volume 231

lecture notes in pure and applied mathematics

commutative ring theory and applications

edited by
Marco Fontana
Salah-Eddine Kabbaj
Sylvia Wiegand

CRC Press
Taylor & Francis Group

commutative ring theory and applications

PURE AND APPLIED MATHEMATICS

A Program of Monographs, Textbooks, and Lecture Notes

LECTURE NOTES IN PURE AND APPLIED MATHEMATICS

1. *N. Jacobson,* Exceptional Lie Algebras
2. *L.-Å. Lindahl and F. Poulsen,* Thin Sets in Harmonic Analysis
3. *I. Satake,* Classification Theory of Semi-Simple Algebraic Groups
4. *F. Hirzebruch et al., Differentiable Manifolds and Quadratic Forms*
5. *I. Chavel,* Riemannian Symmetric Spaces of Rank One
6. *R. B. Burckel,* Characterization of C(X) Among Its Subalgebras
7. *B. R. McDonald et al.,* Ring Theory
8. *Y.-T. Siu,* Techniques of Extension on Analytic Objects
9. *S. R. Caradus et al.,* Calkin Algebras and Algebras of Operators on Banach Spaces
10. *E. O. Roxin et al.,* Differential Games and Control Theory
11. *M. Orzech and C. Small,* The Brauer Group of Commutative Rings
12. *S. Thomier,* Topology and Its Applications
13. *J. M. Lopez and K. A. Ross,* Sidon Sets
14. *W. W. Comfort and S. Negrepontis,* Continuous Pseudometrics
15. *K. McKennon and J. M. Robertson,* Locally Convex Spaces
16. *M. Carmeli and S. Malin,* Representations of the Rotation and Lorentz Groups
17. *G. B. Seligman,* Rational Methods in Lie Algebras
18. *D. G. de Figueiredo,* Functional Analysis
19. *L. Cesari et al.,* Nonlinear Functional Analysis and Differential Equations
20. *J. J. Schäffer,* Geometry of Spheres in Normed Spaces
21. *K. Yano and M. Kon,* Anti-Invariant Submanifolds
22. *W. V. Vasconcelos,* The Rings of Dimension Two
23. *R. E. Chandler,* Hausdorff Compactifications
24. *S. P. Franklin and B. V. S. Thomas,* Topology
25. *S. K. Jain,* Ring Theory
26. *B. R. McDonald and R. A. Morris,* Ring Theory II
27. *R. B. Mura and A. Rhemtulla,* Orderable Groups
28. *J. R. Graef,* Stability of Dynamical Systems
29. *H.-C. Wang,* Homogeneous Branch Algebras
30. *E. O. Roxin et al.,* Differential Games and Control Theory II
31. *R. D. Porter,* Introduction to Fibre Bundles
32. *M. Altman,* Contractors and Contractor Directions Theory and Applications
33. *J. S. Golan,* Decomposition and Dimension in Module Categories
34. *G. Fairweather,* Finite Element Galerkin Methods for Differential Equations
35. *J. D. Sally,* Numbers of Generators of Ideals in Local Rings
36. *S. S. Miller,* Complex Analysis
37. *R. Gordon,* Representation Theory of Algebras
38. *M. Goto and F. D. Grosshans,* Semisimple Lie Algebras
39. *A. I. Arruda et al.,* Mathematical Logic
40. *F. Van Oystaeyen,* Ring Theory
41. *F. Van Oystaeyen and A. Verschoren,* Reflectors and Localization
42. *M. Satyanarayana,* Positively Ordered Semigroups
43. *D. L Russell,* Mathematics of Finite-Dimensional Control Systems
44. *P.-T. Liu and E. Roxin,* Differential Games and Control Theory III
45. *A. Geramita and J. Seberry,* Orthogonal Designs
46. *J. Cigler, V. Losert, and P. Michor,* Banach Modules and Functors on Categories of Banach Spaces
47. *P.-T. Liu and J. G. Sutinen,* Control Theory in Mathematical Economics
48. *C. Byrnes,* Partial Differential Equations and Geometry
49. *G. Klambauer,* Problems and Propositions in Analysis
50. *J. Knopfmacher,* Analytic Arithmetic of Algebraic Function Fields
51. *F. Van Oystaeyen,* Ring Theory
52. *B. Kadem,* Binary Time Series
53. *J. Barros-Neto and R. A. Artino,* Hypoelliptic Boundary-Value Problems
54. *R. L. Sternberg et al.,* Nonlinear Partial Differential Equations in Engineering and Applied Science
55. *B. R. McDonald,* Ring Theory and Algebra III
56. *J. S. Golan,* Structure Sheaves Over a Noncommutative Ring
57. *T. V. Narayana et al.,* Combinatorics, Representation Theory and Statistical Methods in Groups
58. *T. A. Burton,* Modeling and Differential Equations in Biology
59. *K. H. Kim and F. W. Roush,* Introduction to Mathematical Consensus Theory

60. *J. Banas and K. Goebel,* Measures of Noncompactness in Banach Spaces
61. *O. A. Nielson,* Direct Integral Theory
62. *J. E. Smith et al.,* Ordered Groups
63. *J. Cronin,* Mathematics of Cell Electrophysiology
64. *J. W. Brewer,* Power Series Over Commutative Rings
65. *P. K. Kamthan and M. Gupta,* Sequence Spaces and Series
66. *T. G. McLaughlin,* Regressive Sets and the Theory of Isols
67. *T. L. Herdman et al.,* Integral and Functional Differential Equations
68. *R. Draper,* Commutative Algebra
69. *W. G. McKay and J. Patera,* Tables of Dimensions, Indices, and Branching Rules for Representations of Simple Lie Algebras
70. *R. L. Devaney and Z. H. Nitecki,* Classical Mechanics and Dynamical Systems
71. *J. Van Geel,* Places and Valuations in Noncommutative Ring Theory
72. *C. Faith,* Injective Modules and Injective Quotient Rings
73. *A. Fiacco,* Mathematical Programming with Data Perturbations I
74. *P. Schultz et al.,* Algebraic Structures and Applications
75. *L Bican et al.,* Rings, Modules, and Preradicals
76. *D. C. Kay and M. Breen,* Convexity and Related Combinatorial Geometry
77. *P. Fletcher and W. F. Lindgren,* Quasi-Uniform Spaces
78. *C.-C. Yang,* Factorization Theory of Meromorphic Functions
79. *O. Taussky,* Ternary Quadratic Forms and Norms
80. *S. P. Singh and J. H. Burry,* Nonlinear Analysis and Applications
81. *K. B. Hannsgen et al.,* Volterra and Functional Differential Equations
82. *N. L. Johnson et al.,* Finite Geometries
83. *G. I. Zapata,* Functional Analysis, Holomorphy, and Approximation Theory
84. *S. Greco and G. Valla,* Commutative Algebra
85. *A. V. Fiacco,* Mathematical Programming with Data Perturbations II
86. *J.-B. Hiriart-Urruty et al.,* Optimization
87. *A. Figa Talamanca and M. A. Picardello,* Harmonic Analysis on Free Groups
88. *M. Harada,* Factor Categories with Applications to Direct Decomposition of Modules
89. *V. I. Istrătescu,* Strict Convexity and Complex Strict Convexity
90. *V. Lakshmikantham,* Trends in Theory and Practice of Nonlinear Differential Equations
91. *H. L. Manocha and J. B. Srivastava,* Algebra and Its Applications
92. *D. V. Chudnovsky and G. V. Chudnovsky,* Classical and Quantum Models and Arithmetic Problems
93. *J. W. Longley,* Least Squares Computations Using Orthogonalization Methods
94. *L. P. de Alcantara,* Mathematical Logic and Formal Systems
95. *C. E. Aull,* Rings of Continuous Functions
96. *R. Chuaqui,* Analysis, Geometry, and Probability
97. *L. Fuchs and L. Salce,* Modules Over Valuation Domains
98. *P. Fischer and W. R. Smith,* Chaos, Fractals, and Dynamics
99. *W. B. Powell and C. Tsinakis,* Ordered Algebraic Structures
100. *G. M. Rassias and T. M. Rassias,* Differential Geometry, Calculus of Variations, and Their Applications
101. *R.-E. Hoffmann and K. H. Hofmann,* Continuous Lattices and Their Applications
102. *J. H. Lightbourne III and S. M. Rankin III,* Physical Mathematics and Nonlinear Partial Differential Equations
103. *C. A. Baker and L. M. Batten,* Finite Geometrics
104. *J. W. Brewer et al.,* Linear Systems Over Commutative Rings
105. *C. McCrory and T. Shifrin,* Geometry and Topology
106. *D. W. Kueke et al.,* Mathematical Logic and Theoretical Computer Science
107. *B.-L. Lin and S. Simons,* Nonlinear and Convex Analysis
108. *S. J. Lee,* Operator Methods for Optimal Control Problems
109. *V. Lakshmikantham,* Nonlinear Analysis and Applications
110. *S. F. McCormick,* Multigrid Methods
111. *M. C. Tangora,* Computers in Algebra
112. *D. V. Chudnovsky and G. V. Chudnovsky,* Search Theory
113. *D. V. Chudnovsky and R. D. Jenks,* Computer Algebra
114. *M. C. Tangora,* Computers in Geometry and Topology
115. *P. Nelson et al.,* Transport Theory, Invariant Imbedding, and Integral Equations
116. *P. Clément et al.,* Semigroup Theory and Applications
117. *J. Vinuesa,* Orthogonal Polynomials and Their Applications
118. *C. M. Dafermos et al.,* Differential Equations
119. *E. O. Roxin,* Modern Optimal Control
120. *J. C. Díaz,* Mathematics for Large Scale Computing

121. *P. S. Milojevič,* Nonlinear Functional Analysis
122. *C. Sadosky,* Analysis and Partial Differential Equations
123. *R. M. Shortt,* General Topology and Applications
124. *R. Wong,* Asymptotic and Computational Analysis
125. *D. V. Chudnovsky and R. D. Jenks,* Computers in Mathematics
126. *W. D. Wallis et al.,* Combinatorial Designs and Applications
127. *S. Elaydi,* Differential Equations
128. *G. Chen et al.,* Distributed Parameter Control Systems
129. *W. N. Everitt,* Inequalities
130. *H. G. Kaper and M. Garbey,* Asymptotic Analysis and the Numerical Solution of Partial Differential Equations
131. *O. Arino et al.,* Mathematical Population Dynamics
132. *S. Coen,* Geometry and Complex Variables
133. *J. A. Goldstein et al.,* Differential Equations with Applications in Biology, Physics, and Engineering
134. *S. J. Andima et al.,* General Topology and Applications
135. *P Clément et al.,* Semigroup Theory and Evolution Equations
136. *K. Jarosz*, Function Spaces
137. *J. M. Bayod et al.*, *p*-adic Functional Analysis
138. *G. A. Anastassiou*, Approximation Theory
139. *R. S. Rees*, Graphs, Matrices, and Designs
140. *G. Abrams et al.*, Methods in Module Theory
141. *G. L. Mullen and P. J.-S. Shiue*, Finite Fields, Coding Theory, and Advances in Communications and Computing
142. *M. C. Joshi and A. V. Balakrishnan*, Mathematical Theory of Control
143. *G. Komatsu and Y. Sakane*, Complex Geometry
144. *I. J. Bakelman*, Geometric Analysis and Nonlinear Partial Differential Equations
145. *T. Mabuchi and S. Mukai*, Einstein Metrics and Yang–Mills Connections
146. *L. Fuchs and R. Göbel*, Abelian Groups
147. *A. D. Pollington and W. Moran*, Number Theory with an Emphasis on the Markoff Spectrum
148. *G. Dore et al.*, Differential Equations in Banach Spaces
149. *T. West*, Continuum Theory and Dynamical Systems
150. *K. D. Bierstedt et al.*, Functional Analysis
151. *K. G. Fischer et al.*, Computational Algebra
152. *K. D. Elworthy et al.*, Differential Equations, Dynamical Systems, and Control Science
153. *P.-J. Cahen, et al.*, Commutative Ring Theory
154. *S. C. Cooper and W. J. Thron*, Continued Fractions and Orthogonal Functions
155. *P. Clément and G. Lumer*, Evolution Equations, Control Theory, and Biomathematics
156. *M. Gyllenberg and L. Persson*, Analysis, Algebra, and Computers in Mathematical Research
157. *W. O. Bray et al.,* Fourier Analysis
158. *J. Bergen and S. Montgomery*, Advances in Hopf Algebras
159. *A. R. Magid*, Rings, Extensions, and Cohomology
160. *N. H. Pavel*, Optimal Control of Differential Equations
161. *M. Ikawa*, Spectral and Scattering Theory
162. *X. Liu and D. Siegel*, Comparison Methods and Stability Theory
163. *J.-P. Zolésio*, Boundary Control and Variation
164. *M. Křížek et al.*, Finite Element Methods
165. *G. Da Prato and L. Tubaro*, Control of Partial Differential Equations
166. *E. Ballico*, Projective Geometry with Applications
167. *M. Costabel et al.*, Boundary Value Problems and Integral Equations in Nonsmooth Domains
168. *G. Ferreyra, G. R. Goldstein, and F. Neubrander*, Evolution Equations
169. *S. Huggett*, Twistor Theory
170. *H. Cook et al.*, Continua
171. *D. F. Anderson and D. E. Dobbs*, Zero-Dimensional Commutative Rings
172. *K. Jarosz,* Function Spaces
173. *V. Ancona et al.*, Complex Analysis and Geometry
174. *E. Casas*, Control of Partial Differential Equations and Applications
175. *N. Kalton et al.*, Interaction Between Functional Analysis, Harmonic Analysis, and Probability
176. *Z. Deng et al.*, Differential Equations and Control Theory
177. *P. Marcellini et al.* Partial Differential Equations and Applications
178. *A. Kartsatos*, Theory and Applications of Nonlinear Operators of Accretive and Monotone Type
179. *M. Maruyama*, Moduli of Vector Bundles
180. *A. Ursini and P. Aglianò*, Logic and Algebra
181. *X. H. Cao et al.,* Rings, Groups, and Algebras
182. *D. Arnold and R. M. Rangaswamy,* Abelian Groups and Modules
183. *S. R. Chakravarthy and A. S. Alfa*, Matrix-Analytic Methods in Stochastic Models

184. *J. E. Andersen et al.*, Geometry and Physics
185. *P.-J. Cahen et al.*, Commutative Ring Theory
186. *J. A. Goldstein et al.*, Stochastic Processes and Functional Analysis
187. *A. Sorbi*, Complexity, Logic, and Recursion Theory
188. *G. Da Prato and J.-P. Zolésio*, Partial Differential Equation Methods in Control and Shape Analysis
189. *D. D. Anderson*, Factorization in Integral Domains
190. *N. L. Johnson*, Mostly Finite Geometries
191. *D. Hinton and P. W. Schaefer*, Spectral Theory and Computational Methods of Sturm–Liouville Problems
192. *W. H. Schikhof et al.*, *p*-adic Functional Analysis
193. *S. Sertöz*, Algebraic Geometry
194. *G. Caristi and E. Mitidieri*, Reaction Diffusion Systems
195. *A. V. Fiacco*, Mathematical Programming with Data Perturbations
196. *M. Křížek et al.*, Finite Element Methods: Superconvergence, Post-Processing, and A Posteriori Estimates
197. *S. Caenepeel and A. Verschoren*, Rings, Hopf Algebras, and Brauer Groups
198. *V. Drensky et al.*, Methods in Ring Theory
199. *W. B. Jones and A. Sri Ranga*, Orthogonal Functions, Moment Theory, and Continued Fractions
200. *P. E. Newstead*, Algebraic Geometry
201. *D. Dikranjan and L. Salce*, Abelian Groups, Module Theory, and Topology
202. *Z. Chen et al.*, Advances in Computational Mathematics
203. *X. Caicedo and C. H. Montenegro*, Models, Algebras, and Proofs
204. *C. Y. Yıldırım and S. A. Stepanov*, Number Theory and Its Applications
205. *D. E. Dobbs et al.,* Advances in Commutative Ring Theory
206. *F. Van Oystaeyen,* Commutative Algebra and Algebraic Geometry
207. *J. Kakol et al., p*-adic Functional Analysis
208. *M. Boulagouaz and J.-P. Tignol*, Algebra and Number Theory
209. *S. Caenepeel and F. Van Oystaeyen,* Hopf Algebras and Quantum Groups
210. *F. Van Oystaeyen and M. Saorin,* Interactions Between Ring Theory and Representations of Algebras
211. *R. Costa et al.,* Nonassociative Algebra and Its Applications
212. *T.-X. He,* Wavelet Analysis and Multiresolution Methods
213. *H. Hudzik and L. Skrzypczak*, Function Spaces: The Fifth Conference
214. *J. Kajiwara et al.*, Finite or Infinite Dimensional Complex Analysis
215. *G. Lumer and L. Weis*, Evolution Equations and Their Applications in Physical and Life Sciences
216. *J. Cagnol et al.*, Shape Optimization and Optimal Design
217. *J. Herzog and G. Restuccia*, Geometric and Combinatorial Aspects of Commutative Algebra
218. *G. Chen et al.*, Control of Nonlinear Distributed Parameter Systems
219. *F. Ali Mehmeti et al.*, Partial Differential Equations on Multistructures
220. *D. D. Anderson and I. J. Papick*, Ideal Theoretic Methods in Commutative Algebra
221. *Á. Granja et al.*, Ring Theory and Algebraic Geometry
222. *A. K. Katsaras et al., p*-adic Functional Analysis
223. *R. Salvi*, The Navier-Stokes Equations
224. *F. U. Coelho and H. A. Merklen*, Representations of Algebras
225. *S. Aizicovici and N. H. Pavel*, Differential Equations and Control Theory
226. *G. Lyubeznik*, Local Cohomology and Its Applications
227. *G. Da Prato and L. Tubaro*, Stochastic Partial Differential Equations and Applications
228. *W. A. Carnielli et al.*, Paraconsistency
229. *A. Benkirane and A. Touzani*, Partial Differential Equations
230. *A. Illanes et al.*, Continuum Theory
231. *M. Fontana et al.*, Commutative Ring Theory and Applications

Additional Volumes in Preparation

commutative ring theory and applications

proceedings of the fourth international conference

edited by

Marco Fontana
Università degli Studi Roma Tre
Rome, Italy

Salah-Eddine Kabbaj
King Fahd University of Petroleum and Minerals
Dharan, Saudi Arabia

Sylvia Wiegand
University of Nebraska—Lincoln
Lincoln, Nebraska, U.S.A.

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

First issued in hardback 2017

ISBN 13: 978-1-138-40191-4 (hbk)
ISBN 13: 978-0-8247-0855-9 (pbk)

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

Library of Congress Cataloging-in-Publication Data
A catalog record for this book is available from the Library of Congress.

Preface

This volume draws on the contributors' talks at the Fourth International Conference on Commutative Algebra held in Fez, Morocco. The goal of this conference was to present recent progress and new trends in the growing area of commutative algebra, with primary emphasis on commutative ring theory and its applications. The conference also facilitated a fruitful interaction among the participants, whose various mathematical interests shared the same (commutative) algebraic roots.

The book consists of 34 chapters which, while written as separate articles, provide nonetheless a comprehensive report on questions and problems of contemporary interest. Some articles are surveys of their subject, while others present a narrower, in-depth view. All the manuscripts were subject to a strict refereeing process.

This volume encompasses wide-ranging topics in commutative ring theory (along with connections to algebraic number theory, algebraic geometry, homological algebra, and model-theoretic algebra). The topics covered include: algebroid curves, arithmetic rings, chain conditions, class groups, constructions of examples, divisibility and factorization, linear Diophantine equations, the going-down and going-up properties, graded modules and analytic spread, Gröbner bases and computational methods, homological aspects of commutative rings, ideal and module systems, integer-valued polynomials, integral dependence, Krull domains and generalizations, local cohomology, prime spectra and dimension theory, polynomial rings, power series rings, pullbacks, tight closure, ultraproducts, and zero-divisors.

Graduate students and established commutative algebraists will find the book a valuable and reliable source, as will researchers in many other branches of mathematics.

The conference was organized by the University of Fez with the scientific collaboration of the Università degli Studi "Roma Tre," Italy, and the University of Nebraska, U.S.A. Financial support was provided by the Commutative Algebra and Homological Aspects Laboratory, the Faculty of Sciences "Dhar Al-Mehraz," the International Mathematical Union (CDE), the "Espace Sciences & Vie" Association, and the Università degli Studi "Roma Tre."

We wish to express our gratitude to the local organizing committee, especially Professors A. Benkirane, Chairman of the Department of Mathematics, R. Ameziane Hassani, and A. Touzani, as well as to Professor M. H. Kadri and Mr. M. A. Chad, Dean and Secretary-General, respectively, of the Faculty of Sciences "Dhar Al-Mehraz" at Fez. Special thanks are due to Mr. A. Bennani and Mrs. T. Ibn Abdelmoula for their constant help with conference arrangements. The efforts of the contributors and the referees are greatly appreciated; without their work this volume would never have been produced. Last, we thank the editorial staff at Marcel Dekker, Inc., in particular, Maria Allegra and Ana Pacheco, for their patience, hard work, and assistance with this volume.

Marco Fontana
Salah-Eddine Kabbaj
Sylvia Wiegand

Contents

Contributors

D. F. Anderson, Department of Mathematics, The University of Tennessee, Knoxville, Tennessee 37996-1300 USA Email: anderson@math.utk.edu

A. Badawi, Department of Mathematics, Birzeit University, P.O. Box 14, Birzeit, West Bank, Palestine. Email: abring@birzeit.edu

A. Badra, Département de Mathématiques, Université Blaise Pascal, 63177 Aubière, France. Email: badra@ucfma.univ-bpclermont.fr

V. Barucci, Dipartimento di Matematica, Università di Roma "La Sapienza", 00185 Roma, Italy. Email: barucci@mat.uniroma1.it

A. Benhissi, Département de Mathématiques, Faculté des Sciences, B.P. 5000, Monastir, Tunisia. Email: ali.benhissi@fsm.rnu.tn

A. Bouanane, Department of Mathematics, University of Tetouan, P.O. Box 2121, Tetouan, Morocco. Email: abouanane@hotmail.com

J. Brewer, Department of Mathematics, Florida Atlantic University, Boca Raton, FL 33431-6498, USA. Email: brewer@fau.edu

G. W. Chang, Department of Mathematics, Inha University, Incheon, 402-751 Korea

J.-L. Chabert, Département de Mathématiques, Université de Picardie, 80039 Amiens, France. Email: jlchaber@worldnet.fr

T. Coquand, Dept. of Computer Science, Chalmers University, Eklandagatan 86, 412 96 Goteborg, Sweden Email: coquand@cs.chalmers.se

M. D'Anna, Dipartimento di Matematica, Università di Catania, 95125 Catania, Italy. Email: mdanna@dmi.unict.it

D. Delfino, Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd, Piscataway, New Jersey 08854-8019, USA. Email: ddelfino1@ yahoo.com

Y. Diagana, Département de Mathématiques, Université d'Abobo-Adjamé, 03 B.P. 1644, Abidjan, Côte d'Ivoire. Email: y_diagana@yahoo.com

H. Dichi, Département de Mathématiques, Université Blaise Pascal, 63177 Aubière, France. E-mail: henri.dichi@math.univ-bpclermont.fr

D. Dobbs, Department of Mathematics, University of Tennessee, Knoxville, TN 37996-1300, USA. Email: dobbs@math.utk.edu

R. Eddahabi, Department of Mathematics, University of Tetouan, P.O. Box 2121, Tetouan, Morocco.

S. El Baghdadi, Department of Mathematics, Faculté des Sciences et Techniques, P.O. Box 523, Beni Mellal, Morocco. Email: said_ell@yahoo.com

M. Fontana, Dipartimento di Matematica, Università degli Studi "Roma Tre", Largo S.L. Murialdo, 1, 00146 Roma, Italy. Email: fontana@mat.uniroma3.it

R. Fröberg, Department of Mathematics, Stockholm University 106 91 Stockholm, Sweden.
Email: ralff@matematik.su.se

S. Gabelli, Dipartimento di Matematica, Università degli Studi "Roma Tre", Largo S.L. Murialdo, 1, 00146 Roma, Italy. Email: gabelli@mat.uniroma3.it

S. Glaz, Department of Mathematics, University of Connecticut, Storrs, CT 06269-0001, USA. Email: glaz@uconnvm.uconn.edu

F. Halter-Koch, Institute of Mathematics, Graz University, A-8010 Graz, Austria. Email: franz.halterkoch@kfunigraz.ac.at

W. Hassler, Institut fü r Mathematik Karl-Franzens-Universitä t Graz Heinrichstr. 36, A-8010 Graz, Austria. Email: wolfgang.hassler@kfunigraz.ac.at

W. Heinzer, Department of Mathematics, Purdue University, West Lafayette, IN 47907-1968, USA. Email: heinzer@math.purdue.edu

A. Hetzel, Department of Mathematics, University of Tennessee, Knoxville, TN 37996-1300, USA. Email: ahetzel@math.utk.edu

M. Hochster, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1109, USA. Email: hochster@math.lsa.umich.edu

C. Huneke, Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA. Email: huneke@math.ukans.edu

S. Kabbaj, Department of Mathematics, KFUPM, P.O. Box 5046, Dhahran 31261, Saudi Arabia and University of Fez, Fez, Morocco. Email: kabbaj@kfupm.edu.sa

M. Khalis, Department of Mathematics, Faculté des Sciences et Techniques, P.O. Box 577, Settat, Morocco. Email: mohammed_khalis@hotmail.com

L. Klingler, Department of Mathematics, Florida Atlantic University, Boca Raton, FL 33431-6498, USA. Email: klingler@fau.edu

F. Kourki, Department of Mathematics, University of Tetouan, P.O. Box 2121, Tetouan, Morocco. Email: kourki@yahoo.com

Y. Lequain, IMPA, 22.460-320, Rio de Janeiro, Brazil. Email: ylequain@impa.br

H. Lombardi, Equipe de Mathématiques, UMR CNRS 6623, UFR des Sciences et Techniques, Université de Franche-Comté, 25030 Besancon cedex, France.
Email: lombardi@math.univfcomte.fr

T. G. Lucas, Department of Mathematics, University of North Carolina, Charlotte, NC 28223, USA. Email: tglucas@email.uncc.edu

N. Mahdou, Department of Mathematics, FST Fez-Saïss, University of Fez, P.O. Box 2202, Fez, Morocco. Email: mahdou@hotmail.com

A. Mimouni, Department of Mathematics, University of Fez, P.O. Box 1796, Fez, Morocco. Email: a_mimouni@hotmail.com

B. Olberding, Department of Mathematics, University of Louisiana, Monroe LA 71209, USA. E-mail: maolberding@ulm.edu

J. Park, Department of Mathematics, Inha University, Incheon 402-751, Korea Email: jnpark@math.inha.ac.kr

G. Picavet, Département de Mathématiques, Université Blaise Pascal, 63177 Aubière, France E-mail: gabriel.picavet@math.univ-bpclermont.fr

M. Picavet L'Hermitte, Département de Mathématiques, Université Blaise Pascal, 63177 Aubière, France. E-mail: martine.picavet@math.univ-bpclermont.fr

C. Quitté, Laboratoire de Mathématiques, SP2MI, Boulevard 3, Teleport 2, BP 179, 86960 Futuroscope Cedex, France. Email: quitte@mathlabo.univ-poitiers.fr

C. Rotthaus, Department of Mathematics, Michigan State University, East Lansing, MI 48824-0001, USA. Email: rotthaus@math.msu.edu

A. S. Saydam, Department of Mathematics, University of Louisiana, Monroe, LA 71209, USA. Email: masaydam@ulm.edu

I. Swanson, Department of Mathematical Sciences, New Mexico State University - Las Cruces, New Mexico 88003-8001, USA. Email: iswanson@nmsu.Edu

F. Tartarone, Dipartimento di Matematica, Università degli Studi "Roma Tre", Largo S. L. Murialdo, 1, 00146 Roma, Italy. Email: tfrance@mat.uniroma3.it

A. Taylor, Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd, Piscataway, New Jersey 08854-8019, USA. Email: ataylor@math.rutgers.edu

W. V. Vasconcelos, Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd, Piscataway, New Jersey 08854-8019, USA. Email: vasconce@math.rutgers.edu

R. H. Villarreal, Departamento de Matemáticas, Centro de Investigación y de Estudios Avanzados del IPN, Apartado Postal 14–740, 07000, México City, D.F. MEXICO. Email: vila@math.cinvestav.mx

N. Weininger, Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd, Piscataway, New Jersey 08854-8019, USA. Email: nweining@math.rutgers.edu

R. Wiegand, Department of Mathematics, University of Nebraska, Lincoln, NE 68588-0323, USA. Email: rwiegand@math.unl.edu

S. Wiegand, Department of Mathematics, University of Nebraska, Lincoln, NE 68588-0323, USA. Email: swiegand@math.unl.edu

M. Zafrullah, Department of Mathematics, University of Arkansas, Fayetteville AR 72701, USA. Email: mzafrullah@usa.net

$D[X^2, X^3]$ Over an Integral Domain D

DAVID F. ANDERSON Department of Mathematics, The University of Tennessee, Knoxville, Tennessee 37996-1300.

GYU WHAN CHANG Department of Mathematics, Inha University, Incheon, Korea 402-751

JEANAM PARK Department of Mathematics, Inha University, Incheon, Korea 402-751

1. INTRODUCTION

Let D be an integral domain with identity and quotient field K. In this paper, we study the ring $D[X^2, X^3] = D + X^2D[X] \subset D[X]$, and we compare its behavior to its polynomial overring $D[X]$. Of course, $D[X^2, X^3]$ is never integrally closed (or seminormal, root closed, etc.); so in this paper, we are mainly interested in ring-theoretic properties that do not involve "closedness" conditions. Quite often $D[X^2, X^3]$ satisfies a given ring-theoretic property if and only if $D[X]$ satisfies that property. However, in Section 3, the characteristic of D plays an important role. The ring $K[X^2, X^3]$ has proved useful in constructing examples concerning the Picard group (see Theorem 3.4) and nonunique factorization (see [10]). This paper gives several other cases where the ring $D[X^2, X^3]$ can be used to construct interesting, elementary examples (for instance, see Section 3). Many of the results in this paper generalize to monoid domains; we leave this to a future paper.

We first recall some of the properties we will investigate in this paper. An integral domain D is said to be a *weakly factorial domain* (WFD) [4] if each nonzero nonunit of D is a product of primary elements. Following [9], D is called a *generalized weakly factorial domain* (GWFD) if each nonzero prime ideal of D contains a primary element. Clearly, WFDs are GWFDs; however, if D is a Dedekind domain with nonzero torsion divisor class group, then R is a GWFD, but not a WFD (cf. [7, p. 912], [9, Proposition 3.1]). Following [5], D is called a *weakly Krull domain* if $D = \cap_{P\in X^{(1)}(D)} D_P$ and the intersection has finite character, where $X^{(1)}(D)$ is the set of height-one prime ideals of D. A Krull domain is weakly Krull, and a Noetherian domain is weakly Krull if and only if every grade-one prime ideal has height one. In [7, Theorem], it was shown that D is a WFD if and only if D is a weakly Krull domain and $Cl_t(D) = 0$. A Krull domain D is called *almost factorial* if $Cl(D)$ is torsion. As in [5], we say that an integral domain D is an *almost weakly factorial domain (AWFD)* if for each nonzero nonunit $x \in D$, there is an integer $n = n(x) \geq 1$ such that x^n is a product of primary elements. Thus an AWFD is a GWFD. It was shown in [5, Theorem 3.4] that D is an AWFD if and only if D is a weakly Krull domain and $Cl_t(D)$ is torsion. We say that an integral domain D is an *almost GCD-domain (AGCD-domain)* if for all nonzero $x, y \in D$, there exists an integer $n = n(x, y) \geq 1$ such that $(x^n, y^n)_v$ is principal. In [6, Theorem 3.4], it was proved that $Cl_t(D)$ is torsion when D is an AGCD-domain.

Throughout this paper, D denotes an integral domain with quotient field K, $Spec(D)$ its set of prime ideals, and $X^{(1)}(D)$ its set of height-one prime ideals. For $f \in K[X]$, let A_f be the fractional ideal of D generated by the coefficients of f. Recall that for a nonzero fractional ideal A of D, we have $A^{-1} = \{x \in K | xA \subseteq D\}$, $A_v = (A^{-1})^{-1}$, and $A_t = \cup\{(a_1, ..., a_n)_v | 0 \neq (a_1, ..., a_n) \subseteq A\}$. A nonzero fractional ideal A of D is called a *divisorial ideal* (resp., *t-ideal*) if $A_v = A$ (resp., $A_t = A$). We say that D has *t-dimension one*, written t-dim$D = 1$, if each prime t-ideal of D has height one (note that a height-one prime ideal is necessarily a t-ideal). A weakly Krull domain D has t-dim$D = 1$ [5, Lemma 2.1]. An integral ideal of D is said to be a maximal t-ideal if it is maximal with respect to being a t-ideal, and a maximal t-ideal is necessarily a prime ideal.

A nonzero fractional ideal A of D is said to be *t-invertible* if there exists a fractional ideal B of D with $(AB)_t = D$, and in this case we can take $B = A^{-1}$. It is well known that if A is a t-invertible t-ideal, then $A = J_v$ for some finitely generated subideal J of A. The set of t-invertible t-ideals of D forms an abelian group under the t-product $A * B = (AB)_t$. The *t-class group* of D is $Cl_t(D)$ - the group of t-invertible fractional t-ideals of D modulo its subgroup of principal fractional ideals. For D a Krull domain, $Cl_t(D) = Cl(D)$, the divisor class group; while for D a Prüfer domain or one-dimensional integral domain, $Cl_t(D) = C(D) = Pic(D)$, the ideal class group (or Picard group). For a recent survey article on the t-class group, see [8].

2. THE RING $D[X^2, X^3]$

In this section, we study the ring $D[X^2, X^3] = D + X^2D[X]$ and prove some analogs of the polynomial ring $D[X]$. Our first goal is to show that $D[X^2, X^3]$ is a UMT-domain if and only if D is a UMT-domain. The next lemma also holds for monoid domains (cf. [11, Lemma 2.3]).

LEMMA 2.1. *Let I be a nonzero fractional ideal of D. Then*

(1) $(ID[X^2, X^3])^{-1} = I^{-1}D[X^2, X^3]$.

(2) $(ID[X^2, X^3])_v = I_vD[X^2, X^3]$.

(3) $(ID[X^2, X^3])_t = I_tD[X^2, X^3]$.

Proof. (1) It is clear that $I^{-1}D[X^2, X^3] \subseteq (ID[X^2, X^3])^{-1}$. Note that since $I(ID[X^2, X^3])^{-1} \subseteq D[X^2, X^3] \subseteq K[X^2, X^3]$, we have $(ID[X^2, X^3])^{-1} \subseteq K[X^2, X^3]$. If $f \in (ID[X^2, X^3])^{-1}$, then $A_fI \subseteq D$, and hence $A_f \subseteq I^{-1}$. So $f \in A_fD[X^2, X^3] \subseteq I^{-1}D[X^2, X^3]$. Therefore, $(ID[X^2, X^3])^{-1} = I^{-1}D[X^2, X^3]$.

(2) $(ID[X^2, X^3])_v = ((ID[X^2, X^3])^{-1})^{-1} = (I^{-1}D[X^2, X^3])^{-1} = I_vD[X^2, X^3]$ by (1).

(3) It is clear that if $f_1, f_2, \ldots, f_k \in ID[X^2, X^3]$, then

$$(f_1, \ldots, f_k)_v \subseteq ((A_{f_1}, \ldots, A_{f_k})D[X^2, X^3])_v$$

$$= (A_{f_1}, ..., A_{f_k})_v D[X^2, X^3] \subseteq I_t D[X^2, X^3].$$

So $(ID[X^2, X^3])_t \subseteq I_t D[X^2, X^3]$. For the converse, let J be a nonzero finitely generated subideal of I. Then $J_v D[X^2, X^3] = (JD[X^2, X^3])_v \subseteq (ID[X^2, X^3])_t$ by (2). Thus $I_t D[X^2, X^3] \subseteq (ID[X^2, X^3])_t$, and hence $(ID[X^2, X^3])_t = I_t D[X^2, X^3]$. □

LEMMA 2.2. *(cf. [18, Proposition 1.1]) Let Q be a maximal t-ideal of $D[X^2, X^3]$ such that $Q \cap D \neq 0$. Then $Q = (Q \cap D)[X^2, X^3]$. In particular, $Q \cap D$ is a maximal t-ideal of D.*

Proof. It suffices to show that $c(Q)[X^2, X^3] \subseteq Q$, where $c(Q)$ is the ideal of D generated by the coefficients of all the polynomials in Q. If $c(Q) \not\subseteq Q$, then $Q \subsetneq c(Q)[X^2, X^3]$. Since Q is a maximal t-ideal, we have $c(Q)_t[X^2, X^3] = (c(Q)[X^2, X^3])_t = D[X^2, X^3]$. So $c(Q)_t = D$; whence there is a polynomial $f \in Q$ such that $(A_f)_v = D$. Let $0 \neq a \in Q \cap D$.

We claim that $(a, f)^{-1} = D[X^2, X^3]$. First note that $(a, f)^{-1} \subseteq K[X^2, X^3]$ because for $g \in (a, f)^{-1}$, $ag \in D[X^2, X^3] \subseteq K[X^2, X^3]$. Next, if $g \in (a, f)^{-1}$, then there is an integer $m \geq 1$ such that $A_f^{m+1} A_g = A_f^m A_{fg}$ [16, Theorem 28.1]. Thus $(A_f^{m+1} A_g)_v = (A_f^m A_{fg})_v$ and $A_g \subseteq (A_g)_t = ((A_f^{m+1})_v A_g)_v = (A_f^{m+1} A_g)_v = (A_f^m A_{fg})_v = ((A_f^m)_v A_{fg})_v = (A_{fg})_v \subseteq D$. Hence $g \in A_g[X^2, X^3] \subseteq D[X^2, X^3]$. Thus $(a, f)^{-1} = D[X^2, X^3]$, and hence $(a, f)_v = D[X^2, X^3]$, which is a contradiction since Q is a t-ideal. Therefore $c(Q)[X^2, X^3] = Q$, and hence $Q = (Q \cap D)[X^2, X^3]$. □

As in [18], D is called a *UMT-domain* if every upper to zero (a nonzero prime ideal of $D[X]$ which contracts to zero in D) of $D[X]$ is a maximal t-ideal. Recall that $D[X]$ is a UMT-domain if and only if D is a UMT-domain [14, Theorem 3.4]. Thus, as a consequence of our next result, $D[X^2, X^3]$ is a UMT-domain if and only if $D[X]$ is a UMT-domain.

THEOREM 2.3. *$D[X^2, X^3]$ is a UMT-domain if and only if D is a UMT-domain.*

Proof. ($\Rightarrow$) Suppose that $D[X^2, X^3]$ is a UMT-domain. Let P be a maximal t-ideal of D. Then $PD[X^2, X^3]$ is a maximal t-ideal of $D[X^2, X^3]$ by Lemma 2.2. Also, note that $D[X^2, X^3]_{PD[X^2, X^3]} = D[X]_{P[X]}$. Since $D[X^2, X^3]$ is a UMT-domain, $D[X]_{P[X]}$ is a t-linkative UMT-domain [14, Theorem 1.5], and hence D_P

is a t-linkative UMT-domain (see the proof of [14, Theorem 2.4]). Thus D is a UMT-domain [14, Theorem 1.5].

($\Leftarrow$) Suppose that D is a UMT-domain. To show that $D[X^2, X^3]$ is a UMT-domain, it is enough to show that if Q is a maximal t-ideal of $D[X^2, X^3]$, then the integral closure of $D[X^2, X^3]_Q$ is a Prüfer domain [14, Theorem 1.5].

Let Q be a maximal t-ideal of $D[X^2, X^3]$ and let $Q \cap D = P$. If $P \neq 0$, then $Q = P[X^2, X^3]$ by Lemma 2.2. Moreover, since $X^2 \notin P[X^2, X^3]$ we have $D[X^2, X^3]_Q = D[X]_{P[X]}$. Thus the integral closure of $D[X^2, X^3]_Q$ is a Prüfer domain by [14, Theorem 1.5] (note that $D[X]$ is a UMT-domain [14, Theorem 2.4] and $P[X]$ is a prime t-ideal of $D[X]$). If $P = 0$, then $D[X^2, X^3]_Q = K[X^2, X^3]_{QK[X^2,X^3]}$, and hence $D[X^2, X^3]_Q$ is a one-dimensional Noetherian domain. Thus the integral closure of $D[X^2, X^3]_Q$ is a Dedekind domain (cf. [22, Theorem 33.10]), and hence a Prüfer domain. □

LEMMA 2.4. *If Q is a prime ideal of $D[X^2, X^3]$, then there is a unique prime ideal of $D[X]$ lying over Q. Thus the natural map $Spec(D[X]) \to Spec(D[X^2, X^3])$, given by $P \to P \cap D[X^2, X^3]$, is an order-preserving bijection.*

Proof. Let Q be a prime ideal of $D[X^2, X^3]$, $P = Q \cap D$, and $S = \{X^n | n = 0, 2, 3, \dots\}$.

Case 1. $P = 0$. If $QD[X^2, X^3]_S = D[X^2, X^3]_S$, then $Q = XD[X] \cap D[X^2, X^3]$ and $XD[X]$ is the unique prime ideal of $D[X]$ lying over Q. Assume that $QD[X^2, X^3]_S \subsetneq D[X]_S$. Note that $D[X^2, X^3]_S = D[X]_S = D[X, X^{-1}]$. So $QD[X^2, X^3]_S \cap D[X]$ is the unique prime ideal of $D[X]$ lying over Q.

Case 2. $P \neq 0$. If $Q = P[X^2, X^3]$, then $P[X]$ is the unique prime ideal of $D[X]$ lying over Q. Assume that $P[X^2, X^3] \subsetneq Q$. Note that $D[X^2, X^3]/P[X^2, X^3] \cong (D/P)[X^2, X^3]$, $D[X]/P[X] \cong (D/P)[X]$, and $(Q/P[X^2, X^3]) \cap (D/P)[X^2, X^3] = 0$. Thus there is a unique prime ideal of $(D/P)[X]$ lying over $Q/P[X^2, X^3]$ by Case 1. Since every prime ideal of $D[X]$ lying over Q contains $P[X]$, there is a unique prime ideal of $D[X]$ lying over Q. □

We next show that the bijection in Lemma 2.4 preserves t-ideals.

THEOREM 2.5. *Let Q be a prime ideal of $D[X]$ and let $Q' = Q \cap D[X^2, X^3]$. Then Q' is a prime t-ideal of $D[X^2, X^3]$ if and only if Q is a prime t-ideal of $D[X]$.*

Proof. Let $P = Q \cap D = Q' \cap D$ and $S = \{X^n | n = 0, 2, 3, \dots\}$.

Case 1. $P = 0$. Then $\mathrm{ht}Q' = \mathrm{ht}Q = 1$ by Lemma 2.4. Thus Q and Q' are prime t-ideals of $D[X]$ and $D[X^2, X^3]$, respectively.

Case 2. $P \neq 0$. Then $Q = P[X]$ if and only if $Q' = P[X^2, X^3]$ (Lemma 2.4). Thus, by Lemma 2.1, Q is a prime t-ideal of $D[X]$ if and only if P is a prime t-ideal of D, if and only if Q' is a prime t-ideal of $D[X^2, X^3]$.

Case 3. $P \neq 0$. Then $P[X] \subsetneq Q$ if and only if $P[X^2, X^3] \subsetneq Q'$. Note that if either Q or Q' is a t-ideal, then $X \notin Q$. For if $0 \neq a \in P$, then $((a, X)D[X])_v = D[X]$ and $((a, X^2)D[X^2, X^3])_v = D[X^2, X^3]$. Note that $D[X^2, X^3]_S = D[X]_S$, $Q'D[X^2, X^3]_S = QD[X]_S$, $Q = QD[X]_S \cap D[X]$, and $Q' = Q'D[X^2, X^3]_S \cap D[X^2, X^3]$. Thus it suffices to show that if either Q or Q' is a t-ideal, then $Q'D[X^2, X^3]_S = QD[X]_S$ is a t-ideal of $D[X]_S$ by [19, Lemma 3.17].

Let A be a fractional ideal of $D[X]$ such that $A \cap D \neq 0$. We claim that $(AD[X]_S)^{-1} = A^{-1}D[X]_S$. It is clear that $A^{-1}D[X]_S \subseteq (AD[X]_S)^{-1}$. For the converse, let $u \in (AD[X]_S)^{-1}$. Then $uA \subseteq u(AD[X]_S) \subseteq D[X]_S$. Since $A \cap D \neq 0$, $u \in K[X]_S$. Thus $u = \frac{g}{X^m}$ for some $g \in K[X]$ and integer $m \geq 0$. For any $f \in A$, since $uf = (\frac{g}{X^m})f \in D[X]_S$, $fgX^n \in D[X]$ for some integer $n \geq 0$, and hence $fg \in D[X]$. Thus $g \in A^{-1}$ and $u = \frac{g}{X^m} \in A^{-1}D[X]_S$. Hence $(AD[X]_S)^{-1} \subseteq A^{-1}D[X]_S$, and thus $(AD[X]_S)^{-1} = A^{-1}D[X]_S$. A similar argument shows that if A is a fractional ideal of $D[X^2, X^3]$ with $A \cap D \neq 0$, then $(AD[X]_S)^{-1} = A^{-1}D[X]_S$.

Suppose that Q is a t-ideal of $D[X]$ and let B be a finitely generated subideal of Q. Note that $P \neq 0$, and for any $a \in Q$, (B, a) is also a finitely generated subideal of Q and $(BD[X]_S)_v \subseteq ((B, a)D[X]_S)_v$. So we may assume that $B \cap D \neq 0$. By the previous paragraph, we have that $(BD[X]_S)_v = B_vD[X]_S$. Thus $QD[X]_S$ is a t-ideal. Similarly, we have that if Q' is a t-ideal of $D[X^2, X^3]$, then $Q'D[X^2, X^3]_S$ is a t-ideal. Therefore, the proof is completed. $\square$

Recall that D is a *Mori domain* if it satisfies the ascending chain condition on integral divisorial ideals. The class of Mori domains includes Noetherian domains and Krull domains, and is closed under finite intersections. Recall that $D[X]$ is a Mori domain if D is an integrally closed Mori domain [24]. However, an example is given in [25] of a Mori domain D for which $D[X]$ is not a Mori domain. We say that an integral domain D satisfies the *Principal Ideal Theorem (PIT)* if each prime ideal of D which is minimal over a nonzero principal ideal has height one. It follows from [12, Proposition 3.1(b)] that an integral domain D satisfies PIT if and only if each nonzero prime ideal of D is a union of height-one prime ideals. An integral domain D is called an *S-domain* if $htP[X] = 1$ for each prime ideal P of D with $htP = 1$ [20]. Note that $D[X]$ is an S-domain for any integral domain D [2, Theorem 3.2]. Also, note that if $D[X]$ satisfies PIT, then D satisfies PIT and D is an S-domain [12, Proposition 6.1]; but, D satisfies PIT does not imply that $D[X]$ satisfies PIT [12, Remark 6.2]. However, if D is integrally closed, then $D[X]$ satisfies PIT if and only if D satisfies PIT and D is an S-domain [13, Theorem 4].

We next show that $D[X^2, X^3]$ satisfies any of the above three properties if and only if $D[X]$ does.

THEOREM 2.6. *Let D be an integral domain. Then*

(1) *$D[X^2, X^3]$ is an S-domain.*

(2) *$D[X^2, X^3]$ satisfies PIT if and only if $D[X]$ satisfies PIT.*

(3) *$D[X^2, X^3]$ is a Mori domain if and only if $D[X]$ is a Mori domain.*

Proof. (1) Since $D[X]$ is integral over $D[X^2, X^3]$ (or by Lemma 2.4) and $D[X]$ is an S-domain, $D[X^2, X^3]$ is also an S-domain.

(2)($\Rightarrow$) Suppose that $D[X^2, X^3]$ satisfies PIT. Let Q be a prime ideal of $D[X]$ and $P = Q \cap D[X^2, X^3]$. We need to show that $Q = \cup Q_\alpha$, where $\{Q_\alpha\}$ is the set of height-one prime ideals of $D[X]$ contained in Q. Since $D[X^2, X^3]$ satisfies PIT, $P = \cup(Q_\alpha \cap D[X^2, X^3])$ by Lemma 2.4. If $X \in Q$, then $Q = (Q \cap D, X)$. Thus $P = (Q \cap D, X^2, X^3)$. Hence $Q = (Q \cap D, X) \subseteq \cup Q_\alpha$; so $Q = \cup Q_\alpha$. If $X \notin Q$, then $fX^2 \in P$ for any $f \in Q$. Then $fX^2 \in Q_\alpha$ for some Q_α, and hence $f \in Q_\alpha$; so $Q = \cup Q_\alpha$. Thus $D[X]$ satisfies PIT. ($\Leftarrow$) Suppose that $D[X]$ satisfies PIT. Let P be a prime ideal of $D[X^2, X^3]$. By Lemma 2.4, $P = Q \cap D[X^2, X^3]$ for some prime ideal Q of $D[X]$. Since $D[X]$ satisfies PIT, Q is a union of height-one prime ideals of $D[X]$. Since each height-one prime ideals of $D[X]$ contracts to a height-one prime ideal of $D[X^2, X^3]$ by Lemma 2.4, P is thus a union of height-one prime ideals. Hence $D[X^2, X^3]$ satisfies PIT.

(3)($\Rightarrow$) Suppose that $D[X^2, X^3]$ is a Mori domain. Let $S = \{X^n | n = 0, 2, 3, \dots\}$.

Then $D[X] = K[X] \cap D[X^2, X^3]_S$ and $D[X^2, X^3]_S$ is a Mori domain [23, Corollary 3]. Thus $D[X]$ is also a Mori domain. ($\Leftarrow$) This follows since $D[X^2, X^3] = D[X] \cap K[X^2, X^3]$ and $K[X^2, X^3]$ is a one-dimensional Noetherian domain (and hence a Mori domain). $\square$

Our next result is the $D[X^2, X^3]$ analog of [3, Proposition 4.11] that $D[X]$ is a weakly Krull domain if and only if D is a weakly Krull UMT-domain.

PROPOSITION 2.7. *(cf. [3, Proposition 4.11]) $D[X^2, X^3]$ is a weakly Krull domain if and only if D is a weakly Krull UMT-domain.*

Proof. ($\Rightarrow$) Suppose that $D[X^2, X^3]$ is a weakly Krull domain, and hence $D[X^2, X^3]$ has $t-$dimension one. Let P be a prime t-ideal of D. Then $PD[X^2, X^3]$ is a prime t-ideal of $D[X^2, X^3]$, and hence $\mathrm{ht}P = \mathrm{ht}P[DX^2, X^3] = 1$; whence $t\text{-dim}D = 1$. Moreover, if $0 \neq a \in D$, then the number of height-one prime ideals of $D[X^2, X^3]$ that contain a is finite. Hence $\{P \in X^1(D) | a \in P\}$ is finite, and thus D is weakly Krull.

Let $P \in X^1(D)$. Then $\mathrm{ht}P[X^2, X^3] = 1$, and hence $\mathrm{ht}P[X] = 1$, which implies that D is a UMT-domain because $t\text{-dim}D = 1$.

($\Leftarrow$) Suppose that D is a weakly Krull UMT-domain, and let $P \in X^1(D)$. Then $\mathrm{ht}(P[X^2, X^3]) = 1$ by Lemma 2.4. Thus $t\text{-dim}(D[X^2, X^3]) = 1$ by Lemma 2.2 (note that $t\text{-dim}D = 1$ since D is weakly Krull). Hence by [17, Proposition 4] or [19, Proposition 2.8],

$$D[X^2, X^3] = \cap_{Q \in X^1(D[X^2, X^3])} D[X^2, X^3]_Q.$$

Let $0 \neq f \in D[X^2, X^3]$, $A = \{PD[X^2, X^3] | P \in X^1(D) \text{ and } f \in PD[X^2, X^3]\}$, and $B = \{Q \in X^1(D[X^2, X^3]) | Q \cap D = 0 \text{ and } f \in Q\}$. Since D is weakly Krull, A is finite. Moreover, since $K[X^2, X^3]$ is a one-dimensional Noetherian domain, B is also finite. Therefore, $D[X^2, X^3]$ is weakly Krull. $\square$

The final result of this section is the $D[X^2, X^3]$ analog of [24, Lemme 1].

PROPOSITION 2.8. *Let D be integrally closed and $0 \neq f \in K[X^2, X^3]$. Then*

(1) $fK[X^2, X^3] \cap D[X^2, X^3] = fA_f^{-1}[X^2, X^3]$.

(2) $fK[X] \cap D[X^2, X^3] = \begin{cases} fA_f^{-1}[X^2, X^3], & \text{if } f(0) \neq 0 \\ fA_f^{-1}[X], & \text{if } f(0) = 0. \end{cases}$

Proof. (1) Let $fg \in fK[X^2, X^3] \cap D[X^2, X^3]$. Then $A_f A_g \subseteq (A_f A_g)_v = (A_{fg})_v \subseteq D$ because D is integrally closed (cf. [16, Proposition 34.8]). Thus $g \in A_f^{-1}[X^2, X^3]$ and $fK[X^2, X^3] \cap D[X^2, X^3] \subseteq fA_f^{-1}[X^2, X^3]$. The converse is clear.

(2) Case 1. $f(0) = 0$. Let $fg \in fK[X] \cap D[X^2, X^3]$, where $g \in K[X]$. Since D is integrally closed, $A_f A_g \subseteq (A_f A_g)_v = (A_{fg})_v \subseteq D$ (cf. [16, Proposition 34.8]). Thus $g \in (A_f)^{-1}[X]$, and hence $fK[X] \cap D[X^2, X^3] \subseteq fA_f^{-1}[X]$. Moreover, since $f(0) = 0$, we have $fh \in D[X^2, X^3]$ for any $h \in (A_f)^{-1}[X]$. Therefore, $fK[X] \cap D[X^2, X^3] = fA_f^{-1}[X]$ for any $h \in (A_f)^{-1}[X]$.

Case 2. $f(0) \neq 0$. Since $f(0) \neq 0$, $fg \notin K[X^2, X^3]$ for any $g \in K[X] - K[X^2, X^3]$, which implies that the proof is identical to the proof of Case 1. $\square$

3. GENERALIZED WEAKLY FACTORIAL DOMAINS

One of the purposes of this section is to find equivalent conditions for $D[X^2, X^3]$, over an almost factorial domain D, to be a GWFD. The other is to study the t-class group $Cl_t(D[X^2, X^3])$. Recall that a GWFD is weakly Krull and has $t-$dimension one [9, Corollary 2.3], and that an almost factorial domain is a Krull domain with torsion divisor class group.

THEOREM 3.1. *The following statements are equivalent for an almost factorial domain D.*

(1) *$D[X^2, X^3]$ is an AGCD-domain.*
(2) *$D[X^2, X^3]$ is an AWFD.*
(3) *$D[X^2, X^3]$ is a GWFD.*
(4) *char$D = p \neq 0$.*

Proof. (1) $\Rightarrow$ (2): Recall that a Krull domain is a weakly Krull UMT-domain. Thus $D[X^2, X^3]$ is a weakly Krull domain by Proposition 2.7. Also, note that an AGCD-domain has torsion t-class group. Hence $D[X^2, X^3]$ is an AWFD.

(2) $\Rightarrow$ (3): Let Q be a nonzero prime ideal of $D[X^2, X^3]$ and let $0 \neq f \in Q$. By the definition of an AWFD, there is an integer $n \geq 1$ such that f^n is a product of primary elements. Thus Q contains a nonzero primary element of $D[X^2, X^3]$. Therefore, $D[X^2, X^3]$ is a GWFD.

(3) $\Rightarrow$ (4): Since $D[X^2, X^3]$ is a GWFD, $D[X^2, X^3]_{D-\{0\}} = K[X^2, X^3]$ is also a GWFD by [9, Remark 2.5(4)]. Moreover, since char$D$ = char$K$, it suffices to show that char$K \neq 0$.

Suppose that char$K = 0$, and let $Q = (1+X)K[X] \cap K[X^2, X^3]$. Since $K[X^2, X^3]$ is a GWFD, there is a primary element $f \in Q$ such that $Q = \sqrt{fK[X^2, X^3]}$. Let $S = \{X^n | n = 0, 2, 3, \dots\}$. Then $K[X^2, X^3]_S = K[X]_S = K[X, X^{-1}]$. Note that $K[X]_S$ is a PID and $QK[X]_S = (1+X)K[X]_S$. Thus $fK[X]_S = (1+X)^n K[X]_S$ for some integer $n \geq 1$, and hence $f = \frac{u(1+X)^n}{X^m}$ for some integer m and $0 \neq u \in K$.

If $m \geq 0$, then $X^m f = u(1+X)^n$, and hence $m = 0$. Thus $f = u(1+X)^n$, and $u(1+X)^n \in K[X^2, X^3] \Leftrightarrow nuX \in K[X^2, X^3] \Leftrightarrow X \in K[X^2, X^3]$ (note that char$K = 0$), a contradiction. Hence $m < 0$ and $f = u(1+X)^n X^{-m} \in Q \cap (XK[X] \cap K[X^2, X^3])$, which contradicts that f is primary. Thus char$K \neq 0$.

(4) $\Rightarrow$ (1): Let $0 \neq f, g \in D[X^2, X^3]$. Then there is an integer $k \geq 1$ and $h \in K[X]$ such that $(((f,g)D[X])^k)_v = ((f^k, g^k)D[X])_v = hD[X]$ (note that $D[X]$ is a Krull domain with torsion divisor class group)[6, Lemma 3.3]. Thus $f^k = hf_1$ and $g^k = hg_1$ for some $f_1, g_1 \in D[X]$, and $((f_1, g_1)D[X])_v = D[X]$. Since char$D = p \neq 0$, $f_1^p, g_1^p \in D[X^2, X^3]$.

Assume that $(f_1^p, g_1^p)_v \subsetneq D[X^2, X^3]$. Then there is a height-one prime ideal Q of $D[X^2, X^3]$ such that $(f_1^p, g_1^p)_v \subseteq Q$ (note that t-dim$(D[X^2, X^3]) = 1$ since $D[X^2, X^3]$ is weakly Krull). Since $D[X]$ is integral over $D[X^2, X^3]$ (or by Lemma 2.4), there is a height-one prime ideal Q' of $D[X]$ such that $Q' \cap D[X^2, X^3] = Q$. Thus

$$D[X] \supsetneq Q' = Q'_t \supseteq ((f_1^p, g_1^p)D[X])_v = (((f_1, g_1)D[X])^p)_v$$

$$= ((((f_1, g_1)D[X])_v)^p)_v = (D[X]^p)_v = D[X],$$

a contradiction. Hence $(f_1^p, g_1^p)_v = D[X^2, X^3]$. Therefore,

$$(f^{kp}, g^{kp})_v = ((hf_1)^p, (hg_1)^p)_v = h^p (f_1^p, g_1^p)_v = h^p D[X^2, X^3].$$

Thus $D[X^2, X^3]$ is an AGCD-domain. $\square$

COROLLARY 3.2. *The following statements are equivalent for a field K.*

(1) *$K[X^2, X^3]$ is an AGCD-domain.*

(2) *$K[X^2, X^3]$ is an AWFD.*

(3) $K[X^2, X^3]$ *is a GWFD.*

(4) *char*$K = p \neq 0$.

Our next result generalizes Theorem 3.1.

THEOREM 3.3. *(cf. [9, Theorem 3.3]) Let D be an integrally closed domain with char$D = p \neq 0$. Then the following statements are equivalent.*

(1) $D[X^2, X^3]$ *is an AWFD.*

(2) $D[X^2, X^3]$ *is a GWFD.*

(3) $D[X]$ *is an AWFD.*

(4) $D[X]$ *is a GWFD.*

(5) D *is a generalized weakly factorial AGCD-domain.*

(6) D *is an almost weakly factorial AGCD-domain.*

(7) D *is a weakly Krull AGCD-domain.*

Proof. (1) $\Rightarrow$ (2) : This follows from the definitions.

(2) $\Rightarrow$ (4) : By [9, Theorem 2.2], it suffices to show that if Q is a maximal t-ideal of $D[X]$, then $Q = \sqrt{fD[X]}$ for some $f \in D[X]$ because t-dim$D[X] = t$-dim$D[X^2, X^3] = 1$. Let $P = Q \cap D$. If $P \neq 0$, then $Q = P[X]$ and $P[X^2, X^3] = \sqrt{aD[X^2, X^3]}$ for some $a \in P$ (note that $P[X^2, X^3]$ is a height-one prime ideal). Thus $P[X] = \sqrt{aD[X]}$.

Assume that $P = 0$, and let $Q \cap D[X^2, X^3] = \sqrt{fD[X^2, X^3]}$. Note that if $g \in D[X]$, then $g^p \in D[X^2, X^3]$ because char$D = p \neq 0$. Thus $Q = \sqrt{fD[X]}$.

(3) $\Rightarrow$ (1) : Recall that $D[X]$ is a weakly Krull domain $\Leftrightarrow$ D is a weakly Krull UMT-domain $\Leftrightarrow$ $D[X^2, X^3]$ is a weakly Krull domain. Hence it suffices to show that $Cl_t(D[X^2, X^3])$ is torsion.

Let Q be a t-invertible t-ideal of $D[X^2, X^3]$. Then $((QD[X])(Q^{-1}D[X]))_t = (QQ^{-1}D[X])_t \subseteq D[X]$. Since Q is t-invertible, QQ^{-1} is not contained in any height-one prime ideal of $D[X^2, X^3]$ (note that t-dim$D[X] = t$-dim$D[X^2, X^3] = 1$). Thus $(QD[X])(Q^{-1}D[X])$ is not contained in any height-one prime ideal of $D[X]$, and hence $((QD[X])(Q^{-1}D[X]))_t = D[X]$. Since $D[X]$ is an AWFD and thus has torsion t-class group, there is an integer $n \geq 1$ and an $f \in D[X]$ such that $((QD[X])^n)_v = (Q^nD[X])_v = fD[X]$. Since Q is a finite type t-ideal, by the same argument as in the proof of (4) $\Rightarrow$ (1) in Theorem 3.1, we have that $(Q^{pn})_v = f^p D[X^2, X^3]$. Therefore, $D[X^2, X^3]$ is an AWFD.

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7): These implications are in [9, Theorem 3.3]. $\square$

We close this paper with a discussion of $Cl_t(D[X^2, X^3])$. Recall that an integral domain D with quotient field K is *seminormal* if whenever $x^2, x^3 \in D$ for some $x \in K$, then $x \in D$; and that $Pic(D[X]) = Pic(D)$ if and only if D is seminormal. Using the Mayer-Vietoris exact sequence for (U, Pic) (cf. [21, pp. 39-40]), one may show that $Pic(D[X^2, X^3]) = Pic(D) \oplus D$ (as additive abelian groups) when D is seminormal. Also, $Cl_t(D[X]) = Cl_t(D)$ if and only if D is integrally closed [15, Theorem 3.6]. In analogy with the Picard group case, we ask if $Cl_t(D[X^2, X^3]) = Cl_t(D) \oplus K$ (as additive abelian groups) when D is integrally closed. Our final theorem shows that this does hold in the special case when D is a GCD-domain since then $Cl_t(D) = 0$. For example, letting $D = \mathbb{Z}$, we have $Pic(\mathbb{Z}[X^2, X^3]) = \mathbb{Z}$ and $Cl_t(\mathbb{Z}[X^2, X^3]) = \mathbb{Q}$.

THEOREM 3.4. *Let D be a GCD-domain with quotient field K. Then, as additive abelian groups,*

(1) $Pic(D[X^2, X^3]) = D$.

(2) $Cl_t(D[X^2, X^3]) = K$.

Proof. (1) This follows using the Mayer-Vietoris exact sequence for (U, Pic).

(2) Let $S = D - \{0\}$. Then S is an *lcm* splitting set in $D[X^2, X^3]$. Thus

$$Cl_t(D[X^2, X^3]) \cong Cl_t(D[X^2, X^3]_S) = Cl_t(K[X^2, X^3]) = Pic(K[X^2, X^3]) = K$$

by [1, Theorem 4.1]. $\square$

QUESTION 3.5. Compute $Cl_t(D[X^2, X^3])$ for an arbitrary integral domain D with quotient field K. In particular, does $Cl_t(D[X^2, X^3]) = Cl_t(D) \oplus K$ (as additive abelian groups) when D is integrally closed?

ACKNOWLEDGMENTS

The third author's work was supported grant No R02-2000-00016 from the Basic Science Research Program of the Korea Science & Engineering Foundation.

REFERENCES

1. D.D. Anderson, D.F. Anderson, and M. Zafrullah, *Splitting the t-class group*, J. Pure and Appl. Algebra 74 (1991), 17-37.
2. D.D. Anderson, D.F. Anderson, and M. Zafrullah, *Rings between $D[X]$ and $K[X]$*, Houston J. Math. 17 (1991), 109-129.
3. D.D. Anderson, E. Houston, and M. Zafrullah, *t-linked extensions, the t-class group, and Nagata's theorem*, J. Pure and Appl. Algebra 86 (1993), 109-124.
4. D.D. Anderson and L. Mahaney, *On primary factorizations*, J. Pure and Appl. Algebra 54 (1988), 141-154.
5. D.D. Anderson, J. L. Mott, and M. Zafrullah, *Finite character representations for integral domains*, Boll. Un. Mat. Ital. B(7) 6 (1992), 613-630.
6. D.D. Anderson and M. Zafrullah, *Almost Bézout domains*, J. Algebra 142 (1991), 285-309.
7. D.D. Anderson and M. Zafrullah, *Weakly factorial domains and groups of divisibility*, Proc. Amer. Math. Soc. 109 (1990), 907-913.
8. D.F. Anderson, *The class group and local class group of an integral domain*, in Non-Noetherian Ring Theory, Math. Appl., Kluwer Acad. Publ., Dordrecht 520 (2000), 33-55
9. D.F. Anderson, G.W. Chang, and J. Park, *Generalized weakly factorial domains*, Houston J. Math., to appear.
10. D.F. Anderson, S. Chapman, F. Inman, and W.W. Smith, *Factorization in $K[X^2, X^3]$*, Arch. Math., 61(1993), 521-528.
11. S. El Baghdadi, L. Izelgue, and S. Kabbaj, *On the class group of a graded domain*, preprint.
12. V. Barucci, D.F. Anderson, and D. Dobbs, *Coherent Mori domains and the Principal Ideal Theorem*, Comm. Algebra 15 (1987), 1119-1156.
13. G.W. Chang, *On the Principal Ideal Theorem*, Bull. Korean Math. Soc. 36 (1999), 655-660.
14. M. Fontana, S. Gabelli, and E. Houston, *UMT-domains and domains with Prüfer integral closure*, Comm. Algebra 26(4) (1998), 1017-1039.

15. S. Gabelli, *On divisorial ideals in polynomial rings over Mori domains*, Comm. Algebra 15 (1987), 2349–2370

16. R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.

17. M. Griffin, *Some results on v-multiplication rings*, Canad. J. Math., 19 (1967), 710-722.

18. E. Houston and M. Zafrullah, *On t-invertibility II*, Comm. Algebra 17(8) (1989), 1955-1969

19. B.G. Kang, *Prüfer v-multiplication domains and the ring* $R[X]_{N_v}$, J. Algebra 123 (1989), 151-170.

20. I. Kaplansky, *Commutative Rings*, revised edition, Univ. of Chicago Press, 1974.

21. T.Y. Lam, *Lectures on Modules and Rings*, Springer, New York, 1999.

22. M. Nagata, *Local Rings*, Interscience, New York, 1962.

23. J. Querre, *Sur une propriété des anneaux de Krull*, Bull. Sci. Math. 99(2) (1971), 341-354.

24. J. Querre, *Idéaux divisoriels d'un anneaux de polynômes*, J. Algebra 64 (1980), 270-284

25. M. Roitman, *On polynomial extensions of Mori domains over countable fields*, J. Pure and Appl. Algebra 64 (1990), 315-328.

On the Complete Integral Closure of Rings that Admit a ϕ-Strongly Prime Ideal

AYMAN BADAWI, Department of Mathematics, Birzeit University, Box 14, Birzeit, West Bank, Palestine, via Israel. E-Mail: abring@birzeit.edu

ABSTRACT:

Let R be a commutative ring with 1 and $T(R)$ be its total quotient ring such that $Nil(R)$ (the set of all nilpotent elements of R) is a divided prime ideal of R. Then R is called a *ϕ-chained ring (ϕ-CR)* if for every $x, y \in R \setminus Nil(R)$, either $x \mid y$ or $y \mid x$. A prime ideal P of R is said to be a ϕ-strongly prime ideal if for every $a, b \in R \setminus Nil(R)$, either $a \mid b$ or $aP \subset bP$. In this paper, we show that if R admits a regular ϕ-strongly prime ideal, then either R does not admit a minimal regular prime ideal and $c(R)$ (the complete integral closure of R inside $T(R)$) $=$ $T(R)$ is a ϕ-CR or R admits a minimal regular prime ideal Q and $c(R) = (Q : Q)$ is a ϕ-CR with maximal ideal Q. We also prove that the complete integral closure of a conducive domain is a valuation domain.

1 INTRODUCTION

We assume throughout that all rings are commutative with $1 \neq 0$. We begin by recalling some background material. As in [17], an integral domain R, with quotient field K, is called a ***pseudo-valuation domain (PVD)*** in case each prime ideal P of R is *strongly prime*, in the sense that $xy \in P, x \in K, y \in K$ implies that either $x \in P$ or $y \in P$. In [4], Anderson, Dobbs and the author generalized the study of pseudo-valuation domains to the context of arbitrary rings (possibly with nonzero zerodivisors). Recall from [4] that a prime ideal P of R is said to be *strongly prime (in R)* if aP and bR are comparable (under inclusion) for all $a, b \in R$. A ring R is called a ***pseudo-valuation ring (PVR)*** if each prime ideal of R is strongly prime. A PVR is necessarily quasilocal [4, Lemma 1(b)]; a chained ring is a PVR [4, Corollary 4]; and an integral domain is a PVR if and only if it is a PVD (cf. [1, Proposition 3.1], [2, Proposition 4.2], and [6, Proposition 3]). Recall from [7] and [14] that a prime ideal P of R is called *divided* if it is comparable (under inclusion) to every ideal of R. A ring R is called a *divided ring* if every prime ideal of R is divided.

In [8], the author gave another generalization of PVDs to the context of arbitrary rings (possibly with nonzero zerodivisors). As in [8], for a ring R with total quotient ring $T(R)$ such that $Nil(R)$ (the set of all nilpotent elements of R) is a divided prime ideal of R, let $\phi : T(R) \longrightarrow K := R_{Nil(R)}$ such that $\phi(a/b) = a/b$ for every $a \in R$ and every $b \in R \setminus Z(R)$. Then ϕ is a ring homomorphism from $T(R)$ into K, and ϕ restricted to R is also a ring homomorphism from R into K given by $\phi(x) = x/1$ for every $x \in R$. A prime ideal Q of $\phi(R)$ is called a *K-strongly prime* ideal if $xy \in Q$, $x \in K, y \in K$ implies that either $x \in Q$ or $y \in Q$. If each prime ideal of $\phi(R)$ is K-strongly prime, then $\phi(R)$ is called a *K-pseudo-valuation ring (K-PVR)*. A prime ideal P of R is called a *ϕ-strongly prime* ideal if $\phi(P)$ is a K-strongly prime ideal of $\phi(R)$. If a ϕ-strongly prime ideal P of R contains a nonzerodivisor, then we say that P is a regular ϕ-strongly prime ideal. If each prime ideal of R is ϕ-strongly prime, then R is called a *ϕ-pseudo-valuation ring ($\phi - PVR$)*. For an equivalent characterization of a ϕ-PVR, see Proposition 1.1(7). It was shown in [9, Theorem 2.6] that for each $n \geq 0$ there is a ϕ-PVR of Krull dimension n that is not a PVR. Also, recall from [10], that a ring R is called a *ϕ-chained ring (ϕ-CR)* if $Nil(R)$ is a divided prime ideal of R and for every $x \in R_{Nil(R)} \setminus \phi(R)$, we have $x^{-1} \in \phi(R)$. For an equivalent characterization of a ϕ-CR, see Proposition 1.1(9). A ϕ-CR is a divided ring [10, Corollary 3.3(2)], and hence is quasilocal. It was shown in [10, Theorem 2.7] that for each $n \geq 0$ there is a ϕ-CR of Krull dimension n that is not a chained ring.

Suppose that $Nil(R)$ is a divided prime ideal of a commutative ring R such that R admits a regular ϕ-strongly prime. In this paper, we show that $c(R)$ (the complete integral closure of R inside $T(R)$) is a ϕ-chained ring. In fact, we will show that either $c(R) = T(R)$ or $c(R) = (Q : Q) = \{x \in T(R) : xQ \subset Q\}$ for some minimal regular ϕ-strongly prime ideal Q of R.

In the following proposition, we summarize some basic properties of PVRs, ϕ-PVRs, and ϕ-CRs.

PROPOSITION 1.1. *1. An integral domain is a PVR if and only if it is a ϕ-PVR if and only if it is a PVD([1, Proposition 3.1], [2, Proposition 4.2], [6, Proposition 3], and [8]).*

2. *A PVR is a divided ring [4, Lemma 1], and hence is quasilocal.*
3. *A ring R is a PVR if and only if for every $a, b \in R$, either $a \mid b$ in R or $b \mid ac$ in R for each nonunit c in R [4, Theorem 5].*
4. *If R is a PVR, then $Nil(R)$ and $Z(R)$ are divided prime ideals of R ([4], [8]).*
5. *A PVR is a ϕ-PVR [8, Corollary 7(3)].*
6. *If P is a ϕ-strongly prime ideal of R, then P is a divided prime. In particular, if R is a ϕ-PVR, then R is a divided ring [8, Proposition 4], and hence is quasilocal.*
7. *Suppose that $Nil(R)$ is a divided prime ideal of R. Then a prime ideal P of R is ϕ-strongly prime if and only if for every $a, b \in R \setminus Nil(R)$, either $a \mid b$ in R or $aP \subset bP$. In particular, a ring R is a ϕ-PVR if and only if for every $a, b \in R \setminus Nil(R)$, either $a \mid b$ in R or $b \mid ac$ in R for every nonunit $c \in R$ [8, Corollary 7].*
8. *Suppose that $Nil(R)$ is a divided prime ideal of R. If P is a ϕ-strongly prime ideal of R and Q is a prime ideal of R contained in P, then Q is a ϕ-strongly prime ideal of R [8, Proposition 5].*

9. *Suppose that $Nil(R)$ is a divided prime ideal of R. Then a ring R is a ϕ-CR if and only if for every $a, b \in R \setminus Nil(R)$, either $a \mid b$ in R or $b \mid a$ in R [10, Proposition 2.3].*

10. *A ϕ-CR is a ϕ-PVR [10, Corollary 2.3].* □

2 The COMPLETE INTEGRAL CLOSURE OF RINGS THAT ADMIT A REGULAR ϕ-STRONGLY PRIME IDEAL

Throughout this section, $Nil(R)$ denotes the set of all nilpotent elements of R, $Z(R)$ denotes the set of all zerodivisor elements of R, and $c(R)$ denotes the complete integral closure of R inside $T(R)$. The following two lemmas are needed in the proof of Proposition 2.3.

LEMMA 2.1. *Suppose $Nil(R)$ is a divided prime ideal of R and P is a regular ϕ-strongly prime ideal of R. If s is a regular element of R and $z \in Z(R)$, then $s \mid z$ in R. In particular, $Z(R) \subset P$.*

Proof: Let s be a regular element of P and $z \in Z(R)$. Suppose that $s \nmid z$ in R. Then $sP \subset zP$ by Proposition 1.1(7). Since $s \in P$, we have $z \mid s^2$ in R, which is impossible. Hence, $s \mid z$ in R. Thus, $Z(R) \subset P$. Now, suppose that s is a regular element of $R \setminus P$. Since P is divided by Proposition 1.1(6), we conclude that $P \subset (s)$. Hence, since $Z(R) \subset P$, we conclude that $s \mid z$ in R. □

LEMMA 2.2. *Suppose that $Nil(R)$ is a divided prime ideal of R and P is a regular ϕ-strongly prime ideal of R. Then $x^{-1}P \subset P$ for each $x \in T(R) \setminus R$. In particular, if $x \in T(R) \setminus R$, then x is a unit of $T(R)$.*

Proof: First, observe that $Z(R) \subset P$ by Lemma 2.1. Now, let $x = a/b \in T(R) \setminus R$ for some $a \in R$ and for some $b \in R \setminus Z(R)$. Since $b \nmid a$ in R, $Z(R) \subset P$, and P is divided, we conclude that $a \in R \setminus Z(R)$. Hence, $x^{-1} \in T(R)$. Thus, since $b \nmid a$ in R, we have $bP \subset aP$ by Proposition1.1(7). Thus $x^{-1}P = \frac{b}{a}P \subset P$. □

In light of the Lemmas 2.1 and 2.2, we have the following proposition.

PROPOSITION 2.3. *Suppose that $Nil(R)$ is a divided prime ideal of R and P is a regular prime ideal of R. Then the following statements are equivalent:*

1. *P is a ϕ-strongly prime ideal of R.*

2. *$(P : P)$ is a ϕ-CR with maximal ideal P.*

Proof: **(1) $\Longrightarrow$ (2).** First, we show that P is the maximal ideal of $(P : P)$. Let $s \in R \setminus P$. Then s is a regular element of R (because P is a divided regular prime ideal of R, and therefore $Z(R) \subset P$). Hence $1/s \in (P : P)$. Thus, s is a unit of $(P : P)$. Hence, P is the maximal ideal of $(P : P)$. Now, we show that $(P : P)$ is a ϕ-CR. Since $Nil(R)$ is a divided prime ideal of R, $Nil((P : P)) = Nil(R)$. Let $x, y \in (P : P) \setminus Nil(R)$ and suppose that $x \nmid y$ in $(P : P)$. Then $x = a/s$, $y = b/s$

for some $a, b \in R \setminus Nil(R)$, and some $s \in R \setminus Z(R)$. Since $x \not| y$ in $(P : P)$, it is impossible that a be a regular element of R and $b \in Z(R)$. Thus, we consider three cases. Case 1: suppose that $a \in Z(R)$ and $b \in R \setminus Z(R)$. Then $b \mid a$ in R by Lemma 2.1. Hence, $y \mid x$ in $(P : P)$. Case 2: suppose that $a, b \in R \setminus Z(R)$. Since $x \not| y$ in $(P : P)$, we conclude that $w = y/x \in T(R) \setminus R$. Hence, $w^{-1}P = \frac{x}{y}P \subset P$ by Lemma 2.2. Hence, $y \mid x$ in $(P : P)$. Case 3: suppose that $a, b \in Z(R)$. Since $x \not| y$ in $(P : P)$, we conclude that $a \not| b$ in R. Thus, $aP \subset bP$ by Proposition 1.1(7). Let h be a regular element of P. Then $ah = bc$ for some $c \in P$. Suppose that $h \mid c$ in R. Then $b \mid a$ in R. Hence, $y \mid x$ in $(P : P)$. Thus, suppose that $h \not| c$ in R. Then, c is a regular element of P. Hence, $f = c/h \in T(R) \setminus R$. Thus, $f^{-1}P = \frac{h}{c}P \subset P$ by Lemma 2.2. Hence, $f^{-1} \in (P : P)$. Thus, $ah = bc$ implies that $xf^{-1} = y$. Hence, $x \mid y$ in $(P : P)$, a contradiction. Thus, $h \mid c$ in R, and therefore $y \mid x$ in $(P : P)$. Hence, $(P : P)$ is a ϕ-CR by Proposition 1.1(9). **(2) $\Longrightarrow$ (1)**. This is clear by Proposition 1.1(10). □

PROPOSITION 2.4. *Suppose that $Nil(R)$ is a divided prime ideal of R and P is a regular ϕ-strongly prime ideal of R. Then $Q = \cap_{i=1}^{\infty}(s^i)$ is a prime ideal of R for every regular element s of P.*

Proof: Suppose that $xy \in Q$ for some $x, y \in R$. Since $Z(R) \subset (s^i)$ for each $i \geq 1$ by Lemma 2.1, we conclude that $Z(R) \subset Q$. Hence, we may assume that neither $x \in Z(R)$ nor $y \in Z(R)$. Thus, assume that $x \notin Q$. Then $s^n \not| x$ for some $n \geq 1$. Hence, $s^nP \subset xP$ by Proposition 1.1(7). In particular, since $s^n \in P$, we have $s^{2n} \subset xP$. Hence, we have $xy \in (s^{2n+i}) \subset xs^iP \subset (xs^i)$ for every $i \geq 1$. Thus, $y \in (s^i)$ for every $i \geq 1$. Hence, $y \in Q$. □

PROPOSITION 2.5. *Let P be a regular prime ideal of R. Then $(P : P) \subset c(R)$.*

Proof: Let $x \in (P : P)$, and let s be a regular element of P. Then $sx^n \in P$ for every $n \geq 1$. Hence, x is an almost integral element of R. Thus, $x \in c(R)$. □

PROPOSITION 2.6. *Suppose that $Nil(R)$ is a divided prime ideal of R and P is a regular ϕ-strongly prime ideal of R. Then $T(R)$ is a ϕ-CR.*

Proof: First, observe that $Nil(T(R)) = Nil(R)$. Hence, it suffices to show that if $a, b \in R \setminus Nil(R)$, then either $a \mid b$ in $T(R)$ or $b \mid a$ in $T(R)$. Hence, let $a, b \in R \setminus Nil(R)$. Suppose that $a \not| b$ in $T(R)$. Then $a \not| b$ in R. Hence, $aP \subset bP$ by Proposition 1.1(7). Thus, let s be a regular element of P. Then $as = bc$ for some $c \in P$. Thus, $a = b\frac{c}{s}$. Hence, $b \mid a$ in $T(R)$. □

Now, we state our main result in this section

THEOREM 2.7. *Suppose that $Nil(R)$ is a divided prime ideal of R and P is a regular ϕ-strongly prime ideal of R. Then exactly one of the following statements must hold:*

1. *R does not admit a minimal regular prime ideal and $c(R) = T(R)$ is a ϕ-CR.*
2. *R admits a minimal regular prime ideal Q and $c(R) = (Q : Q)$ is a ϕ-CR with maximal ideal Q.*

Proof: **(1)**. Suppose that R does not admit a minimal regular prime ideal. We will show that $1/s \in c(R)$ for every regular element $s \in R$. Hence, let s be a regular element of R. Suppose that $s \in R \setminus P$. Then $1/s \in (P : P)$ because P is a divided prime ideal of R by Proposition 1.1(6). Hence $1/s \in (P : P) \subset c(R)$ by Proposition 2.5. Thus, suppose that $s \in P$. We will show that there is regular prime ideal $H \subset P$ such that $s \notin H$. Deny. Let $F = \{D : D$ is a regular prime ideal of R and $D \subset P\}$ and $N = \cap_{D \in F} D$. Then, $s \in N$. Now, by Proposition 1.1(8) and (6), we conclude that the prime ideals in the set F are linearly ordered. Hence, N is a minimal regular prime ideal of R, which is a contradiction. Thus, there is a regular prime ideal $H \subset P$ such that $s \notin H$. Hence, once again $1/s \in (H : H) \subset c(R)$ by Proposition 2.5. Thus, $c(R) = T(R)$. Now, $T(R)$ is a ϕ-CR by Proposition 2.6.

(2). Suppose that Q is a minimal regular prime ideal of R. First, observe that $Q \subset P$ by Proposition 1.1(6). Thus, Q is a minimal ϕ-strongly prime ideal of R by Proposition 1.1(8). Now, $(Q : Q) \subset c(R)$ by Proposition 2.5. We will show that $c(R) \subset (Q : Q)$. Suppose there is an $x \in c(R) \setminus R$. Then x is a unit of $T(R)$ by Lemma 2.2. We consider three cases. Case 1: suppose that $x^{-1} \in T(R) \setminus R$. Then $xQ \subset Q$ by Lemma 2.2. Hence, $x \in (Q : Q)$. Case 2: suppose that $x^{-1} \in R \setminus Q$. Then $Q \subset (x^{-1})$ by Proposition 1.1(6). Thus, $x \in (Q : Q)$. Case 3: suppose that $x^{-1} \in Q$. This case can not happen, for if $x^{-1} \in Q$, then $D = \cap_{i=1}^{\infty} (x^{-1})^i$ contains a regular element of R because $x \in c(R)$. But D is a prime ideal of R by Proposition 2.4. Hence, D is a regular prime ideal of R that is properly contained in Q. A contradiction, since Q is a minimal regular prime ideal of R. Hence, $c(R) = (Q : Q)$. Now, $c(R) = (Q : Q)$ is a ϕ-CR by Proposition 2.3. □

Suppose that $Nil(R)$ is a divided prime ideal of R and $P \neq Nil(R)$ is a ϕ-strongly prime ideal of R. Then observe that $Nil(\phi(R))$ is a divided prime ideal of $\phi(R)$ and $\phi(P)$ is a regular K-strongly prime ideal of $\phi(R)$ (recall that $K = R_{Nil(R)}$). Now, since $\phi(R)_{Nil(\phi(R))} = K_{Nil(R)}$, we may think of $\phi(P)$ as a ϕ-strongly prime ideal of $\phi(R)$. In light of this argument and Theorem 2.7, we have the following corollary.

COROLLARY 2.8. *Suppose that $Nil(R)$ is a divided prime ideal of R and $P \neq Nil(R)$ is a ϕ-strongly prime ideal of R. Then exactly one of the following statements must hold:*

1. *$\phi(R)$ does not admit a minimal regular prime ideal and $c(\phi(R)) = T(\phi(R)) = K_{Nil(R)}$ is a K-CR.*

2. *$\phi(R)$ admits a minimal regular prime ideal Q and $c(\phi(R)) = (Q : Q)$ is a K-CR.* □

COROLLARY 2.9. *Suppose that R admits a regular strongly prime ideal. Then exactly one of the statements in Theorem 2.7 must hold.* □

COROLLARY 2.10. *Suppose that an integral domain R admits a nonzero strongly prime ideal of R. Then exactly one of the statements in Theorem 2.7 must hold (observe that in this case $c(R)$ is a valuation domain).* □

COROLLARY 2.11. *Suppose that $Nil(R)$ is a divided prime ideal of R and P is a regular ϕ-strongly prime ideal of R. If P contains a finite number, say n, of regular* *prime ideals of R, $P_1 \subset P_2 \subset \cdots \subset P_{n-1} \subset P_n = P$, then $c(R) = (P_1 : P_1)$.* □

Let $J(R)$ denotes the Jacobson radical ideal of R. We have the following result.

COROLLARY 2.12. *Suppose that R is a Prüfer domain such that $J(R)$ contains a nonzero prime ideal of R. Then exactly one of the statements in Theorem 2.7 must hold (once again, observe that in this case $c(R)$ is a valuation domain).*

Proof: Let P be a nonzero prime ideal of R such that $P \subset J(R)$. Then P is a strongly prime ideal by [11, Proposition 1.3, and the proof of Theorem 4.3]. Hence, the claim is now clear. □

It is well-known [17, Proposition 3.2] that if R is a Noetherian pseudo-valuation domain (which is not a field), then R has Krull dimension one. The following is an alternative proof of this fact.

PROPOSITION 2.13. *([17, Proposition 3.2]). If R is a Noetherian pseudo-valuation domain (which is not a field), then R has Krull dimension one.*

Proof: Deny. Let M be the maximal ideal of R. Then there is a nonzero prime ideal P of R such that $P \subset M$ and $M \neq P$. Hence, there is an element $m \in M \setminus P$. Since P is divided, we have $P \subset (m)$. Thus, $1/m \in c(R)$. Since R is Noetherian, $1/m$ is also integral over R, which is impossible. Hence, R has Krull dimension one. □

3 THE COMPLETE INTEGRAL CLOSURE OF CONDUCIVE DOMAINS

Throughout this section, R denotes an integral domain with quotient field K, and $c(R)$ denotes the integral closure of R inside K. If I is a proper ideal of R, then $Rad(I)$ denotes the radical ideal of R. Recall from [11], that Houston and the author defined an ideal I of R to be *powerful* if, whenever $xy \in I$ for elements $x, y \in K$, we have $x \in R$ or $y \in R$. Also, recall that in [13, Theorem 4.5] Bastida and Gilmer proved that a domain R shares an ideal with a valuation domain iff each overring of R which is different from the quotient field K of R has a nonzero conductor to R. Domains with this property, called *conducive domains*, were explicity defined and studied by Dobbs and Fedder [15], and further studied by Barucci, Dobbs, and Fontana [12] and [16]. In [11, Theorem 4.1], Houston and the author proved the following result.

PROPOSITION 3.1. *([11, Theorem 4.1]) An integral domain R is a conducive domain if and only if R admits a powerful ideal.* □

The following proposition is needed in the proof of Theorem 3.2.

PROPOSITION 3.2. *([11, Theorem 1.5 and Lemma 1.1]). Suppose that I is a proper powerful ideal of R. Then $I^2 \subset (s)$ for every $s \in R \setminus Rad(I)$, and $x^{-1}I^2 \subset R$ for every $x \in K \setminus R$.* □

Now, we state the main result of this section.

THEOREM 3.3. *Suppose that R admits a nonzero proper powerful ideal I, that is, R is a conducive domain. Then exactly one of the following two statements must hold:*

1. $\cap_{n=1}^{\infty} I^n \neq 0$ *and exactly one of the following two statements must hold:*
   (a) *R does not admit a minimal regular prime ideal and $c(R) = K$ is a valuation domain.*
   (b) *R admits a minimal regular prime ideal Q and $c(R) = (Q : Q)$ is a valuation domain.*
2. $\cap_{n=1}^{\infty} I^n = 0$ *and $c(R) = \{x \in K : x^{-n} \notin Rad(I)$ for every $n \geq 1\}$ is a valuation domain.*

Proof: (1). Suppose that $P = \cap_{n=1}^{\infty} I^n \neq 0$. Then P is a nonzero strongly prime ideal of R by [11, Proposition 1.8]. Hence, the claim is now clear by Theorem 2.7.

(2) Suppose that $P = \cap_{n=1}^{\infty} I^n = 0$. Let $S = \{x \in K : x^{-n} \notin Rad(I)$ for every $n \geq 1\}$, and let $x \in c(R)$. We will show that $x \in S$. Since $P = 0$ and $x \in c(R)$, $x^{-n} \notin I$ for every $n \geq 1$. Hence, $x \in S$. Thus, $c(R) \subset S$. Now, let $s \in S$. We will show that $s \in c(R)$. Let d be a nonzero element of I^2. Hence, for every $n \geq 1$ we have either $s^{-n} \in K \setminus R$ or $s^{-n} \in R \setminus Rad(I)$. Thus, $ds^n \in R$ for every $n \geq 1$ by Proposition 3.2. Hence, $s \in c(R)$. Thus, $S \subset c(R)$. Therefore, $S = c(R)$. Now, we show that $c(R) = S$ is a valuation domain. Let $x \in K \setminus S$. Then $x^{-n} \in Rad(I)$ for some $n \geq 1$. Hence, $x^n \notin Rad(I)$ for every $n \geq 1$. Thus, $x^{-1} \in S$. Therefore, $c(R) = S$ is a valuation domain. □

References

[1] D. F. Anderson, *Comparability of ideals and valuation overrings,* Houston J. Math. 5(1979), 451-463.

[2] D. F. Anderson, *When the dual of an ideal is a ring,* Houston J. Math. 9(1983), 325-332.

[3] D. F. Anderson, A. Badawi, and D. E. Dobbs, *Pseudo-valuation rings, II,* Boll. Un. Mat. Ital. B(8)3(2000), 535-545.

[4] A. Badawi, D. F. Anderson, and D. E. Dobbs, *Pseudo-valuation rings,* Lecture Notes Pure Appl. Math., Vol. 185(1997), 57-67, Marcel Dekker, New York/Basel.

[5] A. Badawi, *Pseudo-valuation domains: a survey,* to appear in World Scientific, New York/London.

[6] A. Badawi, *On domains which have prime ideals that are linearly ordered,* Comm. Algebra 23(1995), 4365-4373.

[7] A. Badawi, *On divided commutative rings,* Comm. Algebra 27(1999), 1465-1474.

[8] A. Badawi, *On ϕ-pseudo-valuation rings*, Lecture Notes Pure Appl. Math., Vol. 205(1999), 101-110, Marcel Dekker, New York/Basel.

[9] A. Badawi, *On ϕ-pseudo-valuation rings, II*, Houston J. Math. 26(2000), 473-480

[10] A. Badawi, *On ϕ-chained rings and ϕ-pseudo-valuation rings*, Houston J. Math. 27(2001), 725-736

[11] A. Badawi and E. Houston, *Powerful ideals, strongly primary ideals, almost pseudo-valuation domains, and conducive domains*, to appear in Comm. Algebra.

[12] V. Barucci, D. E. Dobbs, and M. Fontana, *Conducive integral domains as pullbacks*, Manuscripta Math. 54(1986), 261-277.

[13] E. Bastida and R. Gilmer, *Overrings and divisorial ideals in rings of the form $D + M$*, Michigan Math. J. 20(1973), 79-95.

[14] D. E. Dobbs, *Divided rings and going-down*, Pacific J. Math. 67(1976), 353-363.

[15] D. E. Dobbs and R. Fedder, *Conducive integral domains*, J. Algebra 86(1984), 494-510.

[16] D. E. Dobbs, V. Barucci, and M. Fontana, *Gorenstein conducive domains*, Comm. Algebra 18(1990), 3889-3903.

[17] J. R. Hedstrom and E. G. Houston, *Pseudo-valuation domains*, Pacific J. Math. 4(1978), 551-567.

[18] J. Huckaba, *Commutative Rings with Zero Divisors*, Marcel Dekker, New York/Basel, 1988.

Frobenius Number of a Linear Diophantine Equation

ABDALLAH BADRA, Université Blaise Pascal, Laboratoire de mathématiques pures, Les Cézeaux, F 63177 Aubière.
e-mail: abdallah.badra@math.univ-bpclermont.fr

ABSTRACT. We denote by $\mathbb{N}_0$ the set of nonnegative integers. Let $d \geq 1$ and $A = \{a_1, \ldots, a_d\}$ a set of positive integers. For every $n \in \mathbb{N}_0$, we write $s(n)$ for the number of solutions $(x_1, \ldots, x_d) \in \mathbb{N}_0^d$ of the equation $a_1x_1 + \cdots + a_dx_d = n$. We set $g(A) = \sup\{n \mid s(n) = 0\} \cup \{-1\}$ the Frobenius number of A. Let $S(A)$ be the subsemigroup of $(\mathbb{N}_0, +)$ generated by A. We set $S'(A) = \mathbb{N}_0 \backslash S(A)$, $N'(A) = \mathrm{Card}S'(A)$ and $N(A) = \mathrm{Card}$ $S(A) \cap \{0, 1, .., g(A)\}$. Let p be a multiple of $\mathrm{lcm}(A)$ and $F_p(t) = \prod_{i=1}^d \sum_{j=0}^{\frac{p}{a_i}-1} t^{ja_i}$. We give an upper bound for $g(A)$ and reduction formulas for $g(A)$, $N'(A)$ and $N(A)$. Characterizations of these invariants as well as numerical symmetric and pseudo-symmetric semigroups in terms of $F_p(t)$, are also obtained.

1 INTRODUCTION

We denote by $\mathbb{N}_0$ (resp. $\mathbb{N}$) the set of nonnegative (resp. positive) integers. Let $d \in \mathbb{N}$ and $A = \{a_1, \ldots, a_d\} \subset \mathbb{N}$. We set $\rho = \gcd(A)$ and $l = \mathrm{lcm}(A)$. For every $n \in \mathbb{N}_0$, we write $s(n)$ for the number of solutions $(x_1, \ldots, x_d) \in \mathbb{N}_0^d$ of the equation $a_1x_1 + \cdots + a_dx_d = n$. We set $g(A) = \sup\{n \mid s(n) = 0\} \cup \{-1\}$ the Frobenius number of A. Let $S(A)$ be the subsemigroup of $(\mathbb{N}_0, +)$ generated by A, $S'(A) = \mathbb{N}_0 \backslash S(A)$, $N'(A) = \mathrm{Card}\ S'(A)$ and $N(A) = \mathrm{Card}\ S(A) \cap \{0, 1, .., g(A)\}$. We say that $S(A)$ is symmetric (resp. pseudo-symmetric) if $\gcd(A) = 1$ and $N'(A) = N(A)$ (resp. $N'(A) = N(A) + 1$). The generating function of the $s(n)$ is

$$\Phi(t) = \frac{1}{\prod_{i=1}^d (1 - t^{a_i})}.$$

Indeed, we have

$$\frac{1}{\prod_{i=1}^d (1 - t^{a_i})} = \prod_{i=1}^d \sum_{j \geq 0} t^{ja_i} = \sum_{n \in S(A)} s(n)t^n.$$

For $p \in I\!\!N$, we define the Frobenius polynomial

$$F_p(t) = \prod_{i=1}^{d} \sum_{j=0}^{\frac{p}{a_i}-1} t^{ja_i} = \frac{(1-t^p)^d}{\prod_{i=1}^{d}(1-t^{a_i})}$$

and we write

$$\Phi(t) = \frac{F_p(t)}{(1-t^p)^d}. \tag{1}$$

In theorem 3.1 we give formulas for $g(A), N'(A)$ and $N(A)$ in terms of $F_p(t)$. As a consequence we obtain an upper bound for the Frobenius number (corollary 3.2) which improves the upper bound given by Chrzastowski-Wachtel and mentioned in [9]. A characterization of numerical symmetric and pseudo-symmetric semigroups (corollary 3.4) is also obtained. In theorem 3.7 we prove reduction formulas for $g(A)$, $N'(A)$ and $N(A)$. The first one generalizes a Raczunas and Chrzastowski-Wachtel theorem [9]. As a consequence (corollary 3.10) we obtain a generalization of a Rödseth formula [10]. It is known that the Hilbert function of a graded module over a polynomial graded ring as well as $s(n)$ are numerical quasi-polynomial functions. In examples 4.9 and 4.10 we give a description of these functions in terms of the Frobenius polynomial.

2 PRELIMINARIES

Given $Q(t) = \sum_j q_j t^j \in \mathbb{Q}[t,t^{-1}]$ and an integer $p \geq 1$, there exists a unique sequence $Q_0, \ldots, Q_{p-1} \in \mathbb{Q}[t,t^{-1}]$ such that $Q(t) = \sum_{r=0}^{p-1} t^r Q_r(t^p)$. Namely, $Q_r(t) = \sum_k q_{r+pk} t^k$. The Q_r are called the *p-components* of Q. We denote by $\omega(Q) = \inf\{j \mid q_j \neq 0\}$ the valuation of Q and $\deg(Q) = \sup\{j \mid q_j \neq 0\}$ the degree of Q, with $\omega(0) = +\infty$ and $\deg(0) = -\infty$. The following invariants will be associated with Q

$$\begin{aligned}
\omega_p(Q) &= \sup\{\omega(t^r Q_r(t^p)) \mid 0 \leq r \leq p-1\} \text{ the } p\text{-valuation of } Q.\\
\delta_p(Q) &= \inf\{\deg(t^r Q_r(t^p)) \mid 0 \leq r \leq p-1\} \text{ the } p\text{-degree of } Q.\\
\Omega_p(Q) &= \sum_{r=0}^{p-1} \omega(Q_r).\\
\Delta_p(Q) &= \sum_{r=0}^{p-1} \deg(Q_r).
\end{aligned}$$

Thus we have
$\omega_p(Q) = +\infty = \Omega_p(Q)$ and $\delta_p(Q) = -\infty = \Delta_p(Q)$ if $Q_r = 0$ for some r.

We fix an integer $n \in \mathbb{Z}$ and we set

$$\widehat{Q}(t) = t^n Q(t^{-1}).$$

So we have $\widehat{\widehat{Q}} = Q$ and

$$\deg(Q) + \omega(\widehat{Q}) = n = \deg(\widehat{Q}) + \omega(Q) \text{ if } Q \neq 0. \tag{2}$$

The p-components $\widehat{Q}_r$ of $\widehat{Q}$ can be deduced from the p-components of Q. Namely, we write $n = p\lambda + \gamma$ with $0 \leq \gamma < p$, so we get

$$\widehat{Q}(t) = \sum_{r=0}^{p-1} t^{p\lambda+\gamma-r} Q_r(t^{-p}) = \sum_{r=0}^{\gamma} t^{\gamma-r}(t^p)^\lambda Q_r(t^{-p}) + \sum_{r=\gamma+1}^{p-1} t^{p+\gamma-r}(t^p)^{\lambda-1} Q_r(t^{-p}).$$

It follows from the uniqueness of the p-components that

$$\widehat{Q}_r(t) = t^\lambda Q_{\gamma-r}(t^{-1}) \text{ for } 0 \leq r \leq \gamma \tag{3}$$

and

$$\widehat{Q}_r(t) = t^{\lambda-1} Q_{p+\gamma-r}(t^{-1}) \text{ for } r > \gamma. \tag{4}$$

So we obtain

$$\widehat{Q}_r = 0 \Leftrightarrow Q_{\gamma-r} = 0 \text{ for } 0 \leq r \leq \gamma \tag{5}$$

and

$$\widehat{Q}_r = 0 \Leftrightarrow Q_{p+\gamma-r} = 0 \text{ for } r > \gamma. \tag{6}$$

If $\widehat{Q}_r \neq 0$, we also deduce from (2)-(4) that

$$\lambda = \deg(\widehat{Q}_r) + \omega(Q_{\gamma-r}) \text{ when } 0 \leq r \leq \gamma \tag{7}$$

and

$$\lambda - 1 = \deg(\widehat{Q}_r) + \omega(Q_{p+\gamma-r}) \text{ when } r > \gamma. \tag{8}$$

Moreover, writing $n = p\lambda + r + (\gamma - r) = p(\lambda - 1) + r + (p + \gamma - r)$ we get

$$n = \deg(t^r \widehat{Q}_r(t^p)) + \omega(t^{\gamma-r} Q_{\gamma-r}(t^p)) \text{ for } 0 \leq r \leq \gamma$$

and

$$n = \deg(t^r \widehat{Q}_r(t^p)) + \omega(t^{p+\gamma-r} Q_{p+\gamma-r}(t^p)) \text{ for } r > \gamma.$$

Hence

$$n = \delta_p(\widehat{Q}) + \omega_p(Q) = \delta_p(Q) + \omega_p(\widehat{Q}). \tag{9}$$

Furthermore, using (7) and (8) we get

$$\sum_{r=0}^{\gamma} \Big(\deg(\widehat{Q}_r) + \omega(Q_{\gamma-r})\Big) + \sum_{r=\gamma+1}^{p-1} \Big(\deg(\widehat{Q}_r) + \omega(Q_{p+\gamma-r})\Big)$$

$$= (\gamma + 1)\lambda + (p - \gamma - 1)(\lambda - 1) = n - p + 1.$$

It follows that

$$\Delta_p(\widehat{Q}) + \Omega_p(Q) = n - p + 1 = \Delta_p(Q) + \Omega_p(\widehat{Q}). \tag{10}$$

Given m, $j \in \mathbb{Z}$, we consider the following polynomials

$$N_{m,j}(t) = \frac{1}{(m-1)!}\prod_{i=1}^{m-1}(t-j+i) \text{ if } m > 1, N_{m,j}(t) = 0 \text{ if } m \leq 0 \text{ and } N_{1,j}(t) = 1.$$

For $Q(t) = \sum_j q_j t^j \in \mathbb{Q}[t, t^{-1}]$ such that $Q(1) \neq 0$, we define

$$V_m(Q,t) = \sum_j q_j N_{m,j}(t).$$

Furthermore, let $Q_0, \dots, Q_{p-1} \in \mathbb{Q}[t, t^{-1}]$ be the p-components of Q. We consider the polynomials $U_0, \dots, U_{p-1} \in \mathbb{Q}[t, t^{-1}]$ defined as follows $U_r = 0$ if $Q_r = 0$ and $Q_r(t) = (1-t)^{i_r} U_r(t)$ with $U_r(1) \neq 0$ otherwise. For all $0 \leq r \leq p-1$, we put $m_r = m - i_r$ and we define the function

$$H_m(Q,.) : \mathbb{Z} \to \mathbb{Q} \text{ by } H_m(Q, r+pk) = V_{m_r}(U_r, k).$$

In order to illustrate these definitions we give the following examples.

EXAMPLE 2.1 Let $Q(t) = F_{12} = \frac{(1-t^{12})^2}{(1-t^2)(1-t^3)} = 1 + t^2 + t^3 + t^4 + t^5 + 2t^6 + t^7 + 2t^8 + 2t^9 + 2t^{10} + 2t^{11} + t^{12} + 2t^{13} + t^{14} + t^{15} + t^{16} + t^{17} + t^{19}$.
We take $p = 12, n = 19$ and $m = 2$.
We write $Q(t) = (1 + t^{12}) + t(2t^{12}) + t^2(1 + t^{12}) + t^3(1 + t^{12}) + t^4(1 + t^{12}) + t^5(1 + t^{12}) + 2t^6 + t^7(1 + t^{12}) + 2t^8 + 2t^9 + 2t^{10} + 2t^{11}$.
We see that the 12-components of $Q(t)$ are $Q_0(t) = Q_2(t) = Q_3(t) = Q_4(t) = Q_5(t) = Q_7(t) = (1+t), Q_1(t) = 2t$ and $Q_6(t) = Q_8(t) = Q_9(t) = Q_{10}(t) = Q_{11}(t) = 2$.
We also have
$\widehat{Q}(t) = t^{19}Q(t^{-1}) = Q(t)$.
$\omega_{12}(Q) = 13,\ \delta_{12}(Q) = 6,\ \Omega_{12}(Q) = 1,\ \Delta_{12}(Q) = 7$.
$N_{2,0}(t) = t+1, N_{2,1}(t) = t$.
$U_r = Q_r$ for all r.
$V_2(U_r, t) = 2t + 1$ for $r \in \{0,2,3,4,5,7\}$, $V_2(U_1, t) = 2t$ and $V_2(U_r, t) = 2(t+1)$ for $r \in \{6,8,9,10,11\}$.
We obtain $H_2(Q, 12k+r) = 2k+1$ for $r \in \{0,2,3,4,5,7\}$, $H_2(Q, 12k+1) = 2k$ and $H_2(Q, 12k+r) = 2(k+1)$ for $r \in \{6,8,9,10,11\}$.

EXAMPLE 2.2 Let $Q(t) = F_6(t) = 1 + t^2 + t^3 + t^4 + t^5 + t^7 = \frac{(1-t^6)^2}{(1-t^2)(1-t^3)}$.
We take $p = 6, n = 7$ and $m = 2$.
We obtain
$\omega_6(Q) = 7,\ \delta_6(Q) = 0,\ \Omega_6(Q) = 1,\ \Delta_6(Q) = 1$.
$U_r = Q_r$ for all r.
$N_{2,0}(t) = t+1, N_{2,1}(t) = t$.
$V_2(U_r, t) = t + 1$ for $r \in \{0,2,3,4,5\}$ and $V_2(U_1, t) = t$.
$H_2(Q, 6k+r) = k+1$ for $r \in \{0,2,3,4,5\}$ and $H_2(Q, 6k+1) = k$.
We observe that $H_2(F_6,.) = H_2(F_{12},.)$.

Given $\Phi(t) \in \mathbb{Q}[[t,t^{-1}]]$, we write $\Phi(t) = \sum_n \varphi(n)t^n$ and we introduce

$$\begin{aligned} g(\Phi) &= \sup\{n \mid \phi(n) = 0\}. \\ S'(\Phi) &= \{n \geq 0 \mid \varphi(n) = 0\}. \\ S(\Phi) &= \{0 \leq n \leq g(\Phi) \mid \varphi(n) \neq 0\}. \\ N'(\Phi) &= \text{Card } S'(\Phi). \\ N(\Phi) &= \text{Card } S(\Phi). \end{aligned}$$

LEMMA 2.3 *Given $m \in \mathbb{Z}$ and $Q(t) = \sum_j q_j t^j \in \mathbb{Q}[t,t^{-1}]$ such that $Q(1) \neq 0$, we consider $\Phi(t) = \sum_n \varphi(n)t^n$ the expansion of $(1-t)^{-m}Q(t)$ as a formal power series. Then, the following conditions hold*

1. *$\varphi(n) = V_m(Q,n)$ for all $n > \deg(Q) - m$.*
2. *We suppose that $m > 0$ and $Q(t)$ has nonnegative coefficients. Then,*
   (a) $\varphi(n) = 0 \Leftrightarrow n < \omega(Q)$.
   (b) $g(\Phi) = \omega(Q) - 1$.
   (c) $N'(\Phi) = \max\{\omega(Q), 0\}$. In particular, $N'(\Phi) = \omega(Q)$ if $Q(t) \in \mathbb{Q}[t]$.

PROOF. 1. Suppose $m > 0$. We have $\Phi(t) = (1-t)^{-m}Q(t) = \left(\sum_j q_j t^j\right) \sum_{j \geq 0} \binom{j+m-1}{m-1} t^j$. So $\varphi(n) = \sum_{j=\omega(Q)}^{n} q_j \binom{n-j+m-1}{m-1}$. Moreover, we have

$$\binom{n-j+m-1}{m-1} = \frac{1}{(m-1)!} \prod_{i=1}^{m-1} (n-j+i) \text{ if } n \geq j.$$

Hence $\varphi(n) = V_m(Q,n)$ if $n \geq \deg(Q)$, in particular, the statement is true for $m = 1$. Now, suppose $m > 1$ and $\deg(Q) - m < n < \deg(Q)$ then $-m < n - \deg(Q) \leq n - j < 0$ for all j such that $n < j \leq \deg(Q)$. It follows that there exists $1 \leq i \leq m-1$ such that $n - j + i = 0$ thus $N_{m,j}(n) = 0$. So we can write

$$V_m(Q,n) = \sum_{j=\omega(Q)}^{n} q_j N_{m,j}(n) = \sum_{j=\omega(Q)}^{n} q_j \binom{n-j+m-1}{m-1} = \varphi(n).$$

Furthermore, if $m \leq 0$ then $\varphi(n) = 0$ for $n > \deg(Q) - m$ because $\Phi(t) \in \mathbb{Q}[t,t^{-1}]$ and $\deg(Q) - m = \deg \Phi(t)$.
2. Follows from the fact that $\varphi(n) = \sum_{j=\omega(Q)}^{n} q_j \binom{n-j+m-1}{m-1} > 0$ if $n \geq \omega(Q)$ and $\varphi(n) = 0$ if $n < \omega(Q)$ □

THEOREM 2.4 *Let $m \in \mathbb{Z}$ and $p \in \mathbb{N}$.* Given $Q(t) = \sum_j q_j t^j \in \mathbb{Q}[t,t^{-1}]$ *such that $Q(1) \neq 0$, we consider $\Phi(t) = \sum_n \varphi(n)t^n$ the expansion of $(1-t^p)^{-m}Q(t)$ as a formal power series. Then the following conditions hold*

1. *$\varphi(n) = H_m(Q,n)$ for all $n > \deg(Q) - mp$.*
2. *We suppose that $m > 0$ and $Q(t)$ has nonnegative coefficients. Then,*
   (a) $\varphi(pk+r) = 0 \Leftrightarrow k < \omega(Q_r)$.

(b) $g(\Phi) = \omega_p(Q) - p = \deg(Q) - p - \delta_p(\hat{Q})$ *where* $\hat{Q}(t) = t^{\deg(Q)}Q(t^{-1})$.

(c) $N'(\Phi) = \sum_{r=0}^{p-1} \max\{\omega(Q_r), 0\}$.
In particular, $N'(\Phi) = \Omega_p(Q)$ if $Q(t) \in \mathbb{Q}[t]$.

PROOF. We write $\Phi(t) = \sum_{r=0}^{p-1} t^r(1-t^p)^{-m}Q_r(t^p) = \sum_{r=0}^{p-1} t^r\ (1-t^p)^{-m_r}U_r(t^p) = \sum_{r=0}^{p-1} t^r\Phi_r(t^p)$ where $\Phi_r(t) = (1-t^p)^{-m_r}U_r(t^p) = \sum_k \varphi_r(k)t^k$. It follows from lemma 2.3.1, that $\varphi(pk+r) = \varphi_r(k) = V_{m_r}(U_r, k)$ for all $k > \deg(U_r) - m_r$. Therefore, $\varphi(n) = H_m(Q, n)$ for $n > \deg(Q) - pm$ because $n = pk + r > \deg(Q) - pm \geq p(\deg(Q_r) - m) + r \Rightarrow k > \deg(Q_r) - m = \deg(U_r) - m_r$.
2 (a) follows from lemma 2.3.2 (a).
b) We have $g(\Phi) = \max\{pg(\Phi_r) + r \mid 0 \leq r \leq p-1\} = \max\{p(\omega(Q_r) - 1) + r \mid 0 \leq r \leq p-1\} = \omega_p(Q) - p$. Moreover, if $Q_r \neq 0$ for all r we have $\omega_p(Q) - p = \deg(Q) - p - \delta_p(\hat{Q})$ by (9). Since $\omega_p(Q) = +\infty = -\delta_p(\hat{Q})$ if $Q_r = 0$ for some r, the equality is still true in this case.
c) Follows from lemma 2.3.2 (c) □

LEMMA 2.5 *Let* $\xi = e^{\frac{2i\pi}{p}}$ *be a primitive p-th root of unity and* $Q(t) = \sum_{r=0}^{p-1} t^r Q_r(t^p) \in \mathbb{Q}[t, t^{-1}]$. *Then, the following conditions are equivalent*

1. $Q(\xi^j) = 0$ *for* $0 < j < p$.

2. $Q(1) = pQ_r(1)$ *for* $0 \leq r \leq p-1$.

PROOF. By successive substitutions of $1, \xi, \dots, \xi^{p-1}$ for t in $Q(t) = \sum_{r=0}^{p-1} t^r Q_r(t^p)$ we obtain a Vandermonde linear system $\sum_{r=0}^{p-1} \xi^{rj}Q_r(1) = Q(\xi^j)$ for $j = 0, \dots, p-1$. If $Q(\xi) = \cdots = Q(\xi^{p-1}) = 0$, the unique solution is $Q_r(1) = \frac{1}{p}Q(1)$ for every $0 \leq r \leq p-1$. Conversely, if $\frac{Q(1)}{p}$ is the common value of the $Q_r(1)$ then $\frac{Q(1)}{p}\sum_{r=0}^{p-1}\xi^{rj} = 0 = Q(\xi^j)$ for $j = 1, \dots, p-1$ □

LEMMA 2.6 *Let* p, q, u *be positive integers and* $Q(t), K(t) \in \mathbb{Q}[t, t^{-1}]$ *such that* $p = qu$ *and* $K(t^u) = Q(t)$. *We denote by* Q_r *(resp.* K_s*) the p-components of* Q *(resp. the q-components of* K*). Then,*

1. $Q_{su} = K_s$ *and* $Q_r = 0$ *for all* $r \notin u\mathbb{Z}$.

2. *We set* $\xi = e^{\frac{2i\pi}{p}}$, *then the following conditions are equivalent*

   (a) $Q(\xi^j) = 0$ *for* $0 < j < q$.

   (b) $Q(\xi^q) = qQ_r(1) = K(1)$ *for all* $r \in u\mathbb{Z}$.

PROOF. We can write $Q(t) = K(t^u) = \sum_{s=0}^{q-1} t^{us}K_s(t^p)$. It follows from the uniqueness of the Q_r that $Q_{su} = K_s$ for $0 \leq s < q$. Now, $Q(\xi^q) = K(1)$ and $Q(\xi^j) = K(\alpha^j)$ with $\alpha = e^{\frac{2i\pi}{q}} = \xi^u$. We apply lemma 2.5 □

For every $p \in I\!N$, we set $F_p(t) = \prod_{i=1}^d \sum_{j=0}^{\frac{p}{a_i}-1} t^{ja_i}$ the *Frobenius polynomial* of A. We write $F_{p,r}$ for the p-components of F_p. It is easy to see that for $n = \deg(F_p) = pd - \sum_{i=1}^d a_i$, we have $\widehat{F}_p(t) = t^n F_p(t^{-1}) = F_p(t)$. Let us write $p = q\rho$ and $a_i = b_i\rho$

for all $1 \leq i \leq d$, where $\rho = \gcd(A)$. So we can write $F_p(t) = K(t^\rho)$ with

$$K(t) = \frac{(1-t^q)^d}{\prod_{i=1}^d (1-t^{b_i})}.$$

Moreover, for $0 < j < q$ the number $\xi^j = e^{\frac{2ij\pi}{q}}$ is a root of $\prod_{i=1}^d (1-t^{b_i})$ of multiplicity $< d$ because $\gcd(b_1, .., b_d) = 1$ whereas ξ^j is a root of $(1-t^q)^d$ of multiplicity $= d$, then $K(\xi^j) = 0$. It follows from lemma 2.6 that $F_{p,r} = K_{\frac{r}{\rho}}$ if $r \in \rho\mathbb{Z}$ and $F_{p,r} = 0$ otherwise. We also deduce that $F_{p,r}(1) = \frac{1}{q}K(1) = \frac{\rho p^{d-1}}{\prod_{i=1}^d a_i}$ if $r \in \rho\mathbb{Z}$ □

3 FROBENIUS NUMBER AND NUMERICAL SEMIGROUPS

In the case of the Frobenius polynomial F_p we set $\omega_p(F_p) = \omega_p(A)$, $\delta_p(F_p) = \delta_p(A)$, $\Omega_p(F_p) = \Omega_p(A)$, $\Delta_p(F) = \Delta_p(A)$.

THEOREM 3.1 *For every $p \in I\!N$, we have*

1. $g(A) = \omega_p(A) - p = p(d-1) - \sum_{i=1}^d a_i - \delta_p(A) = l(d-1) - \sum_{i=1}^d a_i - \delta_l(A)$.
2. $N'(A) = \Omega_p(A) = \Omega_l(A)$.
3. $N(A) = \Delta_p(A) - \delta_p(A) = \Delta_l(A) - \delta_l(A)$.

PROOF. We see that for every $p \in I\!N$, the function $\Phi(t) = (1-t^p)^{-d}F_p(t) = \sum_n s(n)t^n$ is the generating function of the $s(n)$ so $g(A) = g(\Phi)$.
1. follows from theorem 2.4.2 (b).
2. follows from theorem 2.4.2 (c).
3. is a consequence of (10)□

COROLLARY 3.2

1. *For every $p \in I\!N$, we have*

$$g(A) = p(d-1) - \sum_{i=1}^d a_i \text{ if and only if } \delta_p(A) = 0.$$

2. $g(A) = +\infty$ *if and only if* $\rho > 1$.
3. *If* $\rho = 1$, *we have the following upper bound for the Frobenius number*

$$g(A) \leq l(d-1) - \sum_{i=1}^d a_i.$$

4. *If there exists h such that $1 \leq h \leq d$ and* $\gcd(a_1, \ldots, a_h) = 1$ *then* $g(A) \leq \mathrm{lcm}(a_1, \ldots, a_h)(h-1) - \sum_{i=1}^h a_i$.

REMARK 3.3 The upper bound we give in 3) improves the following inequality

$$g(A) \leq l(d-1)$$

proved by Chrzastowski-Wachtel and mentioned in [9].

COROLLARY 3.4 Suppose $\gcd(A) = 1$. Then the following conditions hold

1. $S(A)$ *is symmetric* $\Leftrightarrow \Delta_p(A) = \Omega_p(A) + \delta_p(A)$ for some $p \in l\mathbb{N} \Leftrightarrow \Delta_p(A) = \Omega_p(A) + \delta_p(A)$ for all $p \in l\mathbb{N}$.

2. $S(A)$ *is peudo-symmetric* $\Leftrightarrow \Delta_p(A) + 1 = \Omega_p(A) + \delta_p(A)$ for some $p \in l\mathbb{N} \Leftrightarrow \Delta_p(A) + 1 = \Omega_p(A) + \delta_p(A)$ for all $p \in l\mathbb{N}$.

We suppose $\gcd(A) = 1$. Let $q_1, .., q_d$ be positive integers such that for all $1 \leq i \leq d$, q_i is a divisor of $\gcd(a_1, .., a_{i-1}, a_{i+1}, .., a_d)$. So $\gcd(q_i, q_j) = 1$ for $i \neq j$ because $\gcd(A) = 1$. We set $\hat{q} = \prod_{j=1}^{d} q_j, \hat{q}_i = \prod_{j \neq i} q_j, a_i = b_i \hat{q}_i$ and $B = \{b_1, .., b_d\}$. We have $\gcd(B) = 1$ and $l = \mathrm{lcm}(A) = \hat{q}\mathrm{lcm}(B)$. For $p \in l\mathbb{N}$, we write $p = \hat{q}u$ with $u \in \mathrm{lcm}(B)\mathbb{N}$.

THEOREM 3.5 *The following formulas hold*

1. $\delta_p(A) = \hat{q}\delta_u(B)$.
2. $\omega_p(A) = \hat{q}\omega_u(B) + \sum_{i=1}^{d}(q_i - 1)a_i$.
3. $\Omega_p(A) = \hat{q}\Omega_u(B) + \frac{1}{2}\left(\sum_{i=1}^{d}(q_i - 1)a_i - \hat{q} + 1\right)$.
4. $\Delta_p(A) = \hat{q}\Delta_u(B) + \frac{1}{2}\left(\sum_{i=1}^{d}(q_i - 1)a_i - \hat{q} + 1\right)$.

In order to prove this theorem we need a lemma.

LEMMA 3.6 *Let q and c be two positive integers, $B = \{b_1, .., b_{d-1}, c\}$, and $A = \{a_1, .., a_{d-1}, c\}$ where $a_1 = qb_1, .., a_{d-1} = qb_{d-1}$. Suppose* $\gcd(A) = 1$ *and choose* $p \in \mathrm{lcm}(B)\mathbb{N}$ *so* $\gcd(q, c) = 1$ *and* $qp \in \mathrm{lcm}(A)\mathbb{N}$. *Then, the following formulas hold*

1. $\delta_{qp}(A) = q\delta_p(B)$.
2. $\omega_{qp}(A) = q\omega_p(B) + (q - 1)c$.
3. $\Omega_{qp}(A) = q\Omega_p(B) + \frac{1}{2}(q - 1)(c - 1)$.
4. $\Delta_{qp}(A) = q\Delta_p(B) + \frac{1}{2}(q - 1)(c - 1)$.

PROOF. We denote by

$$F(t) = F_p(t) = \frac{(1 - t^p)^d}{(1 - t^c)\prod_{i=1}^{d-1}(1 - t^{b_i})} = \sum_{r=0}^{p-1} t^r F_r(t^p)$$

the Frobenius polynomial associated with B and

$$G(t) = G_{qp}(t) = \frac{(1 - t^{qp})^d}{(1 - t^c)\prod_{i=1}^{d-1}(1 - t^{a_i})} = \sum_{s=0}^{qp-1} t^s G_s(t^{qp})$$

the Frobenius polynomial associated with A. We see that

$$G(t) = (1 + t^c + .. + t^{(q-1)c})F(t^q) = (1 + t^c + .. + t^{(q-1)c})\sum_{r=0}^{p-1} t^{qr}F_r(t^{qp}).$$

So we obtain

$$G(t) = \sum_{\substack{k=ic+jq \\ 0\leq i\leq q-1}} t^k F_j(t^{qp}) = \sum_{\substack{0\leq k=ic+jq\leq qp-1 \\ 0\leq i\leq q-1}} t^k F_j(t^{qp}) + \sum_{\substack{k>qp-1 \\ 0\leq i\leq q-1}} t^{k-qp}t^{qp}F_j(t^{qp})$$

By identification we deduce that $G_s(t^{qp}) = F_j(t^{qp})$ when $s = ic + jq$ and $G_s(t^{qp}) = t^{qp}F_j(t^{qp})$ when $s = ic + jq - qp = ic - (p-j)q$. In particular, we have $\deg(G_s) = \deg(F_j)$ and $\omega(G_s) = \omega(F_j)$ when $s = ic + jq$ and $\deg(G_s) = 1 + \deg(F_j)$ and $\omega(G_s) = 1 + \omega(F_j)$ when $s = ic + jq - qp$. Therefore, for all s which can be written in the form $s = ic + jq$ we get $\deg(t^sG_s(t^{qp})) = ic + jq + qp\deg(F_j)$ and $\omega(t^sG_s(t^{qp})) = ic + jq + qp\omega(F_j)$. For all s which can be written in the form $s = ic + jq - qp$, we get $\deg(t^sG_s(t^{qp})) = ic + jq - qp + qp(1 + \deg(F_j)) = ic + jq + qp\deg(F_j)$ and $\omega(t^sG_s(t^{qp})) = ic + jq - qp + qp(1 + \omega(F_j)$. It follows that $\delta_{qp}(G) = \min\{ic + jq + qp\deg(F_j)\} = q\min\{j + p\deg(F_j)\} = q\delta_p(F)$ and $\omega_{qp}(G) = \max\{ic+jq+qp\omega(F_j)\ \} = (q-1)c+q\max\{j+p\omega(F_j)\ \} = q\omega_p(F)+(q-1)c$. We also have

$$\Omega_{qp}(G) = \sum_{s=ic+jq} \omega(G_s) + \sum_{s=ic+jq-qp} \omega(G_s) = \sum_{s=ic+jq} \omega(F_j) + \sum_{s=ic-jq} (\omega(F_j) + 1)$$

$= q\Omega_p(F) + N'(c, q) = q\Omega_p(F) + \frac{1}{2}(q-1)(c-1)$. It follows that
$\Delta_{qp}(G) = \Omega_{qp}(G) + \delta_{qp}(G) = q(\Omega_p(F) + \delta_p(F)) + \frac{1}{2}(q-1)(c-1)\square$

PROOF OF THEOREM 3.5. By induction on the number $h = d - k + 1$ such $q_1 = q_2 = .. = q_{k-1} = 1$. If $h = 1$ the result is given by lemma 3.6. Suppose that the result is true when $q_1 = q_2 = .. = q_{k-1} = 1$. We choose $p \in \mathrm{lcm}(A)\mathbb{N}$ and we set $v = \frac{p}{q_k}$, $t_i = q_i$ for $i \neq k$ and $t_k = 1$. Then, we get $\hat{t}_i = \frac{\hat{q}_i}{q_k}$ for all $i \neq k$, $\hat{t}_k = \hat{q}_k$ and $\hat{t} = \frac{\hat{q}}{q_k}$. We also have $\frac{a_i}{q_k} = \frac{b_i\hat{q}_i}{q_k} = b_i\hat{t}_i$ for all $i \neq k$ and $a_k = b_k\hat{t}_k$. We put $c_i = b_i\hat{t}_i$ for all i and $C = \{c_1, .., c_d\}$, thus $a_i = q_kc_i$ for all $i \neq k$ and $a_k = c_k$. It follows from lemma 3.6 and the induction hypothesis that

1) $\delta_p(A) = q_k\delta_v(C) = q_k\hat{t}\delta_u(B) = \hat{q}\delta_u(B)$.

2) $\omega_p(A) = q_k\omega_v(C) + (q_k - 1)c_k = q_k\{\hat{t}\omega_u(B) + \sum_{i=1}^d (t_i - 1)c_i\} + (q_k - 1)c_k = \hat{q}\omega_u(B) + \sum_{i=1}^d (q_i - 1)a_i$.

3) $\Omega_p(A) = q_k\Omega_v(C) + \frac{1}{2}(q_k - 1)(a_k - 1) = q_k\{\hat{t}\Omega_u(B) + \frac{1}{2}(\sum_{i=1}^d (t_i - 1)c_i - \hat{t} + 1)\} + \frac{1}{2}(q_k - 1)(a_k - 1) = \hat{q}\Omega_u(B) + \frac{1}{2}\left(\sum_{i=1}^d (q_i - 1)a_i - \hat{q} + 1\right)$.

4) $\Delta_p(A) = \Omega_p(A) + \delta_p(A) = \hat{q}\Delta_u(B) + \frac{1}{2}\left(\sum_{i=1}^d (q_i - 1)a_i - \hat{q} + 1\right)$ $\square$

THEOREM 3.7 *The following formulas hold*

1. $g(A) = \hat{q}g(B) + \sum_{i=1}^d (q_i - 1)a_i$.
2. $N'(A) = \hat{q}N'(B) + \frac{1}{2}\left(\sum_{i=1}^d (q_i - 1)a_i - \hat{q} + 1\right)$.

3. $N(A) = \hat{q}N(B) + \frac{1}{2}\left(\sum_{i=1}^{d}(q_i - 1)a_i - \hat{q} + 1\right)$.

REMARK 3.8 In formula 1) if we take $q_1 = .. = q_{d-1} = 1$ then we obtain a Brauer and Shockley formula [5] and if we take $q_i = \gcd(A\backslash\{a_i\})$ for all i, we obtain a Raczunas and Chrzastowski-Wachtel formula [9]. Moreover formula 2) is a generalization of a Rödseth formula [10] which is obtained for $q_1 = .. = q_{d-1} = 1$.

THEOREM 3.9 *The following conditions hold*

1. $S(A)$ *is symmetric if and only if* $S(B)$ *is symmetric.*
2. *If* $\hat{q} > 1$ *then* $S(A)$ *is not pseudo-symmetric.*

COROLLARY 3.10 *Suppose there exists* i *such that* $b_i = 1$ (i.e. $a_i = \hat{q}_i$). *Then,* $S(A)$ *is symmetric* and we have

1. (a) $g(A) = \sum_{i=1}^{d}(q_i - 1)a_i - \hat{q}$.
   (b) $N(A) = N'(A) = \frac{1}{2}(\sum_{i=1}^{d}(q_i - 1)a_i - \hat{q} + 1)$.
2. *Suppose, in addition, that* $b_i = 1$ (i.e. $a_i = \hat{q}_i$) *for all* i. *Then, we have*
   (a) $g(A) = l(d-1) - \sum_{i=1}^{d} a_i$.
   (b) $N(A) = N'(A) = \frac{1}{2}(l(d-1) - \sum_{i=1}^{d} a_i + 1)$.

PROOF. Since $1 \in B$, we have $S(B) = \mathbb{N}_0$ then $g(B) = -1$ and $N(B) = N'(B) = 0$. So 1. follows from theorem 3.7. To prove 2., we observe that $q_i a_i = \hat{q} = l = \mathrm{lcm}(A)$ if $a_i = \hat{q}_i$ for all i□

COROLLARY 3.11 *Let* b, d, h, v *be positive integers such that* $b \geq d \geq 2$ *and* $\gcd(b, v) = 1$. *Let* $B = \{b, hb + v, .., hb + (i-1)v, .., hb + (d-1)v\}$, *(*$(b_1, .., b_d)$ *is called an "almost" arithmetic sequence). Then,*
$S(A)$ *is symmetric* $\Leftrightarrow S(B)$ *is symmetric* $\Leftrightarrow d = 2$ or $b \equiv 2 \bmod(\mathrm{d}-1)$.

PROOF. We write $b - 1 = \beta(d-1) + \alpha$ with $0 \leq \alpha < d - 1$, and we use the following known formulas $g(B) = \left(h\left\lfloor\frac{b-2}{d-1}\right\rfloor + h - 1\right)b + bv - v$ [8] and $N'(B) = \frac{1}{2}\{(b-1)(h\beta + v + h - 1) + h\alpha(\beta + 1)\}$ [11] □

EXAMPLE 3.12 Let $A = \{150, 462, 840, 1365\} = \{5(2\times3\times5), 11(2\times3\times7), 12(2\times5\times7), 13(3\times5\times7)\}$. We set $q_1 = 7, q_2 = 5, q_3 = 3, q_4 = 2$ and $B = \{5, 11, 12, 13\}$. This is an almost arithmetic sequence with $b = 5, v = 1, h = 2, d = 4$. We see that $b \equiv 2 \bmod(\mathrm{d}-1)$ hence $S(B)$ is symmetric and we have $g(B) = 19, N'(B) = N(B) = 10$. Moreover, it follows from theorem 3.9 that $S(A)$ is symmetric. Using theorem 3.7 we obtain $g(A) = 210\times19 + 6\times150 + 4\times462 + 2\times840 + 1365 = 9783$. $N'(A) = N(A) = 210\times10 + \frac{1}{2}(6\times150 + 4\times462 + 2\times840 + 1365 - 210 + 1) = 4892$.

4 QUASI-POLYNOMIALS

DEFINITION 4.1 *A quasi-polynomial* P *of period* p *and degree* d *is a sequence* $P = (P_0, \ldots, P_{p-1})$ *with* $P_r \in \mathbb{Q}[t]$ *such that* $d = \sup\{\deg(P_r) \mid 0 \leq r \leq p-1\}$.
A quasi-polynomial P is said to be *uniform* if all the P_r have the same degree d

and the same leading coefficient $c(P)$. Given a function $h : \mathbb{Z} \to \mathbb{Q}$ and $r \in \mathbb{Z}$, we define $h_r : \mathbb{Z} \to \mathbb{Q}$, $k \mapsto h(pk + r)$. We say that h is a quasi-polynomial function if there exists a quasi-polynomial $P = (P_0, \ldots, P_{p-1})$ such that $h_r(k) = P_r(k)$ for all $k \gg 0$ and $0 \leq r \leq p$. We also say that h is P-quasi-polynomial. It is easily seen that a quasi-polynomial function h has a minimal period and every period of h is a multiple of this minimal period. Furthermore, for a fixed period p, h is a P-quasi-polynomial for a unique sequence $P = (P_0, \ldots, P_{p-1})$. A P-quasi-polynomial h is said to be uniform if P is uniform. We write $\deg(h) = \deg(P)$ and $c(h) = c(P)$. We denote by $F(\mathbb{Z})$ the set of all functions $h : \mathbb{Z} \to \mathbb{Q}$. For every integer $i \geq 0$ we consider the operators E^i and Δ_i, which act as follows: $(E^i h)(n) = h(n + i)$, $(\Delta_i h)(n) = h(n + i) - h(n)$. We set $E^0 = I$, $E^1 = E$ and $\Delta_1 = \Delta$ so $\Delta = E - I$, $\Delta_0 = 0$ and $\Delta_i = E^i - I$. For $a \geq 0$ and $n \geq 1$, we have $(I + E^a + \cdots + E^{(n-1)a}) \circ (E^a - I) = E^{na} - I = \Delta_{na}$.

LEMMA 4.2 *Given $h \in F(\mathbb{Z})$, then the following identities hold*

1. $(E^{pi} h)_r = E^i h_r$ *for* $i \geq 0$.
2. $(\Delta_p^m h)_r = \Delta^m h_r$ *for* $m \geq 0$.

PROOF. 1. We write $(E^{pi} h)_r(k) = (E^{pi} h)(pk + r) = h(p(k + i) + r) = h_r(k + i) = (E^i h_r)(k)$.
2. We have $\Delta_p^m = (E^p - I)^m = \sum_{i=0}^m (-1)^{m-i} \binom{m}{i} E^{pi}$. Therefore, $(\Delta_p^m h)_r = \sum_{i=0}^m (-1)^{m-i} \binom{m}{i} (E^{pi} h)_r = \sum_{i=0}^m (-1)^{m-i} \binom{m}{i} E^i h_r = (E - I)^m h_r = \Delta^m h_r$ □

PROPOSITION 4.3 *A function $h \in F(\mathbb{Z})$ is quasi-polynomial of period p and degree d if and only if there exists $(c_0, \ldots, c_{p-1}) \neq (0, \ldots, 0)$ such that $(\Delta_p^d h)_r(k) = c_r$ for all $k \gg 0$ and $0 \leq r \leq p - 1$.*

PROOF. Follows from lemma 4.2 and [6, lemma 4.1.2] □

COROLLARY 4.4 *For $h \in F(\mathbb{Z})$, if $\prod_{i=1}^d (E^{a_i} - I)(h)(n) = 0$ for $n \gg 0$, then h is quasi-polynomial of period $p \in \mathbb{N}$ and degree $< d$.*

PROOF. Follows from $\Delta_p^d = (E^p - I)^d = (\prod_{i=1}^d (\sum_{j=0}^{\frac{p}{a_i}-1} E^{ja_i})) \circ (\prod_{i=1}^d (E^{a_i} - I)$ □

EXAMPLE 4.5 Given $m \in \mathbb{Z}$ and $Q(t) \in \mathbb{Q}[t, t^{-1}]$ such that $Q(1) \neq 0$. The function $H_m(Q, .)$ associated with Q is a P-quasi-polynomial of period p, where $P = (P_0, .., P_{p-1})$ is given by $P_r = V_{m_r}(U_r, .)$.

REMARK 4.6 Suppose $m > 0$. Then, we have

1. $\deg(H_m(Q, .)) = m - 1$.
2. $m_r > 0 \Rightarrow \deg(P_r) = m_r - 1$ and $c(P_r) = \frac{U_r(1)}{(m_r - 1)!}$.
3. If $Q(1) = pQ_r(1) \neq 0$ for all $0 \leq r \leq p-1$, then $H_m(Q, .)$ is uniform of degree $m - 1$ and its leading coefficient is $c(H_m(Q, .)) = \frac{Q_r(1)}{(d-1)!} = \frac{Q(1)}{p(d-1)!}$.