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Nonoscillation and Oscillation: Theory for Functional Differential Equations

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Nonoscillation and Oscillation: Theory for Functional Differential Equations

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Preface

This book is devoted to a rapidly developing branch of the qualitative theory of differential equations with or without delays. It summarizes the most recent contributions of the authors and their colleagues in this area and will be a stimulus to its further development.

There are eight chapters in this book. After the preliminaries in Chapter 1, we present oscillatory and nonoscillatory properties of first order delay differential equations and first order neutral delay differential equations in Chapters 2 and 3, respectively. Classification schemes and existence of positive solutions of neutral delay differential equations with variable coefficients are also considered. In Chapter 4, oscillation and nonoscillation of second order nonlinear differential equations without delays is investigated. Chapter 5 is devoted to classification schemes and existence of positive solutions of second order delay differential equations with or without neutral terms. Nonoscillation and oscillation of higher order delay differential equations is considered in Chapter 6. Chapter 7 features oscillation and nonoscillation for two-dimensional systems of nonlinear differential equations. Finally, in Chapter 8, we give some first results on the oscillation of dynamic equations on time scales. Time scales have been introduced in order to unify continuous and discrete analysis and to extend those theories to cases "in between". Many results given in the first seven chapters of this book may be generalized within the time scales setting (hence accommodating differential equations and difference equations simultaneously), and in this final chapter we present some of those results.

This book is addressed to a wide audience of specialists such as mathematicians, physicists, engineers and biologists. It can be used as a textbook at the graduate level and as a reference book for several disciplines.

Thanks are due to Xiao-Yun Cao for her assistance in typing portions of the book and a very special thank you to Dr. Murat Adıvar, Dr. Elvan Akın-Bohner, Dr. Xiang-Li Fei, and Dr. Hai-Feng Huo for their help in proofreading. Finally, we wish to express our thanks to the staff of Marcel Dekker, Inc., in particular Maria Allegra and Elizabeth Draper, for their cooperation during the preparation of this book for publication.

Ravi Agarwal
Martin Bohner
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Contents

CHAPTER 1

Preliminaries

1.1. Introduction

This chapter is essentially introductory in nature. Its main purpose is to present some basic concepts from the theory of delay differential equations and to sketch some preliminary results which will be used throughout the book. In this respect, this is almost a self-contained monograph. The reader may glance at the material covered in this chapter and then proceed to Chapter 2.

Section 1.2 is concerned with the statement of the basic initial value problems and classification of equations with delays. In Section 1.3 we provide definition of oscillation of solutions with or without delays. Section 1.4 states some fixed point theorems which are important tools in oscillation theory, especially, when one proves the existence of nonoscillatory solutions.

1.2. Initial Value Problems

Let us consider the ordinary differential equation (ODE)

$$x'(t) = f(t, x) \tag{1.1}$$

together with the initial condition

$$x(t_0) = x_0. \tag{1.2}$$

It is well known that under certain assumptions on f the initial value problem (1.1) and (1.2) has a unique solution and is equivalent to the integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} f\Big(s, x(s)\Big)\, ds \quad \text{for} \quad t \geq t_0.$$

Next, we consider a differential equation of the form

$$x'(t) = f\Big(t, x(t), x(t-\tau)\Big) \quad \text{with} \quad \tau > 0 \quad \text{and} \quad t \geq t_0, \tag{1.3}$$

in which the right-hand side depends not only on the instantaneous position $x(t)$, but also on $x(t-\tau)$, the position at τ units back, that is to say, the equation has past memory. Such an equation is called an *ordinary differential equation with delay* or *delay differential equation.* Whenever necessary, we shall consider the integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} f\Big(s, x(s), x(s-\tau)\Big)\, ds,$$

which is equivalent to (1.3). In order to define a solution of (1.3), we need to have a known function φ on $[t_0 - \tau, t_0]$, instead of just the initial condition $x(t_0) = x_0$.

The basic initial value problem for a delay differential equation is posed as follows: On the interval $[t_0, T]$, $T \leq \infty$, we seek a continuous function x that satisfies (1.3) and an initial condition

$$x(t) = \varphi(t) \quad \text{for all} \quad t \in E_{t_0}, \tag{1.4}$$

where t_0 is an initial point, $E_{t_0} = [t_0 - \tau, t_0]$ is the initial set; the known function φ on E_{t_0} is called the *initial function*. Usually, it is assumed that $\varphi(t_0 + 0) = \varphi(t_0)$. We always mean a one-sided derivative when we speak of the derivative at an endpoint of an interval.

Under general assumptions, the existence and uniqueness of solutions to the initial value problem (1.3) and (1.4) can be established (see, for example, Győri and Ladas [**118**]). The solution sometimes is denoted by $x(t, \varphi)$. In the case of a variable delay $\tau = \tau(t) > 0$ in (1.3), it is also required to find a solution of this equation for $t > t_0$ such that on the initial set

$$E_{t_0} = t_0 \cup \left\{ t - \tau(t) : \; t - \tau(t) < t_0, \; t \geq t_0 \right\},$$

x coincides with the given initial function φ. If it is required to determine the solution on the interval $[t_0, T]$, then the initial set is

$$E_{t_0 T} = \{t_0\} \cup \left\{ t - \tau(t) : \; t - \tau(t) < t_0, \; t_0 \leq t \leq T \right\}.$$

Example 1.2.1. For the equation

$$y'(t) = f\left(t, y(t), y(t - \cos^2 t)\right),$$

$t_0 = 0$, $E_0 = [-1, 0]$, and the initial function φ must be given on the interval $[-1, 0]$.

The initial set E_{t_0} depends on the initial point t_0, as can be seen from the following example.

Example 1.2.2. For the equation

$$y'(t) = ay(t/2),$$

we have $\tau(t) = t/2$ so that

$$E_0 = \{0\} \quad \text{and} \quad E_1 = [1/2, 1].$$

Now we consider the differential equation of nth order with l deviating arguments, of the form

$$\begin{aligned} y^{(m_0)}(t) = f\Big(t, y(t), \ldots, y^{(m_0-1)}(t), y(t - \tau_1(t)), \ldots, y^{(m_1-1)}(t - \tau_1(t)), \ldots, \\ y(t - \tau_l(t)), \ldots, y^{(m_l-1)}(t - \tau_l(t))\Big), \end{aligned} \tag{1.5}$$

where the deviations $\tau_i(t) > 0$, and $\max_{0 \leq i \leq l} m_i = n$.

In order to formulate the initial value problem for (1.5), we shall need the following notation. Let t_0 be the given initial point. Each deviation $\tau_i(t)$ defines the initial set $E_{t_0}^{(i)}$ given by

$$E_{t_0}^{(i)} = \{t_0\} \cup \left\{ t - \tau_i(t) : \; t - \tau_i(t) < t_0, \; t \geq t_0 \right\}.$$

We denote $E_{t_0} = \cup_{i=1}^{l} E_{t_0}^{(i)}$, and on E_{t_0} continuous functions φ_k, $k = 0, 1, \ldots, \mu$, must be given, with $\mu = \max_{0 \le i \le l} m_i$. In applications, it is most natural to consider the case where on E_{t_0},

$$\varphi_k(t) = \varphi_0^{(k)}(t) \quad \text{for} \quad k = 0, 1, \ldots, \mu,$$

but it is not generally necessary.

For the nth order differential equation, there should be given initial values $y_0^{(k)}$, $k = 0, 1, 2, \ldots, n-1$. Now let $y_0^{(k)} = \varphi_k(t_0)$, $k = 0, 1, 2, \ldots, \mu$. If $\mu < n-1$, then, in addition, the numbers $y_0^{(\mu+1)}, \ldots, y_0^{(n-1)}$ are given. If the point t_0 is an isolated point of E_{t_0}, then $y_0^{(0)}, \ldots, y_0^{(n)}$ are also given.

For (1.5), the basic initial value problem consists of the determination of an $(n-1)$ times continuously differentiable function y that satisfies (1.5) for $t > t_0$ and the conditions

$$y^{(k)}(t_0 + 0) = y_0^{(k)}$$

for $k = 0, 1, \ldots, n-1$, and

$$y^{(k)}(t - \tau_i(t)) = \varphi_k(t - \tau_i(t)) \quad \text{if} \quad t - \tau_i(t) < t_0$$

for $k = 0, 1, \ldots, \mu$ and $i = 1, 2, \ldots, l$. At the point $t_0 + (k-1)\tau$ the derivative $y^{(k)}(t)$, generally speaking, is discontinuous, but the derivatives of lower order are continuous.

Example 1.2.3. Consider

$$y''(t) = f\left(t, y(t), y'(t), y(t - \cos^2 t), y\left(\frac{t}{2}\right)\right). \tag{1.6}$$

For $t_0 = 0$, we have $n = 2$, $l = 2$, $\mu = 0$, the initial sets $E_0^{(1)} = [-1, 0]$, $E_0^{(2)} = \{0\}$, and $E_0 = [-1, 0]$, on which is given the initial function φ_0, $y_0^{(0)} = \varphi_0(0)$, and $y_0^{(1)}$ is any given number.

For (1.5) a classification method was proposed by Kamenskiĭ [**141**]. We let $\lambda = m_0 - \mu$. If $\lambda > 0$, (1.5) is called an equation with *retarded* arguments or with delay. If $\lambda = 0$, it is called an equation of *neutral* type. If $\lambda < 0$, it is called an equation of *advanced* type.

Example 1.2.4. The equations

$$y'(t) + a(t)y(t - \tau) = 0 \quad \text{with} \quad \tau > 0,$$
$$y'(t) + a(t)y(t + \tau) = 0 \quad \text{with} \quad \tau > 0,$$

and

$$y'(t) + a(t)y(t) + b(t)y'(t - \tau) = 0 \quad \text{with} \quad \tau > 0$$

are of retarded type ($\lambda = 1$), advanced type ($\lambda = -1$), and neutral type ($\lambda = 0$), respectively.

In applications, the equation with retarded arguments is most important; the theory of such equations has been developed extensively. In this book we study mainly equations with or without delays.

1.3. Definition of Oscillation

Before we define oscillation of solutions, let us consider some simple examples.

Example 1.3.1. The equation

$$y'' + y = 0$$

has periodic solutions $y_1(t) = \cos t$ and $y_2(t) = \sin t$.

Example 1.3.2. Consider the equation

$$y''(t) - \frac{1}{t}y'(t) + 4t^2 y(t) = 0,$$

whose solution is $y(t) = \sin t^2$. This solution is not periodic but has an oscillatory property.

Example 1.3.3. Consider the equation

$$y''(t) + \frac{1}{2}y'(t) - \frac{1}{2}y(t-\pi) = 0 \quad \text{for} \quad t \geq 0, \tag{1.7}$$

whose solution $y(t) = 1 - \sin t$ has an infinite sequence of multiple zeros. This solution also has an oscillatory property.

Example 1.3.4. Consider the equation

$$y''(t) - y(-t) = 0.$$

This equation has an oscillatory solution $y_1(t) = \sin t$ and a nonoscillatory solution $y_2(t) = e^t + e^{-t}$.

Let us now restrict our discussion to those solutions y of the equation

$$y''(t) + a(t)y(t - \tau(t)) = 0 \tag{1.8}$$

which exist on some ray $[T_y, \infty)$ and satisfy $\sup\{|y(t)| : t \geq T\} > 0$ for every $T \geq T_y$. In other words, $|y(t)| \not\equiv 0$ on any infinite interval $[T, \infty)$. Such a solution sometimes is said to be a *regular solution.*

We usually assume that $a(t) \geq 0$ or $a(t) \leq 0$ in (1.8), and in doing so we mean to imply that $a(t) \not\equiv 0$ on any infinite interval $[T, \infty)$.

There are various possibilities of defining oscillation of solutions of ODEs (with or without delays). In this section, we give two definitions of oscillation, which are used in the rest of the book; these are the ones most frequently used in the literature.

As we see from the above examples, the definition of oscillation of regular solutions can have two different forms.

Definition 1.3.5. A nontrivial solution y (implying a regular solution always) is said to be *oscillatory* if it has arbitrarily large zeros for $t \geq t_0$, that is, there exists a sequence of zeros $\{t_n\}$ (i.e., $y(t_n) = 0$) of y such that $\lim_{n\to\infty} t_n = \infty$. Otherwise, y is said to be *nonoscillatory.*

For nonoscillatory solutions there exists t_1 such that

$$y(t) \neq 0 \quad \text{for all} \quad t \geq t_1.$$

Definition 1.3.6. A nontrivial solution y is said to be *oscillatory* if it changes sign on (T, ∞), where T is any number.

When $\tau(t) \equiv 0$ and $a(t)$ is continuous in (1.8), the two definitions given above are equivalent. This is because of the fact that the uniqueness of the solution makes multiple zeros impossible. However, as Example 1.2.3 suggests, a differential equation with delay can have solutions with multiple zeros. Then the two definitions are different, especially for higher order ordinary differential equations which may have solutions with multiple zeros.

Definition 1.3.5 is more general than Definition 1.3.6. The solution $y(t) = 1-\sin t$ of (1.7) is oscillatory according to Definition 1.3.5 and is nonoscillatory according to Definition 1.3.6.

In Example 1.3.3, the possibility of multiple zeros of nontrivial solutions is a consequence of the retardation, since if $\tau(t) \equiv 0$, the corresponding equation has no solutions with multiple zeros.

For the system of first order equations with deviating arguments

$$\begin{cases} x' = f_1(t, x, x \circ \tau_1, y, y \circ \tau_2), \\ y' = f_2(t, x, x \circ \tau_1, y, y \circ \tau_2), \end{cases}$$

the solution (x, y) is said to be *strongly* (*weakly*) *oscillatory* if each (at least one) of its components is oscillatory. Otherwise, it is said to be strongly (weakly) nonoscillatory if each (at least one) of its nontrivial components is nonoscillatory.

1.4. Some Fixed Point Theorems

Fixed point theorems are important tools in proving the existence of nonoscillatory solutions. In this section we state some fixed point theorems that we need later. Let us begin with the following notation.

Let S be any fixed set and C_S be the relation of strict inclusion on subsets of S:

$$C_S = \Big\{(A, B) : \ A \subseteq B \subseteq S \text{ and } A \neq B\Big\}$$

We write $A \subset_S B$ instead of the notation $(A, B) \in C_S$.

For the set of real numbers, we have the usual ordering relation $<$. For any distinct real numbers x and y, either $x < y$ or $y < x$.

Definition 1.4.1. A *partial ordering* is a relation R satisfying

(i) if xRy and yRz, then xRz (i.e., R is transitive),
(ii) if xRy and yRx, then $x = y$ (i.e., R is antisymmetric).

If $<$ is such a relation, then we can define $x \leq y$ if either $x < y$ or $x = y$. It is easy to see that $x \leq y < z$ implies $x < z$.

Lemma 1.4.2. *Assume that $<$ is a partial ordering. Then for any x, y, and z, at most one of the three alternatives*

$$x < y, \quad x = y, \quad y < x$$

can hold. Also, $x \leq y \leq x$ implies $x = y$.

Definition 1.4.3. Suppose that $<$ is a partial ordering on A, and consider a subset C of A. An upper bound of C is an element $b \in A$ such that $x \leq b$ for all $x \in C$. Here b may or may not belong to C. If b is the least element of the set of all upper bounds for C, then b is called the *least upper bound* (or *supremum*) of C. We write

$b = \sup C$. Similarly we define the *greatest lower bound* or *infimum* a of C and write $a = \inf C$.

Example 1.4.4. Consider a fixed set S. The set consisting of all subsets of S is denoted by $\mathcal{P}(S)$. Let the partial ordering be $\subset_S$ on S. For A and B in $\mathcal{P}(S)$, the set $\{A, B\}$ has a least upper bound (w.r.t. $\subset_S$), namely $A \cup B$.

Theorem 1.4.5. *Let $<$ be a partial ordering relative to a field A, and suppose that every $B \subseteq A$ has a least upper bound and that* $\inf A \in A$. *Suppose that F maps A into A in such a way that for all $x, y \in A$,*

$$x \leq y \quad \textit{implies} \quad Fx \leq Fy.$$

Then F has a fixed point *in A, i.e., $Fx = x$ for some $x \in A$.*

Definition 1.4.6. A subset S of a normed space X is called *bounded* if there is a number M such that $\|x\| \leq M$ for all $x \in S$.

Definition 1.4.7. A set S in a vector space X is called *convex* if, for any $x, y \in S$, $\lambda x + (1 - \lambda)y \in S$ for all $\lambda \in [0, 1]$.

Definition 1.4.8. Let M, N be normed linear spaces, and $X \subset N$. An operator $T : X \to M$ is called *continuous* at a point $x \in X$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|Tx - Ty\| < \varepsilon$ whenever $y \in X$ with $\|x - y\| < \delta$. The operator T is called continuous on X, or simply continuous, if it is continuous at all points of X.

Theorem 1.4.9. *Every continuous mapping of a closed bounded convex set in $\mathbb{R}^n$ into itself has a fixed point.*

Definition 1.4.10. A subset S of a normed space B is called *compact* if every infinite sequence of elements of S has a subsequence which converges to an element of S.

We can prove that compact sets are closed and bounded, but vice versa this is in general not true.

Lemma 1.4.11. *Continuous mappings take compact sets into compact sets. In other words, if M, N are normed linear spaces, $X \subset M$ is compact, and $T : X \to N$ is continuous, then the image of X under T, i.e., the set $T(X) = \{Tx : x \in X\}$, is compact.*

Definition 1.4.12. A subset S of a normed linear space N is called *relatively compact* if every sequence in S has a subsequence converging to an element of N.

It is obvious that every subset of a compact or relatively compact set is relatively compact.

Lemma 1.4.13. *The closure of a relatively compact set is compact, and a closed and relatively compact set is compact.*

Definition 1.4.14. A function $f : \mathbb{R} \to \mathbb{C}$ is called *bounded* on an interval $I \subset \mathbb{R}$ if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in I$. A family $\mathcal{F}$ of functions is called *uniformly bounded* on I if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in I$ and all $f \in \mathcal{F}$.

Lemma 1.4.15. *Continuous mappings on compact sets are uniformly continuous.*

Definition 1.4.16. A family $\mathcal{F}$ of functions is called *equicontinuous* on an interval $I \subset \mathbb{R}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathcal{F}$, $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in I$ with $|x - y| < \delta$.

Theorem 1.4.17 (Arzelà–Ascoli)**.** *A set of functions in $C[a, b]$ with*

$$\|f\| = \sup_{x \in [a,b]} |f(x)|$$

is relatively compact iff it is uniformly bounded and equicontinuous on $[a, b]$.

Theorem 1.4.18 (Schauder's First Fixed Point Theorem)**.** *If S is a convex and compact subset of a normed linear space, then every continuous mapping of S into itself has a fixed point.*

Theorem 1.4.19 (Schauder's Second Fixed Point Theorem)**.** *If S is a convex closed subset of a normed linear space and R a relatively compact subset of S, then every continuous mapping of S into R has a fixed point.*

Theorem 1.4.19 is the more useful form for the theory of ordinary differential equations or delay differential equations.

Remark 1.4.20. We should point out that we need to use Theorem 1.4.17 carefully, because we usually discuss problems on the infinite interval $[t_0, \infty)$ in the qualitative theory of ODEs. That is, we usually want to prove that the family of functions is uniformly bounded and equicontinuous on $[t_0, \infty)$. Levitan's result [**168**] provides a correct formulation. According to his result, the family of functions is equicontinuous on $[t_0, \infty)$ if for any given $\varepsilon > 0$, the interval $[t_0, \infty)$ can be decomposed into a finite number of subintervals in such a way that on each subinterval all functions of the family have oscillations less than ε.

Definition 1.4.21. A real-valued function ρ defined on a linear space X is called a *seminorm* on X if

(i) $\rho(x + y) \leq \rho(x) + \rho(y)$ for all $x, y \in X$,
(ii) $\rho(\alpha x) = |\alpha|\, \rho(x)$ for all $x \in X$ and all scalars α.

From this definition, we can prove that a seminorm ρ satisfies $\rho(0) = 0$,

$$\rho(x_1 - x_2) \geq |\rho(x_1) - \rho(x_2)|\,,$$

and in particular $\rho(x) \geq 0$. However, it may happen that $\rho(x) = 0$ for $x \neq 0$.

Definition 1.4.22. A family $\mathcal{P}$ of semimorms on X is said to be *separating* if to each $x \neq 0$ there corresponds at least one $\rho \in \mathcal{P}$ with $\rho(x) \neq 0$.

For a separating seminorm family $\mathcal{P}$, if $\rho(x) = 0$ for every $\rho \in \mathcal{P}$, then $x = 0$.

Definition 1.4.23. A topology $\mathcal{T}$ on a linear space E is called *locally convex* if every neighborhood of the element 0 includes a convex neighborhood of 0.

A locally convex topology $\mathcal{T}$ on a linear space E is determined by a family of seminorms $\{\rho_\alpha : \alpha \in I\}$, I being the index set.

Let E be a locally convex space, $x \in E$, $\{x_n\} \subset E$. We say that $x_n \to x$ in E if $\rho_\alpha(x_n - x) \to 0$ as $n \to \infty$ for every $\alpha \in I$.

A set $S \subset E$ is bounded if and only if the set of numbers $\{\rho_\alpha(x) : x \in S\}$ is bounded for every $\alpha \in I$.

Definition 1.4.24. A complete metrizable locally convex space is called a *Fréchet space.*

Theorem 1.4.25 (Schauder–Tychonov Theorem)**.** *Let X be a locally convex topological linear space, C a compact convex subset of X, and $f : C \to C$ a continuous mapping such that $f(C)$ is compact. Then f has a fixed point in C.*

For example, $C([t_0, \infty), \mathbb{R})$ is a locally convex space consisting of the set of all continuous functions. The topology of C is the topology of uniform convergence on every compact interval of $[t_0, \infty)$. The seminorm of the space $C([t_0, \infty), \mathbb{R})$ is defined by

$$\rho_\alpha(x) = \max_{t \in [t_0, \alpha]} |x(t)| \quad \text{for} \quad x \in C \quad \text{and} \quad \alpha \in [t_0, \infty).$$

Let X be any set. A *metric* in X is a function $d : X \times X \to \mathbb{R}$ having the following properties for all $x, y, z \in X$:

(i) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

A *metric space* is a set X together with a given metric in X. A *complete* metric space is a metric space X in which every Cauchy sequence converges to a point in X. A *Banach space* is a normed space that is complete with respect to the metric $d(x, y) = \|x - y\|$ defined by the norm.

Let (X, d) be a metric space and let $T : X \to X$. If there exists a number $L \in [0, 1)$ such that

$$d(Tx, Ty) \leq Ld(x, y) \quad \text{for all} \quad x, y \in X,$$

then we say that T is a *contraction mapping* on X.

Theorem 1.4.26 (Banach's Contraction Mapping Principle)**.** *A contraction mapping on a complete metric space has exactly one fixed point.*

Theorem 1.4.27 (Krasnosel′skiĭ's Fixed Point Theorem)**.** *Let X be a Banach space, let Ω be a bounded closed convex subset of X, and let A, B be maps of Ω into X such that $Ax + By \in \Omega$ for every pair $x, y \in \Omega$. If A is a contraction and B is completely continuous, then the equation*

$$Ax + Bx = x$$

has a solution in Ω.

A nonempty and closed subset K of a Banach space X is called a *cone* if it possesses the following properties:

(i) If $\alpha \in \mathbb{R}^+$ and $x \in K$, then $\alpha x \in K$;
(ii) if $x, y \in K$, then $x + y \in K$;
(iii) if $x \in K \setminus \{0\}$, then $-x \notin K$.

Theorem 1.4.28 (Knaster's Fixed Point Theorem)**.** *Let X be a partially ordered Banach space with ordering $\leq$. Let M be a subset of X with the following properties: The infimum of M belongs to M and every nonempty subset of M has a supremum which belongs to M. Let $T : M \to M$ be an increasing mapping, i.e., $x \leq y$ implies $Tx \leq Ty$. Then T has a fixed point in M.*

Let X be a Banach space, let K be a cone in X, and let $\leq$ be the ordering in X induced by K, i.e., $x \leq y$ if and only if $y - x \in K$. Let D be a subset of K and $T : D \to K$ a mapping.

We denote by $\langle x, y\rangle$ the closed ordered interval between x and y, i.e.,

$$\langle x, y\rangle = \Big\{z \in X : \ x \leq z \leq y\Big\}.$$

We assume that the cone K is normal in X, which implies that ordered intervals are norm bounded. The cones of nonnegative functions are normal in the space of continuous functions with supremum norm and in the space L^p.

Theorem 1.4.29. *Let X be a Banach space, K a normal cone in X, D a subset of K such that if $x, y \in D$ with $x \leq y$, then $\langle x, y\rangle \subset D$, and let $T : D \to K$ be a continuous decreasing mapping which is compact on any closed ordered interval contained in D. Suppose that there exists $x_0 \in D$ such that T^2x_0 is defined (where $T^2x_0 = T(Tx_0)$) and furthermore Tx_0, T^2x_0 are (order) comparable to x_0. Then T has a fixed point in D provided that either*

(i) *$Tx_0 \leq x_0$ and $T^2x_0 \leq x_0$, or $Tx_0 \geq x_0$ and $T^2x_0 \geq x_0$, or*
(ii) *the complete sequence of iterates $\{T^n x_0\}_{n=0}^{\infty}$ is bounded and there exists $y_0 \in D$ such that $Ty_0 \in D$ and $y_0 \leq T^n x_0$ for all $n \in \mathbb{N}_0$.*

Theorem 1.4.30. *Let X be a Banach space and $A : X \to X$ a completely continuous mapping such that $I - A$ is one-to-one. Let Ω be a bounded set with $0 \in (I - A)(\Omega)$. Then the completely continuous mapping $S : \Omega \to X$ has a fixed point in the closure $\overline{\Omega}$ if for any $\lambda \in (0, 1)$, the equation*

$$x = \lambda Sx + (1 - \lambda)Ax$$

has no solution x on the boundary $\partial\Omega$ of Ω.

1.5. Notes

The material in Chapter 1 is based on Erbe, Kong, and Zhang [**92**], Ladde, Lakshmikantham, and Zhang [**166**], and Zhong, Fan, and Chen [**304**].

CHAPTER 2

First Order Delay Differential Equations

2.1. Introduction

In this chapter, we will describe some of the recent developments in oscillation theory of first order delay differential equations. This theory is interesting from the theoretical as well as the practical point of view. It is well known that homogeneous ordinary differential equations (ODEs) of first order do not possess oscillatory solutions. But the presence of deviating arguments can cause oscillation of solutions. In this chapter we will see these phenomena and we will show various techniques used in oscillation and nonoscillation theory of differential equations with delays. We will present some criteria for oscillation and for the existence of positive solutions of delay differential equations of first order.

2.2. Equations with a Single Delay: General Case

We consider linear delay differential inequalities and equations of the form

$$x'(t) + p(t)x\left(\tau(t)\right) \le 0, \tag{2.1}$$

$$x'(t) + p(t)x\left(\tau(t)\right) \ge 0, \tag{2.2}$$

and

$$x'(t) + p(t)x\left(\tau(t)\right) = 0, \tag{2.3}$$

where $p, \tau \in C([t_0, \infty), \mathbb{R}^+)$, $\tau(t) \le t$, and $\lim_{t\to\infty} \tau(t) = \infty$. Set

$$m = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \quad \text{and} \quad M = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds. \tag{2.4}$$

The following lemmas will be used to prove the main results of this section. All inequalities in this section and in the later parts hold eventually if it is not mentioned specifically.

Lemma 2.2.1. *Suppose that* $m > 0$ *and set*

$$\delta(t) = \max\left\{\tau(s):\ s \in [t_0, t]\right\}.$$

Then we have

$$\liminf_{t\to\infty} \int_{\delta(t)}^{t} p(s)ds = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = m. \tag{2.5}$$

Proof. Clearly, $\delta(t) \ge \tau(t)$ and so

$$\int_{\delta(t)}^{t} p(s)ds \le \int_{\tau(t)}^{t} p(s)ds.$$

Hence

$$\liminf_{t\to\infty}\int_{\delta(t)}^{t} p(s)ds \le \liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds.$$

If (2.5) does not hold, then there exist $m' > 0$ and a sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty}\int_{\delta(t_n)}^{t_n} p(s)ds < m' < m.$$

By definition, $\delta(t_n) = \max\{\tau(s) : s \in [t_0, t_n]\}$, and hence there exists $t'_n \in [t_0, t_n]$ such that $\delta(t_n) = \tau(t'_n)$. Hence

$$\int_{\delta(t_n)}^{t_n} p(s)ds = \int_{\tau(t'_n)}^{t_n} p(s)ds > \int_{\tau(t'_n)}^{t'_n} p(s)ds.$$

It follows that $\left\{\int_{\tau(t'_n)}^{t'_n} p(s)ds\right\}_{n=1}^{\infty}$ is a bounded sequence having a convergent subsequence, say

$$\int_{\tau(t'_{n_k})}^{t'_{n_k}} p(s)ds \to c \le m' \quad \text{as} \quad k \to \infty,$$

which implies that

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds \le m',$$

contradicting the first definition in (2.4). □

Lemma 2.2.2. *Let x be an eventually positive solution of* (2.1).

(i) *If $m > \frac{1}{e}$, then*

$$\lim_{t\to\infty}\frac{x(\tau(t))}{x(t)} = \infty. \tag{2.6}$$

(ii) *If $m \le \frac{1}{e}$, then*

$$\lim_{t\to\infty}\frac{x(\tau(t))}{x(t)} \ge \lambda,$$

where λ is the smallest positive root of the equation

$$\lambda = e^{m\lambda}. \tag{2.7}$$

Proof. Let t_1 be a sufficiently large number so that $x(\tau(t)) > 0$ for $t \ge t_1$. Hence x is decreasing on $[t_1, \infty)$ and

$$\frac{x'(t)}{x(t)} \le -p(t)\frac{x(\tau(t))}{x(t)} \le -p(t). \tag{2.8}$$

Integrating (2.8) from $\tau(t)$ to t we have that eventually

$$\frac{x(\tau(t))}{x(t)} \ge \exp\left(\int_{\tau(t)}^{t} p(s)ds\right).$$

Then, for any $\varepsilon > 0$, there exists T_ε such that

$$\frac{x(\tau(t))}{x(t)} \ge e^m - \varepsilon \quad \text{for all} \quad t \ge T_\varepsilon. \tag{2.9}$$

Substituting (2.9) into (2.8) we have $\frac{x'(t)}{x(t)} \le -(e^m - \varepsilon)p(t)$ for $t \ge T_\varepsilon$, and hence

$$\liminf_{t\to\infty} \frac{x(\tau(t))}{x(t)} \ge \exp\left(me^m\right).$$

Set $\lambda_0 = 1$ and recursively $\lambda_n = \exp(m\lambda_{n-1})$ for all $n \in \mathbb{N}$. For a sequence $\{\varepsilon_n\}$ with $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$, there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and

$$\frac{x(\tau(t))}{x(t)} \ge \lambda_n - \varepsilon_n \quad \text{for all} \quad t \ge t_n.$$

If $m > \frac{1}{e}$, then $\lim_{n\to\infty} \lambda_n = \infty$, and (2.6) holds. If $m = \frac{1}{e}$, then $\lim_{n\to\infty} \lambda_n = e$, and if $m < \frac{1}{e}$, then λ_n tends to the smallest root of (2.7). $\square$

Remark 2.2.3. From Theorem 2.2.6 we will see that (2.1) has no eventually positive solutions if $m > \frac{1}{e}$.

Lemma 2.2.4. *Assume τ is nondecreasing, $0 \le m \le \frac{1}{e}$, and x is an eventually positive solution of* (2.1). *Set*

$$r = \liminf_{t\to\infty} \frac{x(t)}{x(\tau(t))}.$$

Then

$$A(m) := \frac{1 - m - \sqrt{1 - 2m - m^2}}{2} \le r \le 1. \tag{2.10}$$

Proof. Assume that $x(t) > 0$ for $t > T_1 \ge t_0$ and that there exists a sequence $\{T_n\}$ such that $T_1 < T_2 < T_3 < \ldots$ and $\tau(t) > T_n$ for $t > T_{n+1}$, $n \in \mathbb{N}$. Hence $x(\tau(t)) > 0$ for $t > T_2$. In view of (2.1), $x'(t) \le 0$ on (T_2, ∞). Clearly, (2.10) holds for $m = 0$. If $0 < m \le \frac{1}{e}$, for any $\varepsilon \in (0, m)$, there exists N_ε such that

$$\int_{\tau(t)}^{t} p(s)ds > m - \varepsilon \quad \text{for} \quad t > N_\varepsilon. \tag{2.11}$$

Let $\varepsilon > 0$ and $t > N_\varepsilon$. Then

$$f(\lambda) := \int_t^\lambda p(s)ds \quad \text{is continuous and} \quad \lim_{\lambda\to\infty} f(\lambda) > m - \varepsilon > 0 = f(t).$$

Hence there exists $\lambda_t > t$ such that $f(\lambda_t) = m - \varepsilon$, i.e.,

$$\int_t^{\lambda_t} p(s)ds = m - \varepsilon$$

holds. From (2.11) we have

$$\int_{\tau(\lambda_t)}^{\lambda_t} p(s)ds > m - \varepsilon = \int_t^{\lambda_t} p(s)ds$$

and therefore $\tau(\lambda_t) < t$.

Integrating (2.1) from $t > \max\{T_4, N_\varepsilon\}$ to λ_t we have

$$x(t) - x(\lambda_t) \ge \int_t^{\lambda_t} p(y)x(\tau(y))dy. \tag{2.12}$$

We see that $\tau(t) \le \tau(y) \le \tau(\lambda_t) < t$ for $t \le y \le \lambda_t$.

Integrating (2.1) from $\tau(y)$ to t we have that for $t \le y \le \lambda_t$

$$
\begin{aligned}
x(\tau(y)) - x(t) &\ge \int_{\tau(y)}^{t} p(u)x(\tau(u))du \\
&\ge x(\tau(t)) \int_{\tau(y)}^{t} p(u)du \\
&= x(\tau(t)) \left(\int_{\tau(y)}^{y} p(u)du - \int_{t}^{y} p(u)du \right) \\
&> x(\tau(t)) \left[(m-\varepsilon) - \int_{t}^{y} p(u)du \right].
\end{aligned} \tag{2.13}
$$

From (2.12) and (2.13) we have

$$
\begin{aligned}
x(t) &\ge x(\lambda_t) + \int_t^{\lambda_t} p(y)x(\tau(y))dy \\
&> x(\lambda_t) + \int_t^{\lambda_t} p(y) \left\{ x(t) + x(\tau(t)) \left[(m-\varepsilon) - \int_t^y p(u)\ du \right] \right\} dy \\
&= x(\lambda_t) + x(t)(m-\varepsilon) + x(\tau(t)) \left[(m-\varepsilon)^2 - \int_t^{\lambda_t} p(y) \int_t^y p(u)dudy \right].
\end{aligned} \tag{2.14}
$$

Noting the known formula

$$
\int_t^{\lambda_t} \int_t^y p(y)p(u)dudy = \int_t^{\lambda_t} \int_u^{\lambda_t} p(y)p(u)dydu = \int_t^{\lambda_t} \int_y^{\lambda_t} p(y)p(u)dudy,
$$

we have

$$
\begin{aligned}
\int_t^{\lambda_t} \int_t^y p(y)p(u)dudy &= \frac{1}{2} \left[\int_t^{\lambda_t} \int_t^y p(y)p(u)dudy + \int_t^{\lambda_t} \int_y^{\lambda_t} p(y)p(u)dudy \right] \\
&= \frac{1}{2} \int_t^{\lambda_t} \int_t^{\lambda_t} p(y)p(u)dudy \\
&= \frac{1}{2} \left[\int_t^{\lambda_t} p(s)ds \right]^2 \\
&= \frac{1}{2}(m-\varepsilon)^2.
\end{aligned}
$$

Substituting this into (2.14) we have

$$
x(t) > x(\lambda_t) + (m-\varepsilon)x(t) + \frac{1}{2}(m-\varepsilon)^2 x(\tau(t)). \tag{2.15}
$$

Hence (note that $1 - m + \varepsilon > 0$)

$$
\frac{x(t)}{x(\tau(t))} > \frac{(m-\varepsilon)^2}{2(1-m+\varepsilon)} =: d_1, \tag{2.16}
$$

and then

$$
x(\lambda_t) > \frac{(m-\varepsilon)^2}{2(1-m+\varepsilon)} x(\tau(\lambda_t)) = d_1 x(\tau(\lambda_t)) \ge d_1 x(t).
$$

Substituting this into (2.15) we obtain

$$
x(t) > (m + d_1 - \varepsilon)x(t) + \frac{1}{2}(m-\varepsilon)^2 x(\tau(t)),
$$

and hence

$$\frac{x(t)}{x(\tau(t))} > \frac{(m-\varepsilon)^2}{2(1-m-d_1+\varepsilon)} =: d_2.$$

In general we have

$$\frac{x(t)}{x(\tau(t))} > \frac{(m-\varepsilon)^2}{2(1-m-d_n+\varepsilon)} =: d_{n+1} \quad \text{for} \quad n \in \mathbb{N}.$$

It is not difficult to see that if ε is small enough, then $1 \geq d_{n+1} > d_n$ for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} d_n = d$ exists and satisfies

$$-2d^2 + 2d(1-m+\varepsilon) = (m-\varepsilon)^2,$$

i.e.,

$$d = \frac{1-m+\varepsilon \pm \sqrt{1-2(m-\varepsilon)-(m-\varepsilon)^2}}{2}.$$

Therefore, for all large t,

$$\frac{x(t)}{x(\tau(t))} \geq \frac{1-m+\varepsilon - \sqrt{1-2(m-\varepsilon)-(m-\varepsilon)^2}}{2}.$$

Letting $\varepsilon \to 0$, we obtain that

$$\frac{x(t)}{x(\tau(t))} \geq \frac{1-m-\sqrt{1-2m-m^2}}{2} = A(m).$$

This shows that (2.10) holds. □

Lemma 2.2.5. *Assume that $M \in (0,1]$ and that τ is nondecreasing. Let x be an eventually positive solution of* (2.1)*. Set*

$$\liminf_{t\to\infty} \frac{x(\tau(t))}{x(t)} = l.$$

Then

$$l \leq B(M) := \left(\frac{1+\sqrt{1-M}}{M}\right)^2. \tag{2.17}$$

Proof. For a given $\varepsilon \in (0, M)$, there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and

$$\int_{\tau(t_n)}^{t_n} p(s)ds > M - \varepsilon, \quad t_n > T, \quad n \in \mathbb{N}.$$

Set $\theta_\varepsilon = 1 - \sqrt{1-(M-\varepsilon)}$. It is easy to see that $0 < \theta_\varepsilon < M - \varepsilon$ for small ε. Hence there exists $\{\lambda_n\}$ such that $\tau(t_n) < \lambda_n < t_n$ and

$$\int_{\lambda_n}^{t_n} p(s)ds = \theta_\varepsilon \quad \text{for} \quad n \in \mathbb{N}.$$

Integrating (2.1) from λ_n to t_n, we obtain

$$x(\lambda_n) - x(t_n) \geq \int_{\lambda_n}^{t_n} p(s)x(\tau(s))ds \geq x(\tau(t_n)) \int_{\lambda_n}^{t_n} p(s)ds = \theta_\varepsilon x(\tau(t_n)).$$

Similarly, we have

$$\begin{aligned} x(\tau(t_n)) - x(\lambda_n) &\geq \int_{\tau(t_n)}^{\lambda_n} p(s)x(\tau(s))ds \\ &\geq x(\tau(\lambda_n)) \int_{\tau(t_n)}^{\lambda_n} p(s)ds \\ &= x(\tau(\lambda_n)) \left[\int_{\tau(t_n)}^{t_n} p(s)ds - \int_{\lambda_n}^{t_n} p(s)ds \right] \\ &> x(\tau(\lambda_n))(M - \varepsilon - \theta_\varepsilon). \end{aligned}$$

From the above inequalities we get

$$x(\lambda_n) > \theta_\varepsilon x(\tau(t_n)) > \theta_\varepsilon \Big(x(\lambda_n) + x(\tau(\lambda_n))(M - \varepsilon - \theta_\varepsilon) \Big)$$

and then

$$\frac{x(\tau(\lambda_n))}{x(\lambda_n)} < \frac{1 - \theta_\varepsilon}{\theta_\varepsilon (M - \varepsilon - \theta_\varepsilon)} \quad \text{for} \quad n \in \mathbb{N},$$

which implies that

$$l \leq \frac{1 - \theta_\varepsilon}{\theta_\varepsilon (M - \varepsilon - \theta_\varepsilon)} \quad \text{for all} \quad \varepsilon \in (0, M).$$

Now, $\theta_\varepsilon \to 1 - \sqrt{1 - M}$ as $\varepsilon \to 0$, and then we obtain

$$l \leq \frac{\sqrt{1 - M}}{(1 - \sqrt{1 - M})(M - 1 + \sqrt{1 - M})} = \left(\frac{1 + \sqrt{1 - M}}{M} \right)^2,$$

which is (2.17). □

We are now in a position to state oscillation criteria for (2.3).

Theorem 2.2.6. *Assume $m > \frac{1}{e}$. Then*

(i) (2.1) *has no eventually positive solutions;*
(ii) (2.2) *has no eventually negative solutions;*
(iii) *every solution of* (2.3) *is oscillatory.*

Proof. It is sufficient to prove (i) as (ii) and (iii) follow from (i). Suppose the contrary is true, and let x be an eventually positive solution of (2.1). In view of Lemma 2.2.1, we may assume that τ is nondecreasing. By Lemma 2.2.2,

$$\liminf_{t \to \infty} \frac{x(\tau(t))}{x(t)} = \infty.$$

On the other hand, from (2.16), $\frac{x(\tau(t))}{x(t)}$ is bounded above. This contradiction proves (i). □

Remark 2.2.7. If τ is nondecreasing and $M \in (0, 1]$, then the condition $m > \frac{1}{e}$ in Theorem 2.2.6 can be replaced by

$$m > \frac{\ln b}{b} \quad \text{with} \quad b = \min\{e, B(M)\}, \tag{2.18}$$

where $B(M)$ is defined in (2.17).

Proof. To see this, let x be a positive solution of (2.1). Set $w(t) = \frac{x(\tau(t))}{x(t)}$. By Lemma 2.2.5, $\liminf_{t\to\infty} w(t) = l \le B(M)$. From (2.1), we obtain

$$-\frac{x'(t)}{x(t)} \ge p(t)w(t) \quad \text{for all} \quad t \ge T,$$

where T is sufficiently large. Integrating from $\tau(t)$ to t we obtain

$$\ln w(t) \ge \int_{\tau(t)}^{t} p(s)w(s)ds = w(\xi_t)\int_{\tau(t)}^{t} p(s)ds$$

for some $\xi_t \in [\tau(t), t]$ and hence

$$\ln l = \liminf_{t\to\infty} \ln w(t) \ge lm$$

and

$$m \le \frac{\ln l}{l} \le \frac{\ln b}{b}$$

which contradicts (2.18). Therefore (2.1) has no eventually positive solutions. $\square$

Theorem 2.2.8. *Assume $0 \le m \le \frac{1}{e}$ and τ is nondecreasing. Furthermore, suppose*

$$M > 1 - A(m), \tag{2.19}$$

where $A(m)$ is defined in (2.10)*, or*

$$M > \frac{\ln\lambda + 1}{\lambda}, \tag{2.20}$$

where λ is the smallest positive root of the equation (2.7)*. Then the conclusions of Theorem 2.2.6 are true.*

Proof. As in Theorem 2.2.6, it is sufficient to show that under our assumptions (2.1) has no eventually positive solutions. We assume that x is an eventually positive solution of (2.1). Integrating (2.1) from $\tau(t)$ to t we obtain

$$x(\tau(t)) - x(t) \ge \int_{\tau(t)}^{t} p(s)x(\tau(s))ds \ge x(\tau(t))\int_{\tau(t)}^{t} p(s)ds.$$

Then if (2.19) holds, by Lemma 2.2.4, we have

$$\begin{aligned} M &= \limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds \le \limsup_{t\to\infty}\left[1 - \frac{x(t)}{x(\tau(t))}\right] \\ &= 1 - \liminf_{t\to\infty}\frac{x(t)}{x(\tau(t))} = 1 - r \le 1 - A(m), \end{aligned} \tag{2.21}$$

which contradicts (2.10).

If (2.20) holds, choose $m' < m$ sufficiently close to m such that

$$M = \limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds > \frac{\ln\lambda' + 1}{\lambda'}, \tag{2.22}$$

where λ' is the smallest root of the equation $\lambda = e^{m'\lambda}$.

Clearly, $\lambda' < \lambda$ and hence $\frac{\ln\lambda'+1}{\lambda'} > \frac{\ln\lambda+1}{\lambda}$. By Lemma 2.2.2, we have

$$\frac{x(\tau(t))}{x(t)} > \lambda' \quad \text{for all large} \quad t. \tag{2.23}$$

From (2.22), there exists t_1 so that (2.23) holds for all $t > \tau(\tau(t_1))$, and

$$\int_{\tau(t_1)}^{t_1} p(s)ds > \frac{\ln \lambda' + 1}{\lambda'}. \tag{2.24}$$

Without loss of generality denote $t_0 = \tau(t_1)$. We shall show that $x(t) > 0$ on $[t_0, t_1]$ will lead to a contradiction. In fact, let $t_2 \in [t_0, t_1]$ be a point at which $x(t_0)/x(t_2) = \lambda'$. If such a point does not exist, take $t_2 = t_1$. Integrating (2.1) over $[t_2, t_1]$ and noting that $x(\tau(t)) \geq x(t_0)$, we have

$$\int_{t_2}^{t_1} p(s)ds \leq \frac{1}{\lambda'}. \tag{2.25}$$

On the other hand, dividing (2.1) by $x(t)$ and integrating it over $[t_0, t_2]$ we find

$$\int_{t_0}^{t_2} p(s)ds \leq -\frac{1}{\lambda'} \int_{t_0}^{t_2} \frac{x'(s)}{x(s)} ds = \frac{\ln \lambda'}{\lambda'}. \tag{2.26}$$

Combining (2.25) and (2.26) we get

$$\int_{t_0}^{t_1} p(s)ds \leq \frac{\ln \lambda' + 1}{\lambda'},$$

which contradicts (2.24). □

Example 2.2.9. Consider the equation

$$x'(t) + \frac{0.85}{a\pi + \sqrt{2}} (2a + \cos t)\, x\left(t - \frac{\pi}{2}\right) = 0, \tag{2.27}$$

where $a = 1.137$. Then (2.27) is in the form (2.3) with

$$p(t) = \frac{0.85}{a\pi + \sqrt{2}} (2a + \cos t) \qquad \text{and} \qquad \tau(t) = t - \frac{\pi}{2}.$$

We have

$$\int_{\tau(t)}^{t} p(s)ds = \frac{0.85}{a\pi + \sqrt{2}} \left(a\pi + \sqrt{2} \cos\left(t - \frac{\pi}{4}\right)\right).$$

Hence

$$m = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = 0.85 \frac{a\pi - \sqrt{2}}{a\pi + \sqrt{2}} = 0.367837 < \frac{1}{e}$$

and

$$M = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = 0.85.$$

It is easy to see that (2.19) holds. Therefore every solution of (2.27) is oscillatory.

In the following we will consider the existence of positive solutions of a linear delay differential equation of the form

$$x'(t) + x\,(t - \tau(t)) = 0, \tag{2.28}$$

where $\tau \in C([t_0, \infty), \mathbb{R}^+)$ and $\lim_{t\to\infty}(t - \tau(t)) = \infty$. Set $T_0 = \inf_{t \geq t_0}\{t - \tau(t)\}$.

Definition 2.2.10. A solution x is called *positive with respect to the initial point* t_0, if x is a solution of (2.28) on (t_0, ∞) and $x(t) > 0$ for all $t \in [T_0, \infty)$.

Theorem 2.2.11. *Equation* (2.28) *has a positive solution with respect to* t_0 *if and only if there exists a real continuous function* λ_0 *on* $[T_0,\infty)$ *such that* $\lambda_0(t)>0$ *for all* $t\ge t_0$ *and*

$$\tau(t)\le t-\Lambda_0^{-1}\left(\Lambda_0(t)-\ln\lambda_0(t)\right)\quad \textit{for all}\quad t\ge t_0, \tag{2.29}$$

where $\Lambda_0(t)=\int_{T_0}^t\lambda_0(s)ds$, *and* Λ_0^{-1} *denotes the inverse function of* Λ_0.

Proof. We first prove necessity. Let x_0 be a positive solution of (2.28) with respect to t_0. Then $x_0(t)>0$ for all $t\in[T_0,\infty)$. Set

$$\lambda_0(t)=\frac{x_0(t-\tau(t))}{x_0(t)}\quad\text{for all}\quad t\ge T_0.$$

Clearly, $\lambda_0(t)>0$ for all $t\ge t_0$ and hence $\Lambda_0(t)=\int_{T_0}^t\lambda_0(s)ds$ defines a function Λ_0 on $[T_0,\infty)$, which is strictly increasing on $[t_0,\infty)$. We have for $t\ge t_0$

$$\begin{aligned}\ln\lambda_0(t)&=\ln\left(\frac{x_0(t-\tau(t))}{x_0(t)}\right)=-\int_{t-\tau(t)}^t\frac{x_0'(s)}{x_0(s)}ds\\&=\int_{t-\tau(t)}^t\lambda_0(s)ds=\Lambda_0(t)-\Lambda_0(t-\tau(t))\end{aligned}$$

and therefore

$$t-\tau(t)=\Lambda_0^{-1}\left(\Lambda_0(t)-\ln\lambda_0(t)\right).$$

Then

$$\tau(t)=t-\Lambda_0^{-1}\left(\Lambda_0(t)-\ln\lambda_0(t)\right)$$

so that (2.29) holds.

Now we prove sufficiency. If there exists a function λ_0 such that (2.29) holds, then

$$\Lambda_0\left(t-\tau(t)\right)\ge\Lambda_0(t)-\ln\lambda_0(t)$$

and

$$\lambda_0(t)\ge\exp\left(\int_{t-\tau(t)}^t\lambda_0(s)ds\right).$$

Define

$$\lambda_1(t)=\begin{cases}\exp\left(\int_{t-\tau(t)}^t\lambda_0(s)ds\right) & \text{if}\quad t\ge t_0\\ \lambda_1(t_0)+\lambda_0(t)-\lambda_0(t_0) & \text{if}\quad t\in[T_0,t_0).\end{cases}$$

Clearly, $\lambda_1(t)\le\lambda_0(t)$ for $t\ge T_0$ and $0\le\lambda_1(t)\le\lambda_0(t)$ for $t\ge t_0$. In general, we define

$$\lambda_n(t)=\begin{cases}\exp\left(\int_{t-\tau(t)}^t\lambda_{n-1}(s)ds\right)a & \text{if}\quad t\ge t_0\\ \lambda_n(t_0)+\lambda_0(t)-\lambda_0(t_0) & \text{if}\quad t\in[T_0,t_0).\end{cases}$$

Thus

$$\lambda_0(t)-\lambda_0(t_0)\le\lambda_n(t)\le\lambda_{n-1}(t)\le\ldots\le\lambda_0(t)\quad\text{for all}\quad t\ge T_0$$

and $\lambda_n(t) \geq 0$ for all $t \geq t_0$. Then $\lim_{n\to\infty} \lambda_n(t) = \lambda(t)$ exists for $t \geq T_0$ and

$$\lim_{n\to\infty} \int_{t-\tau(t)}^{t} \lambda_n(s)ds = \int_{t-\tau(t)}^{t} \lambda(s)ds \quad \text{for all} \quad t \geq t_0.$$

Hence

$$\lambda(t) = \exp\left(\int_{t-\tau(t)}^{t} \lambda(s)ds\right) \quad \text{for all} \quad t \geq t_0.$$

Set

$$x(t) = \exp\left(-\int_{T_0}^{t} \lambda(s)ds\right) \quad \text{for} \quad t \geq T_0.$$

Then x is a positive solution of (2.28) with respect to t_0. □

Remark 2.2.12. If we take $\lambda_0(t) \equiv \lambda > 0$ in Theorem 2.2.11, then condition (2.29) becomes

$$\tau(t) \leq \frac{\ln \lambda}{\lambda} \quad \text{for all} \quad t \geq t_0. \tag{2.30}$$

In particular, if $\lambda = e$, then (2.30) becomes

$$\tau(t) \leq \frac{1}{e} \quad \text{for all} \quad t \geq t_0, \tag{2.31}$$

i.e., (2.31) is a sufficient condition for the existence of positive solutions of (2.28).

Let $t_0 = \frac{1}{2}$ and $\lambda_0(t) = 2t$. Then by Theorem 2.2.11, if

$$\tau(t) = t - \sqrt{t^2 - \ln 2t},$$

then (2.28) has a positive solution with respect to $t_0 = \frac{1}{2}$. In fact, $x(t) = e^{-t^2}$ is such a solution. We note that

$$\tau\left(\frac{e}{2}\right) = \frac{e}{2} - \sqrt{\left(\frac{e}{2}\right)^2 - 1} > \frac{1}{e}.$$

This example shows that (2.31) is not necessary for the existence of a positive solution of (2.28).

We now consider the linear equation of the form

$$x'(t) + p(t)x\left(t - \tau(t)\right) = 0, \tag{2.32}$$

where $p, \tau \in C([t_0, \infty), \mathbb{R}^+)$, $\tau(t) \leq t$, and $\lim_{t\to\infty}(t - \tau(t)) = \infty$. As before, set $T_0 = \inf_{t\geq t_0}\{t - \tau(t)\}$. Similarly as in Theorem 2.2.11 we can prove the following result.

Theorem 2.2.13. *Equation* (2.32) *has a positive solution with respect to t_0 if and only if there exists a continuous function λ_0 on $[T_0, \infty)$ such that $\lambda_0(t) > 0$ for $t \geq t_0$ and*

$$\lambda_0(t) \geq p(t) \exp\left(\int_{t-\tau(t)}^{t} \lambda_0(s)ds\right) \quad \textit{for all} \quad t \geq t_0. \tag{2.33}$$

Remark 2.2.14. If $p(t) > 0$, then (2.33) can be replaced by

$$\tau(t) \leq t - \Lambda_0^{-1}\left(\Lambda_0(t) - \ln \frac{\lambda_0(t)}{p(t)}\right) \quad \text{for all} \quad t \geq t_0.$$

Corollary 2.2.15. *If*

$$\int_{t-\tau(t)}^{t} p(s)ds \leq \frac{1}{e} \quad \textit{for all} \quad t \geq t_0,$$

then (2.32) *has a positive solution with respect to* t_0.

Proof. If we take $\lambda_0(t) = ep(t)$, then (2.33) is satisfied. Then the corollary follows from Theorem 2.2.13. □

Theorem 2.2.16. *Assume that* $\tau(t) \equiv \tau > 0$ *and* $\int_{t_0}^{\infty} p(t)dt = \infty$. *Then* (2.32) *has a positive solution with respect to* t_0 *if and only if there exists a continuous function* λ_0 *on* $[t_0 - \tau, \infty)$ *such that*

$$\int_{t-\tau}^{t} p(s)ds \leq t - \Lambda_0^{-1}\Big(\Lambda_0(t) - \ln \lambda_0(t)\Big) \quad \textit{for all} \quad t \geq t_0. \tag{2.34}$$

Proof. Set $u = P(t) = \int_{t_0}^{t} p(s)ds$ for $t \geq t_0$. Then

$$t - \tau = P^{-1}\left(u - \int_{P^{-1}(u)-\tau}^{P^{-1}(u)} p(s)ds\right).$$

Denote

$$z(u) = x\left(P^{-1}(u)\right).$$

Then (2.32) becomes

$$z'(u) + z\left(u - \int_{P^{-1}(u)-\tau}^{P^{-1}(u)} p(s)ds\right) = 0. \tag{2.35}$$

By Theorem 2.2.11, (2.34) is a necessary and sufficient condition for (2.35) to have a positive solution with respect to 0. From the transformation, it is equivalent to (2.32) having a positive solution with respect to t_0. □

Remark 2.2.17. If we choose $\lambda_0(t) \equiv e$ in (2.34), then we obtain

$$\int_{t-\tau}^{t} p(s)ds \leq \frac{1}{e} \quad \text{for all} \quad t \geq t_0. \tag{2.36}$$

As we have mentioned, (2.36) is a sufficient condition and is not a necessary condition for the existence of a positive solution of (2.32).

Combining Theorem 2.2.6 and (2.36), we obtain the following corollary.

Corollary 2.2.18. *Let* $p(t) \equiv p > 0$ *and* $\tau(t) \equiv \tau > 0$. *Then a necessary and sufficient condition for all solutions of* (2.32) *to be oscillatory is that* $p\tau e > 1$.

Remark 2.2.19. The above techniques can be used on the first order advanced type equations

$$x'(t) = x\left(t + \tau(t)\right) \tag{2.37}$$

and

$$x'(t) = p(t)x\left(t + \tau(t)\right), \tag{2.38}$$