives

$$\begin{aligned}
\|y\|_p &\ge M\,\|k_1\|_p\int_0^T k_2(s)q(s)\psi(x(s))ds\\
&\ge M\|k_1\|_p\int_0^T k_2(s)q(s)\psi(a(s)\beta)ds\\
&> \beta = \|x\|_p
\end{aligned}$$

and thus (2.16) is satisfied. Now apply Theorem 1.1. □

3. $C[0,T]$ SOLUTIONS

In this section we discuss the Fredholm integral inclusion

$$y(t) \in \int_0^T k(t,s)f(s,y(s))ds \quad \text{for } t\in[0,T], \tag{3.1}$$

where k: $[0,T]\times[0,T]\to\mathbf{R}$ and f: $[0,T]\times\mathbf{R}\to K(\mathbf{R})$. We will use Theorem 1.1 to establish the existence of a nonnegative solution $y\in C[0,T]$ to (3.1). We will let $|\cdot|_0$ denote the usual norm on $C[0,T]$ i.e., $|u|_0=\sup_{[0,T]}|u(t)|$ for $u\in C[0,T]$.

Theorem 3.1: *Let* $1\le p<\infty$ *and* q, $1<q\le\infty$, *the conjugate to* p, k: $[0,T]\times[0,T]\to\mathbf{R}$, f: $[0,T]\times\mathbf{R}\to K(\mathbf{R})$ *and assume the following conditions are satisfied*:

$$\text{for each } t\in[0,T],\ \text{the map } s\mapsto k(t,s) \text{ is measurable;} \tag{3.2}$$

$$\sup_{t\in[0,T]}\left(\int_0^T |k(t,s)|^q\,ds\right)^{\frac{1}{q}}<\infty; \tag{3.3}$$

$$\int_0^T |k(t',s)-k(t,s)|^q\,ds\to 0 \quad \text{as } t\to t',\ \text{for each } t'\in[0,T]; \tag{3.4}$$

$$\text{for each } t\in[0,T],\quad k(t,s)\ge 0 \quad \text{for a.e. } s\in[0,T]; \tag{3.5}$$

$$\begin{aligned}&\text{for each measurable } u\text{: } [0,T]\to\mathbf{R},\ \text{the map } t\mapsto f(t,u(t))\\ &\text{has measurable single valued selections;}\end{aligned} \tag{3.6}$$

$$\text{for a.e. } t\in[0,T],\ \text{the map } u\mapsto f(t,u) \text{ is upper semicontinuous;} \tag{3.7}$$

$$\begin{aligned}&\text{for each } r>0, \exists\, h_r\in L^p[0,T] \text{ with } |f(t,y)|\le h_r(t)\\ &\text{for a.e. } t\in[0,T] \text{ and every } y\in\mathbf{R} \text{ with } |y|\le r;\end{aligned} \tag{3.8}$$

$$\text{for a.e. } t\in[0,T] \text{ and all } y\in(0,\infty), u>0 \text{ for all } u\in f(t,y); \tag{3.9}$$

$$\begin{aligned}&\exists\, g\in L^q[0,T] \text{ with } g\text{: } [0,T]\to(0,\infty) \text{ and}\\ &\text{with } k(t,s)\le g(s) \text{ for } t\in[0,T];\end{aligned} \tag{3.10}$$

$$\begin{aligned}&\exists\,\delta,\epsilon, 0\le\delta<\epsilon\le T \text{ and } M, 0<M<1,\\ &\text{with } k(t,s)\ge M\,g(s) \text{ for } t\in[\delta,\epsilon];\end{aligned} \tag{3.11}$$

$$\begin{aligned}&\exists\, h\in L^p[0,T] \text{ with } h\text{: } [0,T]\to(0,\infty),\ \text{and } w\ge 0 \text{ continuous}\\ &\text{and nondecreasing on } (0,\infty) \text{ with } |f(t,y)|\le h(t)\,w(y)\\ &\text{for a.e. } t\in[0,T] \text{ and all } y\in(0,\infty);\end{aligned} \tag{3.12}$$

$$\begin{aligned}&\exists\,\tau\in L^p[\delta,\epsilon] \text{ with } \tau>0 \text{ a.e. on } [\delta,\epsilon] \text{ and with for a.e.}\\ &t\in[\delta,\epsilon] \text{ and } y\in(0,\infty), u\ge\tau(t)\,w(y) \text{ for all } u\in f(t,y);\end{aligned} \tag{3.13}$$

$$\exists\,\alpha>0 \quad \text{with} \quad 1<\frac{\alpha}{w(\alpha)\sup_{t\in[0,T]}\int_0^T k(t,s)h(s)ds} \tag{3.14}$$

and

$$\exists\,\beta > 0,\ \beta \neq \alpha \quad \textit{with} \quad 1 > \frac{\beta}{w(M\beta)\int_\delta^\epsilon \tau(s)\,k(\sigma,s)\,ds}; \tag{3.15}$$

here $\sigma \in [0,T]$ *is such that*

$$\int_\delta^\epsilon \tau(s)k(\sigma,s)ds = \sup_{t\in[0,T]} \int_\delta^\epsilon \tau(s)k(t,s)ds. \tag{3.16}$$

Then (3.1) has at least one nonnegative solution $y \in C[0,T]$ *and either*

(*A*) $\quad 0 < \alpha < |y|_0 < \beta$ *and* $y(t) \geq M\alpha$ *for* $t \in [\delta,\epsilon]$ *if* $\alpha < \beta$

or

(*B*) $\quad 0 < \beta < |y|_0 < \alpha$ *and* $y(t) \geq M\beta$ *for* $t \in [\delta,\epsilon]$ *if* $\beta < \alpha$

holds.

Proof: Let $E = (C[0,T], |\cdot|_0)$ and

$$C = \left\{ y \in C[0,T] : y(t) \geq 0 \text{ for } t \in [0,T] \text{ and } \min_{t\in[\delta,\epsilon]} y(t) \geq M|y|_0 \right\}.$$

Also let $A = K \circ N_f \colon C \to 2^E$, where $K\colon L^p[0,T] \to C[0,T]$ and $N_f\colon C[0,T] \to 2^{L^p[0,T]}$ are given by

$$Ky(t) = \int_0^T k(t,s)y(s)ds$$

and

$$N_f u = \{y \in L^p[0,T] : y(t) \in f(t,u(t)) \quad \text{a.e. } t \in [0,T]\}.$$

Remark 3.1: Note A is well defined since if $x \in C$ then [4,5] guarantee that $N_f x \neq \emptyset$.

We first show $A\colon C \to 2^C$. To see this let $x \in C$ and $y \in Ax$. Then there exists a $v \in N_f x$ with

$$y(t) = \int_0^T k(t,s)v(s)ds \quad \text{for } t \in [0,T].$$

This together with (3.10) yields

$$|y(t)| \leq \int_0^T g(s)v(s)ds \quad \text{for } t \in [0,T]$$

and so

$$|y|_0 \leq \int_0^T g(s)v(s)ds. \tag{3.17}$$

On the other hand (3.11) and (3.17) yields

$$\min_{t\in[\delta,\epsilon]} y(t) = \min_{t\in[\delta,\epsilon]} \int_0^T k(t,s)v(s)ds \geq M \int_0^T g(s)v(s)ds \geq M|y|_0,$$

so $y \in C$. Thus A: $C \to 2^C$. A standard result from the literature [5,7,8,10] guarantees that

$$A\text{: } C \to K(C) \text{ is upper semicontinuous and completely continuous.}$$

Let

$$\Omega_\alpha = \{u \in C[0,T] : |u|_0 < \alpha\} \quad \text{and} \quad \Omega_\beta = \{u \in C[0,T] : |u|_0 < \beta\}.$$

Without loss of generality assume $\beta < \alpha$. If we show

$$|y|_0 < |x|_0 \quad \text{for all } y \in A\,x \text{ and } x \in C \cap \partial\Omega_\alpha \tag{3.18}$$

and

$$|y|_0 > |x|_0 \quad \text{for all } y \in A\,x \text{ and } x \in C \cap \partial\Omega_\beta \tag{3.19}$$

are true, then Theorem 1.1 guarantees the result.

Suppose $x \in C \cap \partial\Omega_\alpha$, so $|x|_0 = \alpha$, and $y \in A\,x$. Then there exists $v \in N_f x$ with

$$y(t) = \int_0^T k(t,s)v(s)\,ds \quad \text{for } t \in [0,T].$$

Now (3.12) implies that for $t \in [0,T]$ we have

$$\begin{aligned} |y(t)| &\le \int_0^T k(t,s)h(s)w(x(s))\,ds \le w(|x|_0) \int_0^T k(t,s)h(s)\,ds \\ &\le \; w(\alpha) \sup_{t\in[0,T]} \int_0^T k(t,s)h(s)\,ds. \end{aligned}$$

This together with (3.14) yields

$$|y|_0 \le w(\alpha) \sup_{t\in[0,T]} \int_0^T k(t,s)h(s)ds < \alpha = |x|_0,$$

so (3.18) holds.

Next suppose $x \in C \cap \partial\Omega_\beta$, so $|x|_0 = \beta$ and $M\beta \le x(t) \le \beta$ for $t \in [\delta, \epsilon]$, and $y \in Ax$. Then there exists $v \in N_f x$ with

$$y(t) = \int_0^T k(t,s)v(s)ds \quad \text{for } t \in [0,T].$$

Notice (3.13) and (3.15) imply

$$\begin{aligned} y(\sigma) &= \int_0^T k(\sigma,s)v(s)ds \ge \int_\delta^\epsilon k(\sigma,s)v(s)ds \\ &\ge \int_\delta^\epsilon k(\sigma,s)\tau(s)w(x(s))ds \ge w(M\beta) \int_\delta^\epsilon k(\sigma,s)\tau(s)ds \\ &> \beta = |x|_0. \end{aligned}$$

Thus $|y|_0 > |x|_0$, so (3.19) holds. Now apply Theorem 1.1. $\square$

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2. A Note on the Structure of the Solution Set for the Cauchy Differential Inclusion in Banach Spaces

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Abstract: In this chapter we investigate the topological structure of the solution set of the Cauchy differential inclusion in Banach spaces. Our multivalued map will be assumed to satisfy a "local" integrably boundedness assumption.

1. INTRODUCTION

Let E be a Banach space, $T > 0$ and F: $[0,T] \times E \to C(E)$ a multivalued map; here $C(E)$ denotes the family of nonempty, closed, convex subsets of E. In this chapter we discuss the topological structure of the solution set of the Cauchy differential inclusion

$$\begin{aligned} & y'(t) \in F(t, y(t)) \quad \text{a.e. } t \in I \equiv [0,T] \\ & y(0) = x_0, \end{aligned} \tag{1.1}$$

using a recent result of Cichon and Kubiaczyk [1]. Throughout this chapter E is a real Banach space with norm $\|\cdot\|$. We denote by $C([0,T],E)$ the space of continuous functions y: $[0,T] \to E$. Let u: $[0,T] \to E$ be a measurable function. By $\int_0^T u(t)dt$ we mean the Bochner integral of u, assuming it exists. We define the Sobolev class $W^{1,1}([0,T],E)$ as the set of continuous functions u such that there exists $v \in L^1([0,T],E)$ with $u(t) - u(0) = \int_0^t v(s)ds$ for all $t \in [0,T]$. Notice if $u \in W^{1,1}([0,T],E)$ then u is differentiable almost everywhere on $[0,T]$, $u' \in L^1([0,T],E)$ and $u(t) - u(0) = \int_0^t u'(s)ds$ for all $t \in [0,T]$. By a solution to (1.1) we mean a function $y \in W^{1,1}([0,T],E)$ satisfying the differential equation in (1.1). Let $S(x_0)$ denote the solution set of (1.1). Recently [1] Cichon and Kubiaczyk proved the following result concerning the topological structure of $S(x_0)$.

Theorem 1.1: *Let $E = (E, \|\cdot\|)$ be a real Banach space, $T > 0$, F: $[0,T] \times E \to C(E)$ with the following conditions satisfied*:

$$F(\cdot, x) \textit{ has a strongly measurable selection for each } x \in E; \tag{1.2}$$

$$F(t,\cdot) \textit{ is upper semicontinuous for each } t \in [0,T]; \tag{1.3}$$

$$\exists\, \eta \in L^1(I) \textit{ with } \|F(t,x)\| = \sup\{|y| : y \in F(t,x)\} \le \eta(t) \textit{ on } I \times E \tag{1.4}$$

and

$$\begin{array}{l} \textit{for any bounded set } \Omega \subseteq E,\ \lim_{h\to 0^+} \alpha(F(I_{t,h} \times \Omega)) \\ \le w(t,\alpha(\Omega)) \textit{ a.e. on } I\textit{; here } I_{t,h} = [t-h,t] \cap I,\ \alpha(\,\cdot\,) \\ \textit{the Kuratowski measure of noncompactness and} \\ w\colon I \times [0,\infty) \to [0,\infty) \textit{ is a Kamke function (i.e., } w \textit{ is a} \\ \textit{Carathéodory map with } \max_{s\in[0,r]} w(t,s) \in L^1(I) \textit{ for all } r > 0, \\ \textit{and } \rho \equiv 0 \textit{ is the only absolutely continuous function satisfying} \\ \rho(0) = 0 \textit{ and } \rho'(t) = w(t,\rho(t)) \textit{ a.e. on I.} \end{array} \tag{1.5}$$

Then $S(x_0)$ is nonempty, compact and connected in $C([0,T],E)$. In fact $S(x_0)$ is a R_δ-set.

Remark 2.1: A set is a R_δ-set if it is the intersection of a decreasing sequence of nonempty, compact, absolute retracts.

Remark 2.2: Deimling and Rao [3] and Tolstonogov [8] showed that $S(x_0)$ is nonempty, compact and connected. Recently Cichon and Kubiaczyk [1] established that $S(x_0)$ is a R_δ-set.

The main goal of this chapter is to remove the "global" integrably boundedness assumption (1.4) on F. By using Theorem 1.1 and a trick involving the Urysohn function we are able to accomplish this if we assume a "local" integrably boundedness assumption on F. This is exactly what is needed from an application viewpoint.

2. SOLUTION SET

First we establish a general existence principle for (1.1). We assume (1.2), (1.3) and (1.5) hold. In addition suppose the following two conditions are satisfied:

$$\begin{array}{l} \textit{for each } r > 0 \textit{ there exists } h_r \in L^1[0,T] \textit{ such that } \|F(t,x)\| \le h_r(t) \\ \textit{for a.e. } t \in [0,T] \textit{ and all } x \in E \textit{ with } \|x\| \le r \end{array} \tag{2.1}$$

and

$$\begin{array}{l} \exists\, M > \|x_0\| \textit{ with } \|y\|_0 = \sup_{t\in[0,T]} \|y(t)\| < M \\ \textit{for any possible solution to (1.1).} \end{array} \tag{2.2}$$

Let $\epsilon > 0$ be given and let $\tau_\epsilon\colon E \to [0,1]$ be the Urysohn function for

$$\big(\overline{B}(0,M), \quad E\backslash B(0,M+\epsilon)\big)$$

such that $\tau_\epsilon(x) = 1$ if $\|x\| \le M$ and $\tau_\epsilon(x) = 0$ if $\|x\| \ge M + \epsilon$. Let $\tilde{F}(t,x) = \tau_\epsilon(x)F(t,x)$ and consider the differential inclusion

$$\begin{array}{l} y'(t) \in \tilde{F}(t,y(t)) \quad \textit{a.e. } t \in [0,T] \\ y(0) = x_0. \end{array} \tag{2.3}$$

We will let $S_\epsilon(x_0)$ denote the solution set of (2.3).

Theorem 2.1: *Let $E = (E, \|\cdot\|)$ be a real Banach space, F: $[0,T] \times E \to C(E)$, and assume (1.2), (1.3), (1.5), (2.1) and (2.2) hold. Let $\epsilon > 0$ and suppose*

$$\|u\|_0 < M \quad \textit{for any possible solution } u \in W^{1,1}([0,T],E) \textit{ to (2.3)} \tag{2.4}$$

is true. Then $S(x_0)$ is a R_δ-set.

Proof: Notice $S(x_0) = S_\epsilon(x_0)$. Also (2.1) and the definition of τ_ϵ guarantees that $\tilde{F}$ satisfies (1.4) (with F replaced by $\tilde{F}$). Notice also if Ω is a bounded subset of E then for a.e. $t \in I$ we have

$$\tilde{F}\big(I_{t,h} \times \Omega\big) \subseteq \mathrm{co}\big(F\big(I_{t,h} \times \Omega\big) \cup \{0\}\big).$$

Now from a standard property of α we have for a.e. $t \in I$ that

$$\alpha\big(\tilde{F}\big(I_{t,h} \times \Omega\big)\big) \le \alpha\big(F\big(I_{t,h} \times \Omega\big)\big)$$

As a result for a.e. $t \in I$ we have

$$\lim_{h \to 0^+} \alpha\big(\tilde{F}\big(I_{t,h} \times \Omega\big)\big) \;\le\; \lim_{h \to 0^+} \alpha\big(F\big(I_{t,h} \times \Omega\big)\big) \le w(t, \alpha(\Omega)).$$

Thus (1.5) is true with F replaced by $\tilde{F}$. Now Theorem 1.1 (applied to $\tilde{F}$) guarantees that $S_\epsilon(x_0)$ is a R_δ-set. □

We now use the existence principle, Theorem 2.1, to establish two applicable results for (1.1). First however recall the following three Lemma's from the literature [5,6,7].

Lemma 2.2: *Let $E = (E, \|\cdot\|)$ be a real Banach space. If $x \in W^{1,1}([0,T],E)$ then $\|x\| \in W^{1,1}([0,T],\mathbf{R})$.*

Lemma 2.3: *Let $E = (E, \|\cdot\|)$ be a real Banach space. Then the following properties hold*:

(*i*) $|\langle x,y\rangle_-| \le \|x\|\,\|y\|$; *here* $x, y \in E$ *and* $\langle x,y\rangle_- = \|x\| \lim_{t \to 0^-} \frac{\|x+ty\|-\|x\|}{t}$;
(*ii*) $\langle \alpha x, \beta y\rangle_- = \alpha\,\beta\,\langle x,y\rangle_-$ *for all* $\alpha\beta \ge 0$ *and* $x, y \in E$;
(*iii*) *if* x: $[0,T] \to E$ *is differentiable at* t *then* $\|x(t)\|\, D^- \,\|x(t)\| = \langle x(t), x'(t)\rangle_-$; *here* D^- *is the left Dini derivative*.

Lemma 2.4: *Let $R \ge 0$, $r \in L^1([0,T],[0,\infty))$ and ψ: $[0,\infty) \to (0,\infty)$ be a Borel function such that*

$$\int_0^T r(s)ds < \int_R^\infty \frac{dx}{\psi(x)}.$$

Let M_0 be such that $\int_0^T r(s)ds = \int_R^{M_0} \frac{dx}{\psi(x)}$. Then for any $[t_0,t_1] \subseteq [0,T]$ and $z \in W^{1,1}([t_0,t_1],[0,\infty))$ with $z'(t) \le r(t)\,\psi(z(t))$ for a.e. $t \in [t_0,t_1]$ and $z(t_0) \le R$, we have $z(t) \le M_0$ for all $t \in [t_0,t_1]$.

Proof: If $z(s) \le R$ for all $s \in [t_0, t_1]$ then the lemma holds. On the other hand if $z(t) > R$ for some $t \in [t_0, t_1]$ then since $z(t_0) \le R$ there exists $\mu \in [t_0, t_1]$ with $z(\mu) = R$ and $z(s) \ge R$ for $s \in [\mu, t]$. Now since $z'(t) \le r(t)\,\psi(z(t))$ for a.e. $t \in [t_0, t_1]$ we may divide by $\psi(z(t))$ and integrate from μ to t to obtain

$$\int_R^{z(t)} \frac{dx}{\psi(x)} \le \int_\mu^t r(s)ds \le \int_0^T r(s)ds = \int_R^{M_0} \frac{dx}{\psi(x)}.$$

Thus $z(t) \le M_0$. $\square$

Theorem 2.5: *Let $E = (E, \|\cdot\|)$ be a real Banach space, F: $[0,T] \times E \to C(E)$, and assume (1.2), (1.3), (1.5) and (2.1) hold. In addition suppose the following conditions are satisfied*:

$$\begin{array}{l}\exists\, q \in L^1([0,T],[0,\infty)) \text{ and } \phi\colon [0,\infty) \to (0,\infty) \text{ a Borel measurable} \\ \text{function such that for a.e. } t \in [0,T] \text{ and all } v \in E \text{ we have} \\ \langle v, z\rangle_- \le q(t)\phi(\|v\|) \text{ for all } z \in F(t,v)\end{array} \tag{2.5}$$

and

$$\int_0^T q(s)ds < \int_{\|x_0\|}^\infty \frac{x}{\phi(x)}\,dx. \tag{2.6}$$

Then $S(x_0)$ is a R_δ-set.

Proof: Let $\epsilon > 0$ be given,

$$I(z) = \int_{\|x_0\|}^z \frac{x}{\phi(x)}dx, \quad M_0 = I^{-1}\left(\int_0^T q(s)ds\right) \quad \textit{and} \quad M = M_0 + 1.$$

We will show any solution u of (1.1) satisfies $\|u\|_0 < M$ and any possible solution y of (2.3) satisfies $\|y\|_0 < M$. If this is true, then Theorem 2.1 guarantees the result.

Suppose u is a possible solution of (1.1). Then

$$\|u(t)\|' = D^- \|u(t)\| \le q(t)\,\frac{\phi(\|u(t)\|)}{\|u(t)\|} \quad \text{a.e. on } \{t : \|u(t)\| > 0\};$$

here we used Lemma 2.2 and Lemma 2.3. Now Lemma 2.4, applied with $R = \|x_0\|$, $\psi(x) = \frac{\phi(x)}{x}$ and $z(t) = \|u(t)\|$, implies $\|u(t)\| \le M_0$ for all $t \in [0,T]$. Consequently $\|u(t)\| < M$ for all $t \in [0,T]$.

Next let y be a possible solution of (2.3). Now for a.e. $t \in [0,T]$ and all $v \in E$ we have, since τ_ϵ: $E \to [0,1]$, that

$$\langle v, z\rangle_- \le q(t)\,\phi(\|v\|)$$

for all $z \in \tilde{F}(t,v) = \tau_\epsilon(v)F(t,v)$. Thus

$$\|y(t)\|' \le q(t)\,\frac{\phi(\|y(t)\|)}{\|y(t)\|} \quad \text{a.e. on } \{t : \|y(t)\| > 0\}.$$

Now Lemma 2.4 implies $\|y(t)\| \leq M_0 < M$ for all $t \in [0, T]$. $\square$

Corollary 2.6: *Let $E = (E, \|\cdot\|)$ be a real Banach space, F: $[0, T] \times E \to C(E)$, and assume (1.2), (1.3), (1.5) and (2.1) hold. In addition suppose the following conditions are satisfied:*

$$\begin{aligned} &\exists\, q \in L^1([0,T],[0,\infty)) \textit{ and } \mu\text{: } [0,\infty) \to (0,\infty) \textit{ a Borel} \\ &\textit{measurable function such that for a.e. } t \in [0,T] \\ &\textit{and all } v \in E \textit{ we have } \|F(t,v)\| \leq q(t)\,\mu(\|v\|) \end{aligned} \tag{2.7}$$

and

$$\int_0^T q(s)ds \; < \int_{\|x_0\|}^{\infty} \frac{dx}{\mu(x)}. \tag{2.8}$$

Then $S(x_0)$ is a R_δ-set.

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3. Fixed Point Theory for Acyclic Maps between Topological Vector Spaces having Sufficiently many Linear Functionals, and Generalized Contractive Maps with Closed Values between Complete Metric Spaces

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Abstract: Two new fixed point theorems for acyclic maps defined on Hausdorff topological spaces with sufficiently many linear functionals and two fixed point theorems for multivalued contractions in the sense of Bose and Mukherjee are presented.

1. INTRODUCTION

This chapter has two main sections. In Section 2 we present two fixed point results for acyclic maps defined on Hausdorff topological vector spaces having sufficiently many linear functionals. In particular we will present an acyclic version of Mönch's fixed point theorem. Recall a map is acyclic if it is a upper semicontinuous multifunction with nonempty, compact, acyclic values. The results of this section contain as a special case the results of Ky Fan (see [5,6] and the references therein). In fact if the maps considered were Kututani maps (i.e., upper semicontinuous multifunction with nonempty, compact, convex values) then the results of this section could be improved considerably. The Kututani maps will be discussed in greater details in a future paper of the authors. In Section 3, we present two fixed point results for multivalued generalized contractive maps (in the sense of Bose and Mukherjee) with closed values defined on complete metric spaces.

2. FIXED POINT THEORY FOR ACYCLIC MAPS

In this section we present two fixed point results for acyclic multivalued maps on Hausdorff topological vector spaces having sufficiently many linear functionals. Recall a topological vector space E is said to have sufficiently many linear functionals if for every $x \in E$ with $x \neq 0$ there exists $l \in E^*$ (the dual space of E) with $l(x) \neq 0$ (Notice from the Hahn-Banach theorem that every Hausdorff locally convex linear topological space has sufficiently many linear functionals). The proof of our results rely on the following result in the literature due to Park [5,6].

Theorem 2.1: *Let Q be a compact, convex subset of a Hausdorff topological vector space having sufficiently many linear functionals and let F: $Q \to AC(Q)$ be an upper semicontinuous map (here $AC(Q)$ denotes the family of nonempty, compact, acyclic subsets of Q). Then F has a fixed point in Q.*

Remark 2.1: Recall a nonempty topological space is acyclic if all of its reduced Čech homology groups over rationals vanish.

Theorem 2.2: *Let Ω be a closed, convex subset of a Hausdorff topological vector space having sufficiently many linear functionals and $x_0 \in \Omega$. Suppose there is a closed map (i.e., has closed graph) F: $\Omega \to AC(\Omega)$ with the properties*:

$$A \subseteq \Omega,\ A = \mathrm{co}(\{x_0\} \cup F(A)) \textit{ implies } \bar{A} \textit{ is compact} \tag{2.1}$$

and

$$\textit{for any } A \subseteq \Omega \textit{ with } \bar{A} \textit{ compact}, \textit{ then } F(\bar{A}) \subseteq \overline{F(A)}. \tag{2.2}$$

Then F has a fixed point in Ω.

Remark 2.2: *If* F is lower semicontinuous then (2.2) holds. Indeed if F is lower semicontinuous then for any $A \subseteq \Omega$ we have $F(\bar{A}) \subseteq \overline{F(A)}$. To see this let $x \in \bar{A}$. We wish to show $F(x) \subseteq \overline{F(A)}$. Let $z \in F(x)$, and let U be an open neighborhood of z. Then since F is lower semicontinuous we have that $F^{-1}(U)$ is an open set containing x. Now since $x \in \bar{A}$ we have $F^{-1}(U) \cap A \neq \emptyset$. Consequently $U \cap F(A) \neq \emptyset$, so $z \in \overline{F(A)}$. We can do this for all $z \in F(x)$ so $F(x) \subseteq \overline{F(A)}$.

Proof: Let

$$D_0 = \{x_0\}, \quad D_n = \mathrm{co}(\{x_0\} \cup F(D_{n-1})) \quad \text{for } n = 1, 2, \ldots \text{ and } D = \bigcup_{n=0}^{\infty} D_n.$$

Now for $n = 0, 1, \ldots$ notice D_n is convex. Also by induction we see that

$$D_0 \subseteq D_1 \subseteq \ldots \subseteq D_{n-1} \subseteq D_n \subseteq \ldots \subseteq \Omega.$$

Consequently D is convex. It is also immediate since (D_n) is increasing that

$$D = \bigcup_{n=1}^{\infty} \mathrm{co}(\{x_0\} \cup F(D_{n-1})) = \mathrm{co}(\{x_0\} \cup F(D)). \tag{2.3}$$

This together with (2.1) implies that $\bar{D}$ is compact. Also from (2.3) we have $F(D) \subseteq D \subseteq \bar{D}$ and this together with (2.2) gives $F(\bar{D}) \subseteq \overline{F(D)} \subseteq \bar{D}$. Consequently F: $\bar{D} \to AC(\bar{D})$ is a closed map. Now [2:pp. 465] implies F: $\bar{D} \to AC(\bar{D})$ is upper semicontinuous and Theorem 2.1 implies that there exists $x \in \bar{D} \subseteq \Omega$ with $x \in F(x)$. $\square$

Next we present a Mönch fixed point theorem for acyclic maps defined on Hausdorff topological vector spaces having sufficiently many linear functionals.

Theorem 2.3: *Let Ω be a closed, convex subset of a Hausdorff topological vector space E having sufficiently many linear functionals and $x_0 \in \Omega$. Suppose there is an upper semicontinuous map $F\colon \Omega \to AC(\Omega)$ with (2.2) holding. In addition assume the following properties hold*:

$$\begin{aligned}&A \subseteq \Omega, A = \text{co}(\{x_0\} \cup F(A)) \textit{ with } \bar{A} = \bar{C}\\ &\textit{and } C \subseteq A \textit{ countable, implies } \bar{A} \textit{ is compact};\end{aligned} \tag{2.4}$$

$$\begin{aligned}&\textit{for any relatively compact subset } A \textit{ of } E \textit{ there}\\ &\textit{exists a countable set } B \subseteq A \textit{ with } \bar{B} = \bar{A}\end{aligned} \tag{2.5}$$

and

$$\textit{if } A \textit{ is a compact subset of } E \textit{ then } \overline{\text{co}}(A) \textit{ is compact.} \tag{2.6}$$

Then F has a fixed point in Ω.

Remark 2.3: If E is metrizable then (2.5) holds since compact metric spaces are separable.

Remark 2.4: If E is a quasicomplete locally convex linear topological space then (2.6) holds.

Proof: Let D_n, $n = 0, 1, \ldots$, and D be as in Theorem 2.2. Now

$$D_0 \subseteq D_1 \subseteq \ldots \subseteq D_{n-1} \subseteq D_n \ldots \subseteq \Omega$$

and D is convex. In addition

$$D = \text{co}(\{x_0\} \cup F(D)). \tag{2.7}$$

We now show D_n is relatively compact for $n = 0, 1, \ldots$ Suppose D_k is relatively compact for some $k \in \{1, 2, \ldots\}$. Then [2] guarantees that $F(\bar{D}_k)$ is compact since F is upper semicontinuous. This together with (2.6) guarantees that D_{k+1} is relatively compact.

Now (2.5) implies that for each $n \in \{0, 1, \ldots\}$ there exists C_n with C_n countable, $C_n \subseteq D_n$, and $\bar{C}_n = \bar{D}_n$. Let $C = \cup_{n=0}^{\infty} C_n$. Now since

$$\bigcup_{n=0}^{\infty} D_n \subseteq \bigcup_{n=0}^{\infty} \bar{D}_n \subseteq \overline{\bigcup_{n=0}^{\infty} D_n},$$

we have

$$\overline{\bigcup_{n=0}^{\infty} \bar{D}_n} = \overline{\bigcup_{n=0}^{\infty} D_n} = \bar{D} \quad \textit{and} \quad \overline{\bigcup_{n=0}^{\infty} \bar{D}_n} = \overline{\bigcup_{n=0}^{\infty} \bar{C}_n} = \overline{\bigcup_{n=0}^{\infty} C_n} = \bar{C}.$$

Thus $\bar{C} = \bar{D}$. This together with (2.4) and (2.7) implies that $\bar{D}$ is compact. Also from (2.7) we have $F(D) \subseteq D \subseteq \bar{D}$ and this together with (2.2) gives $F(\bar{D}) \subseteq \overline{F(D)} \subseteq \bar{D}$. Consequently $F\colon \bar{D} \to AC(\bar{D})$ is upper semicontinuous. Now apply Theorem 2.1. $\square$

Remark 2.5: Suppose in Theorem 2.3 we assumed

$$\text{for any } A \subseteq \Omega \quad \text{we have} \quad F(\bar{A}) \subseteq \overline{F(A)} \tag{2.8}$$

instead of (2.2). Then we could replace (2.4) with

$$C \subseteq \Omega \text{ countable}, \quad \bar{C} = \overline{\text{co}}(\{x_0\} \cup F(C)) \quad \text{implies } \bar{C} \text{ is compact.} \tag{2.9}$$

To see this let $A \subseteq \Omega$, $A = \text{co}\ (\{x_0\} \cup F(A))$ with $\bar{A} = \bar{C}$ and $C \subseteq A$ countable. We must show $\bar{A}(= \bar{C})$ is compact. Now (2.8) implies

$$F(A) \subseteq F(\bar{A}) = F(\bar{C}) \subseteq \overline{F(C)} \subseteq \overline{\text{co}}(\{x_0\} \cup F(C))$$

and so

$$\overline{\text{co}}(\{x_0\} \cup F(A)) \subseteq \overline{\text{co}}(\{x_0\} \cup F(C)).$$

Of course trivially

$$\overline{\text{co}}(\{x_0\} \cup F(C)) \subseteq \overline{\text{co}}(\{x_0\} \cup F(A)),$$

so

$$\overline{\text{co}}(\{x_0\} \cup F(C)) = \overline{\text{co}}(\{x_0\} \cup F(A)).$$

Consequently,

$$\bar{C} = \bar{A} = \overline{\text{co}}(\{x_0\} \cup F(A)) = \overline{\text{co}}(\{x_0\} \cup F(C)),$$

so (2.9) guarantees that $\bar{C}(= \bar{A})$ is compact.

Remark 2.6: It is possible to remove (2.2) in Theorem 2.3 if we assume

$$\begin{array}{l}\text{for any acyclic subset } A \text{ of } \Omega \text{ we have } A \cap \bar{D} \\ \text{is acyclic (where } D \text{ is as in Theorem 2.2).}\end{array} \tag*{(2.2)*}$$

To see this proceed as in Theorem 2.2 to obtain that $\bar{D}$ is compact and $F(D) \subseteq \bar{D}$. Now let $F^*\colon \bar{D} \to AC(\bar{D})$ (see (2.2)*) be given by

$$F^*(x) = F(x) \cap \bar{D}.$$

(Note to check that $F^*(x) \neq \emptyset$ for $x \in \bar{D}$ it is enough to show $\bar{D} \subseteq F^{-1}(\bar{D})$. We have $D \subseteq F^{-1}(\bar{D})$ and since $F^{-1}(\bar{D})$ is closed, since F is upper semicontinuous, we have $\bar{D} \subseteq F^{-1}(\bar{D})$). Now F^* is upper semicontinuous since for any closed set A of $\bar{D}$ its easy to check that $(F^*)^{-1}(A) = F^{-1}(A) \cap \bar{D}$ so $(F^*)^{-1}(A)$ is closed. Now apply Theorem 2.1 (with $F = F^*$ and $Q = \bar{D}$).

Remark 2.7: It is easy to see that the upper semicontinuity assumption in Theorem 2.3 (and Remark 2.6) could be replaced by *graph* (F) is closed and F maps compact sets into relatively compact sets.

3. FIXED POINT THEORY FOR GENERALIZED CONTRACTIVE MAPS

This section presents two new fixed point results (Theorems 3.1 and 3.3) for multivalued maps with closed values defined on a complete metric space X. Our first result (Theorem 3.1) is a local version of a result of Bose and Mukherjee [3] (we will note also in this section that the result of Bose and Mukherjee follows immediately from Theorem 3.1). Our other new result (Theorem 3.3) extends some ideas of Frigon and Granas [4] to the maps considered in this section.

Let (X, d) be a metric space. By $B(x, r)$ we denote the open ball in X centered at x of radius r and by $B(C, r)$ we denote $\cup_{x\in C} B(x, r)$ where C is a subset of X. For C and K two nonempty closed subsets of X we define the generalised Hausdorff distance D to be

$$D(C, K) = \inf\{\epsilon : C \subseteq B(K, \epsilon),\ K \subseteq B(C, \epsilon)\} \in [0, \infty].$$

Theorem 3.1: *Let (X, d) be a complete metric space, $x_0 \in X$ and F: $\overline{B(x_0, r)} \to C(X)$; here $r > 0$ and $C(X)$ denotes the family of nonempty closed subsets of X. Suppose for $x,\ y \in \overline{B(x_0, r)}$ we have*

$$D(F(x), F(y)) \le a_1 \operatorname{dist}(x, F(x)) + a_2 \operatorname{dist}(y, F(y)) + a_3 \operatorname{dist}(y, F(x)) \\ + a_4 \operatorname{dist}(x, F(y)) + a_5\, d(x, y),$$

where $a_1, \dots, a_5$ are nonnegative real numbers with $a_1 + a_2 + a_3 + a_4 + a_5 < 1,\ a_1 + a_4 + a_5 > 0,\ a_2 + a_3 + a_5 > 0$, and with either $a_1 = a_2$ or *$a_3 = a_4$. In addition assume*

$$\operatorname{dist}(x_0, F(x_0)) < \left(\frac{1 - A_1A_2}{1 + A_1}\right) r,$$

where

$$A_1 = \frac{a_1 + a_4 + a_5}{1 - a_2 - a_4} \quad \textit{and} \quad A_2 = \frac{a_2 + a_3 + a_5}{1 - a_1 - a_3}.$$

Then F has a fixed point (i.e., there exists $x \in \overline{B(x_0, r)}$ with $x \in F(x)$).

Remark 3.1: Note if $a_3 = a_4$ then $0 < A_1 < 1$ and $0 < A_2 < 1$, whereas if $a_1 = a_2$ we have $0 < A_1A_2 < 1$. We may if we wish (because of symmetry) take $a_3 = a_4$ and $a_1 = a_2$ (in this case $A_1 = A_2$).

Proof: Choose $x_1 \in F(x_0)$ such that

$$d(x_1, x_0) < \left(\frac{1 - A_1\, A_2}{1 + A_1}\right) r,$$

so $x_1 \in \overline{B(x_0, r)}$.

Next choose $\epsilon > 0$ such that

$$A_1\, d(x_1, x_0) + \frac{\epsilon}{1 - a_2 - a_4} < A_1 \left(\frac{1 - A_1 A_2}{1 + A_1} \right) r. \tag{3.1}$$

Then choose $x_2 \in F(x_1)$ with

$$\begin{aligned} d(x_1, x_2) &\le D(F(x_0), F(x_1)) + \epsilon \\ &\le a_1\, \text{dist}(x_0, F(x_0)) + a_2\, \text{dist}(x_1, F(x_1)) + a_3\, \text{dist}(x_1, F(x_0)) \\ &\quad + a_4\, \text{dist}(x_0, F(x_1)) + a_5\, d(x_0, x_1) + \epsilon \\ &\le a_1\, d(x_0, x_1) + a_2\, d(x_1, x_2) + a_4\, d(x_0, x_2) + a_5\, d(x_0, x_1) + \epsilon \end{aligned}$$

and so

$$d(x_1, x_2) \le A_1\, d(x_0, x_1) + \frac{\epsilon}{1 - a_2 - a_4}.$$

Now with ϵ chosen as in (3.1) we have

$$d(x_1, x_2) < A_1 \left(\frac{1 - A_1 A_2}{1 + A_1} \right) r.$$

Notice

$$x_2 \in \overline{B(x_0, r)}$$

since (we give an argument here which can be used in the general step)

$$\begin{aligned} d(x_0, x_2) &\le \left(\frac{1 - A_1 A_2}{1 + A_1} \right) r + \left(\frac{1 - A_1 A_2}{1 + A_1} \right) A_1\, r \\ &\le \frac{1 - A_1 A_2}{1 + A_1}\, r \{ 1 + A_1 A_2 + (A_1 A_2)^2 + \cdots + A_1 \left[1 + A_1 A_2 + (A_1 A_2)^2 + \cdots \right] \} \\ &= \left(\frac{1 - A_1 A_2}{1 + A_1} \right) r \left[\frac{1 + A_1}{1 - A_1 A_2} \right] = r. \end{aligned}$$

Next choose $\delta > 0$ such that

$$A_2\, d(x_1, x_2) + \frac{\delta}{1 - a_1 - a_3} < A_2 A_1 \left(\frac{1 - A_1 A_2}{1 + A_1} \right) r.$$

Then choose $x_3 \in F(x_2)$ with

$$d(x_2, x_3) \le D(F(x_2), F(x_1)) + \delta.$$

A similar reasoning as above yields

$$d(x_2, x_3) \leq A_2\, d(x_1, x_2) + \frac{\delta}{1 - a_1 - a_3}$$

and so

$$d(x_3, x_2) < A_2 A_1 \left(\frac{1 - A_1 A_2}{1 + A_1}\right) r.$$

Note

$$x_3 \in \overline{B(x_0, r)}$$

since (see the reasoning above)

$$\begin{aligned} d(x_0, x_3) &\leq \left(\frac{1 - A_1 A_2}{1 + A_1}\right) r\, [1 + A_1 + A_1 A_2] \\ &\leq \left(\frac{1 - A_1 A_2}{1 + A_1}\right) r \left[\frac{1 + A_1}{1 - A_1 A_2}\right] = r. \end{aligned}$$

Proceed inductively to obtain $x_n \in F(x_{n-1})$, $n = 4, 5, \cdots$ with $x_n \in \overline{B(x_0, r)}$ and

$$d(x_{2\,j+1}, x_{2\,j+2}) \leq (A_1 A_2)^j\, A_1 \left(\frac{1 - A_1 A_2}{1 + A_1}\right) r, \quad j = 1, 2, \ldots$$

and

$$d(x_{2\,j}, x_{2\,j+1}) \leq (A_1 A_2)^j \left(\frac{1 - A_1 A_2}{1 + A_1}\right) r, \quad j = 2, 3, \ldots.$$

Now it is immediate since $0 < A_1 A_2 < 1$ that (x_n) is Cauchy. Also since X is complete there exists $x \in \overline{B(x_0, r)}$ with $\lim_{n\to\infty} x_n = x$. It remains to show $x \in F(x)$. Notice

$$\begin{aligned} \text{dist}(x, F(x)) &\leq d(x, x_{2\,n+1}) + \text{dist}(x_{2\,n+1}, F(x)) \\ &\leq d(x, x_{2\,n+1}) + D(F(x_{2\,n}), F(x)) \end{aligned}$$

and so

$$\begin{aligned} D(F(x_{2\,n}), F(x)) &\leq a_1\, \text{dist}(x_{2\,n}, F(x_{2\,n})) + a_2\, \text{dist}(x, F(x)) + a_3\, \text{dist}(x, F(x_{2\,n})) \\ &\quad + a_4\, \text{dist}(x_{2\,n}, F(x)) + a_5\, d(x_{2\,n}, x) \\ &\leq a_1\, d(x_{2\,n}, x_{2\,n+1}) + a_2\, [d(x, x_{2\,n+1}) + D(F(x_{2\,n}), F(x))] \\ &\quad + a_3\, d(x, x_{2\,n+1}) + a_4 [d(x_{2\,n}, x_{2\,n+1}) + D(F(x_{2\,n}), F(x))] \\ &\quad + a_5\, d(x_{2\,n}, x). \end{aligned}$$

Consequently,

$$\begin{aligned} D(F(x_{2\,n}), F(x)) \leq \left(\frac{a_1 + a_4}{1 - a_2 - a_4}\right) d(x_{2\,n}, x_{2\,n+1}) + \left(\frac{a_2 + a_3}{1 - a_2 - a_4}\right) d(x, x_{2\,n+1}) \\ + \left(\frac{a_5}{1 - a_2 - a_4}\right) d(x_{2\,n}, x). \end{aligned}$$

As a result we have

$$\begin{aligned}\operatorname{dist}(x,F(x)) &\le \left(\frac{a_1+a_4}{1-a_2-a_4}\right)d(x_{2n},x_{2n+1})+\left(\frac{1+a_3-a_4}{1-a_2-a_4}\right)d(x,x_{2n+1})\\ &\quad+\left(\frac{a_5}{1-a_2-a_4}\right)d(x_{2n},x)\\ &\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Thus $x \in \overline{F(x)} = F(x)$ and we are finished. □

We next note that we obtain Bose and Mukherjee's result [3] as a Corolllary of Theorem 3.1.

Theorem 3.2: *Let (X,d) be a complete metric space, $F\colon X \to C(X)$. Suppose for $x,y \in X$ we have*

$$\begin{aligned}D(F(x),F(y)) &\le a_1 \operatorname{dist}(x,F(x)) + a_2 \operatorname{dist}(y,F(y)) + a_3 \operatorname{dist}(y,F(x))\\ &\quad + a_4 \operatorname{dist}(x,F(y)) + a_5\, d(x,y)\end{aligned}$$

where $a_1,\dots,a_5$ are nonnegative real numbers with $a_1+a_2+a_3+a_4+a_5<1$, $a_1+a_4+a_5>0$, $a_2+a_3+a_5>0$, and with either $a_1=a_2$ or $a_3=a_4$. Then F has a fixed point.

Proof: Fix $x_0 \in X$. Choose $r>0$ so that

$$\operatorname{dist}(x_0,F(x_0)) < \left(\frac{1-A_1A_2}{1+A_1}\right) r.$$

Now Theorem 3.1 guarantees that there exists $x \in \overline{B(x_0,r)}$ with $x \in F(x)$. □

Theorem 3.3: *Let (X,d) be a complete metric space with U an open subset of X. Suppose $H\colon \bar{U}\times[0,1]\to C(X)$ is a closed map (i.e., has closed graph) with the following satisfied:*

- (a) *$x \notin H(x,t)$ for $x \in \partial U$ and $t\in[0,1]$;*
- (b) *for all $t\in[0,1]$ and $x,y\in\bar{U}$, $D(H(x,t),H(y,t)) \le a_1 \operatorname{dist}(x,H(x,t)) + a_2 \operatorname{dist}(y,H(y,t)) + a_3 \operatorname{dist}(y,H(x,t)) + a_4 \operatorname{dist}(x,H(y,t)) + a_5\, d(x,y)$ (here $a_1,\dots,a_5$ are nonnegative real numbers with $a_1+a_2+a_3+a_4+a_5<1$, $a_1+a_4+a_5>0$, $a_2+a_3+a_5>0$, and with either $a_1=a_2$ or $a_3=a_4$); and*
- (c) *there exists a continuous increasing function $\phi\colon[0,1]\to\mathbf{R}$ such that $D(H(x,t),H(x,s)) \le |\phi(t)-\phi(s)|$ for all $t,s\in[0,1]$ and $x\in\bar{U}$.*

Then $H(\cdot,0)$ has a fixed point iff $H(\cdot,1)$ has a fixed point.

Proof: Suppose $H(\cdot,0)$ has a fixed point. Consider

$$Q = \{(t,x)\in[0,1]\times U : x \in H(x,t)\}.$$

Now Q is nonempty since $H(\,\cdot\,,0)$ has a fixed point. On Q define the partial order

$$(t,x)\le(s,y) \quad \text{iff} \quad t\le s \quad \text{and} \quad d(x,y)\le 2\left(\frac{1+A_1}{1-A_1A_2}\right)[\phi(s)-\phi(t)],$$

where

$$A_1 = \frac{a_1 + a_4 + a_5}{1 - a_2 - a_4} \quad \text{and} \quad A_2 = \frac{a_2 + a_3 + a_5}{1 - a_1 - a_3}.$$

Let P be a totally ordered subset of Q and let

$$t^* = \sup\{t : (t, x) \in P\}.$$

Take a sequence $\{(t_n, x_n)\}$ in P such that $(t_n, x_n) \leq (t_{n+1}, x_{n+1})$ and $t_n \to t^*$. We have

$$d(x_m, x_n) \leq 2\left(\frac{1 + A_1}{1 - A_1\, A_2}\right)[\phi(t_m) - \phi(t_n)] \quad \text{for all } m > n,$$

and so (x_m) is a Cauchy sequence, which converges to some $x^* \in \bar{U}$. Now since H is a closed map we have $(t^*, x^*) \in Q$ (note $x^* \in H(x^*, t^*)$ by closedness and (a) implies $x^* \in U$). It is also immediate from the definition of t^* and the fact that P is totally ordered that

$$(t, x) \leq (t^*, x^*) \quad \text{for every } (t, x) \in P.$$

Thus (t^*, x^*) is an upper bound of P. By Zorn's Lemma Q admits a maximal element $(t_0, x_0) \in Q$.

We *claim* $t_0 = 1$ (if our claim is true then we are finished). Suppose our claim is false. Then, choose $r > 0$ and $t \in (t_0, 1]$ with

$$\overline{B(x_0, r)} \subseteq U \quad \text{and} \quad r = 2\left(\frac{1 + A_1}{1 - A_1 A_2}\right)[\phi(t) - \phi(t_0)].$$

Notice

$$\begin{aligned} \text{dist}(x_0, H(x_0, t)) &\leq \text{dist}(x_0, H(x_0, t_0)) + D(H(x_0, t_0)), H(x_0, t))) \\ &\leq \phi(t) - \phi(t_0) = \frac{1}{2}\left(\frac{1 - A_1 A_2}{1 + A_1}\right) r < \left(\frac{1 - A_1 A_2}{1 + A_1}\right) r. \end{aligned}$$

Now Theorem 3.1 guarantees that $H(\cdot, t)$ has a fixed point $x \in \overline{B(x_0, r)}$. Thus $(x, t) \in Q$ and notice since

$$d(x_0, x) \leq r = 2\left(\frac{1 + A_1}{1 - A_1 A_2}\right)[\phi(t) - \phi(t_0)] \quad \text{and} \quad t_0 < t,$$

we have $(t_0, x_0) < (t, x)$. This contradicts the maximality of (t_0, x_0). □

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4. Using the Integral Manifolds to Solvability of Boundary Value Problems*

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Abstract: Acyclic solution sets of quasi-linearized differential systems with constraints are employed for solving nonlinear boundary value problems. A special attention is paid to periodic and anti-periodic solutions. The sufficient criteria are obtained in terms of (not necessarily smooth) bounding functions. The intersection of sublevel sets of these Liapunov-like functions forms a desired bound set with a transversality behaviour on its boundary.

Keywords and Phrases: Solution sets, boundary value problems, topological structure, multi-valued method, bound sets, bounding functions

AMS Subject Classification: 34A60, 34B15, 47H04

1. INTRODUCTION

The aim of our chapter consists in showing how methods of set-valued analysis can be adapted to solve nontraditionally boundary value problems for ordinary differential equations or, more generally, inclusions. More precisely, the solution sets (i.e., integral manifolds, whence the title) of linearized differential systems, satisfying given boundary conditions, give rise to multivalued operators with suitable properties. Thus, the original problem turns out to be equivalent with a fixed-point problem for these operators (cf. [2,3,14,15,25] and the references therein).

This approach requires, besides another, verifying the topological structure of solution sets to linearized systems (in Chapter 3) and the transversality behaviour of solutions on the boundary of a certain set, called the bound set, (in Chapter 5). The latter is guaranteed by constructing a special sort of Liapunov-like functions, called bounding functions, which is always a difficult task. So, the multivalued method developed in [3] (and recalled in Chapter 4) is appropriately elaborated in Chapter 6, where the main results are formulated.

We believe that this method deserves some future interest, because new results might be obtained in this way (cf. [7]), having no analogy by means of a standard (single-valued) manner.

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2. SOME PRELIMINARIES

In the entire text, all topological spaces are metric. Let us recall that a set is *acyclic* (w.r.t. any continuous theory of cohomology) if it is homologically same as a one point space. A nonempty space Y is called an *absolute retract* (AR) if, for any metrizable X and any closed $A \subset X$, every continuous map F: $A \to Y$ is extendable over X. By an R_δ*-set* we mean the intersection of a decreasing sequence of compact AR-spaces. Let us note that any R_δ-set is well-known to be acyclic.

Furthermore, let us recall that a multivalued map with closed values φ: $X \rightsquigarrow Y$ (i.e., φ: $X \to 2^Y \backslash \{\emptyset\}$) is *measurable* if, for any open $U \subset Y$, the set $\{x \in X : \varphi(x) \cap U \neq \emptyset\}$ is measurable. It is *upper semi-continuous* (*u.s.c.*) if $\{x \in X : \varphi(x) \subset U\}$ is open in X, for every open subset $U \subset Y$. Obviously, all u.s.c. maps are measurable. A multivalued map is called *acyclic* (or, in particular, R_δ) if it is an u.s.c. map with nonempty, compact, acyclic (or, in particular, R_δ) values.

At last, all boundary value problems (BVPs) under our consideration will take the form (in general)

$$\begin{aligned} &x' \in F(t,x) \quad \text{for a.a. } t \in I, \\ &x \in S, \end{aligned} \tag{0}$$

where I is a given real compact interval and F: $I \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is an (upper) *Carathéodory function*, i.e.,

(i) the set of values of F is nonempty, compact and convex for all $(t,x) \in I \times \mathbb{R}^n$;
(ii) $F(t, \cdot)$ is u.s.c. for a.a. $t \in I$;
(iii) $F(\cdot, x)$ is measurable for all $x \in \mathbb{R}^n$.

By a *solution* $x(t)$ of (0) we always mean the Carathéodory one, namely an absolutely continuous function $x(t) \in AC(I, \mathbb{R}^n)$ satisfying (0), for a.a. $t \in I$.

For more details and information, we recommend the monograph [21].

3. TOPOLOGICAL STRUCTURE OF SOLUTION SETS

Although the problem of the investigation of structure of solution sets to differential systems comes back to H. Kneser in 1923, they are only rare results related to boundary value problems (see [2,4,5,10–13,17–19,22,26–28]). On the other hand, Cauchy (initial value) problems are treated with this respect rather frequently (see e.g., [3–5,9,18,21] and the references therein). Nevertheless, because of our interest (i.e., BVPs), we recall here only those in [10] and [13] which seem to be the most appropriate for our goal.

In [10], a rather large family of multivalued BVPs has been examined as follows.

Proposition 1: *Consider the problem*

$$\begin{aligned} &x' + A(t)x \in F(t,x) \quad \textit{for a.a. } t \in I, \\ &Lx = \Theta \quad (\Theta \in \mathbb{R}^n), \end{aligned} \tag{1}$$

on a compact interval I, where A: $I \to \mathbb{R}^n \times \mathbb{R}^n$ is a single-valued essentially bounded Lebesgue measurable $(n \times n)$-matrix and F: $I \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a Carathéodory function *which is lipschitzian in x, for a.a. $t \in I$, with a sufficiently small Lipschitz constant k (i.e., $h(F(t,x), F(t,y)) \le k|x-y|$ for all $x, y \in \mathbb{R}^n$ and a.a. $t \in I$, where $h(.,.)$ stands for the Hausdorff metric.*

Let, furthermore, L: $C(I, \mathbb{R}^n) \to \mathbb{R}^n$ be a linear operator such that the associated homogeneous problem,

$$\begin{aligned} &x' + A(t)x = 0 \quad \textit{for a.a. } t \in I, \\ &Lx = 0, \end{aligned}$$

has only the trivial solution on I.

Then the set of solutions of (1) is a (nonempty) compact AR-space (i.e., more than R_δ-set).

Remark 1: In fact, Proposition 1 represents only a particular case of a more general result in [10], where the functional dependence has been also taken into account. Moreover, the covering dimension of solution sets was studied there. On the other hand, because of the applied contraction principle, the assertion reduces just to the uniqueness property for ODEs.

Therefore, we add still the following statements in [13] (cf. Theorems 4 and 5 in [13]).

Proposition 2: *Consider the Floquet problem*

$$\begin{aligned} &x' = f(t,x) \quad \text{for a.a. } t \in [a,b], \\ &x(a) + \mu x(b) = \xi \quad (\mu > 0, \xi \in \mathbb{R}^n), \end{aligned} \tag{2}$$

where f: $I \times \mathbb{R}^n \to \mathbb{R}^n$ is a bounded Carathéodory function. Assume, furthermore, that f satisfies

$$|f(t,x) - f(t,y)| \le p(t)|x-y| \quad \textit{for a.a. } t \in [a,b] \textit{ and } x, y \in \mathbb{R}^n, \tag{3}$$

where p: $[a,b] \to [0,\infty)$ is a Lebesgue integrable function such that

$$\int_a^b p(t)dt \le \sqrt{\pi^2 + \ln^2 \mu}. \tag{4}$$

Then the set of solutions of (2) is an R_δ-set.

Proposition 3: *Consider the Cauchy–Nicoletti problem*

$$\begin{aligned} &x_i' = f_i(t, x_1, \dots, x_n) \quad \text{for a.a. } t \in [a,b], (i = 1, \dots, n), \\ &x_i(t_i) = \xi_i \quad (\xi_i \in \mathbb{R}, t_i \in [a,b], i = 1, \dots, n;\ \xi = (\xi_1, \dots, \xi_n)), \end{aligned} \tag{5}$$

where $f = (f_1, \dots, f_n)$: $[a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a bounded Carathéodory function. Assume, furthermore, that f satisfies (3), where p: $[a,b] \to [0,\infty)$ is a Lebesgue integrable function satisfying this time

$$\int_a^b p(t)dt \le \frac{\pi}{2}. \tag{6}$$

Then the set of solutions of (5) is an R_δ-set.

Remark 2: As pointed out in [13], if the sharp inequalities hold in (4) or (6), then problem (2) or (5) has a unique solution, respectively. On the other hand, for non-sharp inequalities (4) or (6), problem (2) or (5) can possess more solutions, respectively.

To conclude this section, in order to apply the above propositions appropriately in the sequel, let us note that, for a Carathéodory function $G(t,x,y)$: $I \times \mathbb{R}^n \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$, the composed multifunction $G(t,x,q(t))$, with $q \in C(I,\mathbb{R}^m)$, becomes Carathéodory, provided G is product-measurable (see e.g., [8], p. 36). In the single-valued case, it is well-known that this property is satisfied automatically.

Definition 1: For a Carathéodory function $G(t,x,q(t))$, where $q \in C(I,\mathbb{R}^m)$ and $G(t,x,y)$: $I \times \mathbb{R}^n \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is product-measurable and Carathéodory, the sufficient conditions in Propositions 1–3 will be denoted, for the sake of simplicity, as (P1), (P2), (P3), respectively.

4. GENERAL MULTIVALUED METHOD

Now, we shall see how the information about the structure of solution sets to quasi-linearized systems (when applying the Schauder linearization device) can be employed (see condition (i) below) for solving given boundary value problems. The appropriately modified Theorem 2.33 in [3] reads as follows.

Theorem 1: *Consider the boundary value problem*

$$\begin{aligned} &x' \in F(t,x) \quad \textit{for a.a. } t \in I, \\ &x \in S, \end{aligned} \tag{7}$$

where I is a given real compact interval, F: $I \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a Carathéodory function and S is a subset of $AC(I,\mathbb{R}^n)$.

Let G: $I \times \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \rightsquigarrow \mathbb{R}^n$ be a Carathéodory function such that

$$G(t,c,c,1) \subset F(t,c), \quad \textit{for all } (t,c) \in I \times \mathbb{R}^n.$$

Assume that

(*i*) *There exists a bounded retract Q of $C(I,\mathbb{R}^n)$ such that $Q \backslash \partial Q$ is nonempty open and a closed bounded subset S_1 of S such that the associated problem*

$$\begin{aligned} &x' \in G(t,x,q(t),\lambda) \quad \textit{for a.a. } t \in I, \\ &x \in S_1, \end{aligned} \tag{8}$$

is solvable with R_δ-sets of solutions, for each $(q,\lambda) \in Q \times [0,1]$;

(*ii*) *There exists a locally integrable function α: $I \to \mathbb{R}$ such that*

$$|G(t,x(t),q(t),\lambda)| \le \alpha(t) \quad \textit{a.e. in } I,$$

for any $(x,q,\lambda) \in \Gamma_T$ (i.e., from the graph Γ of T), where T denotes the set-valued map which assigns to any $(q,\lambda) \in Q \times [0,1]$ the set of solutions of (8);

(*iii*) $T(Q \times \{0\}) \subset Q$;
(*iv*) *The map T has no fixed points on the boundary ∂Q of Q, for every $(2, \lambda) \in Q \times [0,1]$.*

Then problem (7) admits a solution.

Remark 3: Theorem 1 extends many of its analogies (cf. e.g., [2,14,15] and the references therein). On the other hand, often "only" acyclicity of solution sets is to our disposal (see e.g., [4,12]).

Hence, we add still another statement based on the application of the Eilenberg–Montgomery fixed-point theorem (cf. Corollary 2.35 in [3]).

Theorem 2: *Consider problem (7). Let $G\colon I \times \mathbb{R}^n \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ be a Carathéodory function such that*

$$G(t,c,c) \subset F(t,c), \quad \textit{for all } (t,c) \in I \times R^n.$$

Assume that

(*i*) *There exists a convex closed subset Q of $C(I,\mathbb{R}^n)$ such that the associated problem*

$$\begin{aligned} &x' \in G(t,x,q(t)) \quad \textit{for a.a. } t \in I, \\ &x \in S \cap Q, \end{aligned} \tag{9}$$

has an acyclic set of solutions, for each $q \in Q$;
(*ii*) *There exists a locally integrable function $\alpha\colon I \to \mathbb{R}$ such that*

$$|G(t,x(t),q(t))| \le \alpha(t) \quad \textit{a.e. in } I,$$

for any pair $(q,x) \in \Gamma_T$;
(*iii*) *$T(Q)$ is bounded in $C(I,\mathbb{R}^n)$ and $\overline{T(Q)} \subset S$.*

Then problem (7) admits a solution.

Of course, both theorems can be easily specified w.r.t. concrete BVPs (1) or (2) or (3), when applying properties (P1) or (P2) or (P3) (see Definition 1) in condition (i), respectively. However, condition (iv) in Theorem 1 (or, in particular, condition (iii) in Theorem 2) are still rather implicit. Therefore, in the next section, this condition will be expressed more explicitly in terms of locally lipschitzian bounding functions.

5. BOUND SETS FOR BVPs

In this part, we follow the ideas in [20,24,30] and especially in [29], in order to elaborate them, by means of the techniques in [1,9,23], in a multivalued way. Unlike in all mentioned chapters, but [1], solutions are understood in the Carathéodory sense, which brings some technical difficulties. The details will be published elsewhere (see [6]).

Let us recall some appropriately modified definitions.

Definition 2: By a bound set to problem (7) (or (8)) we mean a bounded subset $\mathcal{B} = \bigcup_{A \in J} \mathcal{B}(A) \subset \mathbb{R}^n$ such that $\mathcal{B}(A)$ is nonempty and open, for each $A \in J$, for which there is no solution $x(t)$ of (7) (or (8)) such that if $x(t) \in \overline{\mathcal{B}(A)}$, for every $t \in J$, then $x(t_0) \in \partial\mathcal{B}(A_0)$, for some $t_0 \in J$.

Obviously, if $\mathcal{B}$ is a bound set to (7) (or (8)) and

$$\Omega := \{x \in S : x(t) \in \mathcal{B}(A), \text{for all } t \in J\},$$

$$\partial\Omega := \{x \in S : x(t) \in \overline{\mathcal{B}(A)}, \text{for all } t \in J, \text{and } x(t_0) \in \partial\mathcal{B}(A_0), \text{for some } t_0 \in J\},$$

then no solution $x(t)$ of (7) (or (8)) can belong to the boundary $\partial\Omega$ of Ω.

As usual, a function $V(t,x) \in C(J \times \mathbb{R}^n, \mathbb{R})$ is said to be *locally lipschitzian at a point* $x_0 \in \mathbb{R}^n$ if there exist a positive constant L and a neighbourhood U of x_0 such that

$$|V(t,x) - V(t,y)| \le L|x-y| \quad \text{for all } x, y \in U, \text{uniformly w.r.t. } t \in J.$$

For such a function V, we can define in a standard manner the *upper right* and the *lower left Dini derivatives* of V at $x_0 \in \mathbb{R}^n$, calculated in $x_1 \in \mathbb{R}^n$, by

$$D^+V(t,x_0)(x_1) := \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, x_0 + hx_1) - V(t,x_0)],$$

and

$$D_-V(t,x_0)(x_1) := \liminf_{h\to 0^-} \frac{1}{h}[V(t+h, x_0 + hx_1) - V(t,x_0)],$$

respectively.

In order to find desired bound sets to (7) (or (8), for verifying condition (iv) in Theorem (1), bounding functions $V(t,x)$ can be constructed as follows.

Lemma 1: *Assume that a nonempty, open, bounded subset $\mathcal{B} \subset \mathbb{R}^n$ exists jointly with a one-parameter family of bounding functions $V_u(t,x) \in C(J \times \mathbb{R}^n, \mathbb{R})$, which are locally lipschitzian in $u \in \bigcup_{A\in J} \partial\mathcal{B}(A)$, such that the following conditions are satisfied:*

$$\begin{aligned}&\forall u \in \cup\partial\mathcal{B}(s)\, \exists r_u > 0 \;\; \textit{such that} \\ &V_u(t,y) \le 0\, \forall y \in \overline{\mathcal{B}} \cap B_u^{r_u}, \;\textit{uniformly w.r.t. } t \in J, \\ &\textit{and } V_u(t,u) \equiv 0,\end{aligned} \tag{B1}$$

where $B_u^r = \{x \in \mathbb{R}^n : |x-u| < r\}$,

$$\begin{aligned}&\forall u \in \bigcup_{A\in J} \partial\mathcal{B}(A), \quad \textit{for a.a. } t \in J, \forall w \in \{F(t,u)\} : \\ &(-\varepsilon, \varepsilon) \not\subset [D^+V_u(t,u)(w), D_-V_u(t,u)(w)],\end{aligned} \tag{B2}$$

where ε is a suitable positive constant.

Then every solution $x(t)$ of a Carathéodory inclusion $x' \in F(t,x)$ with $x(t) \in \overline{\mathcal{B}(A)}$, for every $t \in J$, satisfies $x(t) \in \mathcal{B}(A)$, for every $t \in \operatorname{int} J = \overline{J} \backslash \partial J$.

Proof: Can be done similarly as in [29], where only classical solutions of ODEs have been considered, by means of the arguments used in a slightly different context in [9], (cf. also [23]).

Remark 4: For C^1-functions $V_u(t,x)$, we have

$$D^+V_u(t,u)(w) = D_-V_u(t,u)(w) = \langle \mathrm{grad} V_u(t,u), (1,w) \rangle,$$
$$\text{for every } w \in \{F(t,u)\}.$$

Thus, condition (B2) takes the simple form

$$\forall u \in \partial\mathcal{B}(A), \text{ for a.a. } t \in J, \langle \mathrm{grad} V_u(t,u), (1, F(t,u)) \rangle \geq \varepsilon > 0 \text{ or}$$
$$\langle \mathrm{grad} V_u(t,u), (1, F(t,u)) \rangle \leq -\varepsilon < 0,$$

where $\langle \cdot,\cdot \rangle$ stands for an inner product.

In order $\mathcal{B}$ to be a bound set to (7) (or (8), we must still show that, for each solution $x(t)$ of (7) (or (8)) such that $x(t) \in \overline{\mathcal{B}(A)}$, for every $t \in J = [a,b]$, $x(a) \notin \partial\mathcal{B}(a)$ and $x(b) \notin \partial\mathcal{B}(b)$.

For this, the most convenient problem for our consideration seems to be

$$\begin{aligned} &x' \in F(t,x) \quad \text{for a.a. } t \in [a,b], \\ &x(b) = Mx(a), \end{aligned} \tag{10}$$

where F is a Carathéodory function and M is a regular (nonsingular) $n \times n$-matrix. Observe that, in particular, for $M = -E$, (10) becomes an anti-periodic problem (cf. (2)) and, for $M = E$, a periodic problem.

Let us also recall that a subset $\mathcal{S} \subset \mathbb{R}^n$ is said to be *invariant w.r.t. a subgroup* $\mathcal{H}$ of the group $GL_n(\mathbb{R})$ of the real nonsingular $(n \times n)$-matrices if

$$\forall u \in \mathcal{S}, \forall H \in \mathcal{H} : Hu \in \mathcal{S}.$$

Lemma 2: *Let the assumptions of Lemma 1 be satisfied. Assume, furthermore, that $M\partial\mathcal{B}(a) = \partial\mathcal{B}(b)$ takes place and sufficiently small positive constants δ, ε exist such that*

$$\begin{aligned} &\forall u \in \partial\mathcal{B}(A_a), \textit{for a.a. } t_a \in [a, a+\delta), \textit{for a.a. } t_b \in (b-\delta, b], \\ &\forall w_a \in \{F(t_a,u)\}, \forall w_b \in \{F(t_b, Mu)\} : \\ &(-\varepsilon, \varepsilon) \not\subset [D^+V_u(a,u)(w_a), D_-V_{Mu}(b,Mu)(w_b)]. \end{aligned} \tag{B3}$$

Then $\mathcal{B}$ is a bound set to (10).

Proof: Can be done similarly as in [29], by means of the arguments used in a slightly different context in [9].

Remark 5: For C^1-functions $V_u(t,x)$, condition (B3) takes the form

$$\forall u \in \partial\mathcal{B}(A_a), \text{ for a.a. } t_a \in [a, a+\delta), \text{ for a.a. } t_b \in (b-\delta, b] : (-\varepsilon, \varepsilon) \not\subset$$
$$[\langle \mathrm{grad} V_u(a,u), (1, F(t_a,u)) \rangle, \langle \mathrm{grad} V_{Mu}(b, Mu), (1, F(t_b, Mu)) \rangle].$$

Unfortunately, problems like (5) are much less convenient to handle for the same goal. Therefore, in the remaining part, we restrict ourselves only to (10) and its particular cases $M = \pm E$.

6. MAIN RESULTS

Hence, let us consider the problem

$$\begin{aligned} &x' + A(t)x \in F(t, x) \quad \text{for a.a. } t \in [a, b], \\ &x(b) = Mx(a), \end{aligned} \tag{11}$$

where A: $[a, b] \to \mathbb{R}^n \times \mathbb{R}^n$ is a single-valued essentially bounded Lebesgue measurable $(n \times n)$-matrix, F: $[a, b] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a Carathéodory function and M is a regular $(n \times n)$-matrix.

Summarizing the information from the foregoing sections, we are ready to give the first main result.

Theorem 3: *Consider problem (11) and let* G: $[a, b] \times \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \rightsquigarrow \mathbb{R}^n$ *be a product-measurable Carathéodory function such that*

$$G(t, c, c, 1) \subset F(t, c) \quad \textit{for all } (t, c) \in [a, b] \times \mathbb{R}^n.$$

Assume that

(i) *The associated homogeneous problem*

$$\begin{aligned} &x' + A(t)x = 0 \quad \textit{for a.a. } t \in [a, b], \\ &x(b) = Mx(a), \end{aligned}$$

has only the trivial solution;

(ii) *There exists a bounded retract* Q *of* $C([a, b], \mathbb{R}^n)$ *with the nonempty open* $Q \backslash \partial Q$ *such that* $G(t, x, q(t), \lambda)$ *is lipschitzian in* x *with a sufficiently small Lipschitz constant, for a.a.* $t \in [a, b]$ *and each* $(q, \lambda) \in Q \times [0, 1]$;

(iii) *There exists a Lebesgue integrable function* α: $[a, b] \to \mathbb{R}$ *such that*

$$|G(t, x(t), q(t), \lambda)| \le \alpha(t) \quad \textit{a.e. in } [a, b],$$

for any $(x, q, \lambda) \in \Gamma_T$, *where* T *denotes the set-valued map which assigns to any* $(q, \lambda) \in Q \times [0, 1]$ *the set of solutions of*

$$\begin{aligned} &x' + A(t)x \in G(t, x, q(t), \lambda) \quad \textit{for a.a. } t \in [a, b], \\ &x(b) = Mx(a); \end{aligned}$$

(iv) $T(Q \times \{0\}) \subset Q$ and ∂Q *is fixed-point free*;

(v) *For each* $u \in \bigcup_{A \in J} \partial \mathcal{B}(A)$, *where* $\mathcal{B} = \operatorname{int}\{x(t) \in \mathbb{R}^n : x \in Q\} \neq \phi$ *is the same, for every* $A \in [a, b]$, *there exists a bounding function* $V_u(t, x) \in C([a, b] \times \mathbb{R}^n, \mathbb{R}^n)$, *which is locally lipschitzian in* $u \in \partial \mathcal{B}$, *such that conditions* (B1)–(B3) *hold for any* $(q, \lambda) \in Q \times (0, 1]$, *where* $F := G(t, x, q(t), \lambda) - A(t)x$ *and* $I := [a, b]$;

(*vi*) $M\partial\mathcal{B}(a) = \partial\mathcal{B}(b)$.

Then problem (11) admits a solution.

Proof: Follows immediately from Theorem 1, Proposition 1 and Lemma 2.

If, in particular $M = E$ or $M = -E$, then Theorem 3 significantly simplifies as follows.

Corollary 1: *Consider problem (11), where $M = E$ and $F(t,x) \equiv F(t+(b-a),x)$. Let G: $[a,b]\times\mathbb{R}^n\times\mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ be a product-measurable Carathéodory function such that*

$$G(t,c,c) \subset F(t,c) \quad \textit{for all } (t,c)\in[a,b]\times\mathbb{R}^n.$$

Assume that

(*i*) *A is a piece-wise continuous single-valued bounded $(b-a)$-periodic $(n\times n)$-matrix whose Floquet multipliers lie off the unit circle*;

(*ii*) *There exists a bounded retract Q of $C([a,b],\mathbb{R}^n)$, with $0\in Q\backslash\partial Q$, where $Q\backslash\partial Q$ is nonempty open, such that $G(t,x,q(t))$ is lipschitzian in x with a sufficiently small Lipschitz constant, for a.a. $t\in[a,b]$ and each $q\in Q$*;

(*iii*) *There exists a Lebesgue integrable function α: $[a,b]\to\mathbb{R}$ such that*

$$|G(t,x(t),q(t))| \le \alpha(t) \quad \textit{a.e. in } [a,b],$$

for any $(x,q,\lambda)\in\Gamma_T$, where T denotes the set-valued map which assigns, to any $(q,\lambda)\in Q\times[0,1]$, the set of solutions of

$$\begin{aligned}&x' + A(t)x \in \lambda G(t,x,q(t)) \quad \textit{for a.a. } t\in[a,b],\\ &x(a) = x(b);\end{aligned}$$

(*iv*) *For each $u\in\bigcup_{A\in J}\partial\mathcal{B}(A)$, where $\mathcal{B} = \operatorname{int}\{x(t)\in\mathbb{R}^n : x\in Q\}\neq\phi$ is the same, for every $A\in[a,b]$, there exists a bounding function $V_u(t,x)\in C([a,b]\times\mathbb{R}^n,\mathbb{R}^n)$, which is locally lipschitzian in $u\in\partial\mathcal{B}$, such that conditions* (B1)–(B3) *hold for any $(q,\lambda)\in Q\times(0,1]$, where $F := \lambda G(t,x,q(t)) - A(t)x$ and $I := [a,b]$*;

Then the inclusion $x' + A(t)x\in F(t,x)$ admits a $(b-a)$-periodic solution.

Proof: Follows from Theorem 3, when realizing that condition (i) is implied by the given properties of the related Floquet multipliers (see e.g., [2]). Subsequently, the relation $T(Q\times\{0\})\subset Q$ reduces to the requirement that $0\in Q$. Condition (vi) holds trivially.

Corollary 2: *Consider problem* (11), *where $M = -E$, $A(t)\equiv 0$ and $F(t,x)\equiv -F(t+(b-a),-x)$. Let G: $[a,b]\times\mathbb{R}^n\times\mathbb{R}^n\times[0,1]\rightsquigarrow\mathbb{R}^n$ be a product-measurable Carathéodory function such that*

$$G(t,c,c,1)\subset F(t,c) \quad \textit{for all } (t,c)\in[a,b]\times\mathbb{R}^n.$$

Assume that

(*i*) *There exists a bounded retract Q of $C([a,b],\mathbb{R}^n)$ with the nonempty open $Q\backslash\partial Q$, symmetrical w.r.t. the origin $0\in Q$, such that $G(t,x,q(t),\lambda)$ is lipschitzian in x with a sufficiently small Lipschitz constant, for* a.a. $t\in[a,b]$ *and each* $(q,\lambda)\in Q\times[0,1]$;

(*ii*) *There exists a Lebesgue integrable function* α: $[a,b] \to \mathbb{R}$ *such that*

$$|G(t,x(t),q(t),\lambda)| \leq \alpha(t) \quad \textit{a.e. in } [a,b],$$

for any $(x,q,\lambda) \in \Gamma_T$, *where* T *denotes the set-valued map which assigns, to any* $(q,\lambda) \in Q \times [0,1]$, *the set of solutions of*

$$x' \in G(t,x,q(t),\lambda), \quad \textit{for a.a. } t \in [a,b],$$
$$x(a) = -x(b);$$

(*iii*) $T(Q \times \{0\}) \subset Q$ *and* ∂Q *is fixed-point free;*
(*iv*) *For each* $u \in \partial\mathcal{B}$, *where* $\mathcal{B} = \mathrm{int}\{x(t) \in \mathbb{R}^n : x \in Q\} \neq \phi$ *is the same, for every* $A \in [a,b]$, *there exists a bounding function* $V_u(t,x) \in C([a,b] \times \mathbb{R}^n, \mathbb{R}^n)$, *which is locally lipschitzian in* $u \in \partial\mathcal{B}$, *such that conditions (B1)–(B3) hold for any* $(q,\lambda) \in Q \times (0,1]$, *where* $F := G(t,x,q(t),\lambda)$ *and* $I := [a,b]$;

Then the inclusion $x' \in F(t,x)$ *admits a* $2(b-a)$*-periodic solution* $x(t) \equiv -x(t+(b-a))$.

Proof: Follows from Theorem 3, when realizing that condition (i) and (vi) hold trivially, provided additionally $A(t) \equiv 0$ and a symmetry of Q w.r.t. the origin $0 \in Q$.

In the single-valued case, Corollary 2 can be improved in view of Proposition 2, where $\mu = 1$ and $\xi = 0$, as follows.

Theorem 4: *Consider problem (2), where* $\mu = 1, \xi = 0$ *and* $f(t,x) \equiv -f(t+(b-a), -x)$. *Let* g: $[a,b] \times \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ *be a Carathéodory function such that*

$$g(t,c,c,1) = f(t,c) \quad \textit{for all } (t,c) \in [a,b] \times \mathbb{R}^n.$$

Assume that

(*i*) *There exists a bounded retract* Q of $C([a,b],\mathbb{R}^n)$ *with the nonempty open* $Q \backslash \partial Q$, *symmetrical w.r.t. the origin* $0 \in Q$, *such that*

$$|g(t,x,q(t),\lambda) - g(t,y,q(t),\lambda)| \leq p(t)$$

holds for a.a. $t \in [a,b]$; $x,y, \in \mathbb{R}^n$ *and each* $(q,\lambda) \in Q \times [0,1]$, *where* p: $[a,b] \to [0,\infty)$ *is a Lebesgue integrable function with* (cf. (4))

$$\int_a^b p(t)dt \leq \pi;$$

(*ii*) *There exists a positive constant* α *such that*

$$|g(t,x(t),q(t),\lambda)| \leq \alpha \quad \textit{a.e. in } [a,b],$$

for any $(x,q,\lambda) \in \mathbb{R}^n \times Q \times [0,1]$;
(*iii*) $T(Q \times \{0\}) \subset Q$ *and* ∂Q *is fixed-point free, where* T *denotes the set-valued map which assigns, to any* $(q,\lambda) \in Q \times [0,1]$, *the set of solutions of*

$$x' = g(t,x,q(t),\lambda) \quad \textit{for a.a. } t \in [a,b],$$
$$x(a) = -x(b);$$

(*iv*) *For each* $u \in \partial\mathcal{B}$, *where* $\mathcal{B} = \mathrm{int}\{x(t) \in \mathbb{R}^n : x \in Q\} \neq \phi$ *is the same, for every* $A \in [a, b]$, *there exists a bounding function* $V_u(t, x) \in C([a, b] \times \mathbb{R}^n, \mathbb{R}^n)$, *which is locally lipschitzian in* $u \in \partial\mathcal{B}$, *such that conditions (B1)–(B3) hold for any* $(q, \lambda) \in Q \times (0, 1]$, *where* $f := g(t, x, q(t), \lambda)$, $M = -E$ *and* $I := [a, b]$;

Then the equation $x' = f(t, x)$ *admits a* $2(b-a)$*-periodic solution* $x(t) \equiv -x(t + (b - a))$.

Proof: Is quite analogous to the one of Corollary 2, but when for condition (i) we apply Proposition 2, instead of Proposition 1, i.e., when appropriately modified property (P1) is replaced by (P2).

7. CONCLUDING REMARKS

The usage of integral manifolds in the above spirit is typical for differential inclusions, which are mostly fully (Schauder-like) linearized and then the invariantness of sets Q, under the associated operators T, is ensured, as required in Theorem 2. Those operators are usually at least acyclic (see e.g. [2,16]). Thus, for instance, the Floquet problems can be immediately solved, provided at most linear growth restrictions, with sufficiently small coefficients, of product-measurable Carathéodory right-hand sides F.

On the other hand, to find nonstandard existence criteria for differential equations (e.g., by means of Theorem 4) seems to be still far from obvious (cf. [7]). Nevertheless, we believe (and encourage the reader to this goal) that the application of the above technique can really bring some very delicate new results.

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5. On the Semicontinuity of Nonlinear Spectra

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Abstract: We discuss a fairly general approach to proving the closedness and upper semicontinuity of the multivalued map which associates to each continuous nonlinear operator from a certain operator class a natural spectrum. In this way, we show that the Kachurovskij spectrum for Lipschitz continuous operators, the Furi–Martelli–Vignoli spectrum for quasibounded operators, and the Feng spectrum for so-called k-epi operators are all upper semicontinuous with respect to a suitable normed or locally convex topology. On the other hand, we give a simple counterexample which shows that the Dörfner spectrum for linearly bounded operators has not a closed graph and is not upper semicontinuous either.

Keywords: Nonlinear operator, spectrum, closed multivalued map, upper semicontinuous multivalued map

Classification: 47H04, 47H12

Let X be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Denote by $\mathfrak{L}(X)$ the algebra of all bounded linear operators in X, and by

$$\sigma(L) = \{\lambda : \lambda \in \mathbb{K},\, (\lambda - L)^{-1} \notin \mathfrak{L}(X)\} \tag{1}$$

the spectrum of $L \in \mathfrak{L}(X)$. It is well known (see e.g., [13] or [11, Th. 3.1]) that the map σ: $\mathfrak{L}(X) \to 2^{\mathbb{K}}$ which associates to each operator L its spectrum $\sigma(L)$ is upper semicontinuous but not lower semicontinuous. Roughly speaking, the spectrum $\sigma(L)$ may "collaps", but not "blow up" if the operator L changes continuously.

To see that σ need not be lower semicontinuous, consider, for example, for $\varepsilon \in \mathbb{R}$ the operator $L_\varepsilon \in \mathfrak{L}(l_1(\mathbb{Z}))$ defined on the canonical basis $\{e_k : k \in \mathbb{Z}\}$ in $l_1(\mathbb{Z})$ by $L_\varepsilon e_k = e_{k-1}$ for $k \neq 0$ and $L_\varepsilon e_0 = \varepsilon e_{-1}$. Then we have $\sigma(L_0) = [-1, 1]$, on the one hand, but $\sigma(L_\varepsilon) = \{-1, 1\}$ for $\varepsilon \neq 0$, on the other. Consequently, the map $\varepsilon \mapsto \sigma(L_\varepsilon)$ is not lower semicontinuous at $\varepsilon = 0$.

In view of the importance of spectral theory for linear operators in various fields of mathematics, physics, or engineering, it is not surprising at all that various attempts have been made to define and study spectra also for *nonlinear operators*. Here we mention the

Kachurovskij spectrum [10] for Lipschitz continuous operators, the Neuberger spectrum [12] for Fréchet differentiable operators, the Rhodius spectrum [14] for continuous operators, the Dörfner spectrum [5] for operators with sublinear growth, the Furi–Martelli–Vignoli spectrum [8] for quasibounded operators, and the Feng spectrum [6] for so-called k-epi operators (see [9,15]). In particular, the Furi–Martelli–Vignoli spectrum and the Feng spectrum are upper semicontinuous with respect to a suitable topology. For the other spectra no semicontinuity properties have been studied so far. A comparison of the various advantages and drawbacks of all these spectra may be found in the recent survey [2].

In this note we propose a unified (and quite elementary) approach to proving the upper semicontinuity of the spectrum for several classes of continuous *nonlinear* operators. This approach consists in two parts. First we show that, for f belonging to some class $\mathfrak{M}(X)$ of continuous operators in X, the graph of the multivalued map $\sigma: f \mapsto \sigma(f)$, is *closed* in the product $\mathfrak{M}(X) \times \mathbb{K}$. Afterwards it suffices to use the *local boundedness* of this map, by means of simple upper estimates for the spectral radius, to deduce its upper semicontinuity.

In the first section we recall some semicontinuity definitions for multivalued maps, as well as their connections with closed graphs and local boundedness. This is applied in the second section to give a simple proof of the upper semicontinuity (which is already known) of the Furi–Martelli–Vignoli spectrum and Feng spectrum. Using the same method we show in the third section that also a certain modification of the Furi–Martelli–Vignoli spectrum which was introduced in [3] is upper semicontinuous. In the fourth section we prove the upper semicontinuity of the Kachurovskij spectrum; this gives the upper semicontinuity of the familiar spectrum of a bounded linear operator as a straightforward consequence. Finally, we give a simple counterexample which shows that the Dörfner spectrum has not a closed graph, and is not upper semicontinuous either.

1. SEMICONTINUITY OF MULTIVALUED MAPS

Let $\mathfrak{M}$ be a linear space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ with seminorm p, and let $\sigma: \mathfrak{M} \to 2^{\mathbb{K}}$ be a multivalued functional on $\mathfrak{M}$. For $f \in \mathfrak{M}$ and $\delta > 0$ we denote by $U_\delta(f)$ the p-neighbourhood $\{g : g \in \mathfrak{M},\ p(g - f) < \delta\}$ of f. Recall that σ is called *upper semicontinuous* [resp. *lower semicontinuous*] if, for any $f \in \mathfrak{M}$ and open $V \subseteq \mathbb{K}$ with $\sigma(f) \subseteq V$ [resp. $\sigma(f) \cap V \neq \emptyset$], one can find a $\delta > 0$ such that $\sigma(U_\delta(f)) \subseteq V$ [resp. $\sigma(U_\delta(f)) \cap V \neq \emptyset$]. Moreover, σ is called *closed* if the graph of σ is closed in $\mathfrak{M} \times \mathbb{K}$, i.e., $\lambda_n \in \sigma(f_n)$, $\lambda_n \to \lambda$ and $p(f_n - f) \to 0$ imply that $\lambda \in \sigma(f)$. Obviously, σ is closed if and only if, for every $f \in \mathfrak{M}$ and $\lambda \in \mathbb{K}\backslash\sigma(f)$, there exist $\delta > 0$ and $V_\lambda \subset \mathbb{K}$ open such that $\lambda \in V_\lambda$ and $\sigma(U_\delta(f)) \cap V_\lambda = \emptyset$.

It is easy to see that every upper semicontinuous map with closed values is closed, but not vice versa. For example, the map $\sigma: \mathbb{R} \to 2^{\mathbb{R}}$ defined by $\sigma(t) = [-1/t, 1/t]$ for $t \neq 0$ and $\sigma(0) = 0$ is closed, but not upper semicontinuous at 0. The following proposition provides a simple condition which implies the upper semicontinuity of a closed map.

Lemma 1: *Let $(\mathfrak{M}, p)$ be a seminormed linear space and $\sigma: \mathfrak{M} \to 2^{\mathbb{K}}$ a closed map with bounded values. If*

$$\sup_{\lambda \in \sigma(f)} |\lambda| \leq p(f) \quad (f \in \mathfrak{M}), \tag{2}$$

then σ is upper semicontinuous.