Andrei D. Polyanin
Alexander V. Manzhirov

HANDBOOK OF INTEGRAL EQUATIONS

SECOND EDITION

Chapman & Hall/CRC
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HANDBOOK OF INTEGRAL EQUATIONS

SECOND EDITION

Handbooks of Mathematical Equations

Handbook of Linear Partial Differential Equations for Engineers and Scientists
A. D. Polyanin, 2002

Handbook of First Order Partial Differential Equations
A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux, 2002

Handbook of Exact Solutions for Ordinary Differential Equations, 2nd Edition
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Handbook of Nonlinear Partial Differential Equations
A. D. Polyanin and V. F. Zaitsev, 2004

Handbook of Integral Equations, 2nd Edition
A. D. Polyanin and A. V. Manzhirov, 2008

See also:

Handbook of Mathematics for Engineers and Scientists
A. D. Polyanin and A. V. Manzhirov, 2007

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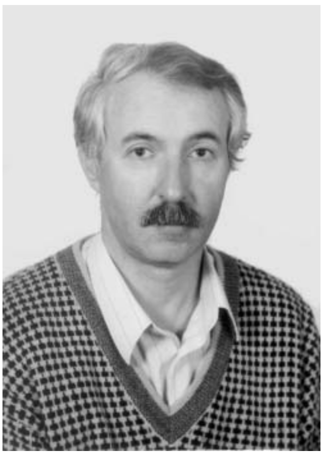

Andrei D. Polyanin, D.Sc., Ph.D., is a well-known scientist of broad interests and is active in various areas of mathematics, mechanics, and chemical engineering sciences. He is one of the most prominent authors in the field of reference literature on mathematics and physics.

Professor Polyanin graduated with honors from the Department of Mechanics and Mathematics of Moscow State University in 1974. He received his Ph.D. degree in 1981 and D.Sc. degree in 1986 at the Institute for Problems in Mechanics of the Russian (former USSR) Academy of Sciences. Since 1975, Professor Polyanin has been working at the Institute for Problems in Mechanics of the Russian Academy of Sciences; he is also Professor of Mathematics at Bauman Moscow State Technical University. He is a member of the Russian National Committee on Theoretical and Applied Mechanics and of the Mathematics and Mechanics Expert Council of the Higher Certification Committee of the Russian Federation.

Professor Polyanin has made important contributions to exact and approximate analytical methods in the theory of differential equations, mathematical physics, integral equations, engineering mathematics, theory of heat and mass transfer, and chemical hydrodynamics. He obtained exact solutions for several thousand ordinary differential, partial differential, and integral equations.

Professor Polyanin is an author of more than 30 books in English, Russian, German, and Bulgarian as well as over 120 research papers and three patents. He has written a number of fundamental handbooks, including A. D. Polyanin and V. F. Zaitsev, *Handbook of Exact Solutions for Ordinary Differential Equations*, CRC Press, 1995 and 2003; A. D. Polyanin and A. V. Manzhirov, *Handbook of Integral Equations*, CRC Press, 1998; A. D. Polyanin, *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, Chapman & Hall/CRC Press, 2002; A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux, *Handbook of First Order Partial Differential Equations*, Taylor & Francis, 2002; A. D. Polyanin and V. F. Zaitsev, *Handbook of Nonlinear Partial Differential Equations*, Chapman & Hall/CRC Press, 2004, and A. D. Polyanin and A. V. Manzhirov, *Handbook of Mathematics for Engineers and Scientists*, Chapman & Hall/CRC Press, 2007.

Professor Polyanin is editor of the book series *Differential and Integral Equations and Their Applications*, Chapman & Hall/CRC Press, London/Boca Raton, and *Physical and Mathematical Reference Literature*, Fizmatlit, Moscow. He is also Editor-in-Chief of the international scientific-educational Website *EqWorld—The World of Mathematical Equations* (http://eqworld.ipmnet.ru), which is visited by over 1700 users a day worldwide. Professor Polyanin is a member of the Editorial Board of the journal *Theoretical Foundations of Chemical Engineering.*

In 1991, Professor Polyanin was awarded a Chaplygin Prize of the Russian Academy of Sciences for his research in mechanics. In 2001, he received an award from the Ministry of Education of the Russian Federation.

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Alexander V. Manzhirov, D.Sc., Ph.D., is a noted scientist in the fields of mechanics and applied mathematics, integral equations, and their applications.

After graduating with honors from the Department of Mechanics and Mathematics of Rostov State University in 1979, Alexander Manzhirov attended postgraduate courses at Moscow Institute of Civil Engineering. He received his Ph.D. degree in 1983 at Moscow Institute of Electronic Engineering Industry and D.Sc. degree in 1993 at the Institute for Problems in Mechanics of the Russian (former USSR) Academy of Sciences. Since 1983, Alexander Manzhirov has been working at the Institute for Problems in Mechanics of the Russian Academy of Sciences. Currently, he is head of the Laboratory for Modeling in Solid Mechanics at the same institute.

Professor Manzhirov is also head of a branch of the Department of Applied Mathematics at Bauman Moscow State Technical University, professor of mathematics at Moscow State University of Engineering and Computer Science, vice-chairman of Mathematics and Mechanics Expert Council of the Higher Certification Committee of the Russian Federation, executive secretary of Solid Mechanics Scientific Council of the Russian Academy of Sciences, and expert in mathematics, mechanics, and computer science of the Russian Foundation for Basic Research. He is a member of the Russian National Committee on Theoretical and Applied Mechanics and the European Mechanics Society (EUROMECH), and member of the editorial board of the journal *Mechanics of Solids* and the international scientific-educational Website *EqWorld—The World of Mathematical Equations* (http://eqworld.ipmnet.ru).

Professor Manzhirov has made important contributions to new mathematical methods for solving problems in the fields of integral equations and their applications, mechanics of growing solids, contact mechanics, tribology, viscoelasticity, and creep theory. He is an author of more than ten books (including *Contact Problems in Mechanics of Growing Solids* [in Russian], Nauka, Moscow, 1991; *Handbook of Integral Equations*, CRC Press, Boca Raton, 1998; *Handbuch der Integralgleichungen: Exacte Lösungen*, Spektrum Akad. Verlag, Heidelberg, 1999; *Contact Problems in the Theory of Creep* [in Russian], National Academy of Sciences of Armenia, Erevan, 1999; A. D. Polyanin and A. V. Manzhirov, *Handbook of Mathematics for Engineers and Scientists*, Chapman & Hall/CRC Press, Boca Raton, 2007), more than 70 research papers, and two patents.

Professor Manzhirov is a winner of the First Competition of the Science Support Foundation 2001, Moscow.

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PREFACE TO THE NEW EDITION

Handbook of Integral Equations, Second Edition, a unique reference for engineers and scientists, contains over 2,500 integral equations with solutions, as well as analytical and numerical methods for solving linear and nonlinear equations. It considers Volterra, Fredholm, Wiener–Hopf, Hammerstein, Urysohn, and other equations, which arise in mathematics, physics, engineering sciences, economics, etc. In total, the number of equations described is an order of magnitude greater than in any other book available.

The second edition has been substantially updated, revised, and extended. It includes new chapters on mixed multidimensional equations, methods of integral equations for ODEs and PDEs, and about 400 new equations with exact solutions. It presents a considerable amount of new material on Volterra, Fredholm, singular, hypersingular, dual, and nonlinear integral equations, integral transforms, and special functions. Many examples were added for illustrative purposes. The new edition has been increased by a total of over 300 pages.

Note that the first part of the book can be used as a database of test problems for numerical and approximate methods for solving linear and nonlinear integral equations.

We would like to express our deep gratitude to Alexei Zhurov and Vasilii Silvestrov for fruitful discussions. We also appreciate the help of Grigory Yosifian in translating new sections of this book and valuable remarks.

The authors hope that the handbook will prove helpful for a wide audience of researchers, college and university teachers, engineers, and students in various fields of applied mathematics, mechanics, physics, chemistry, biology, economics, and engineering sciences.

A. D. Polyanin
A. V. Manzhirov

PREFACE TO THE FIRST EDITION

Integral equations are encountered in various fields of science and numerous applications (in elasticity, plasticity, heat and mass transfer, oscillation theory, fluid dynamics, filtration theory, electrostatics, electrodynamics, biomechanics, game theory, control, queuing theory, electrical engineering, economics, medicine, etc.).

Exact (closed-form) solutions of integral equations play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science. Lots of equations of physics, chemistry, and biology contain functions or parameters which are obtained from experiments and hence are not strictly fixed. Therefore, it is expedient to choose the structure of these functions so that it would be easier to analyze and solve the equation. As a possible selection criterion, one may adopt the requirement that the model integral equation admits a solution in a closed form. Exact solutions can be used to verify the consistency and estimate errors of various numerical, asymptotic, and approximate methods.

More than 2,100 integral equations and their solutions are given in the first part of the book (Chapters 1–6). A lot of new exact solutions to linear and nonlinear equations are included. Special attention is paid to equations of general form, which depend on arbitrary functions. The other equations contain one or more free parameters (the book actually deals with families of integral equations); it is the reader's option to fix these parameters. In total, the number of equations described in this handbook is an order of magnitude greater than in any other book currently available.

The second part of the book (Chapters 7–14) presents exact, approximate analytical, and numerical methods for solving linear and nonlinear integral equations. Apart from the classical methods, some new methods are also described. When selecting the material, the authors have given a pronounced preference to practical aspects of the matter; that is, to methods that allow effectively "constructing" the solution. For the reader's better understanding of the methods, each section is supplied with examples of specific equations. Some sections may be used by lecturers of colleges and universities as a basis for courses on integral equations and mathematical physics equations for graduate and postgraduate students.

For the convenience of a wide audience with different mathematical backgrounds, the authors tried to do their best, wherever possible, to avoid special terminology. Therefore, some of the methods are outlined in a schematic and somewhat simplified manner, with necessary references made to books where these methods are considered in more detail. For some nonlinear equations, only solutions of the simplest form are given. The book does not cover two-, three-, and multidimensional integral equations.

The handbook consists of chapters, sections, and subsections. Equations and formulas are numbered separately in each section. The equations within a section are arranged in increasing order of complexity. The extensive table of contents provides rapid access to the desired equations.

For the reader's convenience, the main material is followed by a number of supplements, where some properties of elementary and special functions are described, tables of indefinite and definite integrals are given, as well as tables of Laplace, Mellin, and other transforms, which are used in the book.

The first and second parts of the book, just as many sections, were written so that they could be read independently from each other. This allows the reader to quickly get to the heart of the matter.

We would like to express our deep gratitude to Rolf Sulanke and Alexei Zhurov for fruitful discussions and valuable remarks. We also appreciate the help of Vladimir Nazaikinskii and Alexander Shtern in translating the second part of this book, and are thankful to Inna Shingareva for her assistance in preparing the camera-ready copy of the book.

The authors hope that the handbook will prove helpful for a wide audience of researchers, college and university teachers, engineers, and students in various fields of mathematics, mechanics, physics, chemistry, biology, economics, and engineering sciences.

A. D. Polyanin
A. V. Manzhirov

SOME REMARKS AND NOTATION

1. In Chapters 1–11, 14, and 18 in the original integral equations, the independent variable is denoted by x, the integration variable by t, and the unknown function by $y = y(x)$.

2. For a function of one variable $f = f(x)$, we use the following notation for the derivatives:

$$f'_x = \frac{df}{dx}, \quad f''_{xx} = \frac{d^2 f}{dx^2}, \quad f'''_{xxx} = \frac{d^3 f}{dx^3}, \quad f''''_{xxxx} = \frac{d^4 f}{dx^4}, \quad \text{and} \quad f^{(n)}_x = \frac{d^n f}{dx^n} \ \text{ for } \ n \geq 5.$$

Occasionally, we use the similar notation for partial derivatives of a function of two variables, for example, $K'_x(x,t) = \dfrac{\partial}{\partial x} K(x,t)$.

3. In some cases, we use the operator notation $\left[f(x)\dfrac{d}{dx}\right]^n g(x)$, which is defined recursively by

$$\left[f(x)\frac{d}{dx}\right]^n g(x) = f(x)\frac{d}{dx}\left\{\left[f(x)\frac{d}{dx}\right]^{n-1} g(x)\right\}.$$

4. It is indicated in the beginning of Chapters 1–8 that $f = f(x)$, $g = g(x)$, $K = K(x)$, etc. are arbitrary functions, and A, B, etc. are free parameters. This means that:

(a) $f = f(x)$, $g = g(x)$, $K = K(x)$, etc. are assumed to be continuous real-valued functions of real arguments;*

(b) if the solution contains derivatives of these functions, then the functions are assumed to be sufficiently differentiable;**

(c) if the solution contains integrals with these functions (in combination with other functions), then the integrals are supposed to converge;

(d) the free parameters A, B, etc. may assume any real values for which the expressions occurring in the equation and the solution make sense (for example, if a solution contains a factor $\dfrac{A}{1-A}$, then it is implied that $A \neq 1$; as a rule, this is not specified in the text).

5. The notations $\operatorname{Re} z$ and $\operatorname{Im} z$ stand, respectively, for the real and the imaginary part of a complex quantity z.

6. In the first part of the book (Chapters 1–8) when referencing a particular equation, we use a notation like 2.3.15, which implies equation 15 from Section 2.3.

7. To highlight portions of the text, the following symbols are used in the book:

► indicates important information pertaining to a group of equations (Chapters 1–8);

⦿ indicates the literature used in the preparation of the text in specific equations (Chapters 1–8) or sections (Chapters 9–18).

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* Less severe restrictions on these functions are presented in the second part of the book.

** Restrictions (b) and (c) imposed on $f = f(x)$, $g = g(x)$, $K = K(x)$, etc. are not mentioned in the text.

Part I

Exact Solutions of Integral Equations

Chapter 1

Linear Equations of the First Kind with Variable Limit of Integration

► ***Notation:*** *$f = f(x)$, $g = g(x)$, $h = h(x)$, $K = K(x)$, and $M = M(x)$ are arbitrary functions (these may be composite functions of the argument depending on two variables x and t); A, B, C, D, E, a, b, c, α, β, γ, λ, and μ are free parameters; and m and n are nonnegative integers.*

► **Preliminary remarks.** For equations of the form

$$\int_a^x K(x,t)y(t)\,dt = f(x), \qquad a \le x \le b,$$

where the functions $K(x,t)$ and $f(x)$ are continuous, the right-hand side must satisfy the following conditions:

1°. If $K(a,a) \neq 0$, then we must have $f(a) = 0$ (for example, the right-hand sides of equations 1.1.1 and 1.2.1 must satisfy this condition).

2°. If $K(a,a) = K'_x(a,a) = \cdots = K_x^{(n-1)}(a,a) = 0$, $0 < \left|K_x^{(n)}(a,a)\right| < \infty$, then the right-hand side of the equation must satisfy the conditions

$$f(a) = f'_x(a) = \cdots = f_x^{(n)}(a) = 0.$$

For example, with $n = 1$, these are constraints for the right-hand side of equation 1.1.2.

3°. If $K(a,a) = K'_x(a,a) = \cdots = K_x^{(n-1)}(a,a) = 0$, $K_x^{(n)}(a,a) = \infty$, then the right-hand side of the equation must satisfy the conditions

$$f(a) = f'_x(a) = \cdots = f_x^{(n-1)}(a) = 0.$$

For example, with $n = 1$, this is a constraint for the right-hand side of equation 1.1.30.

4°. For unbounded $K(x,t)$ with integrable power-law or logarithmic singularity at $x = t$ and continuous $f(x)$, no additional conditions are imposed on the right-hand side of the integral equation (e.g., see Abel's equation 1.1.36).

In the case of a difference kernel, $K(x,t) = K(x-t)$, that can be represented as $x \to t$ in the form

$$K(x-t) = A(x-t)^\lambda + o\big((x-t)^\lambda\big) \qquad (0 < |A| < \infty),$$

the right-hand side of the integral equation, for $\lambda \ge 0$, must satisfy the conditions

$$f(a) = f'_x(a) = \cdots = f_x^{([\lambda])}(a) = 0,$$

where $[\lambda]$ is the integer part of λ. For $-1 < \lambda < 0$, there are no additional conditions imposed on the function $f(x)$.

In Chapter 1, conditions 1°–3° are as a rule not specified.

1.1. Equations Whose Kernels Contain Power-Law Functions

1.1-1. Kernels Linear in the Arguments x and t.

1. $\displaystyle \int_a^x y(t)\,dt = f(x).$

Solution: $y(x) = f'_x(x)$.

2. $\displaystyle \int_a^x (x-t)y(t)\,dt = f(x).$

Solution: $y(x) = f''_{xx}(x)$.

3. $\displaystyle \int_a^x (Ax+Bt+C)y(t)\,dt = f(x).$

This is a special case of equation 1.9.5 with $g(x) = x$.

1°. Solution with $B \neq -A$:

$$y(x) = \frac{d}{dx}\left\{\bigl[(A+B)x+C\bigr]^{-\frac{A}{A+B}} \int_a^x \bigl[(A+B)t+C\bigr]^{-\frac{B}{A+B}} f'_t(t)\,dt\right\}.$$

2°. Solution with $B = -A$:

$$y(x) = \frac{1}{C}\frac{d}{dx}\left[\exp\Bigl(-\frac{A}{C}x\Bigr)\int_a^x \exp\Bigl(\frac{A}{C}t\Bigr) f'_t(t)\,dt\right].$$

1.1-2. Kernels Quadratic in the Arguments x and t.

4. $\displaystyle \int_a^x (x-t)^2 y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = f''_{xx}(a) = 0.$

Solution: $y(x) = \frac{1}{2} f'''_{xxx}(x)$.

5. $\displaystyle \int_a^x (x^2-t^2)y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

This is a special case of equation 1.9.2 with $g(x) = x^2$.

Solution: $\displaystyle y(x) = \frac{1}{2x^2}\bigl[x f''_{xx}(x) - f'_x(x)\bigr].$

6. $\displaystyle \int_a^x \bigl(Ax^2+Bt^2\bigr)y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = x^2$. For $B = -A$, see equation 1.1.5.

Solution: $\displaystyle y(x) = \frac{1}{A+B}\frac{d}{dx}\left[x^{-\frac{2A}{A+B}}\int_a^x t^{-\frac{2B}{A+B}} f'_t(t)\,dt\right].$

7. $\displaystyle \int_a^x \bigl(Ax^2+Bt^2+C\bigr)y(t)\,dt = f(x).$

This is a special case of equation 1.9.5 with $g(x) = x^2$.

Solution:

$$y(x) = \operatorname{sign}\varphi(x)\,\frac{d}{dx}\left\{|\varphi(x)|^{-\frac{A}{A+B}}\int_a^x |\varphi(t)|^{-\frac{B}{A+B}} f'_t(t)\,dt\right\}, \qquad \varphi(x) = (A+B)x^2 + C.$$

8. $\displaystyle\int_a^x \big[Ax^2 + (B-A)xt - Bt^2\big]y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Differentiating with respect to x yields an equation of the form 1.1.3:

$$\int_a^x [2Ax + (B-A)t]y(t)\,dt = f'_x(x).$$

Solution:

$$y(x) = \frac{1}{A+B}\frac{d}{dx}\left[x^{-\frac{2A}{A+B}}\int_a^x t^{\frac{A-B}{A+B}} f''_{tt}(t)\,dt\right].$$

9. $\displaystyle\int_a^x \big(Ax^2 + Bt^2 + Cx + Dt + E\big)y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = Ax^2 + Cx$ and $h(t) = Bt^2 + Dt + E$.

10. $\displaystyle\int_a^x \big(Axt + Bt^2 + Cx + Dt + E\big)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = x$, $h_1(t) = At + C$, $g_2(x) = 1$, and $h_2(t) = Bt^2 + Dt + E$.

11. $\displaystyle\int_a^x \big(Ax^2 + Bxt + Cx + Dt + E\big)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Bx + D$, $h_1(t) = t$, $g_2(x) = Ax^2 + Cx + E$, and $h_2(t) = 1$.

1.1-3. Kernels Cubic in the Arguments x and t.

12. $\displaystyle\int_a^x (x-t)^3 y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = f''_{xx}(a) = f'''_{xxx}(a) = 0.$

Solution: $y(x) = \frac{1}{6} f''''_{xxxx}(x)$.

13. $\displaystyle\int_a^x (x^3 - t^3)y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

This is a special case of equation 1.9.2 with $g(x) = x^3$.

Solution: $y(x) = \dfrac{1}{3x^3}\big[xf'''_{xxx}(x) - 2f'_x(x)\big]$.

14. $\displaystyle\int_a^x \big(Ax^3 + Bt^3\big)y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = x^3$. For $B = -A$, see equation 1.1.13.

Solution with $0 \le a \le x$: $y(x) = \dfrac{1}{A+B}\dfrac{d}{dx}\left[x^{-\frac{3A}{A+B}}\displaystyle\int_a^x t^{-\frac{3B}{A+B}} f'_t(t)\,dt\right]$.

15. $\displaystyle\int_a^x \big(Ax^3 + Bt^3 + C\big)y(t)\,dt = f(x).$

This is a special case of equation 1.9.5 with $g(x) = x^3$.

16. $\displaystyle\int_a^x (x^2t - xt^2)y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

This is a special case of equation 1.9.11 with $g(x) = x^2$ and $h(x) = x$.

Solution: $\displaystyle y(x) = \frac{1}{x}\frac{d^2}{dx^2}\left[\frac{1}{x}f(x)\right].$

17. $\displaystyle\int_a^x (Ax^2t + Bxt^2)y(t)\,dt = f(x).$

This is a special case of equation 1.9.12 with $g(x) = x^2$ and $h(x) = x$. For $B = -A$, see equation 1.1.16.

Solution:

$$y(x) = \frac{1}{(A+B)x}\frac{d}{dx}\left\{x^{-\frac{A}{A+B}}\int_a^x t^{-\frac{B}{A+B}}\frac{d}{dt}\left[\frac{1}{t}f(t)\right]dt\right\}.$$

18. $\displaystyle\int_a^x (Ax^3 + Bxt^2)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^3$, $h_1(t) = 1$, $g_2(x) = Bx$, and $h_2(t) = t^2$.

19. $\displaystyle\int_a^x (Ax^3 + Bx^2t)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^3$, $h_1(t) = 1$, $g_2(x) = Bx^2$, and $h_2(t) = t$.

20. $\displaystyle\int_a^x (Ax^2t + Bt^3)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^2$, $h_1(t) = t$, $g_2(x) = B$, and $h_2(t) = t^3$.

21. $\displaystyle\int_a^x (Axt^2 + Bt^3)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax$, $h_1(t) = t^2$, $g_2(x) = B$, and $h_2(t) = t^3$.

22. $\displaystyle\int_a^x \big(A_3x^3 + B_3t^3 + A_2x^2 + B_2t^2 + A_1x + B_1t + C\big)y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A_3x^3 + A_2x^2 + A_1x + C$ and $h(t) = B_3t^3 + B_2t^2 + B_1t$.

1.1-4. Kernels Containing Higher-Order Polynomials in x and t.

23. $\displaystyle\int_a^x (x - t)^n y(t)\,dt = f(x), \qquad n = 1, 2, \ldots$

It is assumed that the right-hand of the equation satisfies the conditions $f(a) = f'_x(a) = \cdots = f_x^{(n)}(a) = 0$.

Solution: $\displaystyle y(x) = \frac{1}{n!}f_x^{(n+1)}(x).$

Example. For $f(x) = Ax^m$, where m is a positive integer, $m > n$, the solution has the form

$$y(x) = \frac{Am!}{n!\,(m-n-1)!}x^{m-n-1}.$$

24. $\displaystyle\int_a^x (x^n - t^n)y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0, \quad n = 1, 2, \dots$

Solution: $\displaystyle y(x) = \frac{1}{n}\frac{d}{dx}\left[\frac{f'_x(x)}{x^{n-1}}\right].$

25. $\displaystyle\int_a^x \bigl(t^n x^{n+1} - x^n t^{n+1}\bigr)y(t)\,dt = f(x), \qquad n = 2, 3, \dots$

This is a special case of equation 1.9.11 with $g(x) = x^{n+1}$ and $h(x) = x^n$.

Solution: $\displaystyle y(x) = \frac{1}{x^n}\frac{d^2}{dx^2}\left[\frac{f(x)}{x^n}\right].$

1.1-5. Kernels Containing Rational Functions.

26. $\displaystyle\int_0^x \frac{y(t)\,dt}{x+t} = f(x).$

1°. For a polynomial right-hand side, $f(x) = \sum_{n=0}^{N} A_n x^n$, the solution has the form

$$y(x) = \sum_{n=0}^{N} \frac{A_n}{B_n} x^n, \qquad B_n = (-1)^n\left[\ln 2 + \sum_{k=1}^{n} \frac{(-1)^k}{k}\right].$$

2°. For $f(x) = x^\lambda \sum_{n=0}^{N} A_n x^n$, where λ is an arbitrary number ($\lambda > -1$), the solution has the form

$$y(x) = x^\lambda \sum_{n=0}^{N} \frac{A_n}{B_n} x^n, \qquad B_n = \int_0^1 \frac{t^{\lambda+n}\,dt}{1+t}.$$

3°. For $f(x) = \ln x \left(\sum_{n=0}^{N} A_n x^n\right)$, the solution has the form

$$y(x) = \ln x \sum_{n=0}^{N} \frac{A_n}{B_n} x^n + \sum_{n=0}^{N} \frac{A_n I_n}{B_n^2} x^n,$$

$$B_n = (-1)^n\left[\ln 2 + \sum_{k=1}^{n} \frac{(-1)^k}{k}\right], \quad I_n = (-1)^n\left[\frac{\pi^2}{12} + \sum_{k=1}^{n} \frac{(-1)^k}{k^2}\right].$$

4°. For $f(x) = \sum_{n=0}^{N} A_n \bigl(\ln x\bigr)^n$, the solution of the equation has the form

$$y(x) = \sum_{n=0}^{N} A_n Y_n(x),$$

where the functions $Y_n = Y_n(x)$ are given by

$$Y_n(x) = \left\{\frac{d^n}{d\lambda^n}\left[\frac{x^\lambda}{I(\lambda)}\right]\right\}_{\lambda=0}, \qquad I(\lambda) = \int_0^1 \frac{z^\lambda\,dz}{1+z}.$$

5°. For $f(x) = \sum_{n=1}^{N} A_n \cos(\lambda_n \ln x) + \sum_{n=1}^{N} B_n \sin(\lambda_n \ln x)$, the solution of the equation has the form

$$y(x) = \sum_{n=1}^{N} C_n \cos(\lambda_n \ln x) + \sum_{n=1}^{N} D_n \sin(\lambda_n \ln x),$$

where the constants C_n and D_n are found by the method of undetermined coefficients.

6°. For arbitrary $f(x)$, the transformation

$$x = \tfrac{1}{2}e^{2z}, \quad t = \tfrac{1}{2}e^{2\tau}, \quad y(t) = e^{-\tau} w(\tau), \quad f(x) = e^{-z} g(z)$$

leads to an integral equation with difference kernel of the form 1.9.27:

$$\int_{-\infty}^{z} \frac{w(\tau)\,d\tau}{\cosh(z-\tau)} = g(z).$$

27. $\displaystyle\int_0^x \frac{y(t)\,dt}{ax+bt} = f(x), \qquad a > 0, \quad a+b > 0.$

1°. For a polynomial right-hand side, $f(x) = \sum_{n=0}^{N} A_n x^n$, the solution has the form

$$y(x) = \sum_{n=0}^{N} \frac{A_n}{B_n} x^n, \qquad B_n = \int_0^1 \frac{t^n\,dt}{a+bt}.$$

2°. For $f(x) = x^\lambda \sum_{n=0}^{N} A_n x^n$, where λ is an arbitrary number ($\lambda > -1$), the solution has the form

$$y(x) = x^\lambda \sum_{n=0}^{N} \frac{A_n}{B_n} x^n, \qquad B_n = \int_0^1 \frac{t^{\lambda+n}\,dt}{a+bt}.$$

3°. For $f(x) = \ln x \left(\sum_{n=0}^{N} A_n x^n \right)$, the solution has the form

$$y(x) = \ln x \sum_{n=0}^{N} \frac{A_n}{B_n} x^n - \sum_{n=0}^{N} \frac{A_n C_n}{B_n^2} x^n, \qquad B_n = \int_0^1 \frac{t^n\,dt}{a+bt}, \quad C_n = \int_0^1 \frac{t^n \ln t}{a+bt}\,dt.$$

4°. For some other special forms of the right-hand side (see items 4 and 5, equation 1.1.26), the solution may be found by the method of undetermined coefficients.

28. $\displaystyle\int_0^x \frac{y(t)\,dt}{ax^2+bt^2} = f(x), \qquad a > 0, \quad a+b > 0.$

1°. For a polynomial right-hand side, $f(x) = \sum_{n=0}^{N} A_n x^n$, the solution has the form

$$y(x) = \sum_{n=0}^{N} \frac{A_n}{B_n} x^{n+1}, \qquad B_n = \int_0^1 \frac{t^{n+1}\,dt}{a+bt^2}.$$

Example. For $a = b = 1$ and $f(x) = Ax^2 + Bx + C$, the solution of the integral equation is:

$$y(x) = \frac{2A}{1 - \ln 2} x^3 + \frac{4B}{4 - \pi} x^2 + \frac{2C}{\ln 2} x.$$

2°. For $f(x) = x^\lambda \sum_{n=0}^{N} A_n x^n$, where λ is an arbitrary number ($\lambda > -1$), the solution has the form

$$y(x) = x^\lambda \sum_{n=0}^{N} \frac{A_n}{B_n} x^{n+1}, \qquad B_n = \int_0^1 \frac{t^{\lambda+n+1}\,dt}{a + bt^2}.$$

3°. For $f(x) = \ln x \left(\sum_{n=0}^{N} A_n x^n \right)$, the solution has the form

$$y(x) = \ln x \sum_{n=0}^{N} \frac{A_n}{B_n} x^{n+1} - \sum_{n=0}^{N} \frac{A_n C_n}{B_n^2} x^{n+1}, \qquad B_n = \int_0^1 \frac{t^{n+1}\,dt}{a + bt^2}, \quad C_n = \int_0^1 \frac{t^{n+1} \ln t}{a + bt^2}\,dt.$$

29. $\displaystyle \int_0^x \frac{y(t)\,dt}{ax^m + bt^m} = f(x), \qquad a > 0, \quad a + b > 0, \quad m = 1, 2, \ldots$

1°. For a polynomial right-hand side, $f(x) = \sum_{n=0}^{N} A_n x^n$, the solution has the form

$$y(x) = \sum_{n=0}^{N} \frac{A_n}{B_n} x^{m+n-1}, \qquad B_n = \int_0^1 \frac{t^{m+n-1}\,dt}{a + bt^m}.$$

2°. For $f(x) = x^\lambda \sum_{n=0}^{N} A_n x^n$, where λ is an arbitrary number ($\lambda > -1$), the solution has the form

$$y(x) = x^\lambda \sum_{n=0}^{N} \frac{A_n}{B_n} x^{m+n-1}, \qquad B_n = \int_0^1 \frac{t^{\lambda+m+n-1}\,dt}{a + bt^m}.$$

3°. For $f(x) = \ln x \left(\sum_{n=0}^{N} A_n x^n \right)$, the solution has the form

$$y(x) = \ln x \sum_{n=0}^{N} \frac{A_n}{B_n} x^{m+n-1} - \sum_{n=0}^{N} \frac{A_n C_n}{B_n^2} x^{m+n-1},$$

$$B_n = \int_0^1 \frac{t^{m+n-1}\,dt}{a + bt^m}, \quad C_n = \int_0^1 \frac{t^{m+n-1} \ln t}{a + bt^m}\,dt.$$

1.1-6. Kernels Containing Square Roots.

30. $\displaystyle \int_a^x \sqrt{x - t}\, y(t)\,dt = f(x).$

Differentiating with respect to x, we arrive at Abel's equation 1.1.36:

$$\int_a^x \frac{y(t)\,dt}{\sqrt{x - t}} = 2f'_x(x).$$

Solution:

$$y(x) = \frac{2}{\pi} \frac{d^2}{dx^2} \int_a^x \frac{f(t)\,dt}{\sqrt{x - t}}.$$

31. $\displaystyle\int_a^x \big(\sqrt{x}-\sqrt{t}\,\big)y(t)\,dt = f(x).$

This is a special case of equation 1.1.45 with $\mu = \frac{1}{2}$.

Solution: $y(x) = 2\dfrac{d}{dx}\big[\sqrt{x}\, f'_x(x)\big]$.

32. $\displaystyle\int_a^x \big(A\sqrt{x}+B\sqrt{t}\,\big)y(t)\,dt = f(x).$

This is a special case of equation 1.1.46 with $\mu = \frac{1}{2}$.

33. $\displaystyle\int_a^x \big(1+b\sqrt{x-t}\,\big)y(t)\,dt = f(x).$

Differentiating with respect to x, we arrive at Abel's equation of the second kind 2.1.46:

$$y(x) + \frac{b}{2}\int_a^x \frac{y(t)\,dt}{\sqrt{x-t}} = f'_x(x).$$

34. $\displaystyle\int_a^x \big(t\sqrt{x}-x\sqrt{t}\,\big)y(t)\,dt = f(x).$

This is a special case of equation 1.9.11 with $g(x)=\sqrt{x}$ and $h(x)=x$.

35. $\displaystyle\int_a^x \big(At\sqrt{x}+Bx\sqrt{t}\,\big)y(t)\,dt = f(x).$

This is a special case of equation 1.9.12 with $g(x)=\sqrt{x}$ and $h(t)=t$.

36. $\displaystyle\int_a^x \frac{y(t)\,dt}{\sqrt{x-t}} = f(x).$

Abel's equation.

Solution:

$$y(x) = \frac{1}{\pi}\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{\sqrt{x-t}} = \frac{f(a)}{\pi\sqrt{x-a}} + \frac{1}{\pi}\int_a^x \frac{f'_t(t)\,dt}{\sqrt{x-t}}.$$

⊙ Reference: E. T. Whittaker and G. N. Watson (1958).

37. $\displaystyle\int_a^x \left(b+\frac{1}{\sqrt{x-t}}\right)y(t)\,dt = f(x).$

Let us rewrite the equation in the form

$$\int_a^x \frac{y(t)\,dt}{\sqrt{x-t}} = f(x) - b\int_a^x y(t)\,dt.$$

Assuming the right-hand side to be known, we solve this equation as Abel's equation 1.1.36. After some manipulations, we arrive at Abel's equation of the second kind 2.1.46:

$$y(x) + \frac{b}{\pi}\int_a^x \frac{y(t)\,dt}{\sqrt{x-t}} = F(x), \qquad \text{where} \quad F(x) = \frac{1}{\pi}\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{\sqrt{x-t}}.$$

38. $\displaystyle \int_a^x \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{t}}\right) y(t)\,dt = f(x).$

This is a special case of equation 1.1.45 with $\mu = -\frac{1}{2}$.

Solution: $y(x) = -2\left[x^{3/2} f'_x(x)\right]'_x, \qquad a > 0.$

39. $\displaystyle \int_a^x \left(\frac{A}{\sqrt{x}} + \frac{B}{\sqrt{t}}\right) y(t)\,dt = f(x).$

This is a special case of equation 1.1.46 with $\mu = -\frac{1}{2}$.

40. $\displaystyle \int_{-x}^x \sqrt{\frac{x-t}{x+t}}\, y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{\operatorname{sign} x}{2\pi}\left[\frac{d}{dx}\int_0^{|x|} \frac{f(t)-f(-t)}{\sqrt{x^2-t^2}}\,dt - \frac{1}{x}\frac{d}{dx}\int_0^{|x|} \frac{t[f(t)-f(-t)]}{\sqrt{x^2-t^2}}\,dt\right].$$

⦿ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992).

41. $\displaystyle \int_a^x \frac{y(t)\,dt}{\sqrt{x^2-t^2}} = f(x).$

Solution: $\displaystyle y = \frac{2}{\pi}\frac{d}{dx}\int_a^x \frac{tf(t)\,dt}{\sqrt{x^2-t^2}}.$

⦿ Reference: P. P. Zabreyko, A. I. Koshelev, et al. (1975).

42. $\displaystyle \int_0^x \frac{y(t)\,dt}{\sqrt{ax^2+bt^2}} = f(x), \qquad a > 0, \quad a+b > 0.$

1°. For a polynomial right-hand side, $f(x) = \sum\limits_{n=0}^{N} A_n x^n$, the solution has the form

$$y(x) = \sum_{n=0}^{N} \frac{A_n}{B_n} x^n, \qquad B_n = \int_0^1 \frac{t^n\,dt}{\sqrt{a+bt^2}}.$$

2°. For $f(x) = x^\lambda \sum\limits_{n=0}^{N} A_n x^n$, where λ is an arbitrary number ($\lambda > -1$), the solution has the form

$$y(x) = x^\lambda \sum_{n=0}^{N} \frac{A_n}{B_n} x^n, \qquad B_n = \int_0^1 \frac{t^{\lambda+n}\,dt}{\sqrt{a+bt^2}}.$$

3°. For $f(x) = \ln x \left(\sum\limits_{n=0}^{N} A_n x^n\right)$, the solution has the form

$$y(x) = \ln x \sum_{n=0}^{N} \frac{A_n}{B_n} x^n - \sum_{n=0}^{N} \frac{A_n C_n}{B_n^2} x^n, \qquad B_n = \int_0^1 \frac{t^n\,dt}{\sqrt{a+bt^2}}, \quad C_n = \int_0^1 \frac{t^n \ln t}{\sqrt{a+bt^2}}\,dt.$$

4°. For $f(x) = \sum\limits_{n=0}^{N} A_n (\ln x)^n$, the solution of the equation has the form

$$y(x) = \sum_{n=0}^{N} A_n Y_n(x),$$

where the functions $Y_n = Y_n(x)$ are given by

$$Y_n(x) = \left\{ \frac{d^n}{d\lambda^n} \left[\frac{x^\lambda}{I(\lambda)} \right] \right\}_{\lambda=0}, \qquad I(\lambda) = \int_0^1 \frac{z^\lambda \, dz}{\sqrt{a + bz^2}}.$$

5°. For $f(x) = \sum\limits_{n=1}^{N} A_n \cos(\lambda_n \ln x) + \sum\limits_{n=1}^{N} B_n \sin(\lambda_n \ln x)$, the solution of the equation has the form

$$y(x) = \sum_{n=1}^{N} C_n \cos(\lambda_n \ln x) + \sum_{n=1}^{N} D_n \sin(\lambda_n \ln x),$$

where the constants C_n and D_n are found by the method of undetermined coefficients.

1.1-7. Kernels Containing Arbitrary Powers.

43. $\displaystyle\int_a^x (x-t)^\lambda y(t)\,dt = f(x), \qquad f(a) = 0, \quad 0 < \lambda < 1.$

Differentiating with respect to x, we arrive at the generalized Abel equation 1.1.47:

$$\int_a^x \frac{y(t)\,dt}{(x-t)^{1-\lambda}} = \frac{1}{\lambda} f'_x(x).$$

Solution:

$$y(x) = k \frac{d^2}{dx^2} \int_a^x \frac{f(t)\,dt}{(x-t)^\lambda}, \qquad k = \frac{\sin(\pi\lambda)}{\pi\lambda}.$$

⊙ Reference: F. D. Gakhov (1977).

44. $\displaystyle\int_a^x (x-t)^\mu y(t)\,dt = f(x).$

For $\mu = 0, 1, 2, \dots$, see equations 1.1.1, 1.1.2, 1.1.4, 1.1.12, and 1.1.23. For $0 < \mu < 1$, see equation 1.1.43.

Set $\mu = n - \lambda$, where $n = 1, 2, \dots$ and $0 \le \lambda < 1$, and $f(a) = f'_x(a) = \cdots = f_x^{(n-1)}(a) = 0$. On differentiating the equation n times, we arrive at an equation of the form 1.1.47:

$$\int_a^x \frac{y(t)\,d\tau}{(x-t)^\lambda} = \frac{\Gamma(\mu - n + 1)}{\Gamma(\mu+1)} f_x^{(n)}(x),$$

where $\Gamma(\mu)$ is the gamma function.

Example. Set $f(x) = Ax^\beta$, where $\beta \ge 0$, and let $\mu > -1$ and $\mu - \beta \ne 0, 1, 2, \dots$ In this case, the solution has the form $y(x) = \dfrac{A\,\Gamma(\beta+1)}{\Gamma(\mu+1)\,\Gamma(\beta-\mu)} x^{\beta-\mu-1}$.

⊙ Reference: M. L. Krasnov, A. I. Kisilev, and G. I. Makarenko (1971).

45. $\displaystyle\int_a^x (x^\mu - t^\mu) y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = x^\mu$.

Solution: $y(x) = \dfrac{1}{\mu} \left[x^{1-\mu} f'_x(x) \right]'_x$.

46. $\displaystyle\int_a^x \big(Ax^\mu + Bt^\mu\big)y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = x^\mu$. For $B = -A$, see equation 1.1.44.

Solution: $\displaystyle y(x) = \frac{1}{A+B}\frac{d}{dx}\left[x^{-\frac{A\mu}{A+B}}\int_a^x t^{-\frac{B\mu}{A+B}} f_t'(t)\,dt\right].$

47. $\displaystyle\int_a^x \frac{y(t)\,dt}{(x-t)^\lambda} = f(x), \qquad 0 < \lambda < 1.$

The generalized Abel equation.

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi}\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{(x-t)^{1-\lambda}} = \frac{\sin(\pi\lambda)}{\pi}\left[\frac{f(a)}{(x-a)^{1-\lambda}} + \int_a^x \frac{f_t'(t)\,dt}{(x-t)^{1-\lambda}}\right].$$

⊙ Reference: E. T. Whittaker and G. N. Watson (1958).

48. $\displaystyle\int_a^x \left[b + \frac{1}{(x-t)^\lambda}\right]y(t)\,dt = f(x), \qquad 0 < \lambda < 1.$

Rewrite the equation in the form

$$\int_a^x \frac{y(t)\,dt}{(x-t)^\lambda} = f(x) - b\int_a^x y(t)\,dt.$$

Assuming the right-hand side to be known, we solve this equation as the generalized Abel equation 1.1.47. After some manipulations, we arrive at Abel's equation of the second kind 2.1.60:

$$y(x) + \frac{b\sin(\pi\lambda)}{\pi}\int_a^x \frac{y(t)\,dt}{(x-t)^{1-\lambda}} = F(x), \qquad \text{where} \quad F(x) = \frac{\sin(\pi\lambda)}{\pi}\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{(x-t)^{1-\lambda}}.$$

49. $\displaystyle\int_a^x \big(\sqrt{x} - \sqrt{t}\,\big)^\lambda y(t)\,dt = f(x), \qquad 0 < \lambda < 1.$

Solution:

$$y(x) = \frac{k}{\sqrt{x}}\left(\sqrt{x}\frac{d}{dx}\right)^2 \int_a^x \frac{f(t)\,dt}{\sqrt{t}\,\big(\sqrt{x} - \sqrt{t}\,\big)^\lambda}, \qquad k = \frac{\sin(\pi\lambda)}{\pi\lambda}.$$

50. $\displaystyle\int_a^x \frac{y(t)\,dt}{\big(\sqrt{x} - \sqrt{t}\,\big)^\lambda} = f(x), \qquad 0 < \lambda < 1.$

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{2\pi}\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{\sqrt{t}\,\big(\sqrt{x} - \sqrt{t}\,\big)^{1-\lambda}}.$$

51. $\displaystyle\int_a^x \big(Ax^\lambda + Bt^\mu\big)y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = Ax^\lambda$ and $h(t) = Bt^\mu$.

52. $\displaystyle\int_a^x \big[1 + A(x^\lambda t^\mu - x^{\lambda+\mu})\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.13 with $g(x) = Ax^\mu$ and $h(x) = x^\lambda$.

Solution:

$$y(x) = \frac{d}{dx}\left\{\frac{x^\lambda}{\Phi(x)}\int_a^x \big[t^{-\lambda} f(t)\big]'_t \Phi(t)\,dt\right\}, \qquad \Phi(x) = \exp\Big(-\frac{A\mu}{\mu+\lambda}x^{\mu+\lambda}\Big).$$

53. $\displaystyle\int_a^x \big(Ax^\beta t^\gamma + Bx^\delta t^\lambda\big) y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\beta$, $h_1(t) = t^\gamma$, $g_2(x) = Bx^\delta$, and $h_2(t) = t^\lambda$.

54. $\displaystyle\int_a^x \big[Ax^\lambda(t^\mu - x^\mu) + Bx^\beta(t^\gamma - x^\gamma)\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.47 with $g_1(x) = Ax^\lambda$, $h_1(x) = x^\mu$, $g_2(x) = Bx^\beta$, and $h_2(x) = x^\gamma$.

55. $\displaystyle\int_a^x \big[Ax^\lambda t^\mu + Bx^{\lambda+\beta} t^{\mu-\beta} - (A+B)x^{\lambda+\gamma} t^{\mu-\gamma}\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.49 with $g(x) = x$.

56. $\displaystyle\int_a^x t^\sigma (x^\mu - t^\mu)^\lambda y(t)\,dt = f(x), \qquad \sigma > -1, \quad \mu > 0, \quad \lambda > -1.$

The transformation $\tau = t^\mu$, $z = x^\mu$, $w(\tau) = t^{\sigma-\mu+1} y(t)$ leads to an equation of the form 1.1.43:

$$\int_A^z (z-\tau)^\lambda w(\tau)\,d\tau = F(z),$$

where $A = a^\mu$ and $F(z) = \mu f(z^{1/\mu})$.

Solution with $-1 < \lambda < 0$:

$$y(x) = -\frac{\mu\sin(\pi\lambda)}{\pi x^\sigma}\frac{d}{dx}\left[\int_a^x t^{\mu-1}(x^\mu - t^\mu)^{-1-\lambda} f(t)\,dt\right].$$

57. $\displaystyle\int_0^x \frac{y(t)\,dt}{(x+t)^\mu} = f(x).$

This is a special case of equation 1.1.58 with $\lambda = 1$ and $a = b = 1$.

The transformation

$$x = \tfrac{1}{2}e^{2z}, \quad t = \tfrac{1}{2}e^{2\tau}, \quad y(t) = e^{(\mu-2)\tau} w(\tau), \quad f(x) = e^{-\mu z} g(z)$$

leads to an equation with difference kernel of the form 1.9.27:

$$\int_{-\infty}^z \frac{w(\tau)\,d\tau}{\cosh^\mu(z-\tau)} = g(z).$$

58. $\displaystyle \int_0^x \frac{y(t)\,dt}{(ax^\lambda + bt^\lambda)^\mu} = f(x), \qquad a > 0, \quad a + b > 0.$

1°. The substitution $t = xz$ leads to a special case of equation 3.8.45:

$$\int_0^1 \frac{y(xz)\,dz}{(a + bz^\lambda)^\mu} = x^{\lambda\mu-1} f(x). \tag{1}$$

2°. For a polynomial right-hand side, $f(x) = \sum\limits_{m=0}^{n} A_m x^m$, the solution has the form

$$y(x) = x^{\lambda\mu-1} \sum_{m=0}^{n} \frac{A_m}{I_m} x^m, \qquad I_m = \int_0^1 \frac{z^{m+\lambda\mu-1}\,dz}{(a + bz^\lambda)^\mu}.$$

The integrals I_m are supposed to be convergent.

3°. The solution structure for some other right-hand sides of the integral equation may be obtained using (1) and the results presented for the more general equation 3.8.53 (see also equations 3.8.34–3.8.40).

4°. For $a = b$, the equation can be reduced, just as equation 1.1.57, to an integral equation with difference kernel of the form 1.9.27.

59. $\displaystyle \int_a^x \frac{\left(\sqrt{x} + \sqrt{x-t}\right)^{2\lambda} + \left(\sqrt{x} - \sqrt{x-t}\right)^{2\lambda}}{2t^\lambda \sqrt{x-t}}\, y(t)\,dt = f(x).$

The equation can be rewritten in terms of the Gaussian hypergeometric functions in the form

$$\int_a^x (x-t)^{\gamma-1} F\Big(\lambda, -\lambda, \gamma; 1 - \frac{x}{t}\Big) y(t)\,dt = f(x), \qquad \text{where} \quad \gamma = \tfrac{1}{2}.$$

See 1.8.135 for the solution of this equation.

1.1-8. Two-Dimensional Equation of the Abel Type.

60. $\displaystyle \iint_\Delta \frac{u(x,y)\,dx\,dy}{\sqrt{(y_0 - y)^2 - (x_0 - x)^2}} = f(x_0, y_0).$

Here Δ is an isosceles right triangle with apex at the point (x_0, y_0) and base on the x-axis.

Solution:

$$u(x_0, y_0) = \frac{1}{2\pi^2}\left(\frac{\partial^2 g}{\partial x_0^2} - \frac{\partial^2 g}{\partial y_0^2}\right), \qquad g(x_0, y_0) = \iint_\Delta \frac{f(x,y)\,dx\,dy}{\sqrt{(y_0 - y)^2 - (x_0 - x)^2}}.$$

⊙ Reference: P. P. Zabreyko, A. I. Koshelev, et al. (1975).

1.2. Equations Whose Kernels Contain Exponential Functions

1.2-1. Kernels Containing Exponential Functions.

1. $\displaystyle \int_a^x e^{\lambda(x-t)} y(t)\,dt = f(x).$

Solution: $y(x) = f'_x(x) - \lambda f(x)$.

Example. In the special case $a = 0$ and $f(x) = Ax$, the solution has the form $y(x) = A(1 - \lambda x)$.

2. $\displaystyle\int_a^x e^{\lambda x+\beta t} y(t)\,dt = f(x).$

Solution: $y(x) = e^{-(\lambda+\beta)x}\big[f'_x(x) - \lambda f(x)\big].$

Example. In the special case $a = 0$ and $f(x) = A\sin(\gamma x)$, the solution has the form $y(x) = Ae^{-(\lambda+\beta)x} \times [\gamma\cos(\gamma x) - \lambda\sin(\gamma x)]$.

3. $\displaystyle\int_a^x \big[e^{\lambda(x-t)} - 1\big] y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Solution: $y(x) = \frac{1}{\lambda} f''_{xx}(x) - f'_x(x).$

4. $\displaystyle\int_a^x \big[e^{\lambda(x-t)} + b\big] y(t)\,dt = f(x).$

For $b = -1$, see equation 1.2.3. Differentiating with respect to x yields an equation of the form 2.2.1:

$$y(x) + \frac{\lambda}{b+1}\int_a^x e^{\lambda(x-t)} y(t)\,dt = \frac{f'_x(x)}{b+1}.$$

Solution:

$$y(x) = \frac{f'_x(x)}{b+1} - \frac{\lambda}{(b+1)^2}\int_a^x \exp\left[\frac{\lambda b}{b+1}(x-t)\right] f'_t(t)\,dt.$$

5. $\displaystyle\int_a^x \big(e^{\lambda x+\beta t} + b\big) y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = e^{\lambda x}$, $h_1(t) = e^{\beta t}$, $g_2(x) = 1$, and $h_2(t) = b$. For $\beta = -\lambda$, see equation 1.2.4.

6. $\displaystyle\int_a^x \big(e^{\lambda x} - e^{\lambda t}\big) y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

This is a special case of equation 1.9.2 with $g(x) = e^{\lambda x}$.

Solution: $y(x) = e^{-\lambda x}\left[\dfrac{1}{\lambda} f''_{xx}(x) - f'_x(x)\right].$

7. $\displaystyle\int_a^x \big(e^{\lambda x} - e^{\lambda t} + b\big) y(t)\,dt = f(x).$

This is a special case of equation 1.9.3 with $g(x) = e^{\lambda x}$. For $b = 0$, see equation 1.2.6.

Solution:

$$y(x) = \frac{1}{b} f'_x(x) - \frac{\lambda}{b^2} e^{\lambda x}\int_a^x \exp\left(\frac{e^{\lambda t} - e^{\lambda x}}{b}\right) f'_t(t)\,dt.$$

8. $\displaystyle\int_a^x \big(Ae^{\lambda x} + Be^{\lambda t}\big) y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = e^{\lambda x}$. For $B = -A$, see equation 1.2.6.

Solution: $y(x) = \dfrac{1}{A+B}\dfrac{d}{dx}\left[\exp\Big(-\dfrac{A\lambda}{A+B}x\Big)\displaystyle\int_a^x \exp\Big(-\dfrac{B\lambda}{A+B}t\Big) f'_t(t)\,dt\right].$

9. $\displaystyle\int_a^x \big(Ae^{\lambda x} + Be^{\lambda t} + C\big) y(t)\,dt = f(x).$

This is a special case of equation 1.9.5 with $g(x) = e^{\lambda x}$.

10. $\displaystyle\int_a^x \big(Ae^{\lambda x} + Be^{\mu t}\big)y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = Ae^{\lambda x}$ and $h(t) = Be^{\mu t}$. For $\lambda = \mu$, see equation 1.2.8.

11. $\displaystyle\int_a^x \big[e^{\lambda(x-t)} - e^{\mu(x-t)}\big]y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Solution:

$$y(x) = \frac{1}{\lambda-\mu}\big[f''_{xx} - (\lambda+\mu)f'_x + \lambda\mu f\big], \qquad f = f(x).$$

12. $\displaystyle\int_a^x \big[Ae^{\lambda(x-t)} + Be^{\mu(x-t)}\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\lambda x}$, $h_1(t) = e^{-\lambda t}$, $g_2(x) = Be^{\mu x}$, and $h_2(t) = e^{-\mu t}$. For $B = -A$, see equation 1.2.11.

Solution:

$$y(x) = \frac{e^{\lambda x}}{A+B}\frac{d}{dx}\left\{e^{(\mu-\lambda)x}\Phi(x)\int_a^x \left[\frac{f(t)}{e^{\mu t}}\right]'_t \frac{dt}{\Phi(t)}\right\}, \qquad \Phi(x) = \exp\left[\frac{B(\lambda-\mu)}{A+B}x\right].$$

13. $\displaystyle\int_a^x \big[Ae^{\lambda(x-t)} + Be^{\mu(x-t)} + C\big]y(t)\,dt = f(x).$

This is a special case of equation 1.2.14 with $\beta = 0$.

14. $\displaystyle\int_a^x \big[Ae^{\lambda(x-t)} + Be^{\mu(x-t)} + Ce^{\beta(x-t)}\big]y(t)\,dt = f(x).$

Differentiating the equation with respect to x yields

$$(A+B+C)y(x) + \int_a^x \big[A\lambda e^{\lambda(x-t)} + B\mu e^{\mu(x-t)} + C\beta e^{\beta(x-t)}\big]y(t)\,dt = f'_x(x).$$

Eliminating the term with $e^{\beta(x-t)}$ with the aid of the original equation, we arrive at an equation of the form 2.2.10:

$$(A+B+C)y(x) + \int_a^x \big[A(\lambda-\beta)e^{\lambda(x-t)} + B(\mu-\beta)e^{\mu(x-t)}\big]y(t)\,dt = f'_x(x) - \beta f(x).$$

In the special case $A + B + C = 0$, this is an equation of the form 1.2.12.

15. $\displaystyle\int_a^x \big[Ae^{\lambda(x-t)} + Be^{\mu(x-t)} + Ce^{\beta(x-t)} - A - B - C\big]y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Differentiating with respect to x, we arrive at an equation of the form 1.2.14:

$$\int_a^x \big[A\lambda e^{\lambda(x-t)} + B\mu e^{\mu(x-t)} + C\beta e^{\beta(x-t)}\big]y(t)\,dt = f'_x(x).$$

16. $\displaystyle\int_a^x \big(e^{\lambda x+\mu t} - e^{\mu x+\lambda t}\big)y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

This is a special case of equation 1.9.11 with $g(x) = e^{\lambda x}$ and $h(t) = e^{\mu t}$.

Solution:

$$y(x) = \frac{f''_{xx} - (\lambda+\mu)f'_x(x) + \lambda\mu f(x)}{(\lambda-\mu)\exp[(\lambda+\mu)x]}.$$

17. $\displaystyle\int_a^x \bigl(Ae^{\lambda x+\mu t} + Be^{\mu x+\lambda t}\bigr)y(t)\,dt = f(x).$

This is a special case of equation 1.9.12 with $g(x) = e^{\lambda x}$ and $h(t) = e^{\mu t}$. For $B = -A$, see equation 1.2.16.

Solution:

$$y(x) = \frac{1}{(A+B)e^{\mu x}}\frac{d}{dx}\left\{\Phi^A(x)\int_a^x \Phi^B(t)\frac{d}{dt}\left[\frac{f(t)}{e^{\mu t}}\right]dt\right\}, \qquad \Phi(x) = \exp\Bigl(\frac{\mu-\lambda}{A+B}x\Bigr).$$

18. $\displaystyle\int_a^x \bigl(Ae^{\lambda x+\mu t} + Be^{\beta x+\gamma t}\bigr)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\lambda x}$, $h_1(t) = e^{\mu t}$, $g_2(x) = Be^{\beta x}$, and $h_2(t) = e^{\gamma t}$.

19. $\displaystyle\int_a^x \bigl(Ae^{2\lambda x} + Be^{2\beta t} + Ce^{\lambda x} + De^{\beta t} + E\bigr)y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = Ae^{2\lambda x} + Ce^{\lambda x}$ and $h(t) = Be^{2\beta t} + De^{\beta t} + E$.

20. $\displaystyle\int_a^x \bigl(Ae^{\lambda x+\beta t} + Be^{2\beta t} + Ce^{\lambda x} + De^{\beta t} + E\bigr)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = e^{\lambda x}$, $h_1(t) = Ae^{\beta t} + C$, and $g_2(x) = 1$, $h_2(t) = Be^{2\beta t} + De^{\beta t} + E$.

21. $\displaystyle\int_a^x \bigl(Ae^{2\lambda x} + Be^{\lambda x+\beta t} + Ce^{\lambda x} + De^{\beta t} + E\bigr)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Be^{\lambda x} + D$, $h_1(t) = e^{\beta t}$, and $g_2(x) = Ae^{2\lambda x} + Ce^{\lambda x} + E$, $h_2(t) = 1$.

22. $\displaystyle\int_a^x \bigl[1 + Ae^{\lambda x}(e^{\mu t} - e^{\mu x})\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.13 with $g(x) = e^{\mu x}$ and $h(x) = Ae^{\lambda x}$.

Solution:

$$y(x) = \frac{d}{dx}\left\{e^{\lambda x}\Phi(x)\int_a^x \left[\frac{f(t)}{e^{\lambda t}}\right]_t' \frac{dt}{\Phi(t)}\right\}, \qquad \Phi(x) = \exp\left[\frac{A\mu}{\lambda+\mu}e^{(\lambda+\mu)x}\right].$$

23. $\displaystyle\int_a^x \bigl[Ae^{\lambda x}(e^{\mu x} - e^{\mu t}) + Be^{\beta x}(e^{\gamma x} - e^{\gamma t})\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.47 with $g_1(x) = Ae^{\lambda x}$, $h_1(t) = -e^{\mu t}$, $g_2(x) = Be^{\beta x}$, and $h_2(t) = -e^{\gamma t}$.

24. $\displaystyle\int_a^x \bigl\{A\exp(\lambda x + \mu t) + B\exp[(\lambda+\beta)x + (\mu-\beta)t]$

$\qquad\qquad - (A+B)\exp[(\lambda+\gamma)x + (\mu-\gamma)t]\bigr\}y(t)\,dt = f(x).$

This is a special case of equation 1.9.49 with $g_1(x) = e^x$.

25. $\displaystyle \int_a^x \left(e^{\lambda x} - e^{\lambda t}\right)^n y(t)\,dt = f(x), \qquad n = 1, 2, \ldots$

Solution:

$$y(x) = \frac{1}{\lambda^n n!} e^{\lambda x} \left(\frac{1}{e^{\lambda x}} \frac{d}{dx}\right)^{n+1} f(x).$$

26. $\displaystyle \int_a^x \sqrt{e^{\lambda x} - e^{\lambda t}}\, y(t)\,dt = f(x), \qquad \lambda > 0.$

Solution:

$$y(x) = \frac{2}{\pi} e^{\lambda x} \left(e^{-\lambda x} \frac{d}{dx}\right)^2 \int_a^x \frac{e^{\lambda t} f(t)\,dt}{\sqrt{e^{\lambda x} - e^{\lambda t}}}.$$

27. $\displaystyle \int_a^x \frac{y(t)\,dt}{\sqrt{e^{\lambda x} - e^{\lambda t}}} = f(x), \qquad \lambda > 0.$

Solution:

$$y(x) = \frac{\lambda}{\pi} \frac{d}{dx} \int_a^x \frac{e^{\lambda t} f(t)\,dt}{\sqrt{e^{\lambda x} - e^{\lambda t}}}.$$

28. $\displaystyle \int_a^x (e^{\lambda x} - e^{\lambda t})^\mu y(t)\,dt = f(x), \qquad \lambda > 0, \quad 0 < \mu < 1.$

Solution:

$$y(x) = k e^{\lambda x} \left(e^{-\lambda x} \frac{d}{dx}\right)^2 \int_a^x \frac{e^{\lambda t} f(t)\,dt}{(e^{\lambda x} - e^{\lambda t})^\mu}, \qquad k = \frac{\sin(\pi\mu)}{\pi\mu}.$$

29. $\displaystyle \int_a^x \frac{y(t)\,dt}{(e^{\lambda x} - e^{\lambda t})^\mu} = f(x), \qquad \lambda > 0, \quad 0 < \mu < 1.$

Solution:

$$y(x) = \frac{\lambda \sin(\pi\mu)}{\pi} \frac{d}{dx} \int_a^x \frac{e^{\lambda t} f(t)\,dt}{(e^{\lambda x} - e^{\lambda t})^{1-\mu}}.$$

1.2-2. Kernels Containing Power-Law and Exponential Functions.

30. $\displaystyle \int_a^x \left[A(x - t) + B e^{\lambda(x-t)}\right] y(t)\,dt = f(x).$

Differentiating with respect to x, we arrive at an equation of the form 2.2.4:

$$B y(x) + \int_a^x \left[A + B\lambda e^{\lambda(x-t)}\right] y(t)\,dt = f'_x(x).$$

31. $\displaystyle \int_a^x (x - t) e^{\lambda(x-t)} y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Solution: $y(x) = f''_{xx}(x) - 2\lambda f'_x(x) + \lambda^2 f(x)$.

32. $\displaystyle \int_a^x (Ax + Bt + C) e^{\lambda(x-t)} y(t)\,dt = f(x).$

The substitution $u(x) = e^{-\lambda x} y(x)$ leads to an equation of the form 1.1.3:

$$\int_a^x (Ax + Bt + C) u(t)\,dt = e^{-\lambda x} f(x).$$

33. $\displaystyle\int_a^x (Axe^{\lambda t} + Bte^{\mu x})y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax$, $h_1(t) = e^{\lambda t}$, and $g_2(x) = Be^{\mu x}$, $h_2(t) = t$.

34. $\displaystyle\int_a^x \bigl[Axe^{\lambda(x-t)} + Bte^{\mu(x-t)}\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Axe^{\lambda x}$, $h_1(t) = e^{-\lambda t}$, $g_2(x) = Be^{\mu x}$, and $h_2(t) = te^{-\mu t}$.

35. $\displaystyle\int_a^x (x-t)^2 e^{\lambda(x-t)}y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = f''_{xx}(a) = 0.$

Solution: $y(x) = \frac{1}{2}\bigl[f'''_{xxx}(x) - 3\lambda f''_{xx}(x) + 3\lambda^2 f'_x(x) - \lambda^3 f(x)\bigr]$.

36. $\displaystyle\int_a^x (x-t)^n e^{\lambda(x-t)}y(t)\,dt = f(x), \qquad n = 1, 2, \ldots$

It is assumed that $f(a) = f'_x(a) = \cdots = f^{(n)}_x(a) = 0$.

Solution: $y(x) = \dfrac{1}{n!}e^{\lambda x}\dfrac{d^{n+1}}{dx^{n+1}}\bigl[e^{-\lambda x}f(x)\bigr]$.

37. $\displaystyle\int_a^x (Ax^\beta + Be^{\lambda t})y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = Ax^\beta$ and $h(t) = Be^{\lambda t}$.

38. $\displaystyle\int_a^x (Ae^{\lambda x} + Bt^\beta)y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = Ae^{\lambda x}$ and $h(t) = Bt^\beta$.

39. $\displaystyle\int_a^x (Ax^\beta e^{\lambda t} + Bt^\gamma e^{\mu x})y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\beta$, $h_1(t) = e^{\lambda t}$, $g_2(x) = Be^{\mu x}$, and $h_2(t) = t^\gamma$.

40. $\displaystyle\int_a^x e^{\lambda(x-t)}\sqrt{x-t}\,y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi}e^{\lambda x}\frac{d^2}{dx^2}\int_a^x \frac{e^{-\lambda t}f(t)\,dt}{\sqrt{x-t}}.$$

41. $\displaystyle\int_a^x \frac{e^{\lambda(x-t)}}{\sqrt{x-t}}y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{1}{\pi}e^{\lambda x}\frac{d}{dx}\int_a^x \frac{e^{-\lambda t}f(t)\,dt}{\sqrt{x-t}}.$$

42. $\displaystyle\int_a^x (x-t)^\lambda e^{\mu(x-t)} y(t)\,dt = f(x), \qquad 0<\lambda<1.$

Solution:

$$y(x) = ke^{\mu x}\frac{d^2}{dx^2}\int_a^x \frac{e^{-\mu t} f(t)\,dt}{(x-t)^\lambda}, \qquad k = \frac{\sin(\pi\lambda)}{\pi\lambda}.$$

43. $\displaystyle\int_a^x \frac{e^{\lambda(x-t)}}{(x-t)^\mu} y(t)\,dt = f(x), \qquad 0<\mu<1.$

Solution:

$$y(x) = \frac{\sin(\pi\mu)}{\pi} e^{\lambda x}\frac{d}{dx}\int_a^x \frac{e^{-\lambda t} f(t)}{(x-t)^{1-\mu}}\,dt.$$

44. $\displaystyle\int_a^x \bigl(\sqrt{x}-\sqrt{t}\bigr)^\lambda e^{\mu(x-t)} y(t)\,dt = f(x), \qquad 0<\lambda<1.$

The substitution $u(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.1.49:

$$\int_a^x \bigl(\sqrt{x}-\sqrt{t}\bigr)^\lambda u(t)\,dt = e^{-\mu x} f(x).$$

45. $\displaystyle\int_a^x \frac{e^{\mu(x-t)} y(t)\,dt}{\bigl(\sqrt{x}-\sqrt{t}\bigr)^\lambda} = f(x), \qquad 0<\lambda<1.$

The substitution $u(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.1.50:

$$\int_a^x \frac{u(t)\,dt}{(\sqrt{x}-\sqrt{t})^\lambda} = e^{-\mu x} f(x).$$

46. $\displaystyle\int_a^x \frac{e^{\lambda(x-t)}}{\sqrt{x^2-t^2}} y(t)\,dt = f(x).$

Solution: $\displaystyle y = \frac{2}{\pi} e^{\lambda x}\frac{d}{dx}\int_a^x \frac{te^{-\lambda t}}{\sqrt{x^2-t^2}} f(t)\,dt.$

47. $\displaystyle\int_a^x \exp[\lambda(x^2-t^2)] y(t)\,dt = f(x).$

Solution: $y(x) = f'_x(x) - 2\lambda x f(x)$.

48. $\displaystyle\int_a^x [\exp(\lambda x^2) - \exp(\lambda t^2)] y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \exp(\lambda x^2)$.

Solution: $\displaystyle y(x) = \frac{1}{2\lambda}\frac{d}{dx}\left[\frac{f'_x(x)}{x\exp(\lambda x^2)}\right].$

49. $\displaystyle\int_a^x \bigl[A\exp(\lambda x^2) + B\exp(\lambda t^2) + C\bigr] y(t)\,dt = f(x).$

This is a special case of equation 1.9.5 with $g(x) = \exp(\lambda x^2)$.

50. $\displaystyle\int_a^x \bigl[A\exp(\lambda x^2) + B\exp(\mu t^2)\bigr] y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\exp(\lambda x^2)$ and $h(t) = B\exp(\mu t^2)$.

51. $\displaystyle\int_a^x \sqrt{x-t}\,\exp[\lambda(x^2-t^2)]y(t)\,dt = f(x).$

Solution:
$$y(x) = \frac{2}{\pi}\exp(\lambda x^2)\frac{d^2}{dx^2}\int_a^x \frac{\exp(-\lambda t^2)}{\sqrt{x-t}} f(t)\,dt.$$

52. $\displaystyle\int_a^x \frac{\exp[\lambda(x^2-t^2)]}{\sqrt{x-t}}y(t)\,dt = f(x).$

Solution:
$$y(x) = \frac{1}{\pi}\exp(\lambda x^2)\frac{d}{dx}\int_a^x \frac{\exp(-\lambda t^2)}{\sqrt{x-t}} f(t)\,dt.$$

53. $\displaystyle\int_a^x (x-t)^\lambda \exp[\mu(x^2-t^2)]y(t)\,dt = f(x), \qquad 0<\lambda<1.$

Solution:
$$y(x) = k\exp(\mu x^2)\frac{d^2}{dx^2}\int_a^x \frac{\exp(-\mu t^2)}{(x-t)^\lambda} f(t)\,dt, \qquad k = \frac{\sin(\pi\lambda)}{\pi\lambda}.$$

54. $\displaystyle\int_a^x \exp[\lambda(x^\beta-t^\beta)]y(t)\,dt = f(x).$

Solution: $y(x) = f'_x(x) - \lambda\beta x^{\beta-1}f(x).$

55. $\displaystyle\int_0^x (-1)^{[(x-t)/b]}y(t)\,dt = f(x), \qquad f(0) = f'_x(0) = 0.$

Here $b = \text{const}$ and $[A]$ stands for the integer part of the number A.

Solution:
$$y(x) = \frac{1}{2}\int_0^x \left(2\left[\frac{x-t}{b}\right]+1\right) f''_{tt}(t)\,dt.$$

⊙ References: H. M. Srivastava and R. G. Buschman (1977), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 434).

1.3. Equations Whose Kernels Contain Hyperbolic Functions

1.3-1. Kernels Containing Hyperbolic Cosine.

1. $\displaystyle\int_a^x \cosh[\lambda(x-t)]y(t)\,dt = f(x).$

Solution: $\displaystyle y(x) = f'_x(x) - \lambda^2\int_a^x f(x)\,dx.$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 435).

2. $\displaystyle\int_a^x \bigl\{\cosh[\lambda(x-t)]-1\bigr\}y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = f''_{xx}(x) = 0.$

Solution: $\displaystyle y(x) = \frac{1}{\lambda^2}f'''_{xxx}(x) - f'_x(x).$

3. $\displaystyle\int_a^x \big\{\cosh[\lambda(x-t)]+b\big\}y(t)\,dt = f(x).$

For $b=0$, see equation 1.3.1. For $b=-1$, see equation 1.3.2. For $\lambda=0$, see equation 1.1.1. Differentiating the equation with respect to x, we arrive at an equation of the form 2.3.16:

$$y(x)+\frac{\lambda}{b+1}\int_a^x \sinh[\lambda(x-t)]y(t)\,dt=\frac{f'_x(x)}{b+1}.$$

1°. Solution with $b(b+1)<0$:

$$y(x)=\frac{f'_x(x)}{b+1}-\frac{\lambda^2}{k(b+1)^2}\int_a^x \sin[k(x-t)]f'_t(t)\,dt,\qquad \text{where}\quad k=\lambda\sqrt{\frac{-b}{b+1}}.$$

2°. Solution with $b(b+1)>0$:

$$y(x)=\frac{f'_x(x)}{b+1}-\frac{\lambda^2}{k(b+1)^2}\int_a^x \sinh[k(x-t)]f'_t(t)\,dt,\qquad \text{where}\quad k=\lambda\sqrt{\frac{b}{b+1}}.$$

4. $\displaystyle\int_a^x \cosh(\lambda x+\beta t)y(t)\,dt = f(x).$

For $\beta=-\lambda$, see equation 1.3.1.

Differentiating the equation with respect to x twice, we obtain

$$\cosh[(\lambda+\beta)x]y(x)+\lambda\int_a^x \sinh(\lambda x+\beta t)y(t)\,dt=f'_x(x),\tag{1}$$

$$\big\{\cosh[(\lambda+\beta)x]y(x)\big\}'_x+\lambda\sinh[(\lambda+\beta)x]y(x)+\lambda^2\int_a^x \cosh(\lambda x+\beta t)y(t)\,dt=f''_{xx}(x).\tag{2}$$

Eliminating the integral term from (2) with the aid of the original equation, we arrive at the first-order linear ordinary differential equation

$$w'_x+\lambda\tanh[(\lambda+\beta)x]w=f''_{xx}(x)-\lambda^2 f(x),\qquad w=\cosh[(\lambda+\beta)x]y(x).\tag{3}$$

Setting $x=a$ in (1) yields the initial condition $w(a)=f'_x(a)$. On solving equation (3) with this condition, after some manipulations we obtain the solution of the original integral equation in the form

$$\begin{aligned} y(x)={}&\frac{1}{\cosh[(\lambda+\beta)x]}f'_x(x)-\frac{\lambda\sinh[(\lambda+\beta)x]}{\cosh^2[(\lambda+\beta)x]}f(x)\\ &+\frac{\lambda\beta}{\cosh^{k+1}[(\lambda+\beta)x]}\int_a^x f(t)\cosh^{k-2}[(\lambda+\beta)t]\,dt,\qquad k=\frac{\lambda}{\lambda+\beta}.\end{aligned}$$

5. $\displaystyle\int_a^x [\cosh(\lambda x)-\cosh(\lambda t)]y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x)=\cosh(\lambda x)$.

Solution: $\displaystyle y(x)=\frac{1}{\lambda}\frac{d}{dx}\left[\frac{f'_x(x)}{\sinh(\lambda x)}\right].$

6. $\displaystyle\int_a^x [A\cosh(\lambda x)+B\cosh(\lambda t)]y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x)=\cosh(\lambda x)$. For $B=-A$, see equation 1.3.5.

Solution: $\displaystyle y(x)=\frac{1}{A+B}\frac{d}{dx}\left\{\big[\cosh(\lambda x)\big]^{-\frac{A}{A+B}}\int_a^x \big[\cosh(\lambda t)\big]^{-\frac{B}{A+B}} f'_t(t)\,dt\right\}.$

7. $\displaystyle\int_a^x \big[A\cosh(\lambda x) + B\cosh(\mu t) + C\big]\, y(t)\, dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\cosh(\lambda x)$ and $h(t) = B\cosh(\mu t) + C$.

8. $\displaystyle\int_a^x \big\{A_1\cosh[\lambda_1(x-t)] + A_2\cosh[\lambda_2(x-t)]\big\}\, y(t)\, dt = f(x).$

The equation is equivalent to the equation

$$\int_a^x \big\{B_1\sinh[\lambda_1(x-t)] + B_2\sinh[\lambda_2(x-t)]\big\}\, y(t)\, dt = F(x),$$

$$B_1 = \frac{A_1}{\lambda_1}, \quad B_2 = \frac{A_2}{\lambda_2}, \quad F(x) = \int_a^x f(t)\, dt,$$

of the form 1.3.49. (Differentiating this equation yields the original equation.)

9. $\displaystyle\int_a^x \cosh^2[\lambda(x-t)]\, y(t)\, dt = f(x).$

Differentiation yields an equation of the form 2.3.16:

$$y(x) + \lambda\int_a^x \sinh[2\lambda(x-t)]\, y(t)\, dt = f_x'(x).$$

Solution:

$$y(x) = f_x'(x) - \frac{2\lambda^2}{k}\int_a^x \sinh[k(x-t)]\, f_t'(t)\, dt, \qquad \text{where} \quad k = \lambda\sqrt{2}.$$

10. $\displaystyle\int_a^x \big[\cosh^2(\lambda x) - \cosh^2(\lambda t)\big]\, y(t)\, dt = f(x), \qquad f(a) = f_x'(a) = 0.$

Solution: $\displaystyle y(x) = \frac{1}{\lambda}\frac{d}{dx}\left[\frac{f_x'(x)}{\sinh(2\lambda x)}\right].$

11. $\displaystyle\int_a^x \big[A\cosh^2(\lambda x) + B\cosh^2(\lambda t)\big]\, y(t)\, dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \cosh^2(\lambda x)$. For $B = -A$, see equation 1.3.10.

Solution:

$$y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\big[\cosh(\lambda x)\big]^{-\frac{2A}{A+B}}\int_a^x \big[\cosh(\lambda t)\big]^{-\frac{2B}{A+B}} f_t'(t)\, dt\right\}.$$

12. $\displaystyle\int_a^x \big[A\cosh^2(\lambda x) + B\cosh^2(\mu t) + C\big]\, y(t)\, dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\cosh^2(\lambda x)$, and $h(t) = B\cosh^2(\mu t) + C$.

13. $\displaystyle\int_a^x \cosh[\lambda(x-t)]\cosh[\lambda(x+t)]\, y(t)\, dt = f(x).$

Using the formula

$$\cosh(\alpha-\beta)\cosh(\alpha+\beta) = \tfrac{1}{2}[\cosh(2\alpha) + \cosh(2\beta)], \quad \alpha = \lambda x, \quad \beta = \lambda t,$$

we transform the original equation to an equation of the form 1.3.6 with $A = B = 1$:

$$\int_a^x [\cosh(2\lambda x) + \cosh(2\lambda t)]\, y(t)\, dt = 2f(x).$$

Solution:

$$y(x) = \frac{d}{dx}\left[\frac{1}{\sqrt{\cosh(2\lambda x)}}\int_a^x \frac{f_t'(t)\, dt}{\sqrt{\cosh(2\lambda t)}}\right].$$

14. $\displaystyle\int_a^x [\cosh(\lambda x)\cosh(\mu t)+\cosh(\beta x)\cosh(\gamma t)]y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x)=\cosh(\lambda x)$, $h_1(t)=\cosh(\mu t)$, $g_2(x)=\cosh(\beta x)$, and $h_2(t)=\cosh(\gamma t)$.

15. $\displaystyle\int_a^x \cosh^3[\lambda(x-t)]y(t)\,dt = f(x).$

Using the formula $\cosh^3\beta = \frac{1}{4}\cosh 3\beta + \frac{3}{4}\cosh\beta$, we arrive at an equation of the form 1.3.8:

$$\int_a^x \left\{\tfrac{1}{4}\cosh[3\lambda(x-t)] + \tfrac{3}{4}\cosh[\lambda(x-t)]\right\}y(t)\,dt = f(x).$$

16. $\displaystyle\int_a^x \left[\cosh^3(\lambda x)-\cosh^3(\lambda t)\right]y(t)\,dt = f(x), \qquad f(a)=f'_x(a)=0.$

Solution: $\displaystyle y(x)=\frac{1}{3\lambda}\frac{d}{dx}\left[\frac{f'_x(x)}{\sinh(\lambda x)\cosh^2(\lambda x)}\right].$

17. $\displaystyle\int_a^x \left[A\cosh^3(\lambda x)+B\cosh^3(\lambda t)\right]y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x)=\cosh^3(\lambda x)$. For $B=-A$, see equation 1.3.16.

Solution:

$$y(x)=\frac{1}{A+B}\frac{d}{dx}\left\{\left[\cosh(\lambda x)\right]^{-\frac{3A}{A+B}}\int_a^x \left[\cosh(\lambda t)\right]^{-\frac{3B}{A+B}} f'_t(t)\,dt\right\}.$$

18. $\displaystyle\int_a^x \left[A\cosh^2(\lambda x)\cosh(\mu t)+B\cosh(\beta x)\cosh^2(\gamma t)\right]y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x)=A\cosh^2(\lambda x)$, $h_1(t)=\cosh(\mu t)$, $g_2(x)=B\cosh(\beta x)$, and $h_2(t)=\cosh^2(\gamma t)$.

19. $\displaystyle\int_a^x \cosh^4[\lambda(x-t)]y(t)\,dt = f(x).$

Let us transform the kernel of the integral equation using the formula

$$\cosh^4\beta = \tfrac{1}{8}\cosh 4\beta + \tfrac{1}{2}\cosh 2\beta + \tfrac{3}{8}, \quad \text{where} \quad \beta=\lambda(x-t),$$

and differentiate the resulting equation with respect to x. Then we obtain an equation of the form 2.3.18:

$$y(x)+\lambda\int_a^x \left\{\tfrac{1}{2}\sinh[4\lambda(x-t)]+\sinh[2\lambda(x-t)]\right\}y(t)\,dt = f'_x(x).$$

20. $\displaystyle\int_a^x [\cosh(\lambda x)-\cosh(\lambda t)]^n y(t)\,dt = f(x), \qquad n=1,2,\dots$

The right-hand side of the equation is assumed to satisfy the conditions $f(a)=f'_x(a)=\cdots=f^{(n)}_x(a)=0$.

Solution: $\displaystyle y(x)=\frac{\sinh(\lambda x)}{\lambda^n n!}\left[\frac{1}{\sinh(\lambda x)}\frac{d}{dx}\right]^{n+1} f(x).$

21. $\displaystyle\int_a^x \sqrt{\cosh x - \cosh t}\, y(t)\, dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi}\sinh x\left(\frac{1}{\sinh x}\frac{d}{dx}\right)^2 \int_a^x \frac{\sinh t\, f(t)\, dt}{\sqrt{\cosh x - \cosh t}}.$$

22. $\displaystyle\int_a^x \frac{y(t)\, dt}{\sqrt{\cosh x - \cosh t}} = f(x).$

Solution:

$$y(x) = \frac{1}{\pi}\frac{d}{dx}\int_a^x \frac{\sinh t\, f(t)\, dt}{\sqrt{\cosh x - \cosh t}}.$$

23. $\displaystyle\int_a^x (\cosh x - \cosh t)^\lambda y(t)\, dt = f(x), \qquad 0 < \lambda < 1.$

Solution:

$$y(x) = k\sinh x\left(\frac{1}{\sinh x}\frac{d}{dx}\right)^2 \int_a^x \frac{\sinh t\, f(t)\, dt}{(\cosh x - \cosh t)^\lambda}, \qquad k = \frac{\sin(\pi\lambda)}{\pi\lambda}.$$

24. $\displaystyle\int_a^x (\cosh^\mu x - \cosh^\mu t) y(t)\, dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \cosh^\mu x$.

Solution: $\displaystyle y(x) = \frac{1}{\mu}\frac{d}{dx}\left[\frac{f'_x(x)}{\sinh x \cosh^{\mu-1} x}\right].$

25. $\displaystyle\int_a^x \big(A\cosh^\mu x + B\cosh^\mu t\big) y(t)\, dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \cosh^\mu x$. For $B = -A$, see equation 1.3.24.

Solution:

$$y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\big[\cosh(\lambda x)\big]^{-\frac{A\mu}{A+B}} \int_a^x \big[\cosh(\lambda t)\big]^{-\frac{B\mu}{A+B}} f'_t(t)\, dt\right\}.$$

26. $\displaystyle\int_a^x \frac{y(t)\, dt}{(\cosh x - \cosh t)^\lambda} = f(x), \qquad 0 < \lambda < 1.$

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi}\frac{d}{dx}\int_a^x \frac{\sinh t\, f(t)\, dt}{(\cosh x - \cosh t)^{1-\lambda}}.$$

27. $\displaystyle\int_a^x (x - t)\cosh[\lambda(x - t)] y(t)\, dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Differentiating the equation twice yields

$$y(x) + 2\lambda\int_a^x \sinh[\lambda(x - t)]y(t)\, dt + \lambda^2\int_a^x (x - t)\cosh[\lambda(x - t)]y(t)\, dt = f''_{xx}(x).$$

Eliminating the third term on the right-hand side with the aid of the original equation, we arrive at an equation of the form 2.3.16:

$$y(x) + 2\lambda\int_a^x \sinh[\lambda(x - t)]y(t)\, dt = f''_{xx}(x) - \lambda^2 f(x).$$

28. $\displaystyle\int_a^x \frac{\cosh[\lambda(x-t)]}{\sqrt{x-t}}\, y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Solution:

$$y(x) = \frac{2}{\pi\lambda}\int_a^x \frac{\cosh[\lambda(x-t)]}{\sqrt{x-t}}\,[f''_{tt}(t) - \lambda^2 f(t)]\,dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 436).

29. $\displaystyle\int_a^x \sqrt{x-t}\,\cosh\left(\lambda\sqrt{x-t}\,\right) y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{1}{\pi}\int_a^x \frac{\cos\left(\lambda\sqrt{x-t}\,\right)}{\sqrt{x-t}}\, f'_t(t)\,dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 437), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

30. $\displaystyle\int_a^x \frac{\cosh\left(\lambda\sqrt{x-t}\,\right)}{\sqrt{x-t}}\, y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{1}{\pi}\frac{d}{dx}\int_a^x \frac{\cos\left(\lambda\sqrt{x-t}\,\right)}{\sqrt{x-t}}\, f(t)\,dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 437), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

31. $\displaystyle\int_x^\infty \frac{\cosh\left(\lambda\sqrt{t-x}\,\right)}{\sqrt{t-x}}\, y(t)\,dt = f(x).$

Solution:

$$y(x) = -\frac{1}{\pi}\frac{d}{dx}\int_x^\infty \frac{\cos\left(\lambda\sqrt{t-x}\,\right)}{\sqrt{t-x}}\, f(t)\,dt.$$

⊙ References: H. M. Srivastava and R. G. Buschman (1977), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 439), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

32. $\displaystyle\int_0^x \frac{\cosh\left(\lambda\sqrt{x^2-t^2}\,\right)}{\sqrt{x^2-t^2}}\, y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi}\frac{d}{dx}\int_0^x t\,\frac{\cos\left(\lambda\sqrt{x^2-t^2}\,\right)}{\sqrt{x^2-t^2}}\, f(t)\,dt.$$

33. $\displaystyle\int_x^\infty \frac{\cosh\left(\lambda\sqrt{t^2-x^2}\,\right)}{\sqrt{t^2-x^2}}\, y(t)\,dt = f(x).$

Solution:

$$y(x) = -\frac{2}{\pi}\frac{d}{dx}\int_x^\infty t\,\frac{\cos\left(\lambda\sqrt{t^2-x^2}\,\right)}{\sqrt{t^2-x^2}}\, f(t)\,dt.$$

34. $\displaystyle\int_0^x \frac{\cosh\left(\lambda\sqrt{xt-t^2}\right)}{\sqrt{x-t}}\,y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{1}{\pi x}\int_0^x \frac{\cos\left(\lambda\sqrt{x^2-xt}\right)}{\sqrt{x-t}}[f(t)/2 + tf_t'(t)]\,dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 438), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

35. $\displaystyle\int_0^x \frac{\cosh\left(\lambda\sqrt{x^2-xt}\right)}{\sqrt{x-t}}\,y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{\sqrt{x}}{\pi}\frac{d}{dx}\left[\sqrt{x}\int_0^x \frac{\cos\left(\lambda\sqrt{xt-t^2}\right)}{\sqrt{x-t}}f(t)\,dt\right].$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 438), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

36. $\displaystyle\int_a^x \frac{\cosh\left[\lambda\sqrt{(x-t)(x-t+\gamma)}\right]}{\sqrt{x-t}}\,y(t)\,dt = f(x).$

It is assumed that $f(a) = f_x'(a) = f_{xx}''(a) = f_{xxx}'''(a) = 0$.

Solution:

$$y(x) = \frac{2}{\pi\lambda^2}\int_a^x \frac{\sinh\left[\lambda\sqrt{(x-t)(x-t-\gamma)}\right]}{\sqrt{x-t-\gamma}}\int_a^t \sinh[\lambda(t-s)]\left(\frac{d^2}{ds^2}-\lambda^2\right)^2 f(s)\,ds\,dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 438), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

37. $\displaystyle\int_a^x \left[Ax^\beta + B\cosh^\gamma(\lambda t) + C\right]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = Ax^\beta$ and $h(t) = B\cosh^\gamma(\lambda t) + C$.

38. $\displaystyle\int_a^x \left[A\cosh^\gamma(\lambda x) + Bt^\beta + C\right]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\cosh^\gamma(\lambda x)$ and $h(t) = Bt^\beta + C$.

39. $\displaystyle\int_a^x \left(Ax^\lambda\cosh^\mu t + Bt^\beta\cosh^\gamma x\right)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \cosh^\mu t$, $g_2(x) = B\cosh^\gamma x$, and $h_2(t) = t^\beta$.

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40. $\displaystyle\int_a^x \sinh[\lambda(x-t)]y(t)\,dt = f(x), \qquad f(a) = f_x'(a) = 0.$

Solution: $y(x) = \dfrac{1}{\lambda}f_{xx}''(x) - \lambda f(x).$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 435).

41. $\displaystyle\int_a^x \frac{\sinh[\lambda(x-t)]}{\sqrt{x-t}}\,y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Solution:

$$y(x) = \frac{2}{\pi\lambda}\int_a^x \frac{\sinh[\lambda(x-t)]}{\sqrt{x-t}}\left[f''_{tt}(t) - \lambda^2 f(t)\right] dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 436).

42. $\displaystyle\int_a^x \frac{\sinh[\lambda(x-t)]}{(x-t)^{3/2}}\,y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Solution:

$$y(x) = \frac{2}{\pi\lambda}\int_a^x \frac{\sinh[\lambda(x-t)]}{\sqrt{x-t}}\left[f''_{tt}(t) - \lambda^2 f(t) + \frac{f'(t)}{x-t}\right] dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 437).

43. $\displaystyle\int_a^x \bigl\{\sinh[\lambda(x-t)] + b\bigr\}y(t)\,dt = f(x).$

For $b = 0$, see equation 1.3.40. Assume that $b \neq 0$.

Differentiating the equation with respect to x, we arrive at an equation of the form 2.3.3:

$$y(x) + \frac{\lambda}{b}\int_a^x \cosh[\lambda(x-t)]y(t)\,dt = \frac{1}{b}f'_x(x).$$

Solution:

$$y(x) = \frac{1}{b}f'_x(x) + \int_a^x R(x-t)f'_t(t)\,dt,$$

$$R(x) = \frac{\lambda}{b^2}\exp\left(-\frac{\lambda x}{2b}\right)\left[\frac{\lambda}{2bk}\sinh(kx) - \cosh(kx)\right], \quad k = \frac{\lambda\sqrt{1+4b^2}}{2b}.$$

44. $\displaystyle\int_a^x \sinh(\lambda x + \beta t)y(t)\,dt = f(x).$

For $\beta = -\lambda$, see equation 1.3.40. Assume that $\beta \neq -\lambda$.

Differentiating the equation with respect to x twice yields

$$\sinh[(\lambda+\beta)x]y(x) + \lambda\int_a^x \cosh(\lambda x + \beta t)y(t)\,dt = f'_x(x), \tag{1}$$

$$\bigl\{\sinh[(\lambda+\beta)x]y(x)\bigr\}'_x + \lambda\cosh[(\lambda+\beta)x]y(x) + \lambda^2\int_a^x \sinh(\lambda x + \beta t)y(t)\,dt = f''_{xx}(x). \tag{2}$$

Eliminating the integral term from (2) with the aid of the original equation, we arrive at the first-order linear ordinary differential equation

$$w'_x + \lambda\coth[(\lambda+\beta)x]w = f''_{xx}(x) - \lambda^2 f(x), \qquad w = \sinh[(\lambda+\beta)x]y(x). \tag{3}$$

Setting $x = a$ in (1) yields the initial condition $w(a) = f'_x(a)$. On solving equation (3) with this condition, after some manipulations we obtain the solution of the original integral equation in the form

$$y(x) = \frac{1}{\sinh[(\lambda+\beta)x]}f'_x(x) - \frac{\lambda\cosh[(\lambda+\beta)x]}{\sinh^2[(\lambda+\beta)x]}f(x)$$
$$- \frac{\lambda\beta}{\sinh^{k+1}[(\lambda+\beta)x]}\int_a^x f(t)\sinh^{k-2}[(\lambda+\beta)t]\,dt, \qquad k = \frac{\lambda}{\lambda+\beta}.$$

45. $\displaystyle\int_a^x [\sinh(\lambda x) - \sinh(\lambda t)]y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

This is a special case of equation 1.9.2 with $g(x) = \sinh(\lambda x)$.

Solution: $\displaystyle y(x) = \frac{1}{\lambda}\frac{d}{dx}\left[\frac{f'_x(x)}{\cosh(\lambda x)}\right].$

46. $\displaystyle\int_a^x [A\sinh(\lambda x) + B\sinh(\lambda t)]y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \sinh(\lambda x)$. For $B = -A$, see equation 1.3.45.

Solution: $\displaystyle y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\big[\sinh(\lambda x)\big]^{-\frac{A}{A+B}}\int_a^x \big[\sinh(\lambda t)\big]^{-\frac{B}{A+B}} f'_t(t)\,dt\right\}.$

47. $\displaystyle\int_a^x [A\sinh(\lambda x) + B\sinh(\mu t)]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\sinh(\lambda x)$ and $h(t) = B\sinh(\mu t)$.

48. $\displaystyle\int_a^x \big\{\mu\sinh[\lambda(x-t)] - \lambda\sinh[\mu(x-t)]\big\}y(t)\,dt = f(x).$

It is assumed that $f(a) = f'_x(a) = f''_{xx}(a) = f'''_{xxx}(a) = 0$.

Solution:

$$y(x) = \frac{f''''_{xxxx} - (\lambda^2+\mu^2)f''_{xx} + \lambda^2\mu^2 f}{\mu\lambda^3 - \lambda\mu^3}, \qquad f = f(x).$$

49. $\displaystyle\int_a^x \big\{A_1\sinh[\lambda_1(x-t)] + A_2\sinh[\lambda_2(x-t)]\big\}y(t)\,dt = f(x), \quad f(a) = f'_x(a) = 0.$

1°. Introduce the notation

$$I_1 = \int_a^x \sinh[\lambda_1(x-t)]y(t)\,dt, \quad I_2 = \int_a^x \sinh[\lambda_2(x-t)]y(t)\,dt,$$
$$J_1 = \int_a^x \cosh[\lambda_1(x-t)]y(t)\,dt, \quad J_2 = \int_a^x \cosh[\lambda_2(x-t)]y(t)\,dt.$$

Let us successively differentiate the integral equation four times. As a result, we have (the first line is the original equation):

$$A_1I_1 + A_2I_2 = f, \qquad f = f(x), \tag{1}$$
$$A_1\lambda_1J_1 + A_2\lambda_2J_2 = f'_x, \tag{2}$$
$$(A_1\lambda_1 + A_2\lambda_2)y + A_1\lambda_1^2I_1 + A_2\lambda_2^2I_2 = f''_{xx}, \tag{3}$$
$$(A_1\lambda_1 + A_2\lambda_2)y'_x + A_1\lambda_1^3J_1 + A_2\lambda_2^3J_2 = f'''_{xxx}, \tag{4}$$
$$(A_1\lambda_1 + A_2\lambda_2)y''_{xx} + (A_1\lambda_1^3 + A_2\lambda_2^3)y + A_1\lambda_1^4I_1 + A_2\lambda_2^4I_2 = f''''_{xxxx}. \tag{5}$$

Eliminating I_1 and I_2 from (1), (3), and (5), we arrive at the following second-order linear ordinary differential equation with constant coefficients:

$$(A_1\lambda_1 + A_2\lambda_2)y''_{xx} - \lambda_1\lambda_2(A_1\lambda_2 + A_2\lambda_1)y = f''''_{xxxx} - (\lambda_1^2+\lambda_2^2)f''_{xx} + \lambda_1^2\lambda_2^2 f. \tag{6}$$

The initial conditions can be obtained by substituting $x = a$ into (3) and (4):

$$(A_1\lambda_1 + A_2\lambda_2)y(a) = f''_{xx}(a), \quad (A_1\lambda_1 + A_2\lambda_2)y'_x(a) = f'''_{xxx}(a). \tag{7}$$

Solving the differential equation (6) under conditions (7) allows us to find the solution of the integral equation.

2°. Denote

$$\Delta = \lambda_1\lambda_2 \frac{A_1\lambda_2 + A_2\lambda_1}{A_1\lambda_1 + A_2\lambda_2}.$$

2.1. Solution for $\Delta > 0$:

$$(A_1\lambda_1 + A_2\lambda_2)y(x) = f''_{xx}(x) + Bf(x) + C\int_a^x \sinh[k(x-t)]f(t)\,dt,$$

$$k = \sqrt{\Delta}, \quad B = \Delta - \lambda_1^2 - \lambda_2^2, \quad C = \frac{1}{\sqrt{\Delta}}\bigl[\Delta^2 - (\lambda_1^2+\lambda_2^2)\Delta + \lambda_1^2\lambda_2^2\bigr].$$

2.2. Solution for $\Delta < 0$:

$$(A_1\lambda_1 + A_2\lambda_2)y(x) = f''_{xx}(x) + Bf(x) + C\int_a^x \sin[k(x-t)]f(t)\,dt,$$

$$k = \sqrt{-\Delta}, \quad B = \Delta - \lambda_1^2 - \lambda_2^2, \quad C = \frac{1}{\sqrt{-\Delta}}\bigl[\Delta^2 - (\lambda_1^2+\lambda_2^2)\Delta + \lambda_1^2\lambda_2^2\bigr].$$

2.3. Solution for $\Delta = 0$:

$$(A_1\lambda_1 + A_2\lambda_2)y(x) = f''_{xx}(x) - (\lambda_1^2+\lambda_2^2)f(x) + \lambda_1^2\lambda_2^2\int_a^x (x-t)f(t)\,dt.$$

2.4. Solution for $\Delta = \infty$:

$$y(x) = \frac{f''''_{xxxx} - (\lambda_1^2+\lambda_2^2)f''_{xx} + \lambda_1^2\lambda_2^2 f}{A_1\lambda_1^3 + A_2\lambda_2^3}, \qquad f = f(x).$$

In the last case, the relation $A_1\lambda_1 + A_2\lambda_2 = 0$ is valid, and the right-hand side of the integral equation is assumed to satisfy the conditions $f(a) = f'_x(a) = f''_{xx}(a) = f'''_{xxx}(a) = 0$.

50. $\displaystyle\int_a^x \bigl\{A\sinh[\lambda(x-t)] + B\sinh[\mu(x-t)] + C\sinh[\beta(x-t)]\bigr\}y(t)\,dt = f(x).$

It assumed that $f(a) = f'_x(a) = 0$. Differentiating the integral equation twice yields

$$(A\lambda + B\mu + C\beta)y(x) + \int_a^x \bigl\{A\lambda^2\sinh[\lambda(x-t)] + B\mu^2\sinh[\mu(x-t)]\bigr\}y(t)\,dt$$
$$+ C\beta^2\int_a^x \sinh[\beta(x-t)]y(t)\,dt = f''_{xx}(x).$$

Eliminating the last integral with the aid of the original equation, we arrive at an equation of the form 2.3.18:

$$(A\lambda + B\mu + C\beta)y(x)$$
$$+\int_a^x \bigl\{A(\lambda^2-\beta^2)\sinh[\lambda(x-t)] + B(\mu^2-\beta^2)\sinh[\mu(x-t)]\bigr\}y(t)\,dt = f''_{xx}(x) - \beta^2 f(x).$$

In the special case $A\lambda + B\mu + C\beta = 0$, this is an equation of the form 1.3.49.

51. $\displaystyle\int_a^x \sinh^2[\lambda(x-t)]y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = f''_{xx}(a) = 0.$

Differentiating yields an equation of the form 1.3.40:

$$\int_a^x \sinh[2\lambda(x-t)]y(t)\,dt = \frac{1}{\lambda}f'_x(x).$$

Solution: $y(x) = \frac{1}{2}\lambda^{-2}f'''_{xxx}(x) - 2f'_x(x)$.

52. $\displaystyle\int_a^x \big[\sinh^2(\lambda x) - \sinh^2(\lambda t)\big]\, y(t)\, dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Solution: $\displaystyle y(x) = \frac{1}{\lambda}\frac{d}{dx}\left[\frac{f'_x(x)}{\sinh(2\lambda x)}\right].$

53. $\displaystyle\int_a^x \big[A\sinh^2(\lambda x) + B\sinh^2(\lambda t)\big]\, y(t)\, dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \sinh^2(\lambda x)$. For $B = -A$, see equation 1.3.52.

Solution:

$$y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\big[\sinh(\lambda x)\big]^{-\frac{2A}{A+B}} \int_a^x \big[\sinh(\lambda t)\big]^{-\frac{2B}{A+B}} f'_t(t)\, dt\right\}.$$

54. $\displaystyle\int_a^x \big[A\sinh^2(\lambda x) + B\sinh^2(\mu t)\big]\, y(t)\, dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\sinh^2(\lambda x)$ and $h(t) = B\sinh^2(\mu t)$.

55. $\displaystyle\int_a^x \sinh[\lambda(x-t)]\sinh[\lambda(x+t)]\, y(t)\, dt = f(x).$

Using the formula

$$\sinh(\alpha-\beta)\sinh(\alpha+\beta) = \tfrac{1}{2}[\cosh(2\alpha) - \cosh(2\beta)], \quad \alpha = \lambda x, \quad \beta = \lambda t,$$

we reduce the original equation to an equation of the form 1.3.5:

$$\int_a^x [\cosh(2\lambda x) - \cosh(2\lambda t)]\, y(t)\, dt = 2f(x).$$

Solution: $\displaystyle y(x) = \frac{1}{\lambda}\frac{d}{dx}\left[\frac{f'_x(x)}{\sinh(2\lambda x)}\right].$

56. $\displaystyle\int_a^x \big[A\sinh(\lambda x)\sinh(\mu t) + B\sinh(\beta x)\sinh(\gamma t)\big]\, y(t)\, dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = A\sinh(\lambda x)$, $h_1(t) = \sinh(\mu t)$, $g_2(x) = B\sinh(\beta x)$, and $h_2(t) = \sinh(\gamma t)$.

57. $\displaystyle\int_a^x \sinh^3[\lambda(x-t)]\, y(t)\, dt = f(x), \qquad f(a) = f'_x(a) = f''_{xx}(a) = f'''_{xxx}(a) = 0.$

Using the formula $\sinh^3\beta = \frac{1}{4}\sinh 3\beta - \frac{3}{4}\sinh\beta$, we arrive at an equation of the form 1.3.49:

$$\int_a^x \left\{\tfrac{1}{4}\sinh[3\lambda(x-t)] - \tfrac{3}{4}\sinh[\lambda(x-t)]\right\} y(t)\, dt = f(x).$$

58. $\displaystyle\int_a^x \big[\sinh^3(\lambda x) - \sinh^3(\lambda t)\big]\, y(t)\, dt = f(x), \qquad f(a) = f'_x(a) = 0.$

This is a special case of equation 1.9.2 with $g(x) = \sinh^3(\lambda x)$.

59. $\displaystyle\int_a^x \big[A\sinh^3(\lambda x) + B\sinh^3(\lambda t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \sinh^3(\lambda x)$.

Solution:

$$y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\big[\sinh(\lambda x)\big]^{-\frac{3A}{A+B}}\int_a^x \big[\sinh(\lambda t)\big]^{-\frac{3B}{A+B}} f_t'(t)\,dt\right\}.$$

60. $\displaystyle\int_a^x \big[A\sinh^2(\lambda x)\sinh(\mu t) + B\sinh(\beta x)\sinh^2(\gamma t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = A\sinh^2(\lambda x)$, $h_1(t) = \sinh(\mu t)$, $g_2(x) = B\sinh(\beta x)$, and $h_2(t) = \sinh^2(\gamma t)$.

61. $\displaystyle\int_a^x \sinh^4[\lambda(x-t)]y(t)\,dt = f(x).$

It is assumed that $f(a) = f_x'(a) = \cdots = f_{xxxx}''''(a) = 0$.

Let us transform the kernel of the integral equation using the formula

$$\sinh^4\beta = \tfrac{1}{8}\cosh 4\beta - \tfrac{1}{2}\cosh 2\beta + \tfrac{3}{8}, \quad \text{where} \quad \beta = \lambda(x-t),$$

and differentiate the resulting equation with respect to x. Then we arrive at an equation of the form 1.3.49:

$$\lambda\int_a^x \big\{\tfrac{1}{2}\sinh[4\lambda(x-t)] - \sinh[2\lambda(x-t)]\big\}y(t)\,dt = f_x'(x).$$

62. $\displaystyle\int_a^x \sinh^n[\lambda(x-t)]y(t)\,dt = f(x), \qquad n = 2, 3, \ldots$

It is assumed that $f(a) = f_x'(a) = \cdots = f_x^{(n)}(a) = 0$.

1°. Let us differentiate the equation with respect to x twice and transform the kernel of the resulting integral equation using the formula $\cosh^2\beta = 1 + \sinh^2\beta$, where $\beta = \lambda(x-t)$. Then we have

$$\lambda^2 n^2\int_a^x \sinh^n[\lambda(x-t)]y(t)\,dt + \lambda^2 n(n-1)\int_a^x \sinh^{n-2}[\lambda(x-t)]y(t)\,dt = f_{xx}''(x).$$

Eliminating the first term on the left-hand side with the aid of the original equation, we obtain

$$\int_a^x \sinh^{n-2}[\lambda(x-t)]y(t)\,dt = \frac{1}{\lambda^2 n(n-1)}\big[f_{xx}''(x) - \lambda^2 n^2 f(x)\big].$$

This equation has the same form as the original equation, but the exponent of the kernel has been reduced by two.

By applying this technique sufficiently many times, we finally arrive at simple integral equations of the form 1.1.1 (for even n) or 1.3.40 (for odd n).

2°. Solution:

$$y(x) = \frac{1}{\lambda^n n!}\left(\frac{d}{dx} + n\lambda\right)\left(\frac{d}{dx} + (n-2)\lambda\right)\ldots\left(\frac{d}{dx} - n\lambda\right)f(x).$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 436).

63. $\displaystyle\int_a^x \sinh\bigl(\lambda\sqrt{x-t}\,\bigr)y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi\lambda}\frac{d^2}{dx^2}\int_a^x \frac{\cos\bigl(\lambda\sqrt{x-t}\,\bigr)}{\sqrt{x-t}}f(t)\,dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 437), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

64. $\displaystyle\int_x^\infty \sinh\bigl(\lambda\sqrt{t-x}\,\bigr)y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi\lambda}\frac{d^2}{dx^2}\int_x^\infty \frac{\cos\bigl(\lambda\sqrt{t-x}\,\bigr)}{\sqrt{t-x}}f(t)\,dt.$$

⊙ References: H. M. Srivastava and R. G. Buschman (1977), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 439), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

65. $\displaystyle\int_a^x \sqrt{\sinh x - \sinh t}\,y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi}\cosh x\left(\frac{1}{\cosh x}\frac{d}{dx}\right)^2\int_a^x \frac{\cosh t\,f(t)\,dt}{\sqrt{\sinh x - \sinh t}}.$$

66. $\displaystyle\int_a^x \frac{y(t)\,dt}{\sqrt{\sinh x - \sinh t}} = f(x).$

Solution:

$$y(x) = \frac{1}{\pi}\frac{d}{dx}\int_a^x \frac{\cosh t\,f(t)\,dt}{\sqrt{\sinh x - \sinh t}}.$$

67. $\displaystyle\int_a^x (\sinh x - \sinh t)^\lambda y(t)\,dt = f(x), \qquad 0 < \lambda < 1.$

Solution:

$$y(x) = k\cosh x\left(\frac{1}{\cosh x}\frac{d}{dx}\right)^2\int_a^x \frac{\cosh t\,f(t)\,dt}{(\sinh x - \sinh t)^\lambda}, \qquad k = \frac{\sin(\pi\lambda)}{\pi\lambda}.$$

68. $\displaystyle\int_a^x (\sinh^\mu x - \sinh^\mu t)y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \sinh^\mu x$.

Solution: $\displaystyle y(x) = \frac{1}{\mu}\frac{d}{dx}\left[\frac{f'_x(x)}{\cosh x \sinh^{\mu-1} x}\right].$

69. $\displaystyle\int_a^x \bigl[A\sinh^\mu(\lambda x) + B\sinh^\mu(\lambda t)\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \sinh^\mu(\lambda x)$.

Solution with $B \neq -A$:

$$y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\bigl[\sinh(\lambda x)\bigr]^{-\frac{A\mu}{A+B}}\int_a^x \bigl[\sinh(\lambda t)\bigr]^{-\frac{B\mu}{A+B}} f'_t(t)\,dt\right\}.$$

70. $\displaystyle\int_a^x \frac{y(t)\,dt}{(\sinh x - \sinh t)^\lambda} = f(x), \qquad 0 < \lambda < 1.$

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi}\frac{d}{dx}\int_a^x \frac{\cosh t\, f(t)\,dt}{(\sinh x - \sinh t)^{1-\lambda}}.$$

71. $\displaystyle\int_a^x (x-t)\sinh[\lambda(x-t)]y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = f''_{xx}(a) = 0.$

Double differentiation yields

$$2\lambda\int_a^x \cosh[\lambda(x-t)]y(t)\,dt + \lambda^2\int_a^x (x-t)\sinh[\lambda(x-t)]y(t)\,dt = f''_{xx}(x).$$

Eliminating the second term on the left-hand side with the aid of the original equation, we arrive at an equation of the form 1.3.1:

$$\int_a^x \cosh[\lambda(x-t)]y(t)\,dt = \frac{1}{2\lambda}\bigl[f''_{xx}(x) - \lambda^2 f(x)\bigr].$$

Solution:

$$y(x) = \frac{1}{2\lambda}f'''_{xxx}(x) - \lambda f'_x(x) + \tfrac{1}{2}\lambda^3\int_a^x f(t)\,dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 436).

72. $\displaystyle\int_a^x \frac{\sinh[\lambda(x-t)]}{\sqrt{x-t}}\,y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Solution:

$$y(x) = \frac{2}{\pi\lambda}\int_a^x \frac{\sinh[\lambda(x-t)]}{\sqrt{x-t}}[f''_{tt}(t) - \lambda^2 f(t)]\,dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 436).

73. $\displaystyle\int_a^x \frac{\sinh[\lambda(x-t)]}{(x-t)^{3/2}}\,y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Solution:

$$y(x) = \frac{2}{\pi\lambda}\int_a^x \frac{\sinh[\lambda(x-t)]}{\sqrt{x-t}}\left[f''_{tt}(t) - \lambda^2 f(t) + \frac{f'(t)}{x-t}\right]dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 437).

74. $\displaystyle\int_a^x \frac{\sinh\left[\lambda\sqrt{(x-t)(x-t+\gamma)}\right]}{\sqrt{x-t+\gamma}}\,y(t)\,dt = f(x).$

It is assumed that $f(a) = f'_x(a) = f''_{xx}(a) = f'''_{xxx}(a) = 0$.

Solution:

$$y(x) = \frac{2}{\pi\lambda^2}\int_a^x \frac{\cosh\left[\lambda\sqrt{(x-t)(x-t-\gamma)}\right]}{\sqrt{x-t}}\int_a^t \sinh[\lambda(t-s)]\left(\frac{d^2}{ds^2} + \lambda^2\right)^2 f(s)\,ds\,dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 438), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

75. $\displaystyle\int_a^x \big[Ax^\beta + B\sinh^\gamma(\lambda t) + C\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = Ax^\beta$ and $h(t) = B\sinh^\gamma(\lambda t) + C$.

76. $\displaystyle\int_a^x \big[A\sinh^\gamma(\lambda x) + Bt^\beta + C\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\sinh^\gamma(\lambda x)$ and $h(t) = Bt^\beta + C$.

77. $\displaystyle\int_a^x \big(Ax^\lambda \sinh^\mu t + Bt^\beta \sinh^\gamma x\big)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \sinh^\mu t$, $g_2(x) = B\sinh^\gamma x$, and $h_2(t) = t^\beta$.

1.3-3. Kernels Containing Hyperbolic Tangent.

78. $\displaystyle\int_a^x \big[\tanh(\lambda x) - \tanh(\lambda t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \tanh(\lambda x)$.

Solution: $\displaystyle y(x) = \frac{1}{\lambda}\big[\cosh^2(\lambda x) f'_x(x)\big]'_x.$

79. $\displaystyle\int_a^x \big[A\tanh(\lambda x) + B\tanh(\lambda t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \tanh(\lambda x)$. For $B = -A$, see equation 1.3.78.

Solution: $\displaystyle y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\big[\tanh(\lambda x)\big]^{-\frac{A}{A+B}} \int_a^x \big[\tanh(\lambda t)\big]^{-\frac{B}{A+B}} f'_t(t)\,dt\right\}.$

80. $\displaystyle\int_a^x \big[A\tanh(\lambda x) + B\tanh(\mu t) + C\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\tanh(\lambda x)$ and $h(t) = B\tanh(\mu t) + C$.

81. $\displaystyle\int_a^x \big[\tanh^2(\lambda x) - \tanh^2(\lambda t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \tanh^2(\lambda x)$.

Solution: $\displaystyle y(x) = \frac{d}{dx}\left[\frac{\cosh^3(\lambda x) f'_x(x)}{2\lambda\sinh(\lambda x)}\right].$

82. $\displaystyle\int_a^x \big[A\tanh^2(\lambda x) + B\tanh^2(\lambda t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \tanh^2(\lambda x)$. For $B = -A$, see equation 1.3.81.

Solution: $\displaystyle y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\big[\tanh(\lambda x)\big]^{-\frac{2A}{A+B}} \int_a^x \big[\tanh(\lambda t)\big]^{-\frac{2B}{A+B}} f'_t(t)\,dt\right\}.$

83. $\displaystyle\int_a^x \big[A\tanh^2(\lambda x) + B\tanh^2(\mu t) + C\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\tanh^2(\lambda x)$ and $h(t) = B\tanh^2(\mu t) + C$.

84. $\displaystyle\int_a^x \big[\tanh(\lambda x) - \tanh(\lambda t)\big]^n y(t)\,dt = f(x), \qquad n = 1, 2, \ldots$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \cdots = f_x^{(n)}(a) = 0$.

Solution: $\displaystyle y(x) = \frac{1}{\lambda^n n!\cosh^2(\lambda x)}\left[\cosh^2(\lambda x)\frac{d}{dx}\right]^{n+1} f(x).$

85. $\displaystyle\int_a^x \sqrt{\tanh x - \tanh t}\, y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi\cosh^2 x}\left(\cosh^2 x\,\frac{d}{dx}\right)^2 \int_a^x \frac{f(t)\,dt}{\cosh^2 t\,\sqrt{\tanh x - \tanh t}}.$$

86. $\displaystyle\int_a^x \frac{y(t)\,dt}{\sqrt{\tanh x - \tanh t}} = f(x).$

Solution:

$$y(x) = \frac{1}{\pi}\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{\cosh^2 t\,\sqrt{\tanh x - \tanh t}}.$$

87. $\displaystyle\int_a^x (\tanh x - \tanh t)^\lambda y(t)\,dt = f(x), \qquad 0 < \lambda < 1.$

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi\lambda\cosh^2 x}\left(\cosh^2 x\frac{d}{dx}\right)^2 \int_a^x \frac{f(t)\,dt}{\cosh^2 t\,(\tanh x - \tanh t)^\lambda}.$$

88. $\displaystyle\int_a^x (\tanh^\mu x - \tanh^\mu t)y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \tanh^\mu x$.

Solution: $\displaystyle y(x) = \frac{1}{\mu}\frac{d}{dx}\left[\frac{\cosh^{\mu+1} x\, f'_x(x)}{\sinh^{\mu-1} x}\right].$

89. $\displaystyle\int_a^x \big(A\tanh^\mu x + B\tanh^\mu t\big)y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \tanh^\mu x$. For $B = -A$, see equation 1.3.88.

Solution:

$$y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\big[\tanh(\lambda x)\big]^{-\frac{A\mu}{A+B}} \int_a^x \big[\tanh(\lambda t)\big]^{-\frac{B\mu}{A+B}} f'_t(t)\,dt\right\}.$$

90. $\displaystyle\int_a^x \frac{y(t)\,dt}{[\tanh(\lambda x) - \tanh(\lambda t)]^\mu} = f(x), \qquad 0 < \mu < 1.$

This is a special case of equation 1.9.44 with $g(x) = \tanh(\lambda x)$ and $h(x) \equiv 1$.

Solution:

$$y(x) = \frac{\lambda\sin(\pi\mu)}{\pi}\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{\cosh^2(\lambda t)[\tanh(\lambda x) - \tanh(\lambda t)]^{1-\mu}}.$$

91. $\displaystyle\int_a^x \big[Ax^\beta + B\tanh^\gamma(\lambda t) + C\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = Ax^\beta$ and $h(t) = B\tanh^\gamma(\lambda t) + C$.

92. $\displaystyle\int_a^x \big[A\tanh^\gamma(\lambda x) + Bt^\beta + C\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\tanh^\gamma(\lambda x)$ and $h(t) = Bt^\beta + C$.

93. $\displaystyle\int_a^x \big(Ax^\lambda \tanh^\mu t + Bt^\beta \tanh^\gamma x\big)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \tanh^\mu t$, $g_2(x) = B\tanh^\gamma x$, and $h_2(t) = t^\beta$.

1.3-4. Kernels Containing Hyperbolic Cotangent.

94. $\displaystyle\int_a^x \big[\coth(\lambda x) - \coth(\lambda t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \coth(\lambda x)$.

Solution: $\displaystyle y(x) = -\frac{1}{\lambda}\frac{d}{dx}\big[\sinh^2(\lambda x)f'_x(x)\big].$

95. $\displaystyle\int_a^x \big[A\coth(\lambda x) + B\coth(\lambda t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \coth(\lambda x)$. For $B = -A$, see equation 1.3.94.

Solution: $\displaystyle y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\big[\tanh(\lambda x)\big]^{\frac{A}{A+B}}\int_a^x \big[\tanh(\lambda t)\big]^{\frac{B}{A+B}} f'_t(t)\,dt\right\}.$

96. $\displaystyle\int_a^x \big[A\coth(\lambda x) + B\coth(\mu t) + C\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\coth(\lambda x)$ and $h(t) = B\coth(\mu t) + C$.

97. $\displaystyle\int_a^x \big[\coth^2(\lambda x) - \coth^2(\lambda t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \coth^2(\lambda x)$.

Solution: $\displaystyle y(x) = -\frac{d}{dx}\left[\frac{\sinh^3(\lambda x)f'_x(x)}{2\lambda\cosh(\lambda x)}\right].$

98. $\displaystyle\int_a^x \big[A\coth^2(\lambda x) + B\coth^2(\lambda t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \coth^2(\lambda x)$. For $B = -A$, see equation 1.3.97.

Solution: $\displaystyle y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\big[\tanh(\lambda x)\big]^{\frac{2A}{A+B}}\int_a^x \big[\tanh(\lambda t)\big]^{\frac{2B}{A+B}} f'_t(t)\,dt\right\}.$

99. $\displaystyle\int_a^x \big[A\coth^2(\lambda x) + B\coth^2(\mu t) + C\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\coth^2(\lambda x)$ and $h(t) = B\coth^2(\mu t) + C$.

100. $\displaystyle\int_a^x \big[\coth(\lambda x) - \coth(\lambda t)\big]^n y(t)\,dt = f(x), \qquad n = 1, 2, \ldots$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \cdots = f^{(n)}_x(a) = 0$.

Solution: $\displaystyle y(x) = \frac{(-1)^n}{\lambda^n n! \sinh^2(\lambda x)} \left[\sinh^2(\lambda x)\frac{d}{dx}\right]^{n+1} f(x)$.

101. $\displaystyle\int_a^x (\coth^\mu x - \coth^\mu t) y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \coth^\mu x$.

Solution: $\displaystyle y(x) = -\frac{1}{\mu}\frac{d}{dx}\left[\frac{\sinh^{\mu+1} x f'_x(x)}{\cosh^{\mu-1} x}\right]$.

102. $\displaystyle\int_a^x \big(A\coth^\mu x + B\coth^\mu t\big) y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \coth^\mu x$. For $B = -A$, see equation 1.3.101.

Solution:

$$y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\left|\tanh x\right|^{\frac{A\mu}{A+B}} \int_a^x \left|\tanh t\right|^{\frac{B\mu}{A+B}} f'_t(t)\,dt\right\}.$$

103. $\displaystyle\int_a^x \big[Ax^\beta + B\coth^\gamma(\lambda t) + C\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = Ax^\beta$ and $h(t) = B\coth^\gamma(\lambda t) + C$.

104. $\displaystyle\int_a^x \big[A\coth^\gamma(\lambda x) + Bt^\beta + C\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\coth^\gamma(\lambda x)$ and $h(t) = Bt^\beta + C$.

105. $\displaystyle\int_a^x \big(Ax^\lambda \coth^\mu t + Bt^\beta \coth^\gamma x\big) y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \coth^\mu t$, $g_2(x) = B\coth^\gamma x$, and $h_2(t) = t^\beta$.

1.3-5. Kernels Containing Combinations of Hyperbolic Functions.

106. $\displaystyle\int_a^x \big\{\cosh[\lambda(x-t)] + A\sinh[\mu(x-t)]\big\} y(t)\,dt = f(x).$

Let us differentiate the equation with respect to x and then eliminate the integral with the hyperbolic cosine. As a result, we arrive at an equation of the form 2.3.16:

$$y(x) + (\lambda - A^2\mu)\int_a^x \sinh[\mu(x-t)] y(t)\,dt = f'_x(x) - A\mu f(x).$$

107. $\displaystyle\int_a^x \big[A\cosh(\lambda x) + B\sinh(\mu t) + C\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\cosh(\lambda x)$ and $h(t) = B\sinh(\mu t) + C$.

108. $\displaystyle\int_a^x \big[A\cosh^2(\lambda x) + B\sinh^2(\mu t) + C\big]\,y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\cosh^2(\lambda x)$ and $h(t) = B\sinh^2(\mu t) + C$.

109. $\displaystyle\int_a^x \sinh[\lambda(x-t)]\cosh[\lambda(x+t)]y(t)\,dt = f(x).$

Using the formula

$$\sinh(\alpha-\beta)\cosh(\alpha+\beta) = \tfrac{1}{2}\big[\sinh(2\alpha) - \sinh(2\beta)\big], \quad \alpha = \lambda x, \quad \beta = \lambda t,$$

we reduce the original equation to an equation of the form 1.3.45:

$$\int_a^x \big[\sinh(2\lambda x) - \sinh(2\lambda t)\big]\,y(t)\,dt = 2f(x).$$

Solution: $\displaystyle y(x) = \frac{1}{\lambda}\frac{d}{dx}\left[\frac{f'_x(x)}{\cosh(2\lambda x)}\right].$

110. $\displaystyle\int_a^x \cosh[\lambda(x-t)]\sinh[\lambda(x+t)]y(t)\,dt = f(x).$

Using the formula

$$\cosh(\alpha-\beta)\sinh(\alpha+\beta) = \tfrac{1}{2}\big[\sinh(2\alpha) + \sinh(2\beta)\big], \quad \alpha = \lambda x, \quad \beta = \lambda t,$$

we reduce the original equation to an equation of the form 1.3.46 with $A = B = 1$:

$$\int_a^x \big[\sinh(2\lambda x) + \sinh(2\lambda t)\big]\,y(t)\,dt = 2f(x).$$

111. $\displaystyle\int_a^x \big[A\cosh(\lambda x)\sinh(\mu t) + B\cosh(\beta x)\sinh(\gamma t)\big]\,y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = A\cosh(\lambda x)$, $h_1(t) = \sinh(\mu t)$, $g_2(x) = B\cosh(\beta x)$, and $h_2(t) = \sinh(\gamma t)$.

112. $\displaystyle\int_a^x \big[\sinh(\lambda x)\cosh(\mu t) + \sinh(\beta x)\cosh(\gamma t)\big]\,y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = \sinh(\lambda x)$, $h_1(t) = \cosh(\mu t)$, $g_2(x) = \sinh(\beta x)$, and $h_2(t) = \cosh(\gamma t)$.

113. $\displaystyle\int_a^x \big[\cosh(\lambda x)\cosh(\mu t) + \sinh(\beta x)\sinh(\gamma t)\big]\,y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = \cosh(\lambda x)$, $h_1(t) = \cosh(\mu t)$, $g_2(x) = \sinh(\beta x)$, and $h_2(t) = \sinh(\gamma t)$.

114. $\displaystyle\int_a^x \big[A\cosh^\beta(\lambda x) + B\sinh^\gamma(\mu t)\big]\,y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\cosh^\beta(\lambda x)$ and $h(t) = B\sinh^\gamma(\mu t)$.

115. $\displaystyle\int_a^x \big[A\sinh^\beta(\lambda x) + B\cosh^\gamma(\mu t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\sinh^\beta(\lambda x)$ and $h(t) = B\cosh^\gamma(\mu t)$.

116. $\displaystyle\int_a^x \big(Ax^\lambda\cosh^\mu t + Bt^\beta\sinh^\gamma x\big)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \cosh^\mu t$, $g_2(x) = B\sinh^\gamma x$, and $h_2(t) = t^\beta$.

117. $\displaystyle\int_a^x \big\{(x-t)\sinh[\lambda(x-t)] - \lambda(x-t)^2\cosh[\lambda(x-t)]\big\}\,y(t)\,dt = f(x).$

Solution:

$$y(x) = \int_a^x g(t)\,dt,$$

where

$$g(t) = \sqrt{\frac{\pi}{2\lambda}}\,\frac{1}{64\lambda^5}\left(\frac{d^2}{dt^2} - \lambda^2\right)^6 \int_a^t (t-\tau)^{\frac{5}{2}}\, I_{\frac{5}{2}}[\lambda(t-\tau)]\, f(\tau)\,d\tau.$$

118. $\displaystyle\int_a^x \left\{\frac{\sinh[\lambda(x-t)]}{x-t} - \lambda\cosh[\lambda(x-t)]\right\}y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{1}{2\lambda^4}\left(\frac{d^2}{dx^2} - \lambda^2\right)^3 \int_a^x \sinh[\lambda(x-t)]f(t)\,dt.$$

119. $\displaystyle\int_a^x \big[\sinh\big(\lambda\sqrt{x-t}\,\big) - \lambda\sqrt{x-t}\cosh\big(\lambda\sqrt{x-t}\,\big)\big]y(t)\,dt = f(x), \quad f(a) = f'_x(a) = 0.$

Solution:

$$y(x) = -\frac{4}{\pi\lambda^3}\frac{d^3}{dx^3}\int_a^x \frac{\cos\big(\lambda\sqrt{x-t}\,\big)}{\sqrt{x-t}}f(t)\,dt.$$

120. $\displaystyle\int_a^x \big(Ax^\lambda\sinh^\mu t + Bt^\beta\cosh^\gamma x\big)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \sinh^\mu t$, $g_2(x) = B\cosh^\gamma x$, and $h_2(t) = t^\beta$.

121. $\displaystyle\int_a^x \big[A\tanh(\lambda x) + B\coth(\mu t) + C\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\tanh(\lambda x)$ and $h(t) = B\coth(\mu t) + C$.

122. $\displaystyle\int_a^x \big[A\tanh^2(\lambda x) + B\coth^2(\mu t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\tanh^2(\lambda x)$ and $h(t) = B\coth^2(\mu t)$.

123. $\displaystyle\int_a^x \big[\tanh(\lambda x)\coth(\mu t) + \tanh(\beta x)\coth(\gamma t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = \tanh(\lambda x)$, $h_1(t) = \coth(\mu t)$, $g_2(x) = \tanh(\beta x)$, and $h_2(t) = \coth(\gamma t)$.

124. $\displaystyle\int_a^x \big[\coth(\lambda x)\tanh(\mu t) + \coth(\beta x)\tanh(\gamma t)\big]\, y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = \coth(\lambda x)$, $h_1(t) = \tanh(\mu t)$, $g_2(x) = \coth(\beta x)$, and $h_2(t) = \tanh(\gamma t)$.

125. $\displaystyle\int_a^x \big[\tanh(\lambda x)\tanh(\mu t) + \coth(\beta x)\coth(\gamma t)\big]\, y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = \tanh(\lambda x)$, $h_1(t) = \tanh(\mu t)$, $g_2(x) = \coth(\beta x)$, and $h_2(t) = \coth(\gamma t)$.

126. $\displaystyle\int_a^x \big[A\tanh^\beta(\lambda x) + B\coth^\gamma(\mu t)\big]\, y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\tanh^\beta(\lambda x)$ and $h(t) = B\coth^\gamma(\mu t)$.

127. $\displaystyle\int_a^x \big[A\coth^\beta(\lambda x) + B\tanh^\gamma(\mu t)\big]\, y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\coth^\beta(\lambda x)$ and $h(t) = B\tanh^\gamma(\mu t)$.

128. $\displaystyle\int_a^x \big(Ax^\lambda \tanh^\mu t + Bt^\beta \coth^\gamma x\big)\, y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \tanh^\mu t$, $g_2(x) = B\coth^\gamma x$, and $h_2(t) = t^\beta$.

129. $\displaystyle\int_a^x \big(Ax^\lambda \coth^\mu t + Bt^\beta \tanh^\gamma x\big)\, y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \coth^\mu t$, $g_2(x) = B\tanh^\gamma x$, and $h_2(t) = t^\beta$.

1.4. Equations Whose Kernels Contain Logarithmic Functions

1.4-1. Kernels Containing Logarithmic Functions.

1. $\displaystyle\int_a^x (\ln x - \ln t)\, y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \ln x$.

Solution: $y(x) = x f''_{xx}(x) + f'_x(x)$.

2. $\displaystyle\int_0^x \ln(x - t)\, y(t)\,dt = f(x).$

Solution:

$$y(x) = -\int_0^x f''_{tt}(t)\,dt \int_0^\infty \frac{(x-t)^z e^{-\mathcal{C}z}}{\Gamma(z+1)}\,dz - f'_x(0)\int_0^\infty \frac{x^z e^{-\mathcal{C}z}}{\Gamma(z+1)}\,dz,$$

where $\mathcal{C} = \lim\limits_{k\to\infty}\Big(1 + \dfrac{1}{2} + \cdots + \dfrac{1}{k+1} - \ln k\Big) = 0.5772\ldots$ is the Euler constant and $\Gamma(z)$ is the gamma function.

⊙ References: M. L. Krasnov, A. I. Kisilev, and G. I. Makarenko (1971), A. G. Butkovskii (1979).

3. $\displaystyle\int_a^x [\ln(x-t)+A]y(t)\,dt = f(x).$

Solution:

$$y(x) = -\frac{d}{dx}\int_a^x \nu_A(x-t)f(t)\,dt, \qquad \nu_A(x) = \frac{d}{dx}\int_0^\infty \frac{x^z e^{(A-\mathcal{C})z}}{\Gamma(z+1)}\,dz,$$

where $\mathcal{C} = 0.5772\ldots$ is the Euler constant and $\Gamma(z)$ is the gamma function.

For $a = 0$, the solution can be written in the form

$$y(x) = -\int_0^x f''_{tt}(t)\,dt \int_0^\infty \frac{(x-t)^z e^{(A-\mathcal{C})z}}{\Gamma(z+1)}\,dz - f'_x(0)\int_0^\infty \frac{x^z e^{(A-\mathcal{C})z}}{\Gamma(z+1)}\,dz.$$

⊙ Reference: S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

4. $\displaystyle\int_a^x (A\ln x + B\ln t)y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \ln x$. For $B = -A$, see equation 1.4.1.

Solution:

$$y(x) = \frac{\operatorname{sign}(\ln x)}{A+B}\frac{d}{dx}\left\{\left|\ln x\right|^{-\frac{A}{A+B}}\int_a^x \left|\ln t\right|^{-\frac{B}{A+B}} f'_t(t)\,dt\right\}.$$

5. $\displaystyle\int_a^x (A\ln x + B\ln t + C)y(t)\,dt = f(x).$

This is a special case of equation 1.9.5 with $g(x) = x$.

6. $\displaystyle\int_a^x \left[\ln^2(\lambda x) - \ln^2(\lambda t)\right]y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Solution: $\displaystyle y(x) = \frac{d}{dx}\left[\frac{x f'_x(x)}{2\ln(\lambda x)}\right].$

7. $\displaystyle\int_a^x \left[A\ln^2(\lambda x) + B\ln^2(\lambda t)\right]y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \ln^2(\lambda x)$. For $B = -A$, see equation 1.4.6.

Solution:

$$y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\left|\ln(\lambda x)\right|^{-\frac{2A}{A+B}}\int_a^x \left|\ln(\lambda t)\right|^{-\frac{2B}{A+B}} f'_t(t)\,dt\right\}.$$

8. $\displaystyle\int_a^x \left[A\ln^2(\lambda x) + B\ln^2(\mu t) + C\right]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\ln^2(\lambda x)$ and $h(t) = B\ln^2(\mu t) + C$.

9. $\displaystyle\int_a^x \left[\ln(x/t)\right]^n y(t)\,dt = f(x), \qquad n = 1, 2, \ldots$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \cdots = f_x^{(n)}(a) = 0$.

Solution: $\displaystyle y(x) = \frac{1}{n!\,x}\left(x\frac{d}{dx}\right)^{n+1} f(x).$

10. $\displaystyle\int_a^x \left(\ln^2 x - \ln^2 t\right)^n y(t)\,dt = f(x), \qquad n = 1, 2, \ldots$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \cdots = f^{(n)}_x(a) = 0$.

Solution: $\displaystyle y(x) = \frac{\ln x}{2^n n!\, x}\left(\frac{x}{\ln x}\frac{d}{dx}\right)^{n+1} f(x).$

11. $\displaystyle\int_a^x \ln\left(\frac{x+b}{t+b}\right) y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \ln(x+b)$.

Solution: $y(x) = (x+b) f''_{xx}(x) + f'_x(x)$.

12. $\displaystyle\int_a^x \sqrt{\ln(x/t)}\, y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi x}\left(x\frac{d}{dx}\right)^2 \int_a^x \frac{f(t)\,dt}{t\sqrt{\ln(x/t)}}.$$

13. $\displaystyle\int_a^x \frac{y(t)\,dt}{\sqrt{\ln(x/t)}} = f(x).$

Solution:

$$y(x) = \frac{1}{\pi}\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{t\sqrt{\ln(x/t)}}.$$

14. $\displaystyle\int_a^x \left[\ln^\mu(\lambda x) - \ln^\mu(\lambda t)\right] y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \ln^\mu(\lambda x)$.

Solution: $\displaystyle y(x) = \frac{1}{\mu}\frac{d}{dx}\left[x \ln^{1-\mu}(\lambda x) f'_x(x)\right].$

15. $\displaystyle\int_a^x \left[A\ln^\beta(\lambda x) + B\ln^\gamma(\mu t) + C\right] y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\ln^\beta(\lambda x)$ and $h(t) = B\ln^\gamma(\mu t) + C$.

16. $\displaystyle\int_a^x [\ln(x/t)]^\lambda y(t)\,dt = f(x), \qquad 0 < \lambda < 1.$

Solution:

$$y(x) = \frac{k}{x}\left(x\frac{d}{dx}\right)^2 \int_a^x \frac{f(t)\,dt}{t[\ln(x/t)]^\lambda}, \qquad k = \frac{\sin(\pi\lambda)}{\pi\lambda}.$$

17. $\displaystyle\int_a^x \frac{y(t)\,dt}{[\ln(x/t)]^\lambda} = f(x), \qquad 0 < \lambda < 1.$

This is a special case of equation 1.9.44 with $g(x) = \ln x$ and $h(x) \equiv 1$.

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi}\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{t[\ln(x/t)]^{1-\lambda}}.$$

18. $\displaystyle\int_0^x \ln \frac{\sqrt{x}+\sqrt{x-t}}{\sqrt{x}-\sqrt{x-t}}\, y(t)\, dt = f(x).$

Solution:

$$y(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\sqrt{t}}{\sqrt{x-t}}\frac{d}{dt} f(t)\, dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 451).

19. $\displaystyle\int_x^\infty \ln \frac{\sqrt{t}+\sqrt{t-x}}{\sqrt{t}-\sqrt{t-x}}\, y(t)\, dt = f(x).$

Solution:

$$y(x) = \frac{1}{\pi}\frac{1}{\sqrt{x}}\frac{d}{dx}\int_x^\infty \frac{t}{\sqrt{t-x}}\frac{d}{dt} f(t)\, dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 452).

1.4-2. Kernels Containing Power-Law and Logarithmic Functions.

20. $\displaystyle\int_a^x (x-t)\big[\ln(x-t)+A\big]\, y(t)\, dt = f(x).$

Solution:

$$y(x) = -\frac{d^2}{dx^2}\int_a^x \nu_A(x-t) f(t)\, dt, \qquad \nu_A(x) = \frac{d}{dx}\int_0^\infty \frac{x^z e^{(A-\mathcal{C})z}}{\Gamma(z+1)}\, dz,$$

where $\mathcal{C} = 0.5772\ldots$ is the Euler constant and $\Gamma(z)$ is the gamma function.

⊙ Reference: S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

21. $\displaystyle\int_a^x \frac{\ln(x-t)+A}{(x-t)^\lambda}\, y(t)\, dt = f(x), \qquad 0 < \lambda < 1.$

Solution:

$$y(x) = -\frac{\sin(\pi\lambda)}{\pi}\frac{d}{dx}\int_a^x \frac{F(t)\, dt}{(x-t)^{1-\lambda}}, \quad F(x) = \int_a^x \nu_h(x-t) f(t)\, dt,$$
$$\nu_h(x) = \frac{d}{dx}\int_0^\infty \frac{x^z e^{hz}}{\Gamma(z+1)}\, dz, \quad h = A + \psi(1-\lambda),$$

where $\Gamma(z)$ is the gamma function and $\psi(z) = \big[\Gamma(z)\big]'_z$ is the logarithmic derivative of the gamma function.

⊙ Reference: S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

22. $\displaystyle\int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}[\ln(x-t)+A]\, y(t)\, dt = f(x), \qquad \alpha > 0.$

Solution:

$$y(x) = -\frac{1}{\Gamma([\alpha]-\alpha+1)}\left(\frac{d}{dx}\right)^{[\alpha]+1}\int_a^x \frac{F(t)\, dt}{(x-t)^{\alpha-[\alpha]}}, \quad F(x) = \int_a^x \nu_h(x-t) f(t)\, dt,$$
$$\nu_h(x) = \frac{d}{dx}\int_0^\infty \frac{x^z e^{hz}}{\Gamma(z+1)}\, dz, \quad h = A + \psi(\alpha),$$

where $\Gamma(z)$ is the gamma function and $\psi(z) = \big[\Gamma(z)\big]'_z$ is the logarithmic derivative of the gamma function.

⊙ Reference: S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993, p. 483).

23. $\displaystyle\int_a^x \big(t^\beta \ln^\lambda x - x^\beta \ln^\lambda t\big) y(t)\,dt = f(x).$

This is a special case of equation 1.9.11 with $g(x) = \ln^\lambda x$ and $h(t) = t^\beta$.

24. $\displaystyle\int_a^x \big(At^\beta \ln^\lambda x + Bx^\mu \ln^\gamma t\big) y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = A\ln^\lambda x$, $h_1(t) = t^\beta$, $g_2(x) = Bx^\mu$, and $h_2(t) = \ln^\gamma t$.

25. $\displaystyle\int_a^x \ln\left(\frac{x^\mu + b}{ct^\lambda + s}\right) y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = \ln(x^\mu + b)$ and $h(t) = -\ln(ct^\lambda + s)$.

1.5. Equations Whose Kernels Contain Trigonometric Functions

1.5-1. Kernels Containing Cosine.

1. $\displaystyle\int_a^x \cos[\lambda(x-t)] y(t)\,dt = f(x).$

Solution: $\displaystyle y(x) = f'_x(x) + \lambda^2 \int_a^x f(x)\,dx.$

⊙ References: H. M. Srivastava and R. G. Buschman (1977), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 442).

2. $\displaystyle\int_a^x \big\{\cos[\lambda(x-t)] - 1\big\} y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = f''_{xx}(a) = 0.$

Solution: $\displaystyle y(x) = -\frac{1}{\lambda^2} f'''_{xxx}(x) - f'_x(x).$

3. $\displaystyle\int_a^x \big\{\cos[\lambda(x-t)] + b\big\} y(t)\,dt = f(x).$

For $b = 0$, see equation 1.5.1. For $b = -1$, see equation 1.5.2. For $\lambda = 0$, see equation 1.1.1. Differentiating the equation with respect to x, we arrive at an equation of the form 2.5.16:

$$y(x) - \frac{\lambda}{b+1}\int_a^x \sin[\lambda(x-t)] y(t)\,dt = \frac{f'_x(x)}{b+1}.$$

1°. Solution with $b(b+1) > 0$:

$$y(x) = \frac{f'_x(x)}{b+1} + \frac{\lambda^2}{k(b+1)^2}\int_a^x \sin[k(x-t)] f'_t(t)\,dt, \qquad \text{where} \quad k = \lambda\sqrt{\frac{b}{b+1}}.$$

2°. Solution with $b(b+1) < 0$:

$$y(x) = \frac{f'_x(x)}{b+1} + \frac{\lambda^2}{k(b+1)^2}\int_a^x \sinh[k(x-t)] f'_t(t)\,dt, \qquad \text{where} \quad k = \lambda\sqrt{\frac{-b}{b+1}}.$$

4. $\displaystyle\int_a^x \cos(\lambda x + \beta t) y(t)\,dt = f(x).$

Differentiating the equation with respect to x twice yields

$$\cos[(\lambda+\beta)x]y(x) - \lambda \int_a^x \sin(\lambda x + \beta t) y(t)\,dt = f'_x(x), \tag{1}$$

$$\big\{\cos[(\lambda+\beta)x]y(x)\big\}'_x - \lambda \sin[(\lambda+\beta)x]y(x) - \lambda^2 \int_a^x \cos(\lambda x + \beta t) y(t)\,dt = f''_{xx}(x). \tag{2}$$

Eliminating the integral term from (2) with the aid of the original equation, we arrive at the first-order linear ordinary differential equation

$$w'_x - \lambda \tan[(\lambda+\beta)x]w = f''_{xx}(x) + \lambda^2 f(x), \qquad w = \cos[(\lambda+\beta)x]y(x). \tag{3}$$

Setting $x = a$ in (1) yields the initial condition $w(a) = f'_x(a)$. On solving equation (3) under this condition, after some transformations we obtain the solution of the original integral equation in the form

$$\begin{aligned} y(x) = {} & \frac{1}{\cos[(\lambda+\beta)x]} f'_x(x) + \frac{\lambda \sin[(\lambda+\beta)x]}{\cos^2[(\lambda+\beta)x]} f(x) \\ & - \frac{\lambda\beta}{\cos^{k+1}[(\lambda+\beta)x]} \int_a^x f(t) \cos^{k-2}[(\lambda+\beta)t]\,dt, \qquad k = \frac{\lambda}{\lambda+\beta}. \end{aligned}$$

5. $\displaystyle\int_a^x \big[\cos(\lambda x) - \cos(\lambda t)\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \cos(\lambda x)$.

Solution: $\displaystyle y(x) = -\frac{1}{\lambda}\frac{d}{dx}\left[\frac{f'_x(x)}{\sin(\lambda x)}\right].$

6. $\displaystyle\int_a^x \big[A\cos(\lambda x) + B\cos(\lambda t)\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \cos(\lambda x)$. For $B = -A$, see equation 1.5.5.

Solution with $B \neq -A$:

$$y(x) = \frac{\operatorname{sign}\cos(\lambda x)}{A+B}\frac{d}{dx}\left\{\big|\cos(\lambda x)\big|^{-\frac{A}{A+B}} \int_a^x \big|\cos(\lambda t)\big|^{-\frac{B}{A+B}} f'_t(t)\,dt\right\}.$$

7. $\displaystyle\int_a^x \big[A\cos(\lambda x) + B\cos(\mu t) + C\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\cos(\lambda x)$ and $h(t) = B\cos(\mu t) + C$.

8. $\displaystyle\int_a^x \big\{A_1 \cos[\lambda_1(x-t)] + A_2 \cos[\lambda_2(x-t)]\big\} y(t)\,dt = f(x).$

The equation is equivalent to the equation

$$\int_a^x \big\{B_1 \sin[\lambda_1(x-t)] + B_2 \sin[\lambda_2(x-t)]\big\} y(t)\,dt = F(x),$$

$$B_1 = \frac{A_1}{\lambda_1}, \quad B_2 = \frac{A_2}{\lambda_2}, \quad F(x) = \int_a^x f(t)\,dt,$$

which has the form 1.5.41. (Differentiation of this equation yields the original integral equation.)

9. $\displaystyle\int_a^x \cos^2[\lambda(x-t)]y(t)\,dt = f(x).$

Differentiating yields an equation of the form 2.5.16:

$$y(x) - \lambda \int_a^x \sin[2\lambda(x-t)]y(t)\,dt = f_x'(x).$$

Solution:

$$y(x) = f_x'(x) + \frac{2\lambda^2}{k}\int_a^x \sin[k(x-t)]f_t'(t)\,dt, \qquad \text{where} \quad k = \lambda\sqrt{2}.$$

10. $\displaystyle\int_a^x \bigl[\cos^2(\lambda x) - \cos^2(\lambda t)\bigr]y(t)\,dt = f(x), \qquad f(a) = f_x'(a) = 0.$

Solution: $\displaystyle y(x) = -\frac{1}{\lambda}\frac{d}{dx}\left[\frac{f_x'(x)}{\sin(2\lambda x)}\right].$

11. $\displaystyle\int_a^x \bigl[A\cos^2(\lambda x) + B\cos^2(\lambda t)\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \cos^2(\lambda x)$. For $B = -A$, see equation 1.5.10.

Solution:

$$y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\bigl[\cos(\lambda x)\bigr]^{-\frac{2A}{A+B}}\int_a^x \bigl[\cos(\lambda t)\bigr]^{-\frac{2B}{A+B}} f_t'(t)\,dt\right\}.$$

12. $\displaystyle\int_a^x \bigl[A\cos^2(\lambda x) + B\cos^2(\mu t) + C\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\cos^2(\lambda x)$ and $h(t) = B\cos^2(\mu t) + C$.

13. $\displaystyle\int_a^x \cos[\lambda(x-t)]\cos[\lambda(x+t)]y(t)\,dt = f(x).$

Using the trigonometric formula

$$\cos(\alpha-\beta)\cos(\alpha+\beta) = \tfrac{1}{2}\bigl[\cos(2\alpha) + \cos(2\beta)\bigr], \qquad \alpha = \lambda x, \qquad \beta = \lambda t,$$

we reduce the original equation to an equation of the form 1.5.6 with $A = B = 1$:

$$\int_a^x \bigl[\cos(2\lambda x) + \cos(2\lambda t)\bigr]y(t)\,dt = 2f(x).$$

Solution with $\cos(2\lambda x) > 0$:

$$y(x) = \frac{d}{dx}\left[\frac{1}{\sqrt{\cos(2\lambda x)}}\int_a^x \frac{f_t'(t)\,dt}{\sqrt{\cos(2\lambda t)}}\right].$$

14. $\displaystyle\int_a^x \cos[\lambda(x-t)]\cos[\mu(x-t)]y(t)\,dt = f(x), \qquad f(a) = f_x'(a) = 0.$

Solution:

$$y(x) = \frac{1}{\sqrt{\lambda^2-\mu^2}}\left[\frac{d^2}{dx^2} + (\lambda+\mu)^2\right]\left[\frac{d^2}{dx^2} + (\lambda-\mu)^2\right]\int_a^x\int_a^t \sin\bigl[\sqrt{\lambda^2+\mu^2}\,(t-s)\bigr]f(s)\,ds\,dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 444).

15. $\displaystyle\int_a^x \big[A\cos(\lambda x)\cos(\mu t) + B\cos(\beta x)\cos(\gamma t)\big]\,y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = A\cos(\lambda x)$, $h_1(t) = \cos(\mu t)$, $g_2(x) = B\cos(\beta x)$, and $h_2(t) = \cos(\gamma t)$.

16. $\displaystyle\int_a^x \cos^3[\lambda(x-t)]\,y(t)\,dt = f(x).$

Using the formula $\cos^3\beta = \frac{1}{4}\cos 3\beta + \frac{3}{4}\cos\beta$, we arrive at an equation of the form 1.5.8:

$$\int_a^x \left\{\tfrac{1}{4}\cos[3\lambda(x-t)] + \tfrac{3}{4}\cos[\lambda(x-t)]\right\}y(t)\,dt = f(x).$$

17. $\displaystyle\int_a^x \big[\cos^3(\lambda x) - \cos^3(\lambda t)\big]\,y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Solution: $\displaystyle y(x) = -\frac{1}{3\lambda}\frac{d}{dx}\left[\frac{f'_x(x)}{\sin(\lambda x)\cos^2(\lambda x)}\right].$

18. $\displaystyle\int_a^x \big[A\cos^3(\lambda x) + B\cos^3(\lambda t)\big]\,y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \cos^3(\lambda x)$. For $B = -A$, see equation 1.5.17.

Solution:

$$y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\big[\cos(\lambda x)\big]^{-\frac{3A}{A+B}}\int_a^x \big[\cos(\lambda t)\big]^{-\frac{3B}{A+B}} f'_t(t)\,dt\right\}.$$

19. $\displaystyle\int_a^x \big[\cos^2(\lambda x)\cos(\mu t) + \cos(\beta x)\cos^2(\gamma t)\big]\,y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = \cos^2(\lambda x)$, $h_1(t) = \cos(\mu t)$, $g_2(x) = \cos(\beta x)$, and $h_2(t) = \cos^2(\gamma t)$.

20. $\displaystyle\int_a^x \cos^4[\lambda(x-t)]\,y(t)\,dt = f(x).$

Let us transform the kernel of the integral equation using the trigonometric formula $\cos^4\beta = \frac{1}{8}\cos 4\beta + \frac{1}{2}\cos 2\beta + \frac{3}{8}$, where $\beta = \lambda(x-t)$, and differentiate the resulting equation with respect to x. Then we arrive at an equation of the form 2.5.18:

$$y(x) - \lambda\int_a^x \left\{\tfrac{1}{2}\sin[4\lambda(x-t)] + \sin[2\lambda(x-t)]\right\}y(t)\,dt = f'_x(x).$$

21. $\displaystyle\int_a^x \big[\cos(\lambda x) - \cos(\lambda t)\big]^n\,y(t)\,dt = f(x), \qquad n = 1, 2, \ldots$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \cdots = f^{(n)}_x(a) = 0$.

Solution: $\displaystyle y(x) = \frac{(-1)^n}{\lambda^n n!}\sin(\lambda x)\left[\frac{1}{\sin(\lambda x)}\frac{d}{dx}\right]^{n+1} f(x).$

22. $\displaystyle\int_a^x \sqrt{\cos t - \cos x}\, y(t)\, dt = f(x).$

This is a special case of equation 1.9.40 with $g(x) = 1 - \cos x$.

Solution:

$$y(x) = \frac{2}{\pi} \sin x \left(\frac{1}{\sin x} \frac{d}{dx}\right)^2 \int_a^x \frac{\sin t\, f(t)\, dt}{\sqrt{\cos t - \cos x}}.$$

23. $\displaystyle\int_a^x \frac{y(t)\, dt}{\sqrt{\cos t - \cos x}} = f(x).$

Solution:

$$y(x) = \frac{1}{\pi} \frac{d}{dx} \int_a^x \frac{\sin t\, f(t)\, dt}{\sqrt{\cos t - \cos x}}.$$

24. $\displaystyle\int_a^x (\cos t - \cos x)^\lambda y(t)\, dt = f(x), \qquad 0 < \lambda < 1.$

Solution:

$$y(x) = k \sin x \left(\frac{1}{\sin x} \frac{d}{dx}\right)^2 \int_a^x \frac{\sin t\, f(t)\, dt}{(\cos t - \cos x)^\lambda}, \qquad k = \frac{\sin(\pi\lambda)}{\pi\lambda}.$$

25. $\displaystyle\int_a^x (\cos^\mu x - \cos^\mu t) y(t)\, dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \cos^\mu x$.

Solution: $\displaystyle y(x) = -\frac{1}{\mu} \frac{d}{dx} \left[\frac{f'_x(x)}{\sin x \cos^{\mu-1} x}\right].$

26. $\displaystyle\int_a^x \bigl(A \cos^\mu x + B \cos^\mu t\bigr) y(t)\, dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \cos^\mu x$. For $B = -A$, see equation 1.5.25.

Solution:

$$y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ \left|\cos x\right|^{-\frac{A\mu}{A+B}} \int_a^x \left|\cos t\right|^{-\frac{B\mu}{A+B}} f'_t(t)\, dt \right\}.$$

27. $\displaystyle\int_a^x \frac{y(t)\, dt}{(\cos t - \cos x)^\lambda} = f(x), \qquad 0 < \lambda < 1.$

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi} \frac{d}{dx} \int_a^x \frac{\sin t\, f(t)\, dt}{(\cos t - \cos x)^{1-\lambda}}.$$

28. $\displaystyle\int_a^x (x - t) \cos[\lambda(x - t)] y(t)\, dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Differentiating the equation twice yields

$$y(x) - 2\lambda \int_a^x \sin[\lambda(x - t)] y(t)\, dt - \lambda^2 \int_a^x (x - t) \cos[\lambda(x - t)] y(t)\, dt = f''_{xx}(x).$$

Eliminating the third term on the left-hand side with the aid of the original equation, we arrive at an equation of the form 2.5.16:

$$y(x) - 2\lambda \int_a^x \sin[\lambda(x - t)] y(t)\, dt = f''_{xx}(x) + \lambda^2 f(x).$$

29. $\displaystyle\int_a^x \frac{\cos[\lambda(x-t)]}{\sqrt{x-t}}\,y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Solution:

$$y(x) = \frac{2}{\pi\lambda}\int_a^x \frac{\sin[\lambda(x-t)]}{\sqrt{x-t}}\left[f''_{tt}(t) + \lambda^2 f(t)\right] dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 445).

30. $\displaystyle\int_a^x \sqrt{x-t}\,\cos\left(\lambda\sqrt{x-t}\right) y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{1}{\pi}\int_a^x \frac{\cosh\left(\lambda\sqrt{x-t}\right)}{\sqrt{x-t}}\,f'_t(t)\,dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, pp. 445–446), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

31. $\displaystyle\int_a^x \frac{\cos\left(\lambda\sqrt{x-t}\right)}{\sqrt{x-t}}\,y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{1}{\pi}\frac{d}{dx}\int_a^x \frac{\cosh\left(\lambda\sqrt{x-t}\right)}{\sqrt{x-t}}\,f(t)\,dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 446), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

32. $\displaystyle\int_x^\infty \frac{\cos\left(\lambda\sqrt{t-x}\right)}{\sqrt{t-x}}\,y(t)\,dt = f(x).$

Solution:

$$y(x) = -\frac{1}{\pi}\frac{d}{dx}\int_x^\infty \frac{\cosh\left(\lambda\sqrt{t-x}\right)}{\sqrt{t-x}}\,f(t)\,dt.$$

⊙ References: H. M. Srivastava and R. G. Buschman (1977), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 448), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

33. $\displaystyle\int_0^x \frac{\cos\left(\lambda\sqrt{x^2-t^2}\right)}{\sqrt{x^2-t^2}}\,y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi}\frac{d}{dx}\int_0^x t\,\frac{\cosh\left(\lambda\sqrt{x^2-t^2}\right)}{\sqrt{x^2-t^2}}\,f(t)\,dt.$$

34. $\displaystyle\int_x^\infty \frac{\cos\left(\lambda\sqrt{t^2-x^2}\right)}{\sqrt{t^2-x^2}}\,y(t)\,dt = f(x).$

Solution:

$$y(x) = -\frac{2}{\pi}\frac{d}{dx}\int_x^\infty t\,\frac{\cosh\left(\lambda\sqrt{t^2-x^2}\right)}{\sqrt{t^2-x^2}}\,f(t)\,dt.$$

35. $\displaystyle\int_0^x \frac{\cos\left(\lambda\sqrt{xt-t^2}\right)}{\sqrt{x-t}}\,y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{1}{\pi x}\int_0^x \frac{\cosh\left(\lambda\sqrt{x^2-xt}\right)}{\sqrt{x-t}}\left[f(t)/2 + t f'_t(t)\right] dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 446).

36. $\displaystyle \int_0^x \frac{\cos\left(\lambda\sqrt{x^2 - xt}\right)}{\sqrt{x-t}}\, y(t)\, dt = f(x).$

Solution:

$$y(x) = \frac{\sqrt{x}}{\pi}\frac{d}{dx}\left[\sqrt{x}\int_0^x \frac{\cosh\left(\lambda\sqrt{xt - t^2}\right)}{\sqrt{x-t}} f(t)\, dt\right].$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 446).

37. $\displaystyle \int_a^x \frac{\cos\left[\lambda\sqrt{(x-t)(x-t+\gamma)}\right]}{\sqrt{x-t}}\, y(t)\, dt = f(x).$

It is assumed that $f(a) = f'_x(a) = f''_{xx}(a) = f'''_{xxx}(a) = 0$.

Solution:

$$y(x) = \frac{2}{\pi\lambda^2}\int_a^x \frac{\sin\left[\lambda\sqrt{(x-t)(x-t-\gamma)}\right]}{\sqrt{x-t-\gamma}} \int_a^t \sin[\lambda(t-s)]\left(\frac{d^2}{ds^2} + \lambda^2\right)^2 f(s)\, ds\, dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 447).

38. $\displaystyle \int_a^x \left[Ax^\beta + B\cos^\gamma(\lambda t) + C\right] y(t)\, dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = Ax^\beta$ and $h(t) = B\cos^\gamma(\lambda t) + C$.

39. $\displaystyle \int_a^x \left[A\cos^\gamma(\lambda x) + Bt^\beta + C\right] y(t)\, dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\cos^\gamma(\lambda x)$ and $h(t) = Bt^\beta + C$.

40. $\displaystyle \int_a^x \left(Ax^\lambda \cos^\mu t + Bt^\beta \cos^\gamma x\right) y(t)\, dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \cos^\mu t$, $g_2(x) = B\cos^\gamma x$, and $h_2(t) = t^\beta$.

1.5-2. Kernels Containing Sine.

41. $\displaystyle \int_a^x \sin[\lambda(x-t)]\, y(t)\, dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Solution: $y(x) = \dfrac{1}{\lambda} f''_{xx}(x) + \lambda f(x)$.

⊙ References: H. M. Srivastava and R. G. Buschman (1977), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 442).

42. $\displaystyle \int_a^x \left\{\sin[\lambda(x-t)] + b\right\} y(t)\, dt = f(x).$

For $b = 0$, see equation 1.5.41. Assume that $b \neq 0$.

Differentiating the equation with respect to x yields an equation of the form 2.5.3:

$$y(x) + \frac{\lambda}{b}\int_a^x \cos[\lambda(x-t)]\, y(t)\, dt = \frac{1}{b} f'_x(x).$$

43. $\displaystyle\int_a^x \sin(\lambda x+\beta t)y(t)\,dt=f(x).$

For $\beta=-\lambda$, see equation 1.5.41. Assume that $\beta\neq-\lambda$.

Differentiating the equation with respect to x twice yields

$$\sin[(\lambda+\beta)x]y(x)+\lambda\int_a^x \cos(\lambda x+\beta t)y(t)\,dt=f'_x(x), \tag{1}$$

$$\bigl\{\sin[(\lambda+\beta)x]y(x)\bigr\}'_x+\lambda\cos[(\lambda+\beta)x]y(x)-\lambda^2\int_a^x \sin(\lambda x+\beta t)y(t)\,dt=f''_{xx}(x). \tag{2}$$

Eliminating the integral term from (2) with the aid of the original equation, we arrive at the first-order linear ordinary differential equation

$$w'_x+\lambda\cot[(\lambda+\beta)x]w=f''_{xx}(x)+\lambda^2 f(x),\qquad w=\sin[(\lambda+\beta)x]y(x). \tag{3}$$

Setting $x=a$ in (1) yields the initial condition $w(a)=f'_x(a)$. On solving equation (3) under this condition, after some transformation we obtain the solution of the original integral equation in the form

$$\begin{aligned} y(x)=&\frac{1}{\sin[(\lambda+\beta)x]}f'_x(x)-\frac{\lambda\cos[(\lambda+\beta)x]}{\sin^2[(\lambda+\beta)x]}f(x)\\ &-\frac{\lambda\beta}{\sin^{k+1}[(\lambda+\beta)x]}\int_a^x f(t)\sin^{k-2}[(\lambda+\beta)t]\,dt,\qquad k=\frac{\lambda}{\lambda+\beta}.\end{aligned}$$

44. $\displaystyle\int_a^x \bigl[\sin(\lambda x)-\sin(\lambda t)\bigr]y(t)\,dt=f(x).$

This is a special case of equation 1.9.2 with $g(x)=\sin(\lambda x)$.

Solution: $\displaystyle y(x)=\frac{1}{\lambda}\frac{d}{dx}\left[\frac{f'_x(x)}{\cos(\lambda x)}\right].$

45. $\displaystyle\int_a^x \bigl[A\sin(\lambda x)+B\sin(\lambda t)\bigr]y(t)\,dt=f(x).$

This is a special case of equation 1.9.4 with $g(x)=\sin(\lambda x)$. For $B=-A$, see equation 1.5.44.

Solution with $B\neq-A$:

$$y(x)=\frac{\operatorname{sign}\sin(\lambda x)}{A+B}\frac{d}{dx}\left\{\bigl|\sin(\lambda x)\bigr|^{-\frac{A}{A+B}}\int_a^x \bigl|\sin(\lambda t)\bigr|^{-\frac{B}{A+B}}f'_t(t)\,dt\right\}.$$

46. $\displaystyle\int_a^x \bigl[A\sin(\lambda x)+B\sin(\mu t)+C\bigr]y(t)\,dt=f(x).$

This is a special case of equation 1.9.6 with $g(x)=A\sin(\lambda x)$ and $h(t)=B\sin(\mu t)+C$.

47. $\displaystyle\int_a^x \bigl\{\mu\sin[\lambda(x-t)]-\lambda\sin[\mu(x-t)]\bigr\}y(t)\,dt=f(x).$

It is assumed that $f(a)=f'_x(a)=f''_{xx}(a)=f'''_{xxx}(a)=0$.

Solution:

$$y(x)=\frac{f''''_{xxxx}+(\lambda^2+\mu^2)f''_{xx}+\lambda^2\mu^2 f}{\lambda\mu^3-\lambda^3\mu},\qquad f=f(x).$$

48. $\displaystyle\int_a^x \big\{A_1 \sin[\lambda_1(x-t)] + A_2 \sin[\lambda_2(x-t)]\big\} y(t)\,dt = f(x), \quad f(a) = f'_x(a) = 0.$

This equation can be solved in the same manner as equation 1.3.49, i.e., by reducing it to a second-order linear ordinary differential equation with constant coefficients.

Let

$$\Delta = -\lambda_1\lambda_2 \frac{A_1\lambda_2 + A_2\lambda_1}{A_1\lambda_1 + A_2\lambda_2}.$$

1°. Solution for $\Delta > 0$:

$$(A_1\lambda_1 + A_2\lambda_2)y(x) = f''_{xx}(x) + Bf(x) + C\int_a^x \sinh[k(x-t)]f(t)\,dt,$$

$$k = \sqrt{\Delta}, \quad B = \Delta + \lambda_1^2 + \lambda_2^2, \quad C = \frac{1}{\sqrt{\Delta}}\big[\Delta^2 + (\lambda_1^2 + \lambda_2^2)\Delta + \lambda_1^2\lambda_2^2\big].$$

2°. Solution for $\Delta < 0$:

$$(A_1\lambda_1 + A_2\lambda_2)y(x) = f''_{xx}(x) + Bf(x) + C\int_a^x \sin[k(x-t)]f(t)\,dt,$$

$$k = \sqrt{-\Delta}, \quad B = \Delta + \lambda_1^2 + \lambda_2^2, \quad C = \frac{1}{\sqrt{-\Delta}}\big[\Delta^2 + (\lambda_1^2 + \lambda_2^2)\Delta + \lambda_1^2\lambda_2^2\big].$$

3°. Solution for $\Delta = 0$:

$$(A_1\lambda_1 + A_2\lambda_2)y(x) = f''_{xx}(x) + (\lambda_1^2 + \lambda_2^2)f(x) + \lambda_1^2\lambda_2^2\int_a^x (x-t)f(t)\,dt.$$

4°. Solution for $\Delta = \infty$:

$$y(x) = -\frac{f''''_{xxxx} + (\lambda_1^2 + \lambda_2^2)f''_{xx} + \lambda_1^2\lambda_2^2 f}{A_1\lambda_1^3 + A_2\lambda_2^3}, \qquad f = f(x).$$

In the last case, the relation $A_1\lambda_1 + A_2\lambda_2 = 0$ holds and the right-hand side of the integral equation is assumed to satisfy the conditions $f(a) = f'_x(a) = f''_{xx}(a) = f'''_{xxx}(a) = 0$.

Remark. The solution can be obtained from the solution of equation 1.3.49 in which the change of variables $\lambda_k \to i\lambda_k$, $A_k \to -iA_k$, $i^2 = -1$ $(k = 1, 2)$, should be made.

49. $\displaystyle\int_a^x \big\{A \sin[\lambda(x-t)] + B \sin[\mu(x-t)] + C \sin[\beta(x-t)]\big\} y(t)\,dt = f(x).$

It is assumed that $f(a) = f'_x(a) = 0$. Differentiating the integral equation twice yields

$$(A\lambda + B\mu + C\beta)y(x) - \int_a^x \big\{A\lambda^2 \sin[\lambda(x-t)] + B\mu^2 \sin[\mu(x-t)]\big\}y(t)\,dt$$
$$- C\beta^2\int_a^x \sin[\beta(x-t)]y(t)\,dt = f''_{xx}(x).$$

Eliminating the last integral with the aid of the original equation, we arrive at an equation of the form 2.5.18:

$$(A\lambda + B\mu + C\beta)y(x) + \int_a^x \big\{A(\beta^2 - \lambda^2)\sin[\lambda(x-t)]$$
$$+ B(\beta^2 - \mu^2)\sin[\mu(x-t)]\big\}y(t)\,dt = f''_{xx}(x) + \beta^2 f(x).$$

In the special case $A\lambda + B\mu + C\beta = 0$, this is an equation of the form 1.5.41.

50. $\displaystyle\int_a^x \sin^2[\lambda(x-t)]y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = f''_{xx}(a) = 0.$

Differentiation yields an equation of the form 1.5.41:

$$\int_a^x \sin[2\lambda(x-t)]y(t)\,dt = \frac{1}{\lambda}f'_x(x).$$

Solution: $y(x) = \frac{1}{2}\lambda^{-2}f'''_{xxx}(x) + 2f'_x(x)$.

51. $\displaystyle\int_a^x \bigl[\sin^2(\lambda x) - \sin^2(\lambda t)\bigr]y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Solution: $\displaystyle y(x) = \frac{1}{\lambda}\frac{d}{dx}\left[\frac{f'_x(x)}{\sin(2\lambda x)}\right]$.

52. $\displaystyle\int_a^x \bigl[A\sin^2(\lambda x) + B\sin^2(\lambda t)\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \sin^2(\lambda x)$. For $B = -A$, see equation 1.5.51.

Solution:

$$y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\bigl|\sin(\lambda x)\bigr|^{-\frac{2A}{A+B}}\int_a^x \bigl|\sin(\lambda t)\bigr|^{-\frac{2B}{A+B}} f'_t(t)\,dt\right\}.$$

53. $\displaystyle\int_a^x \bigl[A\sin^2(\lambda x) + B\sin^2(\mu t) + C\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\sin^2(\lambda x)$ and $h(t) = B\sin^2(\mu t) + C$.

54. $\displaystyle\int_a^x \sin[\lambda(x-t)]\sin[\lambda(x+t)]y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Using the trigonometric formula

$$\sin(\alpha-\beta)\sin(\alpha+\beta) = \tfrac{1}{2}\bigl[\cos(2\beta) - \cos(2\alpha)\bigr], \qquad \alpha = \lambda x, \qquad \beta = \lambda t,$$

we reduce the original equation to an equation of the form 1.5.5:

$$\int_a^x \bigl[\cos(2\lambda x) - \cos(2\lambda t)\bigr]y(t)\,dt = -2f(x).$$

Solution: $\displaystyle y(x) = \frac{1}{\lambda}\frac{d}{dx}\left[\frac{f'_x(x)}{\sin(2\lambda x)}\right]$.

55. $\displaystyle\int_a^x \sin[\lambda(x-t)]\sin[\mu(x-t)]y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = f''_{xx}(a) = 0.$

Solution:

$$y(x) = \left[\frac{d^2}{dx^2} + (\lambda+\mu)^2\right]\left[\frac{d^2}{dx^2} + (\lambda-\mu)^2\right]\frac{1}{2\lambda\mu}\int_a^x f(t)\,dt.$$

⊙ Reference A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 443).

56. $\displaystyle\int_a^x \big[\sin(\lambda x)\sin(\mu t)+\sin(\beta x)\sin(\gamma t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x)=\sin(\lambda x)$, $h_1(t)=\sin(\mu t)$, $g_2(x)=\sin(\beta x)$, and $h_2(t)=\sin(\gamma t)$.

57. $\displaystyle\int_a^x \sin^3[\lambda(x-t)]y(t)\,dt = f(x).$

It is assumed that $f(a)=f'_x(a)=f''_{xx}(a)=f'''_{xxx}(a)=0$.

Using the formula $\sin^3\beta=-\frac{1}{4}\sin 3\beta+\frac{3}{4}\sin\beta$, we arrive at an equation of the form 1.5.48:

$$\int_a^x \big\{-\tfrac{1}{4}\sin[3\lambda(x-t)]+\tfrac{3}{4}\sin[\lambda(x-t)]\big\}y(t)\,dt = f(x).$$

58. $\displaystyle\int_a^x \big[\sin^3(\lambda x)-\sin^3(\lambda t)\big]y(t)\,dt = f(x), \qquad f(a)=f'_x(a)=0.$

This is a special case of equation 1.9.2 with $g(x)=\sin^3(\lambda x)$.

59. $\displaystyle\int_a^x \big[A\sin^3(\lambda x)+B\sin^3(\lambda t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x)=\sin^3(\lambda x)$. For $B=-A$, see equation 1.5.58.

Solution:

$$y(x)=\frac{\operatorname{sign}\sin(\lambda x)}{A+B}\frac{d}{dx}\left\{\big|\sin(\lambda x)\big|^{-\frac{3A}{A+B}}\int_a^x \big|\sin(\lambda t)\big|^{-\frac{3B}{A+B}} f'_t(t)\,dt\right\}.$$

60. $\displaystyle\int_a^x \big[\sin^2(\lambda x)\sin(\mu t)+\sin(\beta x)\sin^2(\gamma t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x)=\sin^2(\lambda x)$, $h_1(t)=\sin(\mu t)$, $g_2(x)=\sin(\beta x)$, and $h_2(t)=\sin^2(\gamma t)$.

61. $\displaystyle\int_a^x \sin^4[\lambda(x-t)]y(t)\,dt = f(x).$

It is assumed that $f(a)=f'_x(a)=\cdots=f''''_{xxxx}(a)=0$.

Let us transform the kernel of the integral equation using the trigonometric formula $\sin^4\beta=\frac{1}{8}\cos 4\beta-\frac{1}{2}\cos 2\beta+\frac{3}{8}$, where $\beta=\lambda(x-t)$, and differentiate the resulting equation with respect to x. Then we obtain an equation of the form 1.5.48:

$$\lambda\int_a^x \big\{-\tfrac{1}{2}\sin[4\lambda(x-t)]+\sin[2\lambda(x-t)]\big\}y(t)\,dt = f'_x(x).$$

62. $\displaystyle\int_a^x \sin^n[\lambda(x-t)]y(t)\,dt = f(x), \qquad n=2,3,\ldots$

It is assumed that $f(a)=f'_x(a)=\cdots=f^{(n)}_x(a)=0$.

1°. Let us differentiate the equation with respect to x twice and transform the kernel of the resulting integral equation using the formula $\cos^2\beta=1-\sin^2\beta$, where $\beta=\lambda(x-t)$. We have

$$-\lambda^2n^2\int_a^x \sin^n[\lambda(x-t)]y(t)\,dt+\lambda^2n(n-1)\int_a^x \sin^{n-2}[\lambda(x-t)]y(t)\,dt = f''_{xx}(x).$$

Eliminating the first term on the left-hand side with the aid of the original equation, we obtain

$$\int_a^x \sin^{n-2}[\lambda(x-t)]y(t)\,dt = \frac{1}{\lambda^2 n(n-1)}\big[f''_{xx}(x) + \lambda^2 n^2 f(x)\big].$$

This equation has the same form as the original equation, but the degree characterizing the kernel has been reduced by two.

By applying this technique sufficiently many times, we finally arrive at simple integral equations of the form 1.1.1 (for even n) or 1.5.41 (for odd n).

2°. Solution:

$$y(x) = \frac{1}{\lambda^n n!}\left(\frac{d}{dx}\right)^{1-\alpha}\prod_{k=1}^{\beta}\Big[\frac{d^2}{dx^2} + (2k+\alpha)\lambda^2\Big]f(x),$$

where $\alpha = n - 2[n/2]$, $\beta = [(n+1)/2]$, $[A]$ denotes the integer part of number A.

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 443).

63. $\displaystyle\int_a^x (x-t)\sin[\lambda(x-t)]y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = f''_{xx}(a) = 0.$

Solution:

$$y(x) = \frac{1}{2\lambda}\left(\frac{d^2}{dx^2} + \lambda^2\right)^2 \int_a^x f(t)\,dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 444), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

64. $\displaystyle\int_a^x \sin\big(\lambda\sqrt{x-t}\big)y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi\lambda}\frac{d^2}{dx^2}\int_a^x \frac{\cosh\big(\lambda\sqrt{x-t}\big)}{\sqrt{x-t}}f(t)\,dt.$$

See also Example 2 in Section 10.4.

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 445), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

65. $\displaystyle\int_x^\infty \sin(\lambda\sqrt{t-x})y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi\lambda}\frac{d^2}{dx^2}\int_x^\infty \frac{\cos(\lambda\sqrt{t-x}\,)}{\sqrt{t-x}}f(t)\,dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 447).

66. $\displaystyle\int_a^x \frac{\sin[\lambda(x-t)]}{\sqrt{x-t}}\,y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Solution:

$$y(x) = \frac{2}{\pi\lambda}\int_a^x \frac{\cos[\lambda(x-t)]}{\sqrt{x-t}}[f''_{tt}(t) + \lambda^2 f(t)]\,dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 445), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

67. $\displaystyle \int_a^x \frac{\sin[\lambda(x-t)]}{(x-t)^{3/2}}\, y(t)\, dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Solution:

$$y(x) = \frac{2}{\pi\lambda^2} \int_a^x \frac{\sin[\lambda(x-t)]}{\sqrt{x-t}} \left[f''_{tt}(t) + \lambda^2 f(t) + \frac{f'(t)}{x-t} \right] dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 445), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

68. $\displaystyle \int_a^x \frac{\sin\left[\lambda\sqrt{(x-t)(x-t+\gamma)}\,\right]}{\sqrt{x-t+\gamma}}\, y(t)\, dt = f(x).$

It is assumed that $f(a) = f'_x(a) = f''_{xx}(a) = f'''_{xxx}(a) = 0$.

Solution:

$$y(x) = \frac{2}{\pi\lambda^2} \int_a^x \frac{\cos\left[\lambda\sqrt{(x-t)(x-t-\gamma)}\,\right]}{\sqrt{x-t}} \int_a^t \sin[\lambda(t-s)] \left(\frac{d^2}{ds^2} + \lambda^2 \right)^2 f(s)\, ds\, dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 447).

69. $\displaystyle \int_a^x \sqrt{\sin x - \sin t}\; y(t)\, dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi} \cos x \left(\frac{1}{\cos x} \frac{d}{dx} \right)^2 \int_a^x \frac{\cos t\, f(t)\, dt}{\sqrt{\sin x - \sin t}}.$$

70. $\displaystyle \int_a^x \frac{y(t)\, dt}{\sqrt{\sin x - \sin t}} = f(x).$

Solution:

$$y(x) = \frac{1}{\pi} \frac{d}{dx} \int_a^x \frac{\cos t\, f(t)\, dt}{\sqrt{\sin x - \sin t}}.$$

71. $\displaystyle \int_a^x (\sin x - \sin t)^\lambda y(t)\, dt = f(x), \qquad 0 < \lambda < 1.$

Solution:

$$y(x) = k \cos x \left(\frac{1}{\cos x} \frac{d}{dx} \right)^2 \int_a^x \frac{\cos t\, f(t)\, dt}{(\sin x - \sin t)^\lambda}, \qquad k = \frac{\sin(\pi\lambda)}{\pi\lambda}.$$

72. $\displaystyle \int_a^x (\sin^\mu x - \sin^\mu t) y(t)\, dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \sin^\mu x$.

Solution: $\displaystyle y(x) = \frac{1}{\mu} \frac{d}{dx} \left[\frac{f'_x(x)}{\cos x \sin^{\mu-1} x} \right].$

73. $\displaystyle \int_a^x \left\{ A|\sin(\lambda x)|^\mu + B|\sin(\lambda t)|^\mu \right\} y(t)\, dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = |\sin(\lambda x)|^\mu$.

Solution:

$$y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ \left|\sin(\lambda x)\right|^{-\frac{A\mu}{A+B}} \int_a^x \left|\sin(\lambda t)\right|^{-\frac{B\mu}{A+B}} f'_t(t)\, dt \right\}.$$

74. $\displaystyle\int_a^x \frac{y(t)\,dt}{[\sin(\lambda x)-\sin(\lambda t)]^\mu} = f(x), \qquad 0<\mu<1.$

This is a special case of equation 1.9.44 with $g(x)=\sin(\lambda x)$ and $h(x)\equiv 1$.

Solution:

$$y(x) = \frac{\lambda\sin(\pi\mu)}{\pi}\frac{d}{dx}\int_a^x \frac{\cos(\lambda t)f(t)\,dt}{[\sin(\lambda x)-\sin(\lambda t)]^{1-\mu}}.$$

75. $\displaystyle\int_a^x (x-t)\sin[\lambda(x-t)]y(t)\,dt = f(x), \qquad f(a)=f'_x(a)=f''_{xx}(a)=0.$

Double differentiation yields

$$2\lambda\int_a^x \cos[\lambda(x-t)]y(t)\,dt - \lambda^2\int_a^x (x-t)\sin[\lambda(x-t)]y(t)\,dt = f''_{xx}(x).$$

Eliminating the second integral on the left-hand side of this equation with the aid of the original equation, we arrive at an equation of the form 1.5.1:

$$\int_a^x \cos[\lambda(x-t)]y(t)\,dt = \frac{1}{2\lambda}\left[f''_{xx}(x)+\lambda^2 f(x)\right].$$

Solution:

$$y(x) = \frac{1}{2\lambda}f'''_{xxx}(x)+\lambda f'_x(x)+\frac{1}{2}\lambda^3\int_a^x f(t)\,dt.$$

76. $\displaystyle\int_a^x |\sin(\lambda(x-t))|y(t)\,dt = f(x), \qquad f(a)=f'_x(a)=f''_{xx}(a)=0.$

Solution:

$$y(x) = \frac{1}{\lambda}\int_a^x (-1)^{[\lambda(x-t)/\pi]}\bigl(f'''_{ttt}(t)+\lambda^2 f'_t(t)\bigr)\,dt,$$

where $[A]$ denotes the integer part of number A.

⊙ References: H. M. Srivastava and R. G. Buschman (1977), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 443).

77. $\displaystyle\int_a^x \bigl[Ax^\beta + B\sin^\gamma(\lambda t)+C\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x)=Ax^\beta$ and $h(t)=B\sin^\gamma(\lambda t)+C$.

78. $\displaystyle\int_a^x \bigl[A\sin^\gamma(\lambda x) + Bt^\beta+C\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x)=A\sin^\gamma(\lambda x)$ and $h(t)=Bt^\beta+C$.

79. $\displaystyle\int_a^x \bigl(Ax^\lambda\sin^\mu t + Bt^\beta\sin^\gamma x\bigr)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x)=Ax^\lambda$, $h_1(t)=\sin^\mu t$, $g_2(x)=B\sin^\gamma x$, and $h_2(t)=t^\beta$.

1.5-3. Kernels Containing Tangent.

80. $\displaystyle\int_a^x \big[\tan(\lambda x) - \tan(\lambda t)\big]\, y(t)\, dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \tan(\lambda x)$.

Solution: $\displaystyle y(x) = \frac{1}{\lambda}\frac{d}{dx}\big[\cos^2(\lambda x) f'_x(x)\big].$

81. $\displaystyle\int_a^x \big[A\tan(\lambda x) + B\tan(\lambda t)\big]\, y(t)\, dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \tan(\lambda x)$. For $B = -A$, see equation 1.5.80.

Solution: $\displaystyle y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\big[\tan(\lambda x)\big]^{-\frac{A}{A+B}} \int_a^x \big[\tan(\lambda t)\big]^{-\frac{B}{A+B}} f'_t(t)\, dt\right\}.$

82. $\displaystyle\int_a^x \big[A\tan(\lambda x) + B\tan(\mu t) + C\big]\, y(t)\, dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\tan(\lambda x)$ and $h(t) = B\tan(\mu t) + C$.

83. $\displaystyle\int_a^x \big[\tan^2(\lambda x) - \tan^2(\lambda t)\big]\, y(t)\, dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \tan^2(\lambda x)$.

Solution: $\displaystyle y(x) = \frac{d}{dx}\left[\frac{\cos^3(\lambda x) f'_x(x)}{2\lambda\sin(\lambda x)}\right].$

84. $\displaystyle\int_a^x \big[A\tan^2(\lambda x) + B\tan^2(\lambda t)\big]\, y(t)\, dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \tan^2(\lambda x)$. For $B = -A$, see equation 1.5.83.

Solution: $\displaystyle y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\big|\tan(\lambda x)\big|^{-\frac{2A}{A+B}} \int_a^x \big|\tan(\lambda t)\big|^{-\frac{2B}{A+B}} f'_t(t)\, dt\right\}.$

85. $\displaystyle\int_a^x \big[A\tan^2(\lambda x) + B\tan^2(\mu t) + C\big]\, y(t)\, dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\tan^2(\lambda x)$ and $h(t) = B\tan^2(\mu t) + C$.

86. $\displaystyle\int_a^x \big[\tan(\lambda x) - \tan(\lambda t)\big]^n\, y(t)\, dt = f(x), \qquad n = 1, 2, \ldots$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \cdots = f^{(n)}_x(a) = 0$.

Solution: $\displaystyle y(x) = \frac{1}{\lambda^n n!\cos^2(\lambda x)}\left[\cos^2(\lambda x)\frac{d}{dx}\right]^{n+1} f(x).$

87. $\displaystyle\int_a^x \sqrt{\tan x - \tan t}\; y(t)\, dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi\cos^2 x}\left(\cos^2 x\,\frac{d}{dx}\right)^2 \int_a^x \frac{f(t)\, dt}{\cos^2 t\sqrt{\tan x - \tan t}}.$$

88. $\displaystyle\int_a^x \frac{y(t)\,dt}{\sqrt{\tan x - \tan t}} = f(x).$

Solution:

$$y(x) = \frac{1}{\pi}\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{\cos^2 t\sqrt{\tan x - \tan t}}.$$

89. $\displaystyle\int_a^x (\tan x - \tan t)^\lambda y(t)\,dt = f(x), \qquad 0 < \lambda < 1.$

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi\lambda\cos^2 x}\left(\cos^2 x\frac{d}{dx}\right)^2 \int_a^x \frac{f(t)\,dt}{\cos^2 t(\tan x - \tan t)^\lambda}.$$

90. $\displaystyle\int_a^x (\tan^\mu x - \tan^\mu t)y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \tan^\mu x$.

Solution: $\displaystyle y(x) = \frac{1}{\mu}\frac{d}{dx}\left[\frac{\cos^{\mu+1} x f'_x(x)}{\sin^{\mu-1} x}\right].$

91. $\displaystyle\int_a^x \bigl(A\tan^\mu x + B\tan^\mu t\bigr)y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \tan^\mu x$. For $B = -A$, see equation 1.5.90.

Solution:

$$y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\bigl[\tan(\lambda x)\bigr]^{-\frac{A\mu}{A+B}} \int_a^x \bigl[\tan(\lambda t)\bigr]^{-\frac{B\mu}{A+B}} f'_t(t)\,dt\right\}.$$

92. $\displaystyle\int_a^x \frac{y(t)\,dt}{[\tan(\lambda x) - \tan(\lambda t)]^\mu} = f(x), \qquad 0 < \mu < 1.$

This is a special case of equation 1.9.44 with $g(x) = \tan(\lambda x)$ and $h(x) \equiv 1$.

Solution:

$$y(x) = \frac{\lambda\sin(\pi\mu)}{\pi}\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{\cos^2(\lambda t)[\tan(\lambda x) - \tan(\lambda t)]^{1-\mu}}.$$

93. $\displaystyle\int_a^x \bigl[Ax^\beta + B\tan^\gamma(\lambda t) + C\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = Ax^\beta$ and $h(t) = B\tan^\gamma(\lambda t) + C$.

94. $\displaystyle\int_a^x \bigl[A\tan^\gamma(\lambda x) + Bt^\beta + C\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\tan^\gamma(\lambda x)$ and $h(t) = Bt^\beta + C$.

95. $\displaystyle\int_a^x \bigl(Ax^\lambda\tan^\mu t + Bt^\beta\tan^\gamma x\bigr)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \tan^\mu t$, $g_2(x) = B\tan^\gamma x$, and $h_2(t) = t^\beta$.

1.5-4. Kernels Containing Cotangent.

96. $\displaystyle\int_a^x \big[\cot(\lambda x) - \cot(\lambda t)\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \cot(\lambda x)$.

Solution: $\displaystyle y(x) = -\frac{1}{\lambda}\frac{d}{dx}\big[\sin^2(\lambda x) f'_x(x)\big].$

97. $\displaystyle\int_a^x \big[A\cot(\lambda x) + B\cot(\lambda t)\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \cot(\lambda x)$. For $B = -A$, see equation 1.5.96.

Solution: $\displaystyle y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\big[\tan(\lambda x)\big]^{\frac{A}{A+B}}\int_a^x \big[\tan(\lambda t)\big]^{\frac{B}{A+B}} f'_t(t)\,dt\right\}.$

98. $\displaystyle\int_a^x \big[A\cot(\lambda x) + B\cot(\mu t) + C\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\cot(\lambda x)$ and $h(t) = B\cot(\mu t) + C$.

99. $\displaystyle\int_a^x \big[\cot^2(\lambda x) - \cot^2(\lambda t)\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \cot^2(\lambda x)$.

Solution: $\displaystyle y(x) = -\frac{d}{dx}\left[\frac{\sin^3(\lambda x) f'_x(x)}{2\lambda\cos(\lambda x)}\right].$

100. $\displaystyle\int_a^x \big[A\cot^2(\lambda x) + B\cot^2(\lambda t)\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \cot^2(\lambda x)$. For $B = -A$, see equation 1.5.99.

Solution: $\displaystyle y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\big|\tan(\lambda x)\big|^{\frac{2A}{A+B}}\int_a^x \big|\tan(\lambda t)\big|^{\frac{2B}{A+B}} f'_t(t)\,dt\right\}.$

101. $\displaystyle\int_a^x \big[A\cot^2(\lambda x) + B\cot^2(\mu t) + C\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\cot^2(\lambda x)$ and $h(t) = B\cot^2(\mu t) + C$.

102. $\displaystyle\int_a^x \big[\cot(\lambda x) - \cot(\lambda t)\big]^n y(t)\,dt = f(x), \qquad n = 1, 2, \ldots$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \cdots = f^{(n)}_x(a) = 0$.

Solution: $\displaystyle y(x) = \frac{(-1)^n}{\lambda^n n!\,\sin^2(\lambda x)}\left[\sin^2(\lambda x)\frac{d}{dx}\right]^{n+1} f(x).$

103. $\displaystyle\int_a^x (\cot^\mu x - \cot^\mu t) y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \cot^\mu x$.

Solution: $\displaystyle y(x) = -\frac{1}{\mu}\frac{d}{dx}\left[\frac{\sin^{\mu+1} x\, f'_x(x)}{\cos^{\mu-1} x}\right].$

104. $\displaystyle\int_a^x \bigl(A\cot^\mu x + B\cot^\mu t\bigr)y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \cot^\mu x$. For $B = -A$, see equation 1.5.103.

Solution:

$$y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\bigl|\tan x\bigr|^{\frac{A\mu}{A+B}}\int_a^x \bigl|\tan t\bigr|^{\frac{B\mu}{A+B}} f_t'(t)\,dt\right\}.$$

105. $\displaystyle\int_a^x \bigl[Ax^\beta + B\cot^\gamma(\lambda t) + C\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = Ax^\beta$ and $h(t) = B\cot^\gamma(\lambda t) + C$.

106. $\displaystyle\int_a^x \bigl[A\cot^\gamma(\lambda x) + Bt^\beta + C\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\cot^\gamma(\lambda x)$ and $h(t) = Bt^\beta + C$.

107. $\displaystyle\int_a^x \bigl(Ax^\lambda \cot^\mu t + Bt^\beta \cot^\gamma x\bigr)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \cot^\mu t$, $g_2(x) = B\cot^\gamma x$, and $h_2(t) = t^\beta$.

1.5-5. Kernels Containing Combinations of Trigonometric Functions.

108. $\displaystyle\int_a^x \bigl\{\cos[\lambda(x-t)] + A\sin[\mu(x-t)]\bigr\}y(t)\,dt = f(x).$

Differentiating the equation with respect to x followed by eliminating the integral with the cosine yields an equation of the form 2.3.16:

$$y(x) - (\lambda + A^2\mu)\int_a^x \sin[\mu(x-t)]\,y(t)\,dt = f_x'(x) - A\mu f(x).$$

109. $\displaystyle\int_a^x \bigl[A\cos(\lambda x) + B\sin(\mu t) + C\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\cos(\lambda x)$ and $h(t) = B\sin(\mu t) + C$.

110. $\displaystyle\int_a^x \bigl[A\sin(\lambda x) + B\cos(\mu t) + C\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\sin(\lambda x)$ and $h(t) = B\cos(\mu t) + C$.

111. $\displaystyle\int_a^x \bigl[A\cos^2(\lambda x) + B\sin^2(\mu t)\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\cos^2(\lambda x)$ and $h(t) = B\sin^2(\mu t)$.

112. $\displaystyle\int_a^x \sin[\lambda(x-t)]\cos[\lambda(x+t)]y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = 0.$

Using the trigonometric formula

$$\sin(\alpha-\beta)\cos(\alpha+\beta) = \tfrac{1}{2}\big[\sin(2\alpha)-\sin(2\beta)\big], \quad \alpha = \lambda x, \quad \beta = \lambda t,$$

we reduce the original equation to an equation of the form 1.5.44:

$$\int_a^x \big[\sin(2\lambda x)-\sin(2\lambda t)\big]y(t)\,dt = 2f(x).$$

Solution: $y(x) = \dfrac{1}{\lambda}\dfrac{d}{dx}\left[\dfrac{f'_x(x)}{\cos(2\lambda x)}\right].$

113. $\displaystyle\int_a^x \cos[\lambda(x-t)]\sin[\lambda(x+t)]y(t)\,dt = f(x).$

Using the trigonometric formula

$$\cos(\alpha-\beta)\sin(\alpha+\beta) = \tfrac{1}{2}\big[\sin(2\alpha)+\sin(2\beta)\big], \quad \alpha = \lambda x, \quad \beta = \lambda t,$$

we reduce the original equation to an equation of the form 1.5.45 with $A = B = 1$:

$$\int_a^x \big[\sin(2\lambda x)+\sin(2\lambda t)\big]y(t)\,dt = 2f(x).$$

Solution with $\sin(2\lambda x) > 0$:

$$y(x) = \frac{d}{dx}\left[\frac{1}{\sqrt{\sin(2\lambda x)}}\int_a^x \frac{f'_t(t)\,dt}{\sqrt{\sin(2\lambda t)}}\right].$$

114. $\displaystyle\int_a^x \sin[\lambda(x-t)]\cos[\mu(x-t)]y(t)\,dt = f(x), \qquad f(a) = f'_x(a) = f''_{xx}(a) = 0.$

Solution with $\mu < \lambda$:

$$y(x) = \frac{1}{\lambda\sqrt{\lambda^2-\mu^2}}\left[\frac{d^2}{dx^2}+(\lambda+\mu)^2\right]\left[\frac{d^2}{dx^2}+(\lambda-\mu)^2\right]\int_a^x \sin\big[\sqrt{\lambda^2-\mu^2}\,(x-t)\big]f(t)\,dt.$$

Solution with $\mu > \lambda$:

$$y(x) = \frac{1}{\lambda\sqrt{\lambda^2-\mu^2}}\left[\frac{d^2}{dx^2}+(\lambda+\mu)^2\right]\left[\frac{d^2}{dx^2}+(\lambda-\mu)^2\right]\int_a^x \sinh\big[\sqrt{\mu^2-\lambda^2}\,(x-t)\big]f(t)\,dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 444).

115. $\displaystyle\int_a^x \big[A\cos(\lambda x)\sin(\mu t)+B\cos(\beta x)\sin(\gamma t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = A\cos(\lambda x)$, $h_1(t) = \sin(\mu t)$, $g_2(x) = B\cos(\beta x)$, and $h_2(t) = \sin(\gamma t)$.

116. $\displaystyle\int_a^x \big[A\sin(\lambda x)\cos(\mu t) + B\sin(\beta x)\cos(\gamma t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = A\sin(\lambda x)$, $h_1(t) = \cos(\mu t)$, $g_2(x) = B\sin(\beta x)$, and $h_2(t) = \cos(\gamma t)$.

117. $\displaystyle\int_a^x \big[A\cos(\lambda x)\cos(\mu t) + B\sin(\beta x)\sin(\gamma t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = A\cos(\lambda x)$, $h_1(t) = \cos(\mu t)$, $g_2(x) = B\sin(\beta x)$, and $h_2(t) = \sin(\gamma t)$.

118. $\displaystyle\int_a^x \big[A\cos^\beta(\lambda x) + B\sin^\gamma(\mu t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\cos^\beta(\lambda x)$ and $h(t) = B\sin^\gamma(\mu t)$.

119. $\displaystyle\int_a^x \big[A\sin^\beta(\lambda x) + B\cos^\gamma(\mu t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\sin^\beta(\lambda x)$ and $h(t) = B\cos^\gamma(\mu t)$.

120. $\displaystyle\int_a^x \big(Ax^\lambda\cos^\mu t + Bt^\beta\sin^\gamma x\big)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \cos^\mu t$, $g_2(x) = B\sin^\gamma x$, and $h_2(t) = t^\beta$.

121. $\displaystyle\int_a^x \big(Ax^\lambda\sin^\mu t + Bt^\beta\cos^\gamma x\big)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \sin^\mu t$, $g_2(x) = B\cos^\gamma x$, and $h_2(t) = t^\beta$.

122. $\displaystyle\int_a^x \big\{(x-t)\sin[\lambda(x-t)] - \lambda(x-t)^2\cos[\lambda(x-t)]\big\}\,y(t)\,dt = f(x).$

Solution:

$$y(x) = \int_a^x g(t)\,dt,$$

where

$$g(t) = \sqrt{\frac{\pi}{2\lambda}}\,\frac{1}{64\lambda^5}\left(\frac{d^2}{dt^2} + \lambda^2\right)^6 \int_a^t (t-\tau)^{5/2}J_{5/2}[\lambda(t-\tau)]\,f(\tau)\,d\tau.$$

123. $\displaystyle\int_a^x \left\{\frac{\sin[\lambda(x-t)]}{x-t} - \lambda\cos[\lambda(x-t)]\right\}y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{1}{2\lambda^4}\left(\frac{d^2}{dx^2} + \lambda^2\right)^3 \int_a^x \sin[\lambda(x-t)]f(t)\,dt.$$

124. $\displaystyle\int_a^x \big[\sin\big(\lambda\sqrt{x-t}\,\big) - \lambda\sqrt{x-t}\cos\big(\lambda\sqrt{x-t}\,\big)\big]y(t)\,dt = f(x), \quad f(a) = f'_x(a) = 0.$

Solution:

$$y(x) = \frac{4}{\pi\lambda^3}\frac{d^3}{dx^3}\int_a^x \frac{\cosh\big(\lambda\sqrt{x-t}\,\big)}{\sqrt{x-t}}f(t)\,dt.$$

125. $\displaystyle\int_a^x \big[A\tan(\lambda x) + B\cot(\mu t) + C\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\tan(\lambda x)$ and $h(t) = B\cot(\mu t) + C$.

126. $\displaystyle\int_a^x \big[A\tan^2(\lambda x) + B\cot^2(\mu t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\tan^2(\lambda x)$ and $h(t) = B\cot^2(\mu t)$.

127. $\displaystyle\int_a^x \big[\tan(\lambda x)\cot(\mu t) + \tan(\beta x)\cot(\gamma t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = \tan(\lambda x)$, $h_1(t) = \cot(\mu t)$, $g_2(x) = \tan(\beta x)$, and $h_2(t) = \cot(\gamma t)$.

128. $\displaystyle\int_a^x \big[\cot(\lambda x)\tan(\mu t) + \cot(\beta x)\tan(\gamma t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = \cot(\lambda x)$, $h_1(t) = \tan(\mu t)$, $g_2(x) = \cot(\beta x)$, and $h_2(t) = \tan(\gamma t)$.

129. $\displaystyle\int_a^x \big[\tan(\lambda x)\tan(\mu t) + \cot(\beta x)\cot(\gamma t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = \tan(\lambda x)$, $h_1(t) = \tan(\mu t)$, $g_2(x) = \cot(\beta x)$, and $h_2(t) = \cot(\gamma t)$.

130. $\displaystyle\int_a^x \big[A\tan^\beta(\lambda x) + B\cot^\gamma(\mu t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\tan^\beta(\lambda x)$ and $h(t) = B\cot^\gamma(\mu t)$.

131. $\displaystyle\int_a^x \big[A\cot^\beta(\lambda x) + B\tan^\gamma(\mu t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\cot^\beta(\lambda x)$ and $h(t) = B\tan^\gamma(\mu t)$.

132. $\displaystyle\int_a^x \big(Ax^\lambda\tan^\mu t + Bt^\beta\cot^\gamma x\big)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \tan^\mu t$, $g_2(x) = B\cot^\gamma x$, and $h_2(t) = t^\beta$.

133. $\displaystyle\int_a^x \big(Ax^\lambda\cot^\mu t + Bt^\beta\tan^\gamma x\big)y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \cot^\mu t$, $g_2(x) = B\tan^\gamma x$, and $h_2(t) = t^\beta$.

1.6. Equations Whose Kernels Contain Inverse Trigonometric Functions

1.6-1. Kernels Containing Arccosine.

1. $\displaystyle\int_a^x \big[\arccos(\lambda x) - \arccos(\lambda t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \arccos(\lambda x)$.

Solution: $\displaystyle y(x) = -\frac{1}{\lambda}\frac{d}{dx}\left[\sqrt{1-\lambda^2x^2}\,f'_x(x)\right].$

2. $\displaystyle\int_a^x \big[A\arccos(\lambda x) + B\arccos(\lambda t)\big]\,y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \arccos(\lambda x)$. For $B = -A$, see equation 1.6.1.

Solution:

$$y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\big[\arccos(\lambda x)\big]^{-\frac{A}{A+B}}\int_a^x \big[\arccos(\lambda t)\big]^{-\frac{B}{A+B}} f_t'(t)\,dt\right\}.$$

3. $\displaystyle\int_a^x \big[A\arccos(\lambda x) + B\arccos(\mu t) + C\big]\,y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\arccos(\lambda x)$ and $h(t) = B\arccos(\mu t) + C$.

4. $\displaystyle\int_a^x \big[\arccos(\lambda x) - \arccos(\lambda t)\big]^n y(t)\,dt = f(x), \qquad n = 1, 2, \ldots$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f_x'(a) = \cdots = f_x^{(n)}(a) = 0$.

Solution:

$$y(x) = \frac{(-1)^n}{\lambda^n n!\,\sqrt{1-\lambda^2x^2}}\left(\sqrt{1-\lambda^2x^2}\,\frac{d}{dx}\right)^{n+1} f(x).$$

5. $\displaystyle\int_a^x \sqrt{\arccos(\lambda t) - \arccos(\lambda x)}\;y(t)\,dt = f(x).$

This is a special case of equation 1.9.40 with $g(x) = 1 - \arccos(\lambda x)$.

Solution:

$$y(x) = \frac{2}{\pi}\varphi(x)\left(\frac{1}{\varphi(x)}\frac{d}{dx}\right)^2\int_a^x \frac{\varphi(t)f(t)\,dt}{\sqrt{\arccos(\lambda t) - \arccos(\lambda x)}}, \qquad \varphi(x) = \frac{1}{\sqrt{1-\lambda^2x^2}}.$$

6. $\displaystyle\int_a^x \frac{y(t)\,dt}{\sqrt{\arccos(\lambda t) - \arccos(\lambda x)}} = f(x).$

Solution:

$$y(x) = \frac{\lambda}{\pi}\frac{d}{dx}\int_a^x \frac{\varphi(t)f(t)\,dt}{\sqrt{\arccos(\lambda t) - \arccos(\lambda x)}}, \qquad \varphi(x) = \frac{1}{\sqrt{1-\lambda^2x^2}}.$$

7. $\displaystyle\int_a^x \big[\arccos(\lambda t) - \arccos(\lambda x)\big]^\mu y(t)\,dt = f(x), \qquad 0 < \mu < 1.$

Solution:

$$y(x) = k\varphi(x)\left(\frac{1}{\varphi(x)}\frac{d}{dx}\right)^2\int_a^x \frac{\varphi(t)f(t)\,dt}{[\arccos(\lambda t) - \arccos(\lambda x)]^\mu},$$
$$\varphi(x) = \frac{1}{\sqrt{1-\lambda^2x^2}}, \qquad k = \frac{\sin(\pi\mu)}{\pi\mu}.$$

8. $\displaystyle\int_a^x \big[\arccos^\mu(\lambda x) - \arccos^\mu(\lambda t)\big]\,y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \arccos^\mu(\lambda x)$.

Solution: $\displaystyle y(x) = -\frac{1}{\lambda\mu}\frac{d}{dx}\left[\frac{f_x'(x)\sqrt{1-\lambda^2x^2}}{\arccos^{\mu-1}(\lambda x)}\right].$

9. $\displaystyle\int_a^x \frac{y(t)\,dt}{\big[\arccos(\lambda t)-\arccos(\lambda x)\big]^\mu} = f(x), \qquad 0<\mu<1.$

Solution:

$$y(x)=\frac{\lambda\sin(\pi\mu)}{\pi}\frac{d}{dx}\int_a^x \frac{\varphi(t)f(t)\,dt}{[\arccos(\lambda t)-\arccos(\lambda x)]^{1-\mu}}, \qquad \varphi(x)=\frac{1}{\sqrt{1-\lambda^2x^2}}.$$

10. $\displaystyle\int_a^x \big[A\arccos^\beta(\lambda x)+B\arccos^\gamma(\mu t)+C\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x)=A\arccos^\beta(\lambda x)$ and $h(t)=B\arccos^\gamma(\mu t)+C$.

1.6-2. Kernels Containing Arcsine.

11. $\displaystyle\int_a^x \big[\arcsin(\lambda x)-\arcsin(\lambda t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x)=\arcsin(\lambda x)$.

Solution: $y(x)=\dfrac{1}{\lambda}\dfrac{d}{dx}\Big[\sqrt{1-\lambda^2x^2}\,f'_x(x)\Big].$

12. $\displaystyle\int_a^x \big[A\arcsin(\lambda x)+B\arcsin(\lambda t)\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x)=\arcsin(\lambda x)$. For $B=-A$, see equation 1.6.11.

Solution:

$$y(x)=\frac{\operatorname{sign} x}{A+B}\frac{d}{dx}\left\{\big|\arcsin(\lambda x)\big|^{-\frac{A}{A+B}}\int_a^x \big|\arcsin(\lambda t)\big|^{-\frac{B}{A+B}} f'_t(t)\,dt\right\}.$$

13. $\displaystyle\int_a^x \big[A\arcsin(\lambda x)+B\arcsin(\mu t)+C\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x)=A\arcsin(\lambda x)$ and $h(t)=B\arcsin(\mu t)+C$.

14. $\displaystyle\int_a^x \big[\arcsin(\lambda x)-\arcsin(\lambda t)\big]^n y(t)\,dt = f(x), \qquad n=1,2,\dots$

The right-hand side of the equation is assumed to satisfy the conditions $f(a)=f'_x(a)=\cdots=f^{(n)}_x(a)=0$.

Solution:

$$y(x)=\frac{1}{\lambda^n n!\sqrt{1-\lambda^2x^2}}\left(\sqrt{1-\lambda^2x^2}\,\frac{d}{dx}\right)^{n+1}f(x).$$

15. $\displaystyle\int_a^x \sqrt{\arcsin(\lambda x)-\arcsin(\lambda t)}\,y(t)\,dt = f(x).$

Solution:

$$y(x)=\frac{2}{\pi}\varphi(x)\left(\frac{1}{\varphi(x)}\frac{d}{dx}\right)^2\int_a^x \frac{\varphi(t)f(t)\,dt}{\sqrt{\arcsin(\lambda x)-\arcsin(\lambda t)}}, \qquad \varphi(x)=\frac{1}{\sqrt{1-\lambda^2x^2}}.$$

16. $\displaystyle\int_a^x \frac{y(t)\,dt}{\sqrt{\arcsin(\lambda x)-\arcsin(\lambda t)}} = f(x).$

Solution:

$$y(x) = \frac{\lambda}{\pi}\frac{d}{dx}\int_a^x \frac{\varphi(t)f(t)\,dt}{\sqrt{\arcsin(\lambda x)-\arcsin(\lambda t)}}, \qquad \varphi(x) = \frac{1}{\sqrt{1-\lambda^2x^2}}.$$

17. $\displaystyle\int_a^x \bigl[\arcsin(\lambda x)-\arcsin(\lambda t)\bigr]^\mu y(t)\,dt = f(x), \qquad 0<\mu<1.$

Solution:

$$y(x) = k\varphi(x)\left(\frac{1}{\varphi(x)}\frac{d}{dx}\right)^2\int_a^x \frac{\varphi(t)f(t)\,dt}{[\arcsin(\lambda x)-\arcsin(\lambda t)]^\mu},$$
$$\varphi(x) = \frac{1}{\sqrt{1-\lambda^2x^2}}, \quad k = \frac{\sin(\pi\mu)}{\pi\mu}.$$

18. $\displaystyle\int_a^x \bigl[\arcsin^\mu(\lambda x)-\arcsin^\mu(\lambda t)\bigr] y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \arcsin^\mu(\lambda x)$.

Solution: $\displaystyle y(x) = \frac{1}{\lambda\mu}\frac{d}{dx}\left[\frac{f'_x(x)\sqrt{1-\lambda^2x^2}}{\arcsin^{\mu-1}(\lambda x)}\right].$

19. $\displaystyle\int_a^x \frac{y(t)\,dt}{\bigl[\arcsin(\lambda x)-\arcsin(\lambda t)\bigr]^\mu} = f(x), \qquad 0<\mu<1.$

Solution:

$$y(x) = \frac{\lambda\sin(\pi\mu)}{\pi}\frac{d}{dx}\int_a^x \frac{\varphi(t)f(t)\,dt}{[\arcsin(\lambda x)-\arcsin(\lambda t)]^{1-\mu}}, \qquad \varphi(x) = \frac{1}{\sqrt{1-\lambda^2x^2}}.$$

20. $\displaystyle\int_a^x \bigl[A\arcsin^\beta(\lambda x)+B\arcsin^\gamma(\mu t)+C\bigr] y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\arcsin^\beta(\lambda x)$ and $h(t) = B\arcsin^\gamma(\mu t)+C$.

21. $\displaystyle\int_0^x \arcsin\sqrt{1-\frac{t}{x}}\,y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi}\frac{1}{\sqrt{x}}\frac{d}{dx}\int_0^x \frac{t}{\sqrt{x-t}}\frac{d}{dt}f(t)\,dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 452).

22. $\displaystyle\int_x^\infty \arcsin\sqrt{1-\frac{x}{t}}\,y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi}\frac{d}{dx}\int_x^\infty \frac{\sqrt{t}}{\sqrt{t-x}}\frac{d}{dt}f(t)\,dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 453).

1.6-3. Kernels Containing Arctangent.

23. $\displaystyle\int_a^x \big[\arctan(\lambda x) - \arctan(\lambda t)\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \arctan(\lambda x)$.

Solution: $y(x) = \dfrac{1}{\lambda}\dfrac{d}{dx}\big[(1+\lambda^2x^2)\,f'_x(x)\big].$

24. $\displaystyle\int_a^x \big[A\arctan(\lambda x) + B\arctan(\lambda t)\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x)=\arctan(\lambda x)$. For $B=-A$, see equation 1.6.21.

Solution:

$$y(x) = \frac{\operatorname{sign} x}{A+B}\frac{d}{dx}\left\{\big|\arctan(\lambda x)\big|^{-\frac{A}{A+B}}\int_a^x \big|\arctan(\lambda t)\big|^{-\frac{B}{A+B}} f'_t(t)\,dt\right\}.$$

25. $\displaystyle\int_a^x \big[A\arctan(\lambda x) + B\arctan(\mu t) + C\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\arctan(\lambda x)$ and $h(t) = B\arctan(\mu t) + C$.

26. $\displaystyle\int_a^x \big[\arctan(\lambda x) - \arctan(\lambda t)\big]^n y(t)\,dt = f(x), \qquad n = 1, 2, \ldots$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \cdots = f^{(n)}_x(a) = 0$.

Solution:

$$y(x) = \frac{1}{\lambda^n n!\,(1+\lambda^2x^2)}\left((1+\lambda^2x^2)\frac{d}{dx}\right)^{n+1} f(x).$$

27. $\displaystyle\int_a^x \sqrt{\arctan(\lambda x) - \arctan(\lambda t)}\; y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi}\varphi(x)\left(\frac{1}{\varphi(x)}\frac{d}{dx}\right)^2 \int_a^x \frac{\varphi(t)f(t)\,dt}{\sqrt{\arctan(\lambda x) - \arctan(\lambda t)}}, \qquad \varphi(x) = \frac{1}{1+\lambda^2x^2}.$$

28. $\displaystyle\int_a^x \frac{y(t)\,dt}{\sqrt{\arctan(\lambda x) - \arctan(\lambda t)}} = f(x).$

Solution:

$$y(x) = \frac{\lambda}{\pi}\frac{d}{dx}\int_a^x \frac{\varphi(t)f(t)\,dt}{\sqrt{\arctan(\lambda x) - \arctan(\lambda t)}}, \qquad \varphi(x) = \frac{1}{1+\lambda^2x^2}.$$

29. $\displaystyle\int_a^x \sqrt{t}\,\arctan\left(\sqrt{\frac{x-t}{t}}\right) y(t)\,dt = f(x).$

The equation can be rewritten in terms of the Gaussian hypergeometric function in the form

$$\int_a^x (x-t)^{\gamma-1} F\Big(\alpha, \beta, \gamma; 1 - \frac{x}{t}\Big) y(t)\,dt = f(x), \qquad \text{where} \quad \alpha = \tfrac{1}{2}, \quad \beta = 1, \quad \gamma = \tfrac{3}{2}.$$

See 1.8.135 for the solution of this equation.

30. $\displaystyle\int_a^x \big[\arctan(\lambda x) - \arctan(\lambda t)\big]^\mu y(t)\,dt = f(x), \qquad 0 < \mu < 1.$

Solution:

$$y(x) = k\varphi(x)\left(\frac{1}{\varphi(x)}\frac{d}{dx}\right)^2 \int_a^x \frac{\varphi(t)f(t)\,dt}{[\arctan(\lambda x) - \arctan(\lambda t)]^\mu},$$
$$\varphi(x) = \frac{1}{1+\lambda^2x^2}, \quad k = \frac{\sin(\pi\mu)}{\pi\mu}.$$

31. $\displaystyle\int_a^x \big[\arctan^\mu(\lambda x) - \arctan^\mu(\lambda t)\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \arctan^\mu(\lambda x)$.

Solution: $\displaystyle y(x) = \frac{1}{\lambda\mu}\frac{d}{dx}\left[\frac{(1+\lambda^2x^2)f'_x(x)}{\arctan^{\mu-1}(\lambda x)}\right].$

32. $\displaystyle\int_a^x \frac{y(t)\,dt}{\big[\arctan(\lambda x) - \arctan(\lambda t)\big]^\mu} = f(x), \qquad 0 < \mu < 1.$

Solution:

$$y(x) = \frac{\lambda\sin(\pi\mu)}{\pi}\frac{d}{dx}\int_a^x \frac{\varphi(t)f(t)\,dt}{[\arctan(\lambda x) - \arctan(\lambda t)]^{1-\mu}}, \qquad \varphi(x) = \frac{1}{1+\lambda^2x^2}.$$

33. $\displaystyle\int_a^x \big[A\arctan^\beta(\lambda x) + B\arctan^\gamma(\mu t) + C\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\arctan^\beta(\lambda x)$ and $h(t) = B\arctan^\gamma(\mu t) + C$.

1.6-4. Kernels Containing Arccotangent.

34. $\displaystyle\int_a^x \big[\operatorname{arccot}(\lambda x) - \operatorname{arccot}(\lambda t)\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \operatorname{arccot}(\lambda x)$.

Solution: $\displaystyle y(x) = -\frac{1}{\lambda}\frac{d}{dx}\big[(1+\lambda^2x^2)\,f'_x(x)\big].$

35. $\displaystyle\int_a^x \big[A\operatorname{arccot}(\lambda x) + B\operatorname{arccot}(\lambda t)\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \operatorname{arccot}(\lambda x)$. For $B = -A$, see equation 1.6.34.

Solution:

$$y(x) = \frac{1}{A+B}\frac{d}{dx}\left\{\big[\operatorname{arccot}(\lambda x)\big]^{-\frac{A}{A+B}} \int_a^x \big[\operatorname{arccot}(\lambda t)\big]^{-\frac{B}{A+B}} f'_t(t)\,dt\right\}.$$

36. $\displaystyle\int_a^x \big[A\operatorname{arccot}(\lambda x) + B\operatorname{arccot}(\mu t) + C\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A\operatorname{arccot}(\lambda x)$ and $h(t) = B\operatorname{arccot}(\mu t) + C$.

37. $\displaystyle\int_a^x \big[\operatorname{arccot}(\lambda x) - \operatorname{arccot}(\lambda t)\big]^n y(t)\,dt = f(x), \qquad n = 1, 2, \dots$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \cdots = f_x^{(n)}(a) = 0$.

Solution:

$$y(x) = \frac{(-1)^n}{\lambda^n n!\,(1+\lambda^2 x^2)}\left((1+\lambda^2x^2)\frac{d}{dx}\right)^{n+1} f(x).$$

38. $\displaystyle\int_a^x \sqrt{\operatorname{arccot}(\lambda t) - \operatorname{arccot}(\lambda x)}\, y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi}\varphi(x)\left(\frac{1}{\varphi(x)}\frac{d}{dx}\right)^2 \int_a^x \frac{\varphi(t) f(t)\,dt}{\sqrt{\operatorname{arccot}(\lambda t) - \operatorname{arccot}(\lambda x)}}, \qquad \varphi(x) = \frac{1}{1+\lambda^2 x^2}.$$

39. $\displaystyle\int_a^x \frac{y(t)\,dt}{\sqrt{\operatorname{arccot}(\lambda t) - \operatorname{arccot}(\lambda x)}} = f(x).$

Solution:

$$y(x) = \frac{\lambda}{\pi}\frac{d}{dx}\int_a^x \frac{\varphi(t) f(t)\,dt}{\sqrt{\operatorname{arccot}(\lambda t) - \operatorname{arccot}(\lambda x)}}, \qquad \varphi(x) = \frac{1}{1+\lambda^2 x^2}.$$

40. $\displaystyle\int_a^x \big[\operatorname{arccot}(\lambda t) - \operatorname{arccot}(\lambda x)\big]^\mu y(t)\,dt = f(x), \qquad 0 < \mu < 1.$

Solution:

$$y(x) = k\varphi(x)\left(\frac{1}{\varphi(x)}\frac{d}{dx}\right)^2 \int_a^x \frac{\varphi(t) f(t)\,dt}{[\operatorname{arccot}(\lambda t) - \operatorname{arccot}(\lambda x)]^\mu},$$
$$\varphi(x) = \frac{1}{1+\lambda^2 x^2}, \qquad k = \frac{\sin(\pi\mu)}{\pi\mu}.$$

41. $\displaystyle\int_a^x \big[\operatorname{arccot}^\mu(\lambda x) - \operatorname{arccot}^\mu(\lambda t)\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \operatorname{arccot}^\mu(\lambda x)$.

Solution: $\displaystyle y(x) = -\frac{1}{\lambda\mu}\frac{d}{dx}\left[\frac{(1+\lambda^2x^2) f'_x(x)}{\operatorname{arccot}^{\mu-1}(\lambda x)}\right].$

42. $\displaystyle\int_a^x \frac{y(t)\,dt}{\big[\operatorname{arccot}(\lambda t) - \operatorname{arccot}(\lambda x)\big]^\mu} = f(x), \qquad 0 < \mu < 1.$

Solution:

$$y(x) = \frac{\lambda \sin(\pi\mu)}{\pi}\frac{d}{dx}\int_a^x \frac{\varphi(t) f(t)\,dt}{[\operatorname{arccot}(\lambda t) - \operatorname{arccot}(\lambda x)]^{1-\mu}}, \qquad \varphi(x) = \frac{1}{1+\lambda^2 x^2}.$$

43. $\displaystyle\int_a^x \big[A \operatorname{arccot}^\beta(\lambda x) + B \operatorname{arccot}^\gamma(\mu t) + C\big] y(t)\,dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A \operatorname{arccot}^\beta(\lambda x)$ and $h(t) = B \operatorname{arccot}^\gamma(\mu t) + C$.

1.7. Equations Whose Kernels Contain Combinations of Elementary Functions

1.7-1. Kernels Containing Exponential and Hyperbolic Functions.

1. $\displaystyle\int_a^x e^{\mu(x-t)}\big\{A_1\cosh[\lambda_1(x-t)]+A_2\cosh[\lambda_2(x-t)]\big\}y(t)\,dt=f(x).$

The substitution $w(x)=e^{-\mu x}y(x)$ leads to an equation of the form 1.3.8:

$$\int_a^x \big\{A_1\cosh[\lambda_1(x-t)]+A_2\cosh[\lambda_2(x-t)]\big\}w(t)\,dt=e^{-\mu x}f(x).$$

2. $\displaystyle\int_a^x e^{\mu(x-t)}\cosh^2[\lambda(x-t)]y(t)\,dt=f(x).$

Solution:

$$y(x)=\varphi(x)-\frac{2\lambda^2}{k}\int_a^x e^{\mu(x-t)}\sinh[k(x-t)]\varphi(x)\,dt,\qquad k=\lambda\sqrt{2},\quad \varphi(x)=f'_x(x)-\mu f(x).$$

3. $\displaystyle\int_a^x e^{\mu(x-t)}\cosh^3[\lambda(x-t)]y(t)\,dt=f(x).$

The substitution $w(x)=e^{-\mu x}y(x)$ leads to an equation of the form 1.3.15:

$$\int_a^x \cosh^3[\lambda(x-t)]w(t)\,dt=e^{-\mu x}f(x).$$

4. $\displaystyle\int_a^x e^{\mu(x-t)}\cosh^4[\lambda(x-t)]y(t)\,dt=f(x).$

The substitution $w(x)=e^{-\mu x}y(x)$ leads to an equation of the form 1.3.19:

$$\int_a^x \cosh^4[\lambda(x-t)]w(t)\,dt=e^{-\mu x}f(x).$$

5. $\displaystyle\int_a^x e^{\mu(x-t)}\big[\cosh(\lambda x)-\cosh(\lambda t)\big]^n y(t)\,dt=f(x),\qquad n=1,2,\ldots$

Solution:

$$y(x)=\frac{1}{\lambda^n n!}e^{\mu x}\sinh(\lambda x)\left[\frac{1}{\sinh(\lambda x)}\frac{d}{dx}\right]^{n+1}F_\mu(x),\qquad F_\mu(x)=e^{-\mu x}f(x).$$

6. $\displaystyle\int_a^x e^{\mu(x-t)}\sqrt{\cosh x-\cosh t}\,y(t)\,dt=f(x),\qquad f(a)=0.$

Solution:

$$y(x)=\frac{2}{\pi}e^{\mu x}\sinh x\left(\frac{1}{\sinh x}\frac{d}{dx}\right)^2\int_a^x\frac{e^{-\mu t}\sinh t\,f(t)\,dt}{\sqrt{\cosh x-\cosh t}}.$$

7. $\displaystyle\int_a^x \frac{e^{\mu(x-t)}y(t)\,dt}{\sqrt{\cosh x-\cosh t}}=f(x).$

Solution:

$$y(x)=\frac{1}{\pi}e^{\mu x}\frac{d}{dx}\int_a^x\frac{e^{-\mu t}\sinh t\,f(t)\,dt}{\sqrt{\cosh x-\cosh t}}.$$

8. $\displaystyle\int_a^x e^{\mu(x-t)}(\cosh x - \cosh t)^\lambda y(t)\,dt = f(x), \qquad 0 < \lambda < 1.$

The substitution $w(x) = e^{-\mu x}y(x)$ leads to an equation of the form 1.3.23:

$$\int_a^x (\cosh x - \cosh t)^\lambda w(t)\,dt = e^{-\mu x}f(x).$$

9. $\displaystyle\int_a^x \bigl[Ae^{\mu(x-t)} + B\cosh^\lambda x\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B\cosh^\lambda x$, and $h_2(t) = 1$.

10. $\displaystyle\int_a^x \bigl[Ae^{\mu(x-t)} + B\cosh^\lambda t\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B$, and $h_2(t) = \cosh^\lambda t$.

11. $\displaystyle\int_a^x e^{\mu(x-t)}(\cosh^\lambda x - \cosh^\lambda t)y(t)\,dt = f(x).$

The substitution $w(x) = e^{-\mu x}y(x)$ leads to an equation of the form 1.3.24:

$$\int_a^x (\cosh^\lambda x - \cosh^\lambda t)w(t)\,dt = e^{-\mu x}f(x).$$

12. $\displaystyle\int_a^x e^{\mu(x-t)}\bigl(A\cosh^\lambda x + B\cosh^\lambda t\bigr)y(t)\,dt = f(x).$

The substitution $w(x) = e^{-\mu x}y(x)$ leads to an equation of the form 1.3.25:

$$\int_a^x \bigl(A\cosh^\lambda x + B\cosh^\lambda t\bigr)w(t)\,dt = e^{-\mu x}f(x).$$

13. $\displaystyle\int_a^x \frac{e^{\mu(x-t)}y(t)\,dt}{(\cosh x - \cosh t)^\lambda} = f(x), \qquad 0 < \lambda < 1.$

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi}e^{\mu x}\frac{d}{dx}\int_a^x \frac{e^{-\mu t}\,\sinh t\, f(t)\,dt}{(\cosh x - \cosh t)^{1-\lambda}}.$$

14. $\displaystyle\int_a^x e^{\mu(x-t)}\bigl\{A_1\sinh[\lambda_1(x-t)] + A_2\sinh[\lambda_2(x-t)]\bigr\}y(t)\,dt = f(x).$

The substitution $w(x) = e^{-\mu x}y(x)$ leads to an equation of the form 1.3.49:

$$\int_a^x \bigl\{A_1\sinh[\lambda_1(x-t)] + A_2\sinh[\lambda_2(x-t)]\bigr\}w(t)\,dt = e^{-\mu x}f(x).$$

15. $\displaystyle\int_a^x e^{\mu(x-t)}\sinh^2[\lambda(x-t)]y(t)\,dt = f(x).$

The substitution $w(x) = e^{-\mu x}y(x)$ leads to an equation of the form 1.3.51:

$$\int_a^x \sinh^2[\lambda(x-t)]w(t)\,dt = e^{-\mu x}f(x).$$

16. $\displaystyle\int_a^x e^{\mu(x-t)} \sinh^3[\lambda(x-t)]y(t)\,dt = f(x).$

The substitution $w(x) = e^{-\mu x}y(x)$ leads to an equation of the form 1.3.57:

$$\int_a^x \sinh^3[\lambda(x-t)]w(t)\,dt = e^{-\mu x}f(x).$$

17. $\displaystyle\int_a^x e^{\mu(x-t)} \sinh^n[\lambda(x-t)]y(t)\,dt = f(x), \qquad n = 2, 3, \dots$

The substitution $w(x) = e^{-\mu x}y(x)$ leads to an equation of the form 1.3.62:

$$\int_a^x \sinh^n[\lambda(x-t)]w(t)\,dt = e^{-\mu x}f(x).$$

18. $\displaystyle\int_a^x e^{\mu(x-t)} \sinh\bigl(k\sqrt{x-t}\,\bigr)y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi k}e^{\mu x}\frac{d^2}{dx^2}\int_a^x \frac{e^{-\mu t}\cos\bigl(k\sqrt{x-t}\,\bigr)}{\sqrt{x-t}}f(t)\,dt.$$

19. $\displaystyle\int_a^x e^{\mu(x-t)}\sqrt{\sinh x - \sinh t}\,y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi}e^{\mu x}\cosh x\left(\frac{1}{\cosh x}\frac{d}{dx}\right)^2\int_a^x \frac{e^{-\mu t}\cosh t\,f(t)\,dt}{\sqrt{\sinh x - \sinh t}}.$$

20. $\displaystyle\int_a^x \frac{e^{\mu(x-t)}y(t)\,dt}{\sqrt{\sinh x - \sinh t}} = f(x).$

Solution:

$$y(x) = \frac{1}{\pi}e^{\mu x}\frac{d}{dx}\int_a^x \frac{e^{-\mu t}\cosh t\,f(t)\,dt}{\sqrt{\sinh x - \sinh t}}.$$

21. $\displaystyle\int_a^x e^{\mu(x-t)}(\sinh x - \sinh t)^\lambda y(t)\,dt = f(x), \qquad 0 < \lambda < 1.$

The substitution $w(x) = e^{-\mu x}y(x)$ leads to an equation of the form 1.3.67:

$$\int_a^x (\sinh x - \sinh t)^\lambda w(t)\,dt = e^{-\mu x}f(x).$$

22. $\displaystyle\int_a^x e^{\mu(x-t)}(\sinh^\lambda x - \sinh^\lambda t)y(t)\,dt = f(x).$

The substitution $w(x) = e^{-\mu x}y(x)$ leads to an equation of the form 1.3.68:

$$\int_a^x (\sinh^\lambda x - \sinh^\lambda t)w(t)\,dt = e^{-\mu x}f(x).$$

23. $\displaystyle\int_a^x e^{\mu(x-t)}\bigl(A\sinh^\lambda x + B\sinh^\lambda t\bigr)y(t)\,dt = f(x).$

The substitution $w(x) = e^{-\mu x}y(x)$ leads to an equation of the form 1.3.69:

$$\int_a^x \bigl(A\sinh^\lambda x + B\sinh^\lambda t\bigr)w(t)\,dt = e^{-\mu x}f(x).$$

24. $\displaystyle \int_a^x \bigl[Ae^{\mu(x-t)} + B\sinh^\lambda x\bigr] y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B\sinh^\lambda x$, and $h_2(t) = 1$.

25. $\displaystyle \int_a^x \bigl[Ae^{\mu(x-t)} + B\sinh^\lambda t\bigr] y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B$, and $h_2(t) = \sinh^\lambda t$.

26. $\displaystyle \int_a^x \frac{e^{\mu(x-t)} y(t)\,dt}{(\sinh x - \sinh t)^\lambda} = f(x), \qquad 0 < \lambda < 1.$

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi} e^{\mu x} \frac{d}{dx} \int_a^x \frac{e^{-\mu t}\cosh t\, f(t)\,dt}{(\sinh x - \sinh t)^{1-\lambda}}.$$

27. $\displaystyle \int_a^x e^{\mu(x-t)} \bigl(A\tanh^\lambda x + B\tanh^\lambda t\bigr) y(t)\,dt = f(x).$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.3.89:

$$\int_a^x \bigl(A\tanh^\lambda x + B\tanh^\lambda t\bigr) w(t)\,dt = e^{-\mu x} f(x).$$

28. $\displaystyle \int_a^x e^{\mu(x-t)} \bigl(A\tanh^\lambda x + B\tanh^\beta t + C\bigr) y(t)\,dt = f(x).$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.9.6 with $g(x) = A\tanh^\lambda x$, $h(t) = B\tanh^\beta t + C$:

$$\int_a^x \bigl(A\tanh^\lambda x + B\tanh^\beta t + C\bigr) w(t)\,dt = e^{-\mu x} f(x).$$

29. $\displaystyle \int_a^x \bigl[Ae^{\mu(x-t)} + B\tanh^\lambda x\bigr] y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B\tanh^\lambda x$, and $h_2(t) = 1$.

30. $\displaystyle \int_a^x \bigl[Ae^{\mu(x-t)} + B\tanh^\lambda t\bigr] y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B$, and $h_2(t) = \tanh^\lambda t$.

31. $\displaystyle \int_a^x e^{\mu(x-t)} \bigl(A\coth^\lambda x + B\coth^\lambda t\bigr) y(t)\,dt = f(x).$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.3.102:

$$\int_a^x \bigl(A\coth^\lambda x + B\coth^\lambda t\bigr) w(t)\,dt = e^{-\mu x} f(x).$$

32. $\displaystyle\int_a^x e^{\mu(x-t)}\big(A\coth^\lambda x + B\coth^\beta t + C\big)y(t)\,dt = f(x).$

The substitution $w(x) = e^{-\mu x}y(x)$ leads to an equation of the form 1.9.6 with $g(x) = A\coth^\lambda x$, $h(t) = B\coth^\beta t + C$:

$$\int_a^x \big(A\coth^\lambda x + B\coth^\beta t + C\big)w(t)\,dt = e^{-\mu x}f(x).$$

33. $\displaystyle\int_a^x \big[Ae^{\mu(x-t)} + B\coth^\lambda x\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B\coth^\lambda x$, and $h_2(t) = 1$.

34. $\displaystyle\int_a^x \big[Ae^{\mu(x-t)} + B\coth^\lambda t\big]y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B$, and $h_2(t) = \coth^\lambda t$.

1.7-2. Kernels Containing Exponential and Logarithmic Functions.

35. $\displaystyle\int_a^x e^{\lambda(x-t)}(\ln x - \ln t)y(t)\,dt = f(x).$

Solution:

$$y(x) = e^{\lambda x}\big[x\varphi''_{xx}(x) + \varphi'_x(x)\big], \qquad \varphi(x) = e^{-\lambda x}f(x).$$

36. $\displaystyle\int_0^x e^{\lambda(x-t)}\ln(x-t)y(t)\,dt = f(x).$

The substitution $w(x) = e^{-\lambda x}y(x)$ leads to an equation of the form 1.4.2:

$$\int_0^x \ln(x-t)w(t)\,dt = e^{-\lambda x}f(x).$$

37. $\displaystyle\int_a^x e^{\lambda(x-t)}(A\ln x + B\ln t)y(t)\,dt = f(x).$

The substitution $w(x) = e^{-\lambda x}y(x)$ leads to an equation of the form 1.4.4:

$$\int_a^x (A\ln x + B\ln t)w(t)\,dt = e^{-\lambda x}f(x).$$

38. $\displaystyle\int_a^x e^{\mu(x-t)}\big[A\ln^2(\lambda x) + B\ln^2(\lambda t)\big]y(t)\,dt = f(x).$

The substitution $w(x) = e^{-\lambda x}y(x)$ leads to an equation of the form 1.4.7:

$$\int_a^x \big[A\ln^2(\lambda x) + B\ln^2(\lambda t)\big]w(t)\,dt = e^{-\lambda x}f(x).$$

39. $\displaystyle\int_a^x e^{\lambda(x-t)}\bigl[\ln(x/t)\bigr]^n y(t)\,dt = f(x), \qquad n = 1, 2, \dots$

Solution:

$$y(x) = \frac{1}{n!\,x}e^{\lambda x}\left(x\frac{d}{dx}\right)^{n+1}F_\lambda(x), \qquad F_\lambda(x) = e^{-\lambda x}f(x).$$

40. $\displaystyle\int_a^x e^{\lambda(x-t)}\sqrt{\ln(x/t)}\,y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{2e^{\lambda x}}{\pi x}\left(x\frac{d}{dx}\right)^2\int_a^x \frac{e^{-\lambda t}f(t)\,dt}{t\sqrt{\ln(x/t)}}.$$

41. $\displaystyle\int_a^x \frac{e^{\lambda(x-t)}}{\sqrt{\ln(x/t)}}y(t)\,dt = f(x).$

Solution:

$$y(x) = \frac{1}{\pi}e^{\lambda x}\frac{d}{dx}\int_a^x \frac{e^{-\lambda t}f(t)\,dt}{t\sqrt{\ln(x/t)}}.$$

42. $\displaystyle\int_a^x \bigl[Ae^{\mu(x-t)} + B\ln^\nu(\lambda x)\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B\ln^\nu(\lambda x)$, and $h_2(t) = 1$.

43. $\displaystyle\int_a^x \bigl[Ae^{\mu(x-t)} + B\ln^\nu(\lambda t)\bigr]y(t)\,dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B$, and $h_2(t) = \ln^\nu(\lambda t)$.

44. $\displaystyle\int_a^x e^{\mu(x-t)}[\ln(x/t)]^\lambda y(t)\,dt = f(x), \qquad 0 < \lambda < 1.$

The substitution $w(x) = e^{-\mu x}y(x)$ leads to an equation of the form 1.4.16:

$$\int_a^x [\ln(x/t)]^\lambda w(t)\,dt = e^{-\mu x}f(x).$$

45. $\displaystyle\int_a^x \frac{e^{\mu(x-t)}}{[\ln(x/t)]^\lambda}y(t)\,dt = f(x), \qquad 0 < \lambda < 1.$

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi}e^{\mu x}\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{te^{\mu t}[\ln(x/t)]^{1-\lambda}}.$$

1.7-3. Kernels Containing Exponential and Trigonometric Functions.

46. $\displaystyle\int_a^x e^{\mu(x-t)}\cos[\lambda(x-t)]y(t)\,dt = f(x).$

Solution: $\displaystyle y(x) = f'_x(x) - \mu f(x) + \lambda^2\int_a^x e^{\mu(x-t)}f(t)\,dt.$

47. $\displaystyle\int_a^x e^{\mu(x-t)}\big\{A_1\cos[\lambda_1(x-t)]+A_2\cos[\lambda_2(x-t)]\big\}y(t)\,dt=f(x).$

The substitution $w(x)=e^{-\mu x}y(x)$ leads to an equation of the form 1.5.8:

$$\int_a^x \big\{A_1\cos[\lambda_1(x-t)]+A_2\cos[\lambda_2(x-t)]\big\}w(t)\,dt=e^{-\mu x}f(x).$$

48. $\displaystyle\int_a^x e^{\mu(x-t)}\cos^2[\lambda(x-t)]y(t)\,dt=f(x).$

The substitution $w(x)=e^{-\mu x}y(x)$ leads to an equation of the form 1.5.9.

Solution:

$$y(x)=\varphi(x)+\frac{2\lambda^2}{k}\int_a^x e^{\mu(x-t)}\sin[k(x-t)]\varphi(t)\,dt,\qquad k=\lambda\sqrt{2},\quad \varphi(x)=f'_x(x)-\mu f(x).$$

49. $\displaystyle\int_a^x e^{\mu(x-t)}\cos^3[\lambda(x-t)]y(t)\,dt=f(x).$

The substitution $w(x)=e^{-\mu x}y(x)$ leads to an equation of the form 1.5.16:

$$\int_a^x \cos^3[\lambda(x-t)]w(t)\,dt=e^{-\mu x}f(x).$$

50. $\displaystyle\int_a^x e^{\mu(x-t)}\cos^4[\lambda(x-t)]y(t)\,dt=f(x).$

The substitution $w(x)=e^{-\mu x}y(x)$ leads to an equation of the form 1.5.20:

$$\int_a^x \cos^4[\lambda(x-t)]w(t)\,dt=e^{-\mu x}f(x).$$

51. $\displaystyle\int_a^x e^{\mu(x-t)}\big[\cos(\lambda x)-\cos(\lambda t)\big]^n y(t)\,dt=f(x),\qquad n=1,2,\dots$

The right-hand side of the equation is assumed to satisfy the conditions $f(a)=f'_x(a)=\cdots=f^{(n)}_x(a)=0$.

Solution:

$$y(x)=\frac{(-1)^n}{\lambda^n n!}e^{\mu x}\sin(\lambda x)\left[\frac{1}{\sin(\lambda x)}\frac{d}{dx}\right]^{n+1}F_\mu(x),\qquad F_\mu(x)=e^{-\mu x}f(x).$$

52. $\displaystyle\int_a^x e^{\mu(x-t)}\sqrt{\cos t-\cos x}\;y(t)\,dt=f(x).$

Solution:

$$y(x)=\frac{2}{\pi}e^{\mu x}\sin x\left(\frac{1}{\sin x}\frac{d}{dx}\right)^2\int_a^x \frac{e^{-\mu t}\sin t\,f(t)\,dt}{\sqrt{\cos t-\cos x}}.$$

53. $\displaystyle\int_a^x \frac{e^{\mu(x-t)}y(t)\,dt}{\sqrt{\cos t-\cos x}}=f(x).$

Solution:

$$y(x)=\frac{1}{\pi}e^{\mu x}\frac{d}{dx}\int_a^x \frac{e^{-\mu t}\sin t\,f(t)\,dt}{\sqrt{\cos t-\cos x}}.$$

54. $\displaystyle\int_a^x e^{\mu(x-t)}(\cos t-\cos x)^{\lambda}y(t)\,dt=f(x),\qquad 0<\lambda<1.$

Solution:

$$y(x)=ke^{\mu x}\sin x\left(\frac{1}{\sin x}\frac{d}{dx}\right)^2\int_a^x\frac{e^{-\mu t}\sin t\,f(t)\,dt}{(\cos t-\cos x)^{\lambda}},\qquad k=\frac{\sin(\pi\lambda)}{\pi\lambda}.$$

55. $\displaystyle\int_a^x e^{\mu(x-t)}(\cos^{\lambda}x-\cos^{\lambda}t)y(t)\,dt=f(x).$

The substitution $w(x)=e^{-\mu x}y(x)$ leads to an equation of the form 1.5.25:

$$\int_a^x(\cos^{\lambda}x-\cos^{\lambda}t)w(t)\,dt=e^{-\mu x}f(x).$$

56. $\displaystyle\int_a^x e^{\mu(x-t)}\big(A\cos^{\lambda}x+B\cos^{\lambda}t\big)y(t)\,dt=f(x).$

The substitution $w(x)=e^{-\mu x}y(x)$ leads to an equation of the form 1.5.26:

$$\int_a^x\big(A\cos^{\lambda}x+B\cos^{\lambda}t\big)w(t)\,dt=e^{-\mu x}f(x).$$

57. $\displaystyle\int_a^x\frac{e^{\mu(x-t)}y(t)\,dt}{(\cos t-\cos x)^{\lambda}}=f(x),\qquad 0<\lambda<1.$

The substitution $w(x)=e^{-\mu x}y(x)$ leads to an equation of the form 1.5.27:

$$\int_a^x\frac{w(t)\,dt}{(\cos t-\cos x)^{\lambda}}=e^{-\mu x}f(x).$$

58. $\displaystyle\int_a^x\big[Ae^{\mu(x-t)}+B\cos^{\nu}(\lambda x)\big]y(t)\,dt=f(x).$

This is a special case of equation 1.9.15 with $g_1(x)=Ae^{\mu x}$, $h_1(t)=e^{-\mu t}$, $g_2(x)=B\cos^{\nu}(\lambda x)$, and $h_2(t)=1$.

59. $\displaystyle\int_a^x\big[Ae^{\mu(x-t)}+B\cos^{\nu}(\lambda t)\big]y(t)\,dt=f(x).$

This is a special case of equation 1.9.15 with $g_1(x)=Ae^{\mu x}$, $h_1(t)=e^{-\mu t}$, $g_2(x)=B$, and $h_2(t)=\cos^{\nu}(\lambda t)$.

60. $\displaystyle\int_a^x e^{\mu(x-t)}\sin[\lambda(x-t)]y(t)\,dt=f(x),\qquad f(a)=f'_x(a)=0.$

Solution: $y(x)=\frac{1}{\lambda}\big[f''_{xx}(x)-2\mu f'_x(x)+(\lambda^2+\mu^2)f(x)\big]$.

61. $\displaystyle\int_a^x e^{\mu(x-t)}\big\{A_1\sin[\lambda_1(x-t)]+A_2\sin[\lambda_2(x-t)]\big\}y(t)\,dt=f(x).$

The substitution $w(x)=e^{-\mu x}y(x)$ leads to an equation of the form 1.5.48:

$$\int_a^x\big\{A_1\sin[\lambda_1(x-t)]+A_2\sin[\lambda_2(x-t)]\big\}w(t)\,dt=e^{-\mu x}f(x).$$