

**Andrei D. Polyanin
Alexander V. Manzhirov**



HANDBOOK OF INTEGRAL EQUATIONS

SECOND EDITION

 **Chapman & Hall/CRC**
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**HANDBOOK OF
INTEGRAL
EQUATIONS**

SECOND EDITION

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CONTENTS

Authors	xxix
Preface	xxxi
Some Remarks and Notation	xxxiii

Part I. Exact Solutions of Integral Equations

1. Linear Equations of the First Kind with Variable Limit of Integration	3
1.1. Equations Whose Kernels Contain Power-Law Functions	4
1.1-1. Kernels Linear in the Arguments x and t	4
1.1-2. Kernels Quadratic in the Arguments x and t	4
1.1-3. Kernels Cubic in the Arguments x and t	5
1.1-4. Kernels Containing Higher-Order Polynomials in x and t	6
1.1-5. Kernels Containing Rational Functions	7
1.1-6. Kernels Containing Square Roots	9
1.1-7. Kernels Containing Arbitrary Powers	12
1.1-8. Two-Dimensional Equation of the Abel Type	15
1.2. Equations Whose Kernels Contain Exponential Functions	15
1.2-1. Kernels Containing Exponential Functions	15
1.2-2. Kernels Containing Power-Law and Exponential Functions	19
1.3. Equations Whose Kernels Contain Hyperbolic Functions	22
1.3-1. Kernels Containing Hyperbolic Cosine	22
1.3-2. Kernels Containing Hyperbolic Sine	28
1.3-3. Kernels Containing Hyperbolic Tangent	36
1.3-4. Kernels Containing Hyperbolic Cotangent	38
1.3-5. Kernels Containing Combinations of Hyperbolic Functions	39
1.4. Equations Whose Kernels Contain Logarithmic Functions	42
1.4-1. Kernels Containing Logarithmic Functions	42
1.4-2. Kernels Containing Power-Law and Logarithmic Functions	45
1.5. Equations Whose Kernels Contain Trigonometric Functions	46
1.5-1. Kernels Containing Cosine	46
1.5-2. Kernels Containing Sine	52
1.5-3. Kernels Containing Tangent	60
1.5-4. Kernels Containing Cotangent	62
1.5-5. Kernels Containing Combinations of Trigonometric Functions	63
1.6. Equations Whose Kernels Contain Inverse Trigonometric Functions	66
1.6-1. Kernels Containing Arccosine	66
1.6-2. Kernels Containing Arcsine	68
1.6-3. Kernels Containing Arctangent	70
1.6-4. Kernels Containing Arccotangent	71

1.7. Equations Whose Kernels Contain Combinations of Elementary Functions	73
1.7-1. Kernels Containing Exponential and Hyperbolic Functions	73
1.7-2. Kernels Containing Exponential and Logarithmic Functions	77
1.7-3. Kernels Containing Exponential and Trigonometric Functions	78
1.7-4. Kernels Containing Hyperbolic and Logarithmic Functions	83
1.7-5. Kernels Containing Hyperbolic and Trigonometric Functions	84
1.7-6. Kernels Containing Logarithmic and Trigonometric Functions	85
1.8. Equations Whose Kernels Contain Special Functions	86
1.8-1. Kernels Containing Error Function or Exponential Integral	86
1.8-2. Kernels Containing Sine and Cosine Integrals	87
1.8-3. Kernels Containing Fresnel Integrals	87
1.8-4. Kernels Containing Incomplete Gamma Functions	88
1.8-5. Kernels Containing Bessel Functions	88
1.8-6. Kernels Containing Modified Bessel Functions	97
1.8-7. Kernels Containing Legendre Polynomials	105
1.8-8. Kernels Containing Associated Legendre Functions	107
1.8-9. Kernels Containing Confluent Hypergeometric Functions	107
1.8-10. Kernels Containing Hermite Polynomials	108
1.8-11. Kernels Containing Chebyshev Polynomials	109
1.8-12. Kernels Containing Laguerre Polynomials	110
1.8-13. Kernels Containing Jacobi Theta Functions	110
1.8-14. Kernels Containing Other Special Functions	111
1.9. Equations Whose Kernels Contain Arbitrary Functions	111
1.9-1. Equations with Degenerate Kernel: $K(x, t) = g_1(x)h_1(t) + g_2(x)h_2(t)$	111
1.9-2. Equations with Difference Kernel: $K(x, t) = K(x - t)$	114
1.9-3. Other Equations	122
1.10. Some Formulas and Transformations	124
2. Linear Equations of the Second Kind with Variable Limit of Integration	127
2.1. Equations Whose Kernels Contain Power-Law Functions	127
2.1-1. Kernels Linear in the Arguments x and t	127
2.1-2. Kernels Quadratic in the Arguments x and t	129
2.1-3. Kernels Cubic in the Arguments x and t	132
2.1-4. Kernels Containing Higher-Order Polynomials in x and t	133
2.1-5. Kernels Containing Rational Functions	136
2.1-6. Kernels Containing Square Roots and Fractional Powers	138
2.1-7. Kernels Containing Arbitrary Powers	139
2.2. Equations Whose Kernels Contain Exponential Functions	144
2.2-1. Kernels Containing Exponential Functions	144
2.2-2. Kernels Containing Power-Law and Exponential Functions	151
2.3. Equations Whose Kernels Contain Hyperbolic Functions	154
2.3-1. Kernels Containing Hyperbolic Cosine	154
2.3-2. Kernels Containing Hyperbolic Sine	156
2.3-3. Kernels Containing Hyperbolic Tangent	161
2.3-4. Kernels Containing Hyperbolic Cotangent	162
2.3-5. Kernels Containing Combinations of Hyperbolic Functions	164
2.4. Equations Whose Kernels Contain Logarithmic Functions	164
2.4-1. Kernels Containing Logarithmic Functions	164
2.4-2. Kernels Containing Power-Law and Logarithmic Functions	165

2.5. Equations Whose Kernels Contain Trigonometric Functions	166
2.5-1. Kernels Containing Cosine	166
2.5-2. Kernels Containing Sine	169
2.5-3. Kernels Containing Tangent	174
2.5-4. Kernels Containing Cotangent	175
2.5-5. Kernels Containing Combinations of Trigonometric Functions	176
2.6. Equations Whose Kernels Contain Inverse Trigonometric Functions	176
2.6-1. Kernels Containing Arccosine	176
2.6-2. Kernels Containing Arcsine	177
2.6-3. Kernels Containing Arctangent	178
2.6-4. Kernels Containing Arccotangent	178
2.7. Equations Whose Kernels Contain Combinations of Elementary Functions	179
2.7-1. Kernels Containing Exponential and Hyperbolic Functions	179
2.7-2. Kernels Containing Exponential and Logarithmic Functions	180
2.7-3. Kernels Containing Exponential and Trigonometric Functions	181
2.7-4. Kernels Containing Hyperbolic and Logarithmic Functions	185
2.7-5. Kernels Containing Hyperbolic and Trigonometric Functions	186
2.7-6. Kernels Containing Logarithmic and Trigonometric Functions	187
2.8. Equations Whose Kernels Contain Special Functions	187
2.8-1. Kernels Containing Bessel Functions	187
2.8-2. Kernels Containing Modified Bessel Functions	189
2.9. Equations Whose Kernels Contain Arbitrary Functions	191
2.9-1. Equations with Degenerate Kernel: $K(x, t) = g_1(x)h_1(t) + \cdots + g_n(x)h_n(t)$	191
2.9-2. Equations with Difference Kernel: $K(x, t) = K(x - t)$	203
2.9-3. Other Equations	212
2.10. Some Formulas and Transformations	215
3. Linear Equations of the First Kind with Constant Limits of Integration	217
3.1. Equations Whose Kernels Contain Power-Law Functions	217
3.1-1. Kernels Linear in the Arguments x and t	217
3.1-2. Kernels Quadratic in the Arguments x and t	219
3.1-3. Kernels Containing Integer Powers of x and t or Rational Functions	220
3.1-4. Kernels Containing Square Roots	222
3.1-5. Kernels Containing Arbitrary Powers	223
3.1-6. Equations Containing the Unknown Function of a Complicated Argument	227
3.1-7. Singular Equations	228
3.2. Equations Whose Kernels Contain Exponential Functions	231
3.2-1. Kernels Containing Exponential Functions of the Form $e^{\lambda x-t }$	231
3.2-2. Kernels Containing Exponential Functions of the Forms $e^{\lambda x}$ and $e^{\mu t}$	234
3.2-3. Kernels Containing Exponential Functions of the Form $e^{\lambda xt}$	234
3.2-4. Kernels Containing Power-Law and Exponential Functions	236
3.2-5. Kernels Containing Exponential Functions of the Form $e^{\lambda(x \pm t)^2}$	236
3.2-6. Other Kernels	237
3.3. Equations Whose Kernels Contain Hyperbolic Functions	238
3.3-1. Kernels Containing Hyperbolic Cosine	238
3.3-2. Kernels Containing Hyperbolic Sine	238
3.3-3. Kernels Containing Hyperbolic Tangent	241
3.3-4. Kernels Containing Hyperbolic Cotangent	242

3.4. Equations Whose Kernels Contain Logarithmic Functions	242
3.4-1. Kernels Containing Logarithmic Functions	242
3.4-2. Kernels Containing Power-Law and Logarithmic Functions	244
3.4-3. Equation Containing the Unknown Function of a Complicated Argument	246
3.5. Equations Whose Kernels Contain Trigonometric Functions	246
3.5-1. Kernels Containing Cosine	246
3.5-2. Kernels Containing Sine	247
3.5-3. Kernels Containing Tangent	251
3.5-4. Kernels Containing Cotangent	252
3.5-5. Kernels Containing a Combination of Trigonometric Functions	252
3.5-6. Equations Containing the Unknown Function of a Complicated Argument	254
3.5-7. Singular Equations	255
3.6. Equations Whose Kernels Contain Combinations of Elementary Functions	255
3.6-1. Kernels Containing Hyperbolic and Logarithmic Functions	255
3.6-2. Kernels Containing Logarithmic and Trigonometric Functions	256
3.6-3. Kernels Containing Combinations of Exponential and Other Elementary Functions	257
3.7. Equations Whose Kernels Contain Special Functions	258
3.7-1. Kernels Containing Error Function, Exponential Integral or Logarithmic Integral	258
3.7-2. Kernels Containing Sine Integrals, Cosine Integrals, or Fresnel Integrals	258
3.7-3. Kernels Containing Gamma Functions	260
3.7-4. Kernels Containing Incomplete Gamma Functions	260
3.7-5. Kernels Containing Bessel Functions of the First Kind	261
3.7-6. Kernels Containing Bessel Functions of the Second Kind	264
3.7-7. Kernels Containing Combinations of the Bessel Functions	265
3.7-8. Kernels Containing Modified Bessel Functions of the First Kind	266
3.7-9. Kernels Containing Modified Bessel Functions of the Second Kind	266
3.7-10. Kernels Containing a Combination of Bessel and Modified Bessel Functions	269
3.7-11. Kernels Containing Legendre Functions	270
3.7-12. Kernels Containing Associated Legendre Functions	271
3.7-13. Kernels Containing Kummer Confluent Hypergeometric Functions	272
3.7-14. Kernels Containing Tricomi Confluent Hypergeometric Functions	274
3.7-15. Kernels Containing Whittaker Confluent Hypergeometric Functions	274
3.7-16. Kernels Containing Gauss Hypergeometric Functions	276
3.7-17. Kernels Containing Parabolic Cylinder Functions	276
3.7-18. Kernels Containing Other Special Functions	277
3.8. Equations Whose Kernels Contain Arbitrary Functions	278
3.8-1. Equations with Degenerate Kernel	278
3.8-2. Equations Containing Modulus	279
3.8-3. Equations with Difference Kernel: $K(x, t) = K(x - t)$	284
3.8-4. Other Equations of the Form $\int_a^b K(x, t)y(t) dt = F(x)$	285
3.8-5. Equations of the Form $\int_a^b K(x, t)y(\cdot \cdot) dt = F(x)$	289
3.9. Dual Integral Equations of the First Kind	295
3.9-1. Kernels Containing Trigonometric Functions	295
3.9-2. Kernels Containing Bessel Functions of the First Kind	297
3.9-3. Kernels Containing Bessel Functions of the Second Kind	299
3.9-4. Kernels Containing Legendre Spherical Functions of the First Kind, $i^2 = -1$	299

4. Linear Equations of the Second Kind with Constant Limits of Integration	301
4.1. Equations Whose Kernels Contain Power-Law Functions	301
4.1-1. Kernels Linear in the Arguments x and t	301
4.1-2. Kernels Quadratic in the Arguments x and t	304
4.1-3. Kernels Cubic in the Arguments x and t	307
4.1-4. Kernels Containing Higher-Order Polynomials in x and t	311
4.1-5. Kernels Containing Rational Functions	314
4.1-6. Kernels Containing Arbitrary Powers	317
4.1-7. Singular Equations	319
4.2. Equations Whose Kernels Contain Exponential Functions	320
4.2-1. Kernels Containing Exponential Functions	320
4.2-2. Kernels Containing Power-Law and Exponential Functions	326
4.3. Equations Whose Kernels Contain Hyperbolic Functions	327
4.3-1. Kernels Containing Hyperbolic Cosine	327
4.3-2. Kernels Containing Hyperbolic Sine	329
4.3-3. Kernels Containing Hyperbolic Tangent	332
4.3-4. Kernels Containing Hyperbolic Cotangent	333
4.3-5. Kernels Containing Combination of Hyperbolic Functions	334
4.4. Equations Whose Kernels Contain Logarithmic Functions	334
4.4-1. Kernels Containing Logarithmic Functions	334
4.4-2. Kernels Containing Power-Law and Logarithmic Functions	335
4.5. Equations Whose Kernels Contain Trigonometric Functions	335
4.5-1. Kernels Containing Cosine	335
4.5-2. Kernels Containing Sine	337
4.5-3. Kernels Containing Tangent	342
4.5-4. Kernels Containing Cotangent	343
4.5-5. Kernels Containing Combinations of Trigonometric Functions	344
4.5-6. Singular Equation	344
4.6. Equations Whose Kernels Contain Inverse Trigonometric Functions	344
4.6-1. Kernels Containing Arccosine	344
4.6-2. Kernels Containing Arcsine	345
4.6-3. Kernels Containing Arctangent	346
4.6-4. Kernels Containing Arccotangent	347
4.7. Equations Whose Kernels Contain Combinations of Elementary Functions	348
4.7-1. Kernels Containing Exponential and Hyperbolic Functions	348
4.7-2. Kernels Containing Exponential and Logarithmic Functions	349
4.7-3. Kernels Containing Exponential and Trigonometric Functions	349
4.7-4. Kernels Containing Hyperbolic and Logarithmic Functions	351
4.7-5. Kernels Containing Hyperbolic and Trigonometric Functions	352
4.7-6. Kernels Containing Logarithmic and Trigonometric Functions	353
4.8. Equations Whose Kernels Contain Special Functions	353
4.8-1. Kernels Containing Bessel Functions	353
4.8-2. Kernels Containing Modified Bessel Functions	355
4.9. Equations Whose Kernels Contain Arbitrary Functions	357
4.9-1. Equations with Degenerate Kernel: $K(x, t) = g_1(x)h_1(t) + \cdots + g_n(x)h_n(t)$	357
4.9-2. Equations with Difference Kernel: $K(x, t) = K(x - t)$	372
4.9-3. Other Equations of the Form $y(x) + \int_a^b K(x, t)y(t) dt = F(x)$	374
4.9-4. Equations of the Form $y(x) + \int_a^b K(x, t)y(\cdots) dt = F(x)$	381
4.10. Some Formulas and Transformations	390

5. Nonlinear Equations of the First Kind with Variable Limit of Integration	393
5.1. Equations with Quadratic Nonlinearity That Contain Arbitrary Parameters	393
5.1-1. Equations of the Form $\int_0^x y(t)y(x-t) dt = f(x)$	393
5.1-2. Equations of the Form $\int_0^x K(x,t)y(t)y(x-t) dt = f(x)$	395
5.1-3. Equations of the Form $\int_0^x y(t)y(\cdots) dt = f(x)$	396
5.2. Equations with Quadratic Nonlinearity That Contain Arbitrary Functions	397
5.2-1. Equations of the Form $\int_a^x K(x,t)[Ay(t) + By^2(t)] dt = f(x)$	397
5.2-2. Equations of the Form $\int_a^x K(x,t)y(t)y(ax + bt) dt = f(x)$	398
5.3. Equations with Nonlinearity of General Form	399
5.3-1. Equations of the Form $\int_a^x K(x,t)f(t,y(t)) dt = g(x)$	399
5.3-2. Other Equations	401
6. Nonlinear Equations of the Second Kind with Variable Limit of Integration	403
6.1. Equations with Quadratic Nonlinearity That Contain Arbitrary Parameters	403
6.1-1. Equations of the Form $y(x) + \int_a^x K(x,t)y^2(t) dt = F(x)$	403
6.1-2. Equations of the Form $y(x) + \int_a^x K(x,t)y(t)y(x-t) dt = F(x)$	406
6.2. Equations with Quadratic Nonlinearity That Contain Arbitrary Functions	406
6.2-1. Equations of the Form $y(x) + \int_a^x K(x,t)y^2(t) dt = F(x)$	406
6.2-2. Other Equations	407
6.3. Equations with Power-Law Nonlinearity	408
6.3-1. Equations Containing Arbitrary Parameters	408
6.3-2. Equations Containing Arbitrary Functions	410
6.4. Equations with Exponential Nonlinearity	411
6.4-1. Equations Containing Arbitrary Parameters	411
6.4-2. Equations Containing Arbitrary Functions	413
6.5. Equations with Hyperbolic Nonlinearity	414
6.5-1. Integrands with Nonlinearity of the Form $\cosh[\beta y(t)]$	414
6.5-2. Integrands with Nonlinearity of the Form $\sinh[\beta y(t)]$	415
6.5-3. Integrands with Nonlinearity of the Form $\tanh[\beta y(t)]$	416
6.5-4. Integrands with Nonlinearity of the Form $\coth[\beta y(t)]$	418
6.6. Equations with Logarithmic Nonlinearity	419
6.6-1. Integrands Containing Power-Law Functions of x and t	419
6.6-2. Integrands Containing Exponential Functions of x and t	419
6.6-3. Other Integrands	420
6.7. Equations with Trigonometric Nonlinearity	420
6.7-1. Integrands with Nonlinearity of the Form $\cos[\beta y(t)]$	420
6.7-2. Integrands with Nonlinearity of the Form $\sin[\beta y(t)]$	422
6.7-3. Integrands with Nonlinearity of the Form $\tan[\beta y(t)]$	423
6.7-4. Integrands with Nonlinearity of the Form $\cot[\beta y(t)]$	424
6.8. Equations with Nonlinearity of General Form	425
6.8-1. Equations of the Form $y(x) + \int_a^x K(x,t)G(y(t)) dt = F(x)$	425
6.8-2. Equations of the Form $y(x) + \int_a^x K(x-t)G(t,y(t)) dt = F(x)$	428
6.8-3. Other Equations	431
7. Nonlinear Equations of the First Kind with Constant Limits of Integration	433
7.1. Equations with Quadratic Nonlinearity That Contain Arbitrary Parameters	433
7.1-1. Equations of the Form $\int_a^b K(t)y(x)y(t) dt = F(x)$	433
7.1-2. Equations of the Form $\int_a^b K(t)y(t)y(xt) dt = F(x)$	435
7.1-3. Other Equations	436

7.2. Equations with Quadratic Nonlinearity That Contain Arbitrary Functions	437
7.2-1. Equations of the Form $\int_a^b K(t)y(t)y(\cdots) dt = F(x)$	437
7.2-2. Equations of the Form $\int_a^b [K(x, t)y(t) + M(x, t)y^2(t)] dt = F(x)$	443
7.3. Equations with Power-Law Nonlinearity That Contain Arbitrary Functions	444
7.3-1. Equations of the Form $\int_a^b K(t)y^\mu(x)y^\gamma(t) dt = F(x)$	444
7.3-2. Equations of the Form $\int_a^b K(t)y^\gamma(t)y(xt) dt = F(x)$	444
7.3-3. Equations of the Form $\int_a^b K(t)y^\gamma(t)y(x + \beta t) dt = F(x)$	445
7.3-4. Equations of the Form $\int_a^b [K(x, t)y(t) + M(x, t)y^\gamma(t)] dt = f(x)$	446
7.3-5. Other Equations	446
7.4. Equations with Nonlinearity of General Form	447
7.4-1. Equations of the Form $\int_a^b \varphi(y(x))K(t, y(t)) dt = F(x)$	447
7.4-2. Equations of the Form $\int_a^b y(xt)K(t, y(t)) dt = F(x)$	447
7.4-3. Equations of the Form $\int_a^b y(x + \beta t)K(t, y(t)) dt = F(x)$	449
7.4-4. Equations of the Form $\int_a^b [K(x, t)y(t) + \varphi(x)\Psi(t, y(t))] dt = F(x)$	450
7.4-5. Other Equations	451
8. Nonlinear Equations of the Second Kind with Constant Limits of Integration	453
8.1. Equations with Quadratic Nonlinearity That Contain Arbitrary Parameters	453
8.1-1. Equations of the Form $y(x) + \int_a^b K(x, t)y^2(t) dt = F(x)$	453
8.1-2. Equations of the Form $y(x) + \int_a^b K(x, t)y(x)y(t) dt = F(x)$	454
8.1-3. Equations of the Form $y(x) + \int_a^b K(t)y(t)y(\cdots) dt = F(x)$	455
8.2. Equations with Quadratic Nonlinearity That Contain Arbitrary Functions	456
8.2-1. Equations of the Form $y(x) + \int_a^b K(x, t)y^2(t) dt = F(x)$	456
8.2-2. Equations of the Form $y(x) + \int_a^b \sum K_{nm}(x, t)y^n(x)y^m(t) dt = F(x), n + m \leq 2$	457
8.2-3. Equations of the Form $y(x) + \int_a^b K(t)y(t)y(\cdots) dt = F(x)$	460
8.3. Equations with Power-Law Nonlinearity	464
8.3-1. Equations of the Form $y(x) + \int_a^b K(x, t)y^\beta(t) dt = F(x)$	464
8.3-2. Other Equations	465
8.4. Equations with Exponential Nonlinearity	467
8.4-1. Integrands with Nonlinearity of the Form $\exp[\beta y(t)]$	467
8.4-2. Other Integrands	468
8.5. Equations with Hyperbolic Nonlinearity	468
8.5-1. Integrands with Nonlinearity of the Form $\cosh[\beta y(t)]$	468
8.5-2. Integrands with Nonlinearity of the Form $\sinh[\beta y(t)]$	469
8.5-3. Integrands with Nonlinearity of the Form $\tanh[\beta y(t)]$	469
8.5-4. Integrands with Nonlinearity of the Form $\coth[\beta y(t)]$	470
8.5-5. Other Integrands	471
8.6. Equations with Logarithmic Nonlinearity	472
8.6-1. Integrands with Nonlinearity of the Form $\ln[\beta y(t)]$	472
8.6-2. Other Integrands	473
8.7. Equations with Trigonometric Nonlinearity	473
8.7-1. Integrands with Nonlinearity of the Form $\cos[\beta y(t)]$	473
8.7-2. Integrands with Nonlinearity of the Form $\sin[\beta y(t)]$	474
8.7-3. Integrands with Nonlinearity of the Form $\tan[\beta y(t)]$	475
8.7-4. Integrands with Nonlinearity of the Form $\cot[\beta y(t)]$	475
8.7-5. Other Integrands	476

8.8. Equations with Nonlinearity of General Form	477
8.8-1. Equations of the Form $y(x) + \int_a^b K(x-t)G(y(t)) dt = F(x)$	477
8.8-2. Equations of the Form $y(x) + \int_a^b K(x,t)G(t, y(t)) dt = F(x)$	479
8.8-3. Equations of the Form $y(x) + \int_a^b G(x, t, y(t)) dt = F(x)$	483
8.8-4. Equations of the Form $y(x) + \int_a^b y(xt)G(t, y(t)) dt = F(x)$	485
8.8-5. Equations of the Form $y(x) + \int_a^b y(x + \beta t)G(t, y(t)) dt = F(x)$	487
8.8-6. Other Equations	494

Part II. Methods for Solving Integral Equations

9. Main Definitions and Formulas. Integral Transforms	501
9.1. Some Definitions, Remarks, and Formulas	501
9.1-1. Some Definitions	501
9.1-2. Structure of Solutions to Linear Integral Equations	502
9.1-3. Integral Transforms	503
9.1-4. Residues. Calculation Formulas. Cauchy's Residue Theorem	504
9.1-5. Jordan Lemma	505
9.2. Laplace Transform	505
9.2-1. Definition. Inversion Formula	505
9.2-2. Inverse Transforms of Rational Functions	506
9.2-3. Inversion of Functions with Finitely Many Singular Points	507
9.2-4. Convolution Theorem. Main Properties of the Laplace Transform	507
9.2-5. Limit Theorems	507
9.2-6. Representation of Inverse Transforms as Convergent Series	509
9.2-7. Representation of Inverse Transforms as Asymptotic Expansions as $x \rightarrow \infty$	509
9.2-8. Post-Widder Formula	510
9.3. Mellin Transform	510
9.3-1. Definition. Inversion Formula	510
9.3-2. Main Properties of the Mellin Transform	511
9.3-3. Relation Among the Mellin, Laplace, and Fourier Transforms	511
9.4. Fourier Transform	512
9.4-1. Definition. Inversion Formula	512
9.4-2. Asymmetric Form of the Transform	512
9.4-3. Alternative Fourier Transform	512
9.4-4. Convolution Theorem. Main Properties of the Fourier Transforms	513
9.5. Fourier Cosine and Sine Transforms	514
9.5-1. Fourier Cosine Transform	514
9.5-2. Fourier Sine Transform	514
9.6. Other Integral Transforms	515
9.6-1. Hankel Transform	515
9.6-2. Meijer Transform	516
9.6-3. Kontorovich-Lebedev Transform	516
9.6-4. Y-transform	516
9.6-5. Summary Table of Integral Transforms	517
10. Methods for Solving Linear Equations of the Form $\int_a^x K(x, t)y(t) dt = f(x)$	519
10.1. Volterra Equations of the First Kind	519
10.1-1. Equations of the First Kind. Function and Kernel Classes	519
10.1-2. Existence and Uniqueness of a Solution	520
10.1-3. Some Problems Leading to Volterra Integral Equations of the First Kind	520

10.2. Equations with Degenerate Kernel: $K(x, t) = g_1(x)h_1(t) + \cdots + g_n(x)h_n(t)$	522
10.2-1. Equations with Kernel of the Form $K(x, t) = g_1(x)h_1(t) + g_2(x)h_2(t)$	522
10.2-2. Equations with General Degenerate Kernel	523
10.3. Reduction of Volterra Equations of the First Kind to Volterra Equations of the Second Kind	524
10.3-1. First Method	524
10.3-2. Second Method	524
10.4. Equations with Difference Kernel: $K(x, t) = K(x - t)$	524
10.4-1. Solution Method Based on the Laplace Transform	524
10.4-2. Case in Which the Transform of the Solution is a Rational Function	525
10.4-3. Convolution Representation of a Solution	526
10.4-4. Application of an Auxiliary Equation	527
10.4-5. Reduction to Ordinary Differential Equations	527
10.4-6. Reduction of a Volterra Equation to a Wiener–Hopf Equation	528
10.5. Method of Fractional Differentiation	529
10.5-1. Definition of Fractional Integrals	529
10.5-2. Definition of Fractional Derivatives	529
10.5-3. Main Properties	530
10.5-4. Solution of the Generalized Abel Equation	531
10.5-5. Erdélyi–Kober Operators	532
10.6. Equations with Weakly Singular Kernel	532
10.6-1. Method of Transformation of the Kernel	532
10.6-2. Kernel with Logarithmic Singularity	533
10.7. Method of Quadratures	534
10.7-1. Quadrature Formulas	534
10.7-2. General Scheme of the Method	535
10.7-3. Algorithm Based on the Trapezoidal Rule	536
10.7-4. Algorithm for an Equation with Degenerate Kernel	536
10.8. Equations with Infinite Integration Limit	537
10.8-1. Equation of the First Kind with Variable Lower Limit of Integration	537
10.8-2. Reduction to a Wiener–Hopf Equation of the First Kind	538
11. Methods for Solving Linear Equations of the Form $y(x) - \int_a^x K(x, t)y(t) dt = f(x)$ 539	
11.1. Volterra Integral Equations of the Second Kind	539
11.1-1. Preliminary Remarks. Equations for the Resolvent	539
11.1-2. Relationship Between Solutions of Some Integral Equations	540
11.2. Equations with Degenerate Kernel: $K(x, t) = g_1(x)h_1(t) + \cdots + g_n(x)h_n(t)$	540
11.2-1. Equations with Kernel of the Form $K(x, t) = \varphi(x) + \psi(x)(x - t)$	540
11.2-2. Equations with Kernel of the Form $K(x, t) = \varphi(t) + \psi(t)(t - x)$	541
11.2-3. Equations with Kernel of the Form $K(x, t) = \sum_{m=1}^n \varphi_m(x)(x - t)^{m-1}$	542
11.2-4. Equations with Kernel of the Form $K(x, t) = \sum_{m=1}^n \varphi_m(t)(t - x)^{m-1}$	543
11.2-5. Equations with Degenerate Kernel of the General Form	543
11.3. Equations with Difference Kernel: $K(x, t) = K(x - t)$	544
11.3-1. Solution Method Based on the Laplace Transform	544
11.3-2. Method Based on the Solution of an Auxiliary Equation	546
11.3-3. Reduction to Ordinary Differential Equations	547
11.3-4. Reduction to a Wiener–Hopf Equation of the Second Kind	547
11.3-5. Method of Fractional Integration for the Generalized Abel Equation	548
11.3-6. Systems of Volterra Integral Equations	549

11.4. Operator Methods for Solving Linear Integral Equations	549
11.4-1. Application of a Solution of a “Truncated” Equation of the First Kind	549
11.4-2. Application of the Auxiliary Equation of the Second Kind	551
11.4-3. Method for Solving “Quadratic” Operator Equations	552
11.4-4. Solution of Operator Equations of Polynomial Form	553
11.4-5. Some Generalizations	554
11.5. Construction of Solutions of Integral Equations with Special Right-Hand Side	555
11.5-1. General Scheme	555
11.5-2. Generating Function of Exponential Form	555
11.5-3. Power-Law Generating Function	557
11.5-4. Generating Function Containing Sines and Cosines	558
11.6. Method of Model Solutions	559
11.6-1. Preliminary Remarks	559
11.6-2. Description of the Method	560
11.6-3. Model Solution in the Case of an Exponential Right-Hand Side	561
11.6-4. Model Solution in the Case of a Power-Law Right-Hand Side	562
11.6-5. Model Solution in the Case of a Sine-Shaped Right-Hand Side	562
11.6-6. Model Solution in the Case of a Cosine-Shaped Right-Hand Side	563
11.6-7. Some Generalizations	563
11.7. Method of Differentiation for Integral Equations	564
11.7-1. Equations with Kernel Containing a Sum of Exponential Functions	564
11.7-2. Equations with Kernel Containing a Sum of Hyperbolic Functions	564
11.7-3. Equations with Kernel Containing a Sum of Trigonometric Functions	564
11.7-4. Equations Whose Kernels Contain Combinations of Various Functions	565
11.8. Reduction of Volterra Equations of the Second Kind to Volterra Equations of the First Kind	565
11.8-1. First Method	565
11.8-2. Second Method	566
11.9. Successive Approximation Method	566
11.9-1. General Scheme	566
11.9-2. Formula for the Resolvent	567
11.10. Method of Quadratures	568
11.10-1. General Scheme of the Method	568
11.10-2. Application of the Trapezoidal Rule	568
11.10-3. Case of a Degenerate Kernel	569
11.11. Equations with Infinite Integration Limit	569
11.11-1. Equation of the Second Kind with Variable Lower Integration Limit	570
11.11-2. Reduction to a Wiener–Hopf Equation of the Second Kind	571
12. Methods for Solving Linear Equations of the Form $\int_a^b K(x, t)y(t) dt = f(x)$	573
12.1. Some Definition and Remarks	573
12.1-1. Fredholm Integral Equations of the First Kind	573
12.1-2. Integral Equations of the First Kind with Weak Singularity	574
12.1-3. Integral Equations of Convolution Type	574
12.1-4. Dual Integral Equations of the First Kind	575
12.1-5. Some Problems Leading to Integral Equations of the First Kind	575
12.2. Integral Equations of the First Kind with Symmetric Kernel	577
12.2-1. Solution of an Integral Equation in Terms of Series in Eigenfunctions of Its Kernel	577
12.2-2. Method of Successive Approximations	579

12.3. Integral Equations of the First Kind with Nonsymmetric Kernel	580
12.3-1. Representation of a Solution in the Form of Series. General Description	580
12.3-2. Special Case of a Kernel That is a Generating Function	580
12.3-3. Special Case of the Right-Hand Side Represented in Terms of Orthogonal Functions	582
12.3-4. General Case. Galerkin's Method	582
12.3-5. Utilization of the Schmidt Kernels for the Construction of Solutions of Equations	582
12.4. Method of Differentiation for Integral Equations	583
12.4-1. Equations with Modulus	583
12.4-2. Other Equations. Some Generalizations	585
12.5. Method of Integral Transforms	586
12.5-1. Equation with Difference Kernel on the Entire Axis	586
12.5-2. Equations with Kernel $K(x, t) = K(x/t)$ on the Semiaxis	587
12.5-3. Equation with Kernel $K(x, t) = K(xt)$ and Some Generalizations	587
12.6. Krein's Method and Some Other Exact Methods for Integral Equations of Special Types	588
12.6-1. Krein's Method for an Equation with Difference Kernel with a Weak Singularity	588
12.6-2. Kernel is the Sum of a Nondegenerate Kernel and an Arbitrary Degenerate Kernel	589
12.6-3. Reduction of Integral Equations of the First Kind to Equations of the Second Kind	591
12.7. Riemann Problem for the Real Axis	592
12.7-1. Relationships Between the Fourier Integral and the Cauchy Type Integral	592
12.7-2. One-Sided Fourier Integrals	593
12.7-3. Analytic Continuation Theorem and the Generalized Liouville Theorem	595
12.7-4. Riemann Boundary Value Problem	595
12.7-5. Problems with Rational Coefficients	601
12.7-6. Exceptional Cases. The Homogeneous Problem	602
12.7-7. Exceptional Cases. The Nonhomogeneous Problem	604
12.8. Carleman Method for Equations of the Convolution Type of the First Kind	606
12.8-1. Wiener-Hopf Equation of the First Kind	606
12.8-2. Integral Equations of the First Kind with Two Kernels	607
12.9. Dual Integral Equations of the First Kind	610
12.9-1. Carleman Method for Equations with Difference Kernels	610
12.9-2. General Scheme of Finding Solutions of Dual Integral Equations	611
12.9-3. Exact Solutions of Some Dual Equations of the First Kind	613
12.9-4. Reduction of Dual Equations to a Fredholm Equation	615
12.10. Asymptotic Methods for Solving Equations with Logarithmic Singularity	618
12.10-1. Preliminary Remarks	618
12.10-2. Solution for Large λ	619
12.10-3. Solution for Small λ	620
12.10-4. Integral Equation of Elasticity	621
12.11. Regularization Methods	621
12.11-1. Lavrentiev Regularization Method	621
12.11-2. Tikhonov Regularization Method	622
12.12. Fredholm Integral Equation of the First Kind as an Ill-Posed Problem	623
12.12-1. General Notions of Well-Posed and Ill-Posed Problems	623
12.12-2. Integral Equation of the First Kind is an Ill-Posed Problem	624

13. Methods for Solving Linear Equations of the Form $y(x) - \int_a^b K(x, t)y(t) dt = f(x)$	625
13.1. Some Definition and Remarks	625
13.1-1. Fredholm Equations and Equations with Weak Singularity of the Second Kind	625
13.1-2. Structure of the Solution	626
13.1-3. Integral Equations of Convolution Type of the Second Kind	626
13.1-4. Dual Integral Equations of the Second Kind	627
13.2. Fredholm Equations of the Second Kind with Degenerate Kernel. Some Generalizations	627
13.2-1. Simplest Degenerate Kernel	627
13.2-2. Degenerate Kernel in the General Case	628
13.2-3. Kernel is the Sum of a Nondegenerate Kernel and an Arbitrary Degenerate Kernel	631
13.3. Solution as a Power Series in the Parameter. Method of Successive Approximations	632
13.3-1. Iterated Kernels	632
13.3-2. Method of Successive Approximations	633
13.3-3. Construction of the Resolvent	633
13.3-4. Orthogonal Kernels	634
13.4. Method of Fredholm Determinants	635
13.4-1. Formula for the Resolvent	635
13.4-2. Recurrent Relations	636
13.5. Fredholm Theorems and the Fredholm Alternative	637
13.5-1. Fredholm Theorems	637
13.5-2. Fredholm Alternative	638
13.6. Fredholm Integral Equations of the Second Kind with Symmetric Kernel	639
13.6-1. Characteristic Values and Eigenfunctions	639
13.6-2. Bilinear Series	640
13.6-3. Hilbert–Schmidt Theorem	641
13.6-4. Bilinear Series of Iterated Kernels	642
13.6-5. Solution of the Nonhomogeneous Equation	642
13.6-6. Fredholm Alternative for Symmetric Equations	643
13.6-7. Resolvent of a Symmetric Kernel	644
13.6-8. Extremal Properties of Characteristic Values and Eigenfunctions	644
13.6-9. Kellog’s Method for Finding Characteristic Values in the Case of Symmetric Kernel	645
13.6-10. Trace Method for the Approximation of Characteristic Values	646
13.6-11. Integral Equations Reducible to Symmetric Equations	647
13.6-12. Skew-Symmetric Integral Equations	647
13.6-13. Remark on Nonsymmetric Kernels	647
13.7. Integral Equations with Nonnegative Kernels	648
13.7-1. Positive Principal Eigenvalues. Generalized Jentzch Theorem	648
13.7-2. Positive Solutions of a Nonhomogeneous Integral Equation	649
13.7-3. Estimates for the Spectral Radius	649
13.7-4. Basic Definition and Theorems for Oscillating Kernels	651
13.7-5. Stochastic Kernels	654
13.8. Operator Method for Solving Integral Equations of the Second Kind	655
13.8-1. Simplest Scheme	655
13.8-2. Solution of Equations of the Second Kind on the Semiaxis	655

13.9. Methods of Integral Transforms and Model Solutions	656
13.9-1. Equation with Difference Kernel on the Entire Axis	656
13.9-2. Equation with the Kernel $K(x, t) = t^{-1}Q(x/t)$ on the Semiaxis	657
13.9-3. Equation with the Kernel $K(x, t) = t^{\beta}Q(xt)$ on the Semiaxis	658
13.9-4. Method of Model Solutions for Equations on the Entire Axis	659
13.10. Carleman Method for Integral Equations of Convolution Type of the Second Kind ..	660
13.10-1. Wiener–Hopf Equation of the Second Kind	660
13.10-2. Integral Equation of the Second Kind with Two Kernels	664
13.10-3. Equations of Convolution Type with Variable Integration Limit	668
13.10-4. Dual Equation of Convolution Type of the Second Kind	670
13.11. Wiener–Hopf Method	671
13.11-1. Some Remarks	671
13.11-2. Homogeneous Wiener–Hopf Equation of the Second Kind	673
13.11-3. General Scheme of the Method. The Factorization Problem	676
13.11-4. Nonhomogeneous Wiener–Hopf Equation of the Second Kind	677
13.11-5. Exceptional Case of a Wiener–Hopf Equation of the Second Kind	678
13.12. Krein’s Method for Wiener–Hopf Equations	679
13.12-1. Some Remarks. The Factorization Problem	679
13.12-2. Solution of the Wiener–Hopf Equations of the Second Kind	681
13.12-3. Hopf–Fock Formula	683
13.13. Methods for Solving Equations with Difference Kernels on a Finite Interval	683
13.13-1. Krein’s Method	683
13.13-2. Kernels with Rational Fourier Transforms	685
13.13-3. Reduction to Ordinary Differential Equations	686
13.14. Method of Approximating a Kernel by a Degenerate One	687
13.14-1. Approximation of the Kernel	687
13.14-2. Approximate Solution	688
13.15. Bateman Method	689
13.15-1. General Scheme of the Method	689
13.15-2. Some Special Cases	690
13.16. Collocation Method	692
13.16-1. General Remarks	692
13.16-2. Approximate Solution	693
13.16-3. Eigenfunctions of the Equation	694
13.17. Method of Least Squares	695
13.17-1. Description of the Method	695
13.17-2. Construction of Eigenfunctions	696
13.18. Bubnov–Galerkin Method	697
13.18-1. Description of the Method	697
13.18-2. Characteristic Values	697
13.19. Quadrature Method	698
13.19-1. General Scheme for Fredholm Equations of the Second Kind	698
13.19-2. Construction of the Eigenfunctions	699
13.19-3. Specific Features of the Application of Quadrature Formulas	700
13.20. Systems of Fredholm Integral Equations of the Second Kind	701
13.20-1. Some Remarks	701
13.20-2. Method of Reducing a System of Equations to a Single Equation	701

13.21. Regularization Method for Equations with Infinite Limits of Integration	702
13.21-1. Basic Equation and Fredholm Theorems	702
13.21-2. Regularizing Operators	703
13.21-3. Regularization Method	704
14. Methods for Solving Singular Integral Equations of the First Kind	707
14.1. Some Definitions and Remarks	707
14.1-1. Integral Equations of the First Kind with Cauchy Kernel	707
14.1-2. Integral Equations of the First Kind with Hilbert Kernel	707
14.2. Cauchy Type Integral	708
14.2-1. Definition of the Cauchy Type Integral	708
14.2-2. Hölder Condition	709
14.2-3. Principal Value of a Singular Integral	709
14.2-4. Multivalued Functions	711
14.2-5. Principal Value of a Singular Curvilinear Integral	712
14.2-6. Poincaré–Bertrand Formula	714
14.3. Riemann Boundary Value Problem	714
14.3-1. Principle of Argument. The Generalized Liouville Theorem	714
14.3-2. Hermite Interpolation Polynomial	716
14.3-3. Notion of the Index	716
14.3-4. Statement of the Riemann Problem	718
14.3-5. Solution of the Homogeneous Problem	720
14.3-6. Solution of the Nonhomogeneous Problem	721
14.3-7. Riemann Problem with Rational Coefficients	723
14.3-8. Riemann Problem for a Half-Plane	725
14.3-9. Exceptional Cases of the Riemann Problem	727
14.3-10. Riemann Problem for a Multiply Connected Domain	731
14.3-11. Riemann Problem for Open Curves	734
14.3-12. Riemann Problem with a Discontinuous Coefficient	739
14.3-13. Riemann Problem in the General Case	741
14.3-14. Hilbert Boundary Value Problem	742
14.4. Singular Integral Equations of the First Kind	743
14.4-1. Simplest Equation with Cauchy Kernel	743
14.4-2. Equation with Cauchy Kernel on the Real Axis	743
14.4-3. Equation of the First Kind on a Finite Interval	744
14.4-4. General Equation of the First Kind with Cauchy Kernel	745
14.4-5. Equations of the First Kind with Hilbert Kernel	746
14.5. Multhopp–Kalandiya Method	747
14.5-1. Solution That is Unbounded at the Endpoints of the Interval	747
14.5-2. Solution Bounded at One Endpoint of the Interval	749
14.5-3. Solution Bounded at Both Endpoints of the Interval	750
14.6. Hypersingular Integral Equations	751
14.6-1. Hypersingular Integral Equations with Cauchy- and Hilbert-Type Kernels	751
14.6-2. Definition of Hypersingular Integrals	751
14.6-3. Exact Solution of the Simplest Hypersingular Equation with Cauchy-Type Kernel	753
14.6-4. Exact Solution of the Simplest Hypersingular Equation with Hilbert-Type Kernel	754
14.6-5. Numerical Methods for Hypersingular Equations	754

15. Methods for Solving Complete Singular Integral Equations	757
15.1. Some Definitions and Remarks	757
15.1-1. Integral Equations with Cauchy Kernel	757
15.1-2. Integral Equations with Hilbert Kernel	759
15.1-3. Fredholm Equations of the Second Kind on a Contour	759
15.2. Carleman Method for Characteristic Equations	761
15.2-1. Characteristic Equation with Cauchy Kernel	761
15.2-2. Transposed Equation of a Characteristic Equation	764
15.2-3. Characteristic Equation on the Real Axis	765
15.2-4. Exceptional Case of a Characteristic Equation	767
15.2-5. Characteristic Equation with Hilbert Kernel	769
15.2-6. Tricomi Equation	769
15.3. Complete Singular Integral Equations Solvable in a Closed Form	770
15.3-1. Closed-Form Solutions in the Case of Constant Coefficients	770
15.3-2. Closed-Form Solutions in the General Case	771
15.4. Regularization Method for Complete Singular Integral Equations	772
15.4-1. Certain Properties of Singular Operators	772
15.4-2. Regularizer	774
15.4-3. Methods of Left and Right Regularization	775
15.4-4. Problem of Equivalent Regularization	776
15.4-5. Fredholm Theorems	777
15.4-6. Carleman–Vekua Approach to the Regularization	778
15.4-7. Regularization in Exceptional Cases	779
15.4-8. Complete Equation with Hilbert Kernel	780
15.5. Analysis of Solutions Singularities for Complete Integral Equations with Generalized Cauchy Kernels	783
15.5-1. Statement of the Problem and Preliminary Remarks	783
15.5-2. Auxiliary Results	784
15.5-3. Equations for the Exponents of Singularity of a Solution	787
15.5-4. Analysis of Equations for Singularity Exponents	789
15.5-5. Application to an Equation Arising in Fracture Mechanics	791
15.6. Direct Numerical Solution of Singular Integral Equations with Generalized Kernels ..	792
15.6-1. Preliminary Remarks	792
15.6-2. Quadrature Formulas for Integrals with the Jacobi Weight Function	793
15.6-3. Approximation of Solutions in Terms of a System of Orthogonal Polynomials ..	795
15.6-4. Some Special Functions and Their Calculations	797
15.6-5. Numerical Solution of Singular Integral Equations	799
15.6-6. Numerical Solutions of Singular Integral Equations of Bueckner Type	801
16. Methods for Solving Nonlinear Integral Equations	805
16.1. Some Definitions and Remarks	805
16.1-1. Nonlinear Equations with Variable Limit of Integration (Volterra Equations) ..	805
16.1-2. Nonlinear Equations with Constant Integration Limits (Urysohn Equations) ..	806
16.1-3. Some Special Features of Nonlinear Integral Equations	807
16.2. Exact Methods for Nonlinear Equations with Variable Limit of Integration	809
16.2-1. Method of Integral Transforms	809
16.2-2. Method of Differentiation for Nonlinear Equations with Degenerate Kernel ..	810

16.3. Approximate and Numerical Methods for Nonlinear Equations with Variable Limit of Integration	811
16.3-1. Successive Approximation Method	811
16.3-2. Newton–Kantorovich Method	813
16.3-3. Collocation Method	815
16.3-4. Quadrature Method	816
16.4. Exact Methods for Nonlinear Equations with Constant Integration Limits	817
16.4-1. Nonlinear Equations with Degenerate Kernels	817
16.4-2. Method of Integral Transforms	819
16.4-3. Method of Differentiating for Integral Equations	820
16.4-4. Method for Special Urysohn Equations of the First Kind	821
16.4-5. Method for Special Urysohn Equations of the Second Kind	822
16.4-6. Some Generalizations	824
16.5. Approximate and Numerical Methods for Nonlinear Equations with Constant Integration Limits	826
16.5-1. Successive Approximation Method	826
16.5-2. Newton–Kantorovich Method	827
16.5-3. Quadrature Method	829
16.5-4. Tikhonov Regularization Method	829
16.6 Existence and Uniqueness Theorems for Nonlinear Equations	830
16.6-1. Hammerstein Equations	830
16.6-2. Urysohn Equations	832
16.7. Nonlinear Equations with a Parameter: Eigenfunctions, Eigenvalues, Bifurcation Points	834
16.7-1. Eigenfunctions and Eigenvalues of Nonlinear Integral Equations	834
16.7-2. Local Solutions of a Nonlinear Integral Equation with a Parameter	835
16.7-3. Bifurcation Points of Nonlinear Integral Equations	835
17. Methods for Solving Multidimensional Mixed Integral Equations	839
17.1. Some Definition and Remarks	839
17.1-1. Basic Classes of Functions	839
17.1-2. Mixed Equations on a Finite Interval	840
17.1-3. Mixed Equation on a Ring-Shaped (Circular) Domain	841
17.1-4. Mixed Equations on a Closed Bounded Set	842
17.2. Methods of Solution of Mixed Integral Equations on a Finite Interval	843
17.2-1. Equation with a Hilbert–Schmidt Kernel and a Given Right-Hand Side	843
17.2-2. Equation with Hilbert–Schmidt Kernel and Auxiliary Conditions	845
17.2-3. Equation with a Schmidt Kernel and a Given Right-Hand Side on an Interval	848
17.2-4. Equation with a Schmidt Kernel and Auxiliary Conditions	851
17.3. Methods of Solving Mixed Integral Equations on a Ring-Shaped Domain	855
17.3-1. Equation with a Hilbert–Schmidt Kernel and a Given Right-Hand Side	855
17.3-2. Equation with a Hilbert–Schmidt Kernel and Auxiliary Conditions	856
17.3-3. Equation with a Schmidt Kernel and a Given Right-Hand Side	859
17.3-4. Equation with a Schmidt Kernel and Auxiliary Conditions on Ring-Shaped Domain	862
17.4. Projection Method for Solving Mixed Equations on a Bounded Set	866
17.4-1. Mixed Operator Equation with a Given Right-Hand Side	866
17.4-2. Mixed Operator Equations with Auxiliary Conditions	869
17.4-3. General Projection Problem for Operator Equation	873

18. Application of Integral Equations for the Investigation of Differential Equations . .	875
18.1. Reduction of the Cauchy Problem for ODEs to Integral Equations	875
18.1-1. Cauchy Problem for First-Order ODEs. Uniqueness and Existence Theorems	875
18.1-2. Cauchy Problem for First-Order ODEs. Method of Successive Approximations	876
18.1-3. Cauchy Problem for Second-Order ODEs. Method of Successive Approximations	876
18.1-4. Cauchy Problem for a Special n -Order Linear ODE	876
18.2. Reduction of Boundary Value Problems for ODEs to Volterra Integral Equations. Calculation of Eigenvalues	877
18.2-1. Reduction of Differential Equations to Volterra Integral Equations	877
18.2-2. Application of Volterra Equations to the Calculation of Eigenvalues	879
18.3. Reduction of Boundary Value Problems for ODEs to Fredholm Integral Equations with the Help of the Green's Function	881
18.3-1. Linear Ordinary Differential Equations. Fundamental Solutions	881
18.3-2. Boundary Value Problems for n th Order Differential Equations. Green's Function	882
18.3-3. Boundary Value Problems for Second-Order Differential Equations. Green's Function	883
18.3-4. Nonlinear Problem of Nonisothermal Flow in Plane Channel	884
18.4. Reduction of PDEs with Boundary Conditions of the Third Kind to Integral Equations	887
18.4-1. Usage of Particular Solutions of PDEs for the Construction of Other Solutions	887
18.4-2. Mass Transfer to a Particle in Fluid Flow Complicated by a Surface Reaction	888
18.4-3. Integral Equations for Surface Concentration and Diffusion Flux	890
18.4-4. Method of Numerical Integration of the Equation for Surface Concentration .	891
18.5. Representation of Linear Boundary Value Problems in Terms of Potentials	892
18.5-1. Basic Types of Potentials for the Laplace Equation and Their Properties	892
18.5-2. Integral Identities. Green's Formula	895
18.5-3. Reduction of Interior Dirichlet and Neumann Problems to Integral Equations .	895
18.5-4. Reduction of Exterior Dirichlet and Neumann Problems to Integral Equations	896
18.6. Representation of Solutions of Nonlinear PDEs in Terms of Solutions of Linear Integral Equations (Inverse Scattering)	898
18.6-1. Description of the Zakharov–Shabat Method	898
18.6-2. Korteweg–de Vries Equation and Other Nonlinear Equations	899

Supplements

Supplement 1. Elementary Functions and Their Properties	905
1.1. Power, Exponential, and Logarithmic Functions	905
1.1-1. Properties of the Power Function	905
1.1-2. Properties of the Exponential Function	905
1.1-3. Properties of the Logarithmic Function	906
1.2. Trigonometric Functions	907
1.2-1. Simplest Relations	907
1.2-2. Reduction Formulas	907
1.2-3. Relations Between Trigonometric Functions of Single Argument	908
1.2-4. Addition and Subtraction of Trigonometric Functions	908
1.2-5. Products of Trigonometric Functions	908
1.2-6. Powers of Trigonometric Functions	908
1.2-7. Addition Formulas	909

1.2-8. Trigonometric Functions of Multiple Arguments	909
1.2-9. Trigonometric Functions of Half Argument	909
1.2-10. Differentiation Formulas	910
1.2-11. Integration Formulas	910
1.2-12. Expansion in Power Series	910
1.2-13. Representation in the Form of Infinite Products	910
1.2-14. Euler and de Moivre Formulas. Relationship with Hyperbolic Functions	911
1.3. Inverse Trigonometric Functions	911
1.3-1. Definitions of Inverse Trigonometric Functions	911
1.3-2. Simplest Formulas	912
1.3-3. Some Properties	912
1.3-4. Relations Between Inverse Trigonometric Functions	912
1.3-5. Addition and Subtraction of Inverse Trigonometric Functions	912
1.3-6. Differentiation Formulas	913
1.3-7. Integration Formulas	913
1.3-8. Expansion in Power Series	913
1.4. Hyperbolic Functions	913
1.4-1. Definitions of Hyperbolic Functions	913
1.4-2. Simplest Relations	913
1.4-3. Relations Between Hyperbolic Functions of Single Argument ($x \geq 0$)	914
1.4-4. Addition and Subtraction of Hyperbolic Functions	914
1.4-5. Products of Hyperbolic Functions	914
1.4-6. Powers of Hyperbolic Functions	914
1.4-7. Addition Formulas	915
1.4-8. Hyperbolic Functions of Multiple Argument	915
1.4-9. Hyperbolic Functions of Half Argument	915
1.4-10. Differentiation Formulas	916
1.4-11. Integration Formulas	916
1.4-12. Expansion in Power Series	916
1.4-13. Representation in the Form of Infinite Products	916
1.4-14. Relationship with Trigonometric Functions	916
1.5. Inverse Hyperbolic Functions	917
1.5-1. Definitions of Inverse Hyperbolic Functions	917
1.5-2. Simplest Relations	917
1.5-3. Relations Between Inverse Hyperbolic Functions	917
1.5-4. Addition and Subtraction of Inverse Hyperbolic Functions	917
1.5-5. Differentiation Formulas	917
1.5-6. Integration Formulas	918
1.5-7. Expansion in Power Series	918
Supplement 2. Finite Sums and Infinite Series	919
2.1. Finite Numerical Sums	919
2.1-1. Progressions	919
2.1-2. Sums of Powers of Natural Numbers Having the Form $\sum k^m$	919
2.1-3. Alternating Sums of Powers of Natural Numbers, $\sum (-1)^k k^m$	920
2.1-4. Other Sums Containing Integers	920
2.1-5. Sums Containing Binomial Coefficients	920
2.1-6. Other Numerical Sums	921

2.2. Finite Functional Sums	922
2.2-1. Sums Involving Hyperbolic Functions	922
2.2-2. Sums Involving Trigonometric Functions	922
2.3. Infinite Numerical Series	924
2.3-1. Progressions	924
2.3-2. Other Numerical Series	924
2.4. Infinite Functional Series	925
2.4-1. Power Series	925
2.4-2. Trigonometric Series in One Variable Involving Sine	927
2.4-3. Trigonometric Series in One Variable Involving Cosine	928
2.4-4. Trigonometric Series in Two Variables	930
Supplement 3. Tables of Indefinite Integrals	933
3.1. Integrals Involving Rational Functions	933
3.1-1. Integrals Involving $a + bx$	933
3.1-2. Integrals Involving $a + x$ and $b + x$	933
3.1-3. Integrals Involving $a^2 + x^2$	934
3.1-4. Integrals Involving $a^2 - x^2$	935
3.1-5. Integrals Involving $a^3 + x^3$	936
3.1-6. Integrals Involving $a^3 - x^3$	936
3.1-7. Integrals Involving $a^4 \pm x^4$	937
3.2. Integrals Involving Irrational Functions	937
3.2-1. Integrals Involving $x^{1/2}$	937
3.2-2. Integrals Involving $(a + bx)^{p/2}$	938
3.2-3. Integrals Involving $(x^2 + a^2)^{1/2}$	938
3.2-4. Integrals Involving $(x^2 - a^2)^{1/2}$	938
3.2-5. Integrals Involving $(a^2 - x^2)^{1/2}$	939
3.2-6. Integrals Involving Arbitrary Powers. Reduction Formulas	939
3.3. Integrals Involving Exponential Functions	940
3.4. Integrals Involving Hyperbolic Functions	940
3.4-1. Integrals Involving $\cosh x$	940
3.4-2. Integrals Involving $\sinh x$	941
3.4-3. Integrals Involving $\tanh x$ or $\coth x$	942
3.5. Integrals Involving Logarithmic Functions	943
3.6. Integrals Involving Trigonometric Functions	944
3.6-1. Integrals Involving $\cos x$ ($n = 1, 2, \dots$)	944
3.6-2. Integrals Involving $\sin x$ ($n = 1, 2, \dots$)	945
3.6-3. Integrals Involving $\sin x$ and $\cos x$	947
3.6-4. Reduction Formulas	947
3.6-5. Integrals Involving $\tan x$ and $\cot x$	947
3.7. Integrals Involving Inverse Trigonometric Functions	948
Supplement 4. Tables of Definite Integrals	951
4.1. Integrals Involving Power-Law Functions	951
4.1-1. Integrals Over a Finite Interval	951
4.1-2. Integrals Over an Infinite Interval	952
4.2. Integrals Involving Exponential Functions	954
4.3. Integrals Involving Hyperbolic Functions	955
4.4. Integrals Involving Logarithmic Functions	955

4.5. Integrals Involving Trigonometric Functions	956
4.5-1. Integrals Over a Finite Interval	956
4.5-2. Integrals Over an Infinite Interval	957
4.6. Integrals Involving Bessel Functions	958
4.6-1. Integrals Over an Infinite Interval	958
4.6-2. Other Integrals	959
Supplement 5. Tables of Laplace Transforms	961
5.1. General Formulas	961
5.2. Expressions with Power-Law Functions	963
5.3. Expressions with Exponential Functions	963
5.4. Expressions with Hyperbolic Functions	964
5.5. Expressions with Logarithmic Functions	965
5.6. Expressions with Trigonometric Functions	966
5.7. Expressions with Special Functions	967
Supplement 6. Tables of Inverse Laplace Transforms	969
6.1. General Formulas	969
6.2. Expressions with Rational Functions	971
6.3. Expressions with Square Roots	975
6.4. Expressions with Arbitrary Powers	977
6.5. Expressions with Exponential Functions	978
6.6. Expressions with Hyperbolic Functions	979
6.7. Expressions with Logarithmic Functions	980
6.8. Expressions with Trigonometric Functions	981
6.9. Expressions with Special Functions	981
Supplement 7. Tables of Fourier Cosine Transforms	983
7.1. General Formulas	983
7.2. Expressions with Power-Law Functions	983
7.3. Expressions with Exponential Functions	984
7.4. Expressions with Hyperbolic Functions	985
7.5. Expressions with Logarithmic Functions	985
7.6. Expressions with Trigonometric Functions	986
7.7. Expressions with Special Functions	987
Supplement 8. Tables of Fourier Sine Transforms	989
8.1. General Formulas	989
8.2. Expressions with Power-Law Functions	989
8.3. Expressions with Exponential Functions	990
8.4. Expressions with Hyperbolic Functions	991
8.5. Expressions with Logarithmic Functions	992
8.6. Expressions with Trigonometric Functions	992
8.7. Expressions with Special Functions	993

Supplement 9. Tables of Mellin Transforms	997
9.1. General Formulas	997
9.2. Expressions with Power-Law Functions	998
9.3. Expressions with Exponential Functions	998
9.4. Expressions with Logarithmic Functions	999
9.5. Expressions with Trigonometric Functions	999
9.6. Expressions with Special Functions	1000
Supplement 10. Tables of Inverse Mellin Transforms	1001
10.1. Expressions with Power-Law Functions	1001
10.2. Expressions with Exponential and Logarithmic Functions	1002
10.3. Expressions with Trigonometric Functions	1003
10.4. Expressions with Special Functions	1004
Supplement 11. Special Functions and Their Properties	1007
11.1. Some Coefficients, Symbols, and Numbers	1007
11.1-1. Binomial Coefficients	1007
11.1-2. Pochhammer Symbol	1007
11.1-3. Bernoulli Numbers	1008
11.1-4. Euler Numbers	1008
11.2. Error Functions. Exponential and Logarithmic Integrals	1009
11.2-1. Error Function and Complementary Error Function	1009
11.2-2. Exponential Integral	1010
11.2-3. Logarithmic Integral	1010
11.3. Sine Integral and Cosine Integral. Fresnel Integrals	1011
11.3-1. Sine Integral	1011
11.3-2. Cosine Integral	1011
11.3-3. Fresnel Integrals and Generalized Fresnel Integrals	1012
11.4. Gamma Function, Psi Function, and Beta Function	1012
11.4-1. Gamma Function	1012
11.4-2. Psi Function (Digamma Function)	1013
11.4-3. Beta Function	1014
11.5. Incomplete Gamma and Beta Functions	1014
11.5-1. Incomplete Gamma Function	1014
11.5-2. Incomplete Beta Function	1015
11.6. Bessel Functions (Cylindrical Functions)	1016
11.6-1. Definitions and Basic Formulas	1016
11.6-2. Integral Representations and Asymptotic Expansions	1017
11.6-3. Zeros of Bessel Functions	1019
11.6-4. Orthogonality Properties of Bessel Functions	1019
11.6-5. Hankel Functions (Bessel Functions of the Third Kind)	1020
11.7. Modified Bessel Functions	1021
11.7-1. Definitions. Basic Formulas	1021
11.7-2. Integral Representations and Asymptotic Expansions	1022
11.8. Airy Functions	1023
11.8-1. Definition and Basic Formulas	1023
11.8-2. Power Series and Asymptotic Expansions	1023

11.9. Confluent Hypergeometric Functions	1024
11.9-1. Kummer and Tricomi Confluent Hypergeometric Functions	1024
11.9-2. Integral Representations and Asymptotic Expansions	1027
11.9-3. Whittaker Confluent Hypergeometric Functions	1027
11.10. Gauss Hypergeometric Functions	1028
11.10-1. Various Representations of the Gauss Hypergeometric Function	1028
11.10-2. Basic Properties	1028
11.11. Legendre Polynomials, Legendre Functions, and Associated Legendre Functions	1030
11.11-1. Legendre Polynomials and Legendre Functions	1030
11.11-2. Associated Legendre Functions with Integer Indices and Real Argument	1031
11.11-3. Associated Legendre Functions. General Case	1032
11.12. Parabolic Cylinder Functions	1034
11.12-1. Definitions. Basic Formulas	1034
11.12-2. Integral Representations, Asymptotic Expansions, and Linear Relations	1035
11.13. Elliptic Integrals	1035
11.13-1. Complete Elliptic Integrals	1035
11.13-2. Incomplete Elliptic Integrals (Elliptic Integrals)	1037
11.14. Elliptic Functions	1038
11.14-1. Jacobi Elliptic Functions	1039
11.14-2. Weierstrass Elliptic Function	1042
11.15. Jacobi Theta Functions	1043
11.15-1. Series Representation of the Jacobi Theta Functions. Simplest Properties	1043
11.15-2. Various Relations and Formulas. Connection with Jacobi Elliptic Functions	1044
11.16. Mathieu Functions and Modified Mathieu Functions	1045
11.16-1. Mathieu Functions	1045
11.16-2. Modified Mathieu Functions	1046
11.17. Orthogonal Polynomials	1047
11.17-1. Laguerre Polynomials and Generalized Laguerre Polynomials	1047
11.17-2. Chebyshev Polynomials and Functions	1048
11.17-3. Hermite Polynomials and Functions	1050
11.17-4. Jacobi Polynomials	1051
11.17-5. Gegenbauer Polynomials	1051
11.18. Nonorthogonal Polynomials	1052
11.18-1. Bernoulli Polynomials	1052
11.18-2. Euler Polynomials	1053
Supplement 12. Some Notions of Functional Analysis	1055
12.1. Functions of Bounded Variation	1055
12.1-1. Definition of a Function of Bounded Variation	1055
12.1-2. Classes of Functions of Bounded Variation	1056
12.1-3. Properties of Functions of Bounded Variation	1056
12.1-4. Criteria for Functions to Have Bounded Variation	1057
12.1-5. Properties of Continuous Functions of Bounded Variation	1057
12.2. Stieltjes Integral	1057
12.2-1. Basic Definitions	1057
12.2-2. Properties of the Stieltjes Integral	1058
12.2-3. Existence Theorems for the Stieltjes Integral	1058

12.3. Lebesgue Integral	1059
12.3-1. Riemann Integral and the Lebesgue Integral	1059
12.3-2. Sets of Zero Measure. Notion of “Almost Everywhere”	1060
12.3-3. Step Functions and Measurable Functions	1060
12.3-4. Definition and Properties of the Lebesgue Integral	1061
12.3-5. Measurable Sets	1062
12.3-6. Integration Over Measurable Sets	1063
12.3-7. Case of an Infinite Interval	1063
12.3-8. Case of Several Variables	1064
12.3-9. Spaces L_p	1064
12.4. Linear Normed Spaces	1065
12.4-1. Linear Spaces	1065
12.4-2. Linear Normed Spaces	1065
12.4-3. Space of Continuous Functions $C(a, b)$	1066
12.4-4. Lebesgue Space $L_p(a, b)$	1066
12.4-5. Hölder Space $C_\alpha(0, 1)$	1066
12.4-6. Space of Functions of Bounded Variation $V(0, 1)$	1066
12.5. Euclidean and Hilbert Spaces. Linear Operators in Hilbert Spaces	1067
12.5-1. Preliminary Remarks	1067
12.5-2. Euclidean and Hilbert Spaces	1067
12.5-3. Linear Operators in Hilbert Spaces	1068
References	1071
Index	1081

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PREFACE TO THE NEW EDITION

Handbook of Integral Equations, Second Edition, a unique reference for engineers and scientists, contains over 2,500 integral equations with solutions, as well as analytical and numerical methods for solving linear and nonlinear equations. It considers Volterra, Fredholm, Wiener–Hopf, Hammerstein, Urysohn, and other equations, which arise in mathematics, physics, engineering sciences, economics, etc. In total, the number of equations described is an order of magnitude greater than in any other book available.

The second edition has been substantially updated, revised, and extended. It includes new chapters on mixed multidimensional equations, methods of integral equations for ODEs and PDEs, and about 400 new equations with exact solutions. It presents a considerable amount of new material on Volterra, Fredholm, singular, hypersingular, dual, and nonlinear integral equations, integral transforms, and special functions. Many examples were added for illustrative purposes. The new edition has been increased by a total of over 300 pages.

Note that the first part of the book can be used as a database of test problems for numerical and approximate methods for solving linear and nonlinear integral equations.

We would like to express our deep gratitude to Alexei Zhurov and Vasilii Silvestrov for fruitful discussions. We also appreciate the help of Grigory Yosifian in translating new sections of this book and valuable remarks.

The authors hope that the handbook will prove helpful for a wide audience of researchers, college and university teachers, engineers, and students in various fields of applied mathematics, mechanics, physics, chemistry, biology, economics, and engineering sciences.

A. D. Polyinin
A. V. Manzhirov

PREFACE TO THE FIRST EDITION

Integral equations are encountered in various fields of science and numerous applications (in elasticity, plasticity, heat and mass transfer, oscillation theory, fluid dynamics, filtration theory, electrostatics, electrodynamics, biomechanics, game theory, control, queuing theory, electrical engineering, economics, medicine, etc.).

Exact (closed-form) solutions of integral equations play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science. Lots of equations of physics, chemistry, and biology contain functions or parameters which are obtained from experiments and hence are not strictly fixed. Therefore, it is expedient to choose the structure of these functions so that it would be easier to analyze and solve the equation. As a possible selection criterion, one may adopt the requirement that the model integral equation admits a solution in a closed form. Exact solutions can be used to verify the consistency and estimate errors of various numerical, asymptotic, and approximate methods.

More than 2,100 integral equations and their solutions are given in the first part of the book (Chapters 1–6). A lot of new exact solutions to linear and nonlinear equations are included. Special attention is paid to equations of general form, which depend on arbitrary functions. The other equations contain one or more free parameters (the book actually deals with families of integral

equations); it is the reader's option to fix these parameters. In total, the number of equations described in this handbook is an order of magnitude greater than in any other book currently available.

The second part of the book (Chapters 7–14) presents exact, approximate analytical, and numerical methods for solving linear and nonlinear integral equations. Apart from the classical methods, some new methods are also described. When selecting the material, the authors have given a pronounced preference to practical aspects of the matter; that is, to methods that allow effectively “constructing” the solution. For the reader's better understanding of the methods, each section is supplied with examples of specific equations. Some sections may be used by lecturers of colleges and universities as a basis for courses on integral equations and mathematical physics equations for graduate and postgraduate students.

For the convenience of a wide audience with different mathematical backgrounds, the authors tried to do their best, wherever possible, to avoid special terminology. Therefore, some of the methods are outlined in a schematic and somewhat simplified manner, with necessary references made to books where these methods are considered in more detail. For some nonlinear equations, only solutions of the simplest form are given. The book does not cover two-, three-, and multidimensional integral equations.

The handbook consists of chapters, sections, and subsections. Equations and formulas are numbered separately in each section. The equations within a section are arranged in increasing order of complexity. The extensive table of contents provides rapid access to the desired equations.

For the reader's convenience, the main material is followed by a number of supplements, where some properties of elementary and special functions are described, tables of indefinite and definite integrals are given, as well as tables of Laplace, Mellin, and other transforms, which are used in the book.

The first and second parts of the book, just as many sections, were written so that they could be read independently from each other. This allows the reader to quickly get to the heart of the matter.

We would like to express our deep gratitude to Rolf Sulanke and Alexei Zhurov for fruitful discussions and valuable remarks. We also appreciate the help of Vladimir Nazaikinskii and Alexander Shtern in translating the second part of this book, and are thankful to Inna Shingareva for her assistance in preparing the camera-ready copy of the book.

The authors hope that the handbook will prove helpful for a wide audience of researchers, college and university teachers, engineers, and students in various fields of mathematics, mechanics, physics, chemistry, biology, economics, and engineering sciences.

A. D. Polyinin

A. V. Manzhurov

SOME REMARKS AND NOTATION

1. In Chapters 1–11, 14, and 18 in the original integral equations, the independent variable is denoted by x , the integration variable by t , and the unknown function by $y = y(x)$.

2. For a function of one variable $f = f(x)$, we use the following notation for the derivatives:

$$f'_x = \frac{df}{dx}, \quad f''_{xx} = \frac{d^2 f}{dx^2}, \quad f'''_{xxx} = \frac{d^3 f}{dx^3}, \quad f^{(4)}_{xxxx} = \frac{d^4 f}{dx^4}, \quad \text{and} \quad f^{(n)}_x = \frac{d^n f}{dx^n} \quad \text{for } n \geq 5.$$

Occasionally, we use the similar notation for partial derivatives of a function of two variables, for example, $K'_x(x, t) = \frac{\partial}{\partial x} K(x, t)$.

3. In some cases, we use the operator notation $\left[f(x) \frac{d}{dx} \right]^n g(x)$, which is defined recursively by

$$\left[f(x) \frac{d}{dx} \right]^n g(x) = f(x) \frac{d}{dx} \left\{ \left[f(x) \frac{d}{dx} \right]^{n-1} g(x) \right\}.$$

4. It is indicated in the beginning of Chapters 1–8 that $f = f(x)$, $g = g(x)$, $K = K(x)$, etc. are arbitrary functions, and A , B , etc. are free parameters. This means that:

- (a) $f = f(x)$, $g = g(x)$, $K = K(x)$, etc. are assumed to be continuous real-valued functions of real arguments;*
- (b) if the solution contains derivatives of these functions, then the functions are assumed to be sufficiently differentiable;**
- (c) if the solution contains integrals with these functions (in combination with other functions), then the integrals are supposed to converge;
- (d) the free parameters A , B , etc. may assume any real values for which the expressions occurring in the equation and the solution make sense (for example, if a solution contains a factor $\frac{A}{1-A}$, then it is implied that $A \neq 1$; as a rule, this is not specified in the text).

5. The notations $\operatorname{Re} z$ and $\operatorname{Im} z$ stand, respectively, for the real and the imaginary part of a complex quantity z .

6. In the first part of the book (Chapters 1–8) when referencing a particular equation, we use a notation like 2.3.15, which implies equation 15 from Section 2.3.

7. To highlight portions of the text, the following symbols are used in the book:

- indicates important information pertaining to a group of equations (Chapters 1–8);
- ⊙ indicates the literature used in the preparation of the text in specific equations (Chapters 1–8) or sections (Chapters 9–18).

* Less severe restrictions on these functions are presented in the second part of the book.

** Restrictions (b) and (c) imposed on $f = f(x)$, $g = g(x)$, $K = K(x)$, etc. are not mentioned in the text.

Part I

Exact Solutions of Integral Equations

Chapter 1

Linear Equations of the First Kind with Variable Limit of Integration

► **Notation:** $f = f(x)$, $g = g(x)$, $h = h(x)$, $K = K(x)$, and $M = M(x)$ are arbitrary functions (these may be composite functions of the argument depending on two variables x and t); $A, B, C, D, E, a, b, c, \alpha, \beta, \gamma, \lambda$, and μ are free parameters; and m and n are nonnegative integers.

► **Preliminary remarks.** For equations of the form

$$\int_a^x K(x, t)y(t) dt = f(x), \quad a \leq x \leq b,$$

where the functions $K(x, t)$ and $f(x)$ are continuous, the right-hand side must satisfy the following conditions:

1°. If $K(a, a) \neq 0$, then we must have $f(a) = 0$ (for example, the right-hand sides of equations 1.1.1 and 1.2.1 must satisfy this condition).

2°. If $K(a, a) = K'_x(a, a) = \dots = K_x^{(n-1)}(a, a) = 0$, $0 < |K_x^{(n)}(a, a)| < \infty$, then the right-hand side of the equation must satisfy the conditions

$$f(a) = f'_x(a) = \dots = f_x^{(n)}(a) = 0.$$

For example, with $n = 1$, these are constraints for the right-hand side of equation 1.1.2.

3°. If $K(a, a) = K'_x(a, a) = \dots = K_x^{(n-1)}(a, a) = 0$, $K_x^{(n)}(a, a) = \infty$, then the right-hand side of the equation must satisfy the conditions

$$f(a) = f'_x(a) = \dots = f_x^{(n-1)}(a) = 0.$$

For example, with $n = 1$, this is a constraint for the right-hand side of equation 1.1.30.

4°. For unbounded $K(x, t)$ with integrable power-law or logarithmic singularity at $x = t$ and continuous $f(x)$, no additional conditions are imposed on the right-hand side of the integral equation (e.g., see Abel's equation 1.1.36).

In the case of a difference kernel, $K(x, t) = K(x - t)$, that can be represented as $x \rightarrow t$ in the form

$$K(x - t) = A(x - t)^\lambda + o((x - t)^\lambda) \quad (0 < |A| < \infty),$$

the right-hand side of the integral equation, for $\lambda \geq 0$, must satisfy the conditions

$$f(a) = f'_x(a) = \dots = f_x^{([\lambda])}(a) = 0,$$

where $[\lambda]$ is the integer part of λ . For $-1 < \lambda < 0$, there are no additional conditions imposed on the function $f(x)$.

In Chapter 1, conditions 1°–3° are as a rule not specified.

1.1. Equations Whose Kernels Contain Power-Law Functions

1.1-1. Kernels Linear in the Arguments x and t .

1. $\int_a^x y(t) dt = f(x).$

Solution: $y(x) = f'_x(x).$

2. $\int_a^x (x-t)y(t) dt = f(x).$

Solution: $y(x) = f''_{xx}(x).$

3. $\int_a^x (Ax + Bt + C)y(t) dt = f(x).$

This is a special case of equation 1.9.5 with $g(x) = x$.

1°. Solution with $B \neq -A$:

$$y(x) = \frac{d}{dx} \left\{ [(A+B)x + C]^{-\frac{A}{A+B}} \int_a^x [(A+B)t + C]^{-\frac{B}{A+B}} f'_t(t) dt \right\}.$$

2°. Solution with $B = -A$:

$$y(x) = \frac{1}{C} \frac{d}{dx} \left[\exp\left(-\frac{A}{C}x\right) \int_a^x \exp\left(\frac{A}{C}t\right) f'_t(t) dt \right].$$

1.1-2. Kernels Quadratic in the Arguments x and t .

4. $\int_a^x (x-t)^2 y(t) dt = f(x), \quad f(a) = f'_x(a) = f''_{xx}(a) = 0.$

Solution: $y(x) = \frac{1}{2} f'''_{xxx}(x).$

5. $\int_a^x (x^2 - t^2)y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$

This is a special case of equation 1.9.2 with $g(x) = x^2$.

Solution: $y(x) = \frac{1}{2x^2} [x f''_{xx}(x) - f'_x(x)].$

6. $\int_a^x (Ax^2 + Bt^2)y(t) dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = x^2$. For $B = -A$, see equation 1.1.5.

Solution: $y(x) = \frac{1}{A+B} \frac{d}{dx} \left[x^{-\frac{2A}{A+B}} \int_a^x t^{-\frac{2B}{A+B}} f'_t(t) dt \right].$

7. $\int_a^x (Ax^2 + Bt^2 + C)y(t) dt = f(x).$

This is a special case of equation 1.9.5 with $g(x) = x^2$.

Solution:

$$y(x) = \text{sign } \varphi(x) \frac{d}{dx} \left\{ |\varphi(x)|^{-\frac{A}{A+B}} \int_a^x |\varphi(t)|^{-\frac{B}{A+B}} f'_t(t) dt \right\}, \quad \varphi(x) = (A+B)x^2 + C.$$

$$8. \quad \int_a^x [Ax^2 + (B - A)xt - Bt^2] y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

Differentiating with respect to x yields an equation of the form 1.1.3:

$$\int_a^x [2Ax + (B - A)t] y(t) dt = f'_x(x).$$

Solution:

$$y(x) = \frac{1}{A + B} \frac{d}{dx} \left[x^{-\frac{2A}{A+B}} \int_a^x t^{\frac{A-B}{A+B}} f''_{tt}(t) dt \right].$$

$$9. \quad \int_a^x (Ax^2 + Bt^2 + Cx + Dt + E) y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = Ax^2 + Cx$ and $h(t) = Bt^2 + Dt + E$.

$$10. \quad \int_a^x (Axt + Bt^2 + Cx + Dt + E) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = x$, $h_1(t) = At + C$, $g_2(x) = 1$, and $h_2(t) = Bt^2 + Dt + E$.

$$11. \quad \int_a^x (Ax^2 + Bxt + Cx + Dt + E) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Bx + D$, $h_1(t) = t$, $g_2(x) = Ax^2 + Cx + E$, and $h_2(t) = 1$.

1.1-3. Kernels Cubic in the Arguments x and t .

$$12. \quad \int_a^x (x - t)^3 y(t) dt = f(x), \quad f(a) = f'_x(a) = f''_{xx}(a) = f'''_{xxx}(a) = 0.$$

Solution: $y(x) = \frac{1}{6} f'''_{xxx}(x)$.

$$13. \quad \int_a^x (x^3 - t^3) y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

This is a special case of equation 1.9.2 with $g(x) = x^3$.

Solution: $y(x) = \frac{1}{3x^3} [x f'''_{xxx}(x) - 2f'_x(x)]$.

$$14. \quad \int_a^x (Ax^3 + Bt^3) y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = x^3$. For $B = -A$, see equation 1.1.13.

Solution with $0 \leq a \leq x$: $y(x) = \frac{1}{A + B} \frac{d}{dx} \left[x^{-\frac{3A}{A+B}} \int_a^x t^{-\frac{3B}{A+B}} f'_t(t) dt \right]$.

$$15. \quad \int_a^x (Ax^3 + Bt^3 + C) y(t) dt = f(x).$$

This is a special case of equation 1.9.5 with $g(x) = x^3$.

$$16. \int_a^x (x^2 t - x t^2) y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

This is a special case of equation 1.9.11 with $g(x) = x^2$ and $h(x) = x$.

$$\text{Solution: } y(x) = \frac{1}{x} \frac{d^2}{dx^2} \left[\frac{1}{x} f(x) \right].$$

$$17. \int_a^x (A x^2 t + B x t^2) y(t) dt = f(x).$$

This is a special case of equation 1.9.12 with $g(x) = x^2$ and $h(x) = x$. For $B = -A$, see equation 1.1.16.

Solution:

$$y(x) = \frac{1}{(A+B)x} \frac{d}{dx} \left\{ x^{-\frac{A}{A+B}} \int_a^x t^{-\frac{B}{A+B}} \frac{d}{dt} \left[\frac{1}{t} f(t) \right] dt \right\}.$$

$$18. \int_a^x (A x^3 + B x t^2) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = A x^3$, $h_1(t) = 1$, $g_2(x) = B x$, and $h_2(t) = t^2$.

$$19. \int_a^x (A x^3 + B x^2 t) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = A x^3$, $h_1(t) = 1$, $g_2(x) = B x^2$, and $h_2(t) = t$.

$$20. \int_a^x (A x^2 t + B t^3) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = A x^2$, $h_1(t) = t$, $g_2(x) = B$, and $h_2(t) = t^3$.

$$21. \int_a^x (A x t^2 + B t^3) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = A x$, $h_1(t) = t^2$, $g_2(x) = B$, and $h_2(t) = t^3$.

$$22. \int_a^x (A_3 x^3 + B_3 t^3 + A_2 x^2 + B_2 t^2 + A_1 x + B_1 t + C) y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A_3 x^3 + A_2 x^2 + A_1 x + C$ and $h(t) = B_3 t^3 + B_2 t^2 + B_1 t$.

1.1-4. Kernels Containing Higher-Order Polynomials in x and t .

$$23. \int_a^x (x-t)^n y(t) dt = f(x), \quad n = 1, 2, \dots$$

It is assumed that the right-hand of the equation satisfies the conditions $f(a) = f'_x(a) = \dots = f_x^{(n)}(a) = 0$.

$$\text{Solution: } y(x) = \frac{1}{n!} f_x^{(n+1)}(x).$$

Example. For $f(x) = A x^m$, where m is a positive integer, $m > n$, the solution has the form

$$y(x) = \frac{A m!}{n! (m-n-1)!} x^{m-n-1}.$$

$$24. \quad \int_a^x (x^n - t^n) y(t) dt = f(x), \quad f(a) = f'_x(a) = 0, \quad n = 1, 2, \dots$$

$$\text{Solution: } y(x) = \frac{1}{n} \frac{d}{dx} \left[\frac{f'_x(x)}{x^{n-1}} \right].$$

$$25. \quad \int_a^x (t^n x^{n+1} - x^n t^{n+1}) y(t) dt = f(x), \quad n = 2, 3, \dots$$

This is a special case of equation 1.9.11 with $g(x) = x^{n+1}$ and $h(x) = x^n$.

$$\text{Solution: } y(x) = \frac{1}{x^n} \frac{d^2}{dx^2} \left[\frac{f(x)}{x^n} \right].$$

1.1-5. Kernels Containing Rational Functions.

$$26. \quad \int_0^x \frac{y(t) dt}{x+t} = f(x).$$

1°. For a polynomial right-hand side, $f(x) = \sum_{n=0}^N A_n x^n$, the solution has the form

$$y(x) = \sum_{n=0}^N \frac{A_n}{B_n} x^n, \quad B_n = (-1)^n \left[\ln 2 + \sum_{k=1}^n \frac{(-1)^k}{k} \right].$$

2°. For $f(x) = x^\lambda \sum_{n=0}^N A_n x^n$, where λ is an arbitrary number ($\lambda > -1$), the solution has the form

$$y(x) = x^\lambda \sum_{n=0}^N \frac{A_n}{B_n} x^n, \quad B_n = \int_0^1 \frac{t^{\lambda+n} dt}{1+t}.$$

3°. For $f(x) = \ln x \left(\sum_{n=0}^N A_n x^n \right)$, the solution has the form

$$y(x) = \ln x \sum_{n=0}^N \frac{A_n}{B_n} x^n + \sum_{n=0}^N \frac{A_n I_n}{B_n^2} x^n, \\ B_n = (-1)^n \left[\ln 2 + \sum_{k=1}^n \frac{(-1)^k}{k} \right], \quad I_n = (-1)^n \left[\frac{\pi^2}{12} + \sum_{k=1}^n \frac{(-1)^k}{k^2} \right].$$

4°. For $f(x) = \sum_{n=0}^N A_n (\ln x)^n$, the solution of the equation has the form

$$y(x) = \sum_{n=0}^N A_n Y_n(x),$$

where the functions $Y_n = Y_n(x)$ are given by

$$Y_n(x) = \left\{ \frac{d^n}{d\lambda^n} \left[\frac{x^\lambda}{I(\lambda)} \right] \right\}_{\lambda=0}, \quad I(\lambda) = \int_0^1 \frac{z^\lambda dz}{1+z}.$$

5°. For $f(x) = \sum_{n=1}^N A_n \cos(\lambda_n \ln x) + \sum_{n=1}^N B_n \sin(\lambda_n \ln x)$, the solution of the equation has the form

$$y(x) = \sum_{n=1}^N C_n \cos(\lambda_n \ln x) + \sum_{n=1}^N D_n \sin(\lambda_n \ln x),$$

where the constants C_n and D_n are found by the method of undetermined coefficients.

6°. For arbitrary $f(x)$, the transformation

$$x = \frac{1}{2}e^{2z}, \quad t = \frac{1}{2}e^{2\tau}, \quad y(t) = e^{-\tau}w(\tau), \quad f(x) = e^{-z}g(z)$$

leads to an integral equation with difference kernel of the form 1.9.27:

$$\int_{-\infty}^z \frac{w(\tau) d\tau}{\cosh(z - \tau)} = g(z).$$

27. $\int_0^x \frac{y(t) dt}{ax + bt} = f(x), \quad a > 0, \quad a + b > 0.$

1°. For a polynomial right-hand side, $f(x) = \sum_{n=0}^N A_n x^n$, the solution has the form

$$y(x) = \sum_{n=0}^N \frac{A_n}{B_n} x^n, \quad B_n = \int_0^1 \frac{t^n dt}{a + bt}.$$

2°. For $f(x) = x^\lambda \sum_{n=0}^N A_n x^n$, where λ is an arbitrary number ($\lambda > -1$), the solution has the form

$$y(x) = x^\lambda \sum_{n=0}^N \frac{A_n}{B_n} x^n, \quad B_n = \int_0^1 \frac{t^{\lambda+n} dt}{a + bt}.$$

3°. For $f(x) = \ln x \left(\sum_{n=0}^N A_n x^n \right)$, the solution has the form

$$y(x) = \ln x \sum_{n=0}^N \frac{A_n}{B_n} x^n - \sum_{n=0}^N \frac{A_n C_n}{B_n^2} x^n, \quad B_n = \int_0^1 \frac{t^n dt}{a + bt}, \quad C_n = \int_0^1 \frac{t^n \ln t}{a + bt} dt.$$

4°. For some other special forms of the right-hand side (see items 4 and 5, equation 1.1.26), the solution may be found by the method of undetermined coefficients.

28. $\int_0^x \frac{y(t) dt}{ax^2 + bt^2} = f(x), \quad a > 0, \quad a + b > 0.$

1°. For a polynomial right-hand side, $f(x) = \sum_{n=0}^N A_n x^n$, the solution has the form

$$y(x) = \sum_{n=0}^N \frac{A_n}{B_n} x^{n+1}, \quad B_n = \int_0^1 \frac{t^{n+1} dt}{a + bt^2}.$$

Example. For $a = b = 1$ and $f(x) = Ax^2 + Bx + C$, the solution of the integral equation is:

$$y(x) = \frac{2A}{1 - \ln 2}x^3 + \frac{4B}{4 - \pi}x^2 + \frac{2C}{\ln 2}x.$$

2°. For $f(x) = x^\lambda \sum_{n=0}^N A_n x^n$, where λ is an arbitrary number ($\lambda > -1$), the solution has the form

$$y(x) = x^\lambda \sum_{n=0}^N \frac{A_n}{B_n} x^{n+1}, \quad B_n = \int_0^1 \frac{t^{\lambda+n+1} dt}{a + bt^2}.$$

3°. For $f(x) = \ln x \left(\sum_{n=0}^N A_n x^n \right)$, the solution has the form

$$y(x) = \ln x \sum_{n=0}^N \frac{A_n}{B_n} x^{n+1} - \sum_{n=0}^N \frac{A_n C_n}{B_n^2} x^{n+1}, \quad B_n = \int_0^1 \frac{t^{n+1} dt}{a + bt^2}, \quad C_n = \int_0^1 \frac{t^{n+1} \ln t}{a + bt^2} dt.$$

29.
$$\int_0^x \frac{y(t) dt}{ax^m + bt^m} = f(x), \quad a > 0, \quad a + b > 0, \quad m = 1, 2, \dots$$

1°. For a polynomial right-hand side, $f(x) = \sum_{n=0}^N A_n x^n$, the solution has the form

$$y(x) = \sum_{n=0}^N \frac{A_n}{B_n} x^{m+n-1}, \quad B_n = \int_0^1 \frac{t^{m+n-1} dt}{a + bt^m}.$$

2°. For $f(x) = x^\lambda \sum_{n=0}^N A_n x^n$, where λ is an arbitrary number ($\lambda > -1$), the solution has the form

$$y(x) = x^\lambda \sum_{n=0}^N \frac{A_n}{B_n} x^{m+n-1}, \quad B_n = \int_0^1 \frac{t^{\lambda+m+n-1} dt}{a + bt^m}.$$

3°. For $f(x) = \ln x \left(\sum_{n=0}^N A_n x^n \right)$, the solution has the form

$$y(x) = \ln x \sum_{n=0}^N \frac{A_n}{B_n} x^{m+n-1} - \sum_{n=0}^N \frac{A_n C_n}{B_n^2} x^{m+n-1},$$

$$B_n = \int_0^1 \frac{t^{m+n-1} dt}{a + bt^m}, \quad C_n = \int_0^1 \frac{t^{m+n-1} \ln t}{a + bt^m} dt.$$

1.1-6. Kernels Containing Square Roots.

30.
$$\int_a^x \sqrt{x-t} y(t) dt = f(x).$$

Differentiating with respect to x , we arrive at Abel's equation 1.1.36:

$$\int_a^x \frac{y(t) dt}{\sqrt{x-t}} = 2f'_x(x).$$

Solution:

$$y(x) = \frac{2}{\pi} \frac{d^2}{dx^2} \int_a^x \frac{f(t) dt}{\sqrt{x-t}}.$$

$$31. \int_a^x (\sqrt{x} - \sqrt{t}) y(t) dt = f(x).$$

This is a special case of equation 1.1.45 with $\mu = \frac{1}{2}$.

$$\text{Solution: } y(x) = 2 \frac{d}{dx} [\sqrt{x} f'_x(x)].$$

$$32. \int_a^x (A\sqrt{x} + B\sqrt{t}) y(t) dt = f(x).$$

This is a special case of equation 1.1.46 with $\mu = \frac{1}{2}$.

$$33. \int_a^x (1 + b\sqrt{x-t}) y(t) dt = f(x).$$

Differentiating with respect to x , we arrive at Abel's equation of the second kind 2.1.46:

$$y(x) + \frac{b}{2} \int_a^x \frac{y(t) dt}{\sqrt{x-t}} = f'_x(x).$$

$$34. \int_a^x (t\sqrt{x} - x\sqrt{t}) y(t) dt = f(x).$$

This is a special case of equation 1.9.11 with $g(x) = \sqrt{x}$ and $h(x) = x$.

$$35. \int_a^x (At\sqrt{x} + Bx\sqrt{t}) y(t) dt = f(x).$$

This is a special case of equation 1.9.12 with $g(x) = \sqrt{x}$ and $h(t) = t$.

$$36. \int_a^x \frac{y(t) dt}{\sqrt{x-t}} = f(x).$$

Abel's equation.

Solution:

$$y(x) = \frac{1}{\pi} \frac{d}{dx} \int_a^x \frac{f(t) dt}{\sqrt{x-t}} = \frac{f(a)}{\pi\sqrt{x-a}} + \frac{1}{\pi} \int_a^x \frac{f'_t(t) dt}{\sqrt{x-t}}.$$

⊙ Reference: E. T. Whittaker and G. N. Watson (1958).

$$37. \int_a^x \left(b + \frac{1}{\sqrt{x-t}} \right) y(t) dt = f(x).$$

Let us rewrite the equation in the form

$$\int_a^x \frac{y(t) dt}{\sqrt{x-t}} = f(x) - b \int_a^x y(t) dt.$$

Assuming the right-hand side to be known, we solve this equation as Abel's equation 1.1.36. After some manipulations, we arrive at Abel's equation of the second kind 2.1.46:

$$y(x) + \frac{b}{\pi} \int_a^x \frac{y(t) dt}{\sqrt{x-t}} = F(x), \quad \text{where } F(x) = \frac{1}{\pi} \frac{d}{dx} \int_a^x \frac{f(t) dt}{\sqrt{x-t}}.$$

$$38. \int_a^x \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{t}} \right) y(t) dt = f(x).$$

This is a special case of equation 1.1.45 with $\mu = -\frac{1}{2}$.

Solution: $y(x) = -2 \left[x^{3/2} f'_x(x) \right]'_x, \quad a > 0.$

$$39. \int_a^x \left(\frac{A}{\sqrt{x}} + \frac{B}{\sqrt{t}} \right) y(t) dt = f(x).$$

This is a special case of equation 1.1.46 with $\mu = -\frac{1}{2}$.

$$40. \int_{-x}^x \sqrt{\frac{x-t}{x+t}} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{\text{sign } x}{2\pi} \left[\frac{d}{dx} \int_0^{|x|} \frac{f(t) - f(-t)}{\sqrt{x^2 - t^2}} dt - \frac{1}{x} \frac{d}{dx} \int_0^{|x|} \frac{t[f(t) - f(-t)]}{\sqrt{x^2 - t^2}} dt \right].$$

● Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992).

$$41. \int_a^x \frac{y(t) dt}{\sqrt{x^2 - t^2}} = f(x).$$

Solution: $y = \frac{2}{\pi} \frac{d}{dx} \int_a^x \frac{t f(t) dt}{\sqrt{x^2 - t^2}}.$

● Reference: P. P. Zabreyko, A. I. Koshelev, et al. (1975).

$$42. \int_0^x \frac{y(t) dt}{\sqrt{ax^2 + bt^2}} = f(x), \quad a > 0, \quad a + b > 0.$$

1°. For a polynomial right-hand side, $f(x) = \sum_{n=0}^N A_n x^n$, the solution has the form

$$y(x) = \sum_{n=0}^N \frac{A_n}{B_n} x^n, \quad B_n = \int_0^1 \frac{t^n dt}{\sqrt{a + bt^2}}.$$

2°. For $f(x) = x^\lambda \sum_{n=0}^N A_n x^n$, where λ is an arbitrary number ($\lambda > -1$), the solution has the form

$$y(x) = x^\lambda \sum_{n=0}^N \frac{A_n}{B_n} x^n, \quad B_n = \int_0^1 \frac{t^{\lambda+n} dt}{\sqrt{a + bt^2}}.$$

3°. For $f(x) = \ln x \left(\sum_{n=0}^N A_n x^n \right)$, the solution has the form

$$y(x) = \ln x \sum_{n=0}^N \frac{A_n}{B_n} x^n - \sum_{n=0}^N \frac{A_n C_n}{B_n^2} x^n, \quad B_n = \int_0^1 \frac{t^n dt}{\sqrt{a + bt^2}}, \quad C_n = \int_0^1 \frac{t^n \ln t}{\sqrt{a + bt^2}} dt.$$

4°. For $f(x) = \sum_{n=0}^N A_n (\ln x)^n$, the solution of the equation has the form

$$y(x) = \sum_{n=0}^N A_n Y_n(x),$$

where the functions $Y_n = Y_n(x)$ are given by

$$Y_n(x) = \left\{ \frac{d^n}{d\lambda^n} \left[\frac{x^\lambda}{I(\lambda)} \right] \right\}_{\lambda=0}, \quad I(\lambda) = \int_0^1 \frac{z^\lambda dz}{\sqrt{a+bz^2}}.$$

5°. For $f(x) = \sum_{n=1}^N A_n \cos(\lambda_n \ln x) + \sum_{n=1}^N B_n \sin(\lambda_n \ln x)$, the solution of the equation has the form

$$y(x) = \sum_{n=1}^N C_n \cos(\lambda_n \ln x) + \sum_{n=1}^N D_n \sin(\lambda_n \ln x),$$

where the constants C_n and D_n are found by the method of undetermined coefficients.

1.1-7. Kernels Containing Arbitrary Powers.

43. $\int_a^x (x-t)^\lambda y(t) dt = f(x), \quad f(a) = 0, \quad 0 < \lambda < 1.$

Differentiating with respect to x , we arrive at the generalized Abel equation 1.1.47:

$$\int_a^x \frac{y(t) dt}{(x-t)^{1-\lambda}} = \frac{1}{\lambda} f'_x(x).$$

Solution:

$$y(x) = k \frac{d^2}{dx^2} \int_a^x \frac{f(t) dt}{(x-t)^\lambda}, \quad k = \frac{\sin(\pi\lambda)}{\pi\lambda}.$$

⊙ Reference: F. D. Gakhov (1977).

44. $\int_a^x (x-t)^\mu y(t) dt = f(x).$

For $\mu = 0, 1, 2, \dots$, see equations 1.1.1, 1.1.2, 1.1.4, 1.1.12, and 1.1.23. For $0 < \mu < 1$, see equation 1.1.43.

Set $\mu = n - \lambda$, where $n = 1, 2, \dots$ and $0 \leq \lambda < 1$, and $f(a) = f'_x(a) = \dots = f_x^{(n-1)}(a) = 0$.

On differentiating the equation n times, we arrive at an equation of the form 1.1.47:

$$\int_a^x \frac{y(t) d\tau}{(x-t)^\lambda} = \frac{\Gamma(\mu - n + 1)}{\Gamma(\mu + 1)} f_x^{(n)}(x),$$

where $\Gamma(\mu)$ is the gamma function.

Example. Set $f(x) = Ax^\beta$, where $\beta \geq 0$, and let $\mu > -1$ and $\mu - \beta \neq 0, 1, 2, \dots$. In this case, the solution has the form $y(x) = \frac{A\Gamma(\beta+1)}{\Gamma(\mu+1)\Gamma(\beta-\mu)} x^{\beta-\mu-1}$.

⊙ Reference: M. L. Krasnov, A. I. Kisilev, and G. I. Makarenko (1971).

45. $\int_a^x (x^\mu - t^\mu) y(t) dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = x^\mu$.

Solution: $y(x) = \frac{1}{\mu} [x^{1-\mu} f'_x(x)]'_x.$

46. $\int_a^x (Ax^\mu + Bt^\mu)y(t) dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = x^\mu$. For $B = -A$, see equation 1.1.44.

Solution: $y(x) = \frac{1}{A+B} \frac{d}{dx} \left[x^{-\frac{A\mu}{A+B}} \int_a^x t^{-\frac{B\mu}{A+B}} f'_t(t) dt \right].$

47. $\int_a^x \frac{y(t) dt}{(x-t)^\lambda} = f(x), \quad 0 < \lambda < 1.$

The generalized Abel equation.

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi} \frac{d}{dx} \int_a^x \frac{f(t) dt}{(x-t)^{1-\lambda}} = \frac{\sin(\pi\lambda)}{\pi} \left[\frac{f(a)}{(x-a)^{1-\lambda}} + \int_a^x \frac{f'_t(t) dt}{(x-t)^{1-\lambda}} \right].$$

⊙ Reference: E. T. Whittaker and G. N. Watson (1958).

48. $\int_a^x \left[b + \frac{1}{(x-t)^\lambda} \right] y(t) dt = f(x), \quad 0 < \lambda < 1.$

Rewrite the equation in the form

$$\int_a^x \frac{y(t) dt}{(x-t)^\lambda} = f(x) - b \int_a^x y(t) dt.$$

Assuming the right-hand side to be known, we solve this equation as the generalized Abel equation 1.1.47. After some manipulations, we arrive at Abel's equation of the second kind 2.1.60:

$$y(x) + \frac{b \sin(\pi\lambda)}{\pi} \int_a^x \frac{y(t) dt}{(x-t)^{1-\lambda}} = F(x), \quad \text{where} \quad F(x) = \frac{\sin(\pi\lambda)}{\pi} \frac{d}{dx} \int_a^x \frac{f(t) dt}{(x-t)^{1-\lambda}}.$$

49. $\int_a^x (\sqrt{x} - \sqrt{t})^\lambda y(t) dt = f(x), \quad 0 < \lambda < 1.$

Solution:

$$y(x) = \frac{k}{\sqrt{x}} \left(\sqrt{x} \frac{d}{dx} \right)^2 \int_a^x \frac{f(t) dt}{\sqrt{t} (\sqrt{x} - \sqrt{t})^\lambda}, \quad k = \frac{\sin(\pi\lambda)}{\pi\lambda}.$$

50. $\int_a^x \frac{y(t) dt}{(\sqrt{x} - \sqrt{t})^\lambda} = f(x), \quad 0 < \lambda < 1.$

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{2\pi} \frac{d}{dx} \int_a^x \frac{f(t) dt}{\sqrt{t} (\sqrt{x} - \sqrt{t})^{1-\lambda}}.$$

51. $\int_a^x (Ax^\lambda + Bt^\mu)y(t) dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = Ax^\lambda$ and $h(t) = Bt^\mu$.

$$52. \int_a^x [1 + A(x^\lambda t^\mu - x^{\lambda+\mu})] y(t) dt = f(x).$$

This is a special case of equation 1.9.13 with $g(x) = Ax^\mu$ and $h(x) = x^\lambda$.

Solution:

$$y(x) = \frac{d}{dx} \left\{ \frac{x^\lambda}{\Phi(x)} \int_a^x [t^{-\lambda} f(t)]'_t \Phi(t) dt \right\}, \quad \Phi(x) = \exp\left(-\frac{A\mu}{\mu + \lambda} x^{\mu+\lambda}\right).$$

$$53. \int_a^x (Ax^\beta t^\gamma + Bx^\delta t^\lambda) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\beta$, $h_1(t) = t^\gamma$, $g_2(x) = Bx^\delta$, and $h_2(t) = t^\lambda$.

$$54. \int_a^x [Ax^\lambda(t^\mu - x^\mu) + Bx^\beta(t^\gamma - x^\gamma)] y(t) dt = f(x).$$

This is a special case of equation 1.9.47 with $g_1(x) = Ax^\lambda$, $h_1(x) = x^\mu$, $g_2(x) = Bx^\beta$, and $h_2(x) = x^\gamma$.

$$55. \int_a^x [Ax^\lambda t^\mu + Bx^{\lambda+\beta} t^{\mu-\beta} - (A+B)x^{\lambda+\gamma} t^{\mu-\gamma}] y(t) dt = f(x).$$

This is a special case of equation 1.9.49 with $g(x) = x$.

$$56. \int_a^x t^\sigma (x^\mu - t^\mu)^\lambda y(t) dt = f(x), \quad \sigma > -1, \quad \mu > 0, \quad \lambda > -1.$$

The transformation $\tau = t^\mu$, $z = x^\mu$, $w(\tau) = t^{\sigma-\mu+1} y(t)$ leads to an equation of the form 1.1.43:

$$\int_A^z (z - \tau)^\lambda w(\tau) d\tau = F(z),$$

where $A = a^\mu$ and $F(z) = \mu f(z^{1/\mu})$.

Solution with $-1 < \lambda < 0$:

$$y(x) = -\frac{\mu \sin(\pi\lambda)}{\pi x^\sigma} \frac{d}{dx} \left[\int_a^x t^{\mu-1} (x^\mu - t^\mu)^{-1-\lambda} f(t) dt \right].$$

$$57. \int_0^x \frac{y(t) dt}{(x+t)^\mu} = f(x).$$

This is a special case of equation 1.1.58 with $\lambda = 1$ and $a = b = 1$.

The transformation

$$x = \frac{1}{2} e^{2z}, \quad t = \frac{1}{2} e^{2\tau}, \quad y(t) = e^{(\mu-2)\tau} w(\tau), \quad f(x) = e^{-\mu z} g(z)$$

leads to an equation with difference kernel of the form 1.9.27:

$$\int_{-\infty}^z \frac{w(\tau) d\tau}{\cosh^\mu(z - \tau)} = g(z).$$

$$58. \int_0^x \frac{y(t) dt}{(ax^\lambda + bt^\lambda)^\mu} = f(x), \quad a > 0, \quad a + b > 0.$$

1°. The substitution $t = xz$ leads to a special case of equation 3.8.45:

$$\int_0^1 \frac{y(xz) dz}{(a + bz^\lambda)^\mu} = x^{\lambda\mu-1} f(x). \quad (1)$$

2°. For a polynomial right-hand side, $f(x) = \sum_{m=0}^n A_m x^m$, the solution has the form

$$y(x) = x^{\lambda\mu-1} \sum_{m=0}^n \frac{A_m}{I_m} x^m, \quad I_m = \int_0^1 \frac{z^{m+\lambda\mu-1} dz}{(a + bz^\lambda)^\mu}.$$

The integrals I_m are supposed to be convergent.

3°. The solution structure for some other right-hand sides of the integral equation may be obtained using (1) and the results presented for the more general equation 3.8.53 (see also equations 3.8.34–3.8.40).

4°. For $a = b$, the equation can be reduced, just as equation 1.1.57, to an integral equation with difference kernel of the form 1.9.27.

$$59. \int_a^x \frac{(\sqrt{x} + \sqrt{x-t})^{2\lambda} + (\sqrt{x} - \sqrt{x-t})^{2\lambda}}{2t^\lambda \sqrt{x-t}} y(t) dt = f(x).$$

The equation can be rewritten in terms of the Gaussian hypergeometric functions in the form

$$\int_a^x (x-t)^{\gamma-1} F\left(\lambda, -\lambda, \gamma; 1 - \frac{x}{t}\right) y(t) dt = f(x), \quad \text{where } \gamma = \frac{1}{2}.$$

See 1.8.135 for the solution of this equation.

1.1-8. Two-Dimensional Equation of the Abel Type.

$$60. \iint_{\Delta} \frac{u(x, y) dx dy}{\sqrt{(y_0 - y)^2 - (x_0 - x)^2}} = f(x_0, y_0).$$

Here Δ is an isosceles right triangle with apex at the point (x_0, y_0) and base on the x -axis.

Solution:

$$u(x_0, y_0) = \frac{1}{2\pi^2} \left(\frac{\partial^2 g}{\partial x_0^2} - \frac{\partial^2 g}{\partial y_0^2} \right), \quad g(x_0, y_0) = \iint_{\Delta} \frac{f(x, y) dx dy}{\sqrt{(y_0 - y)^2 - (x_0 - x)^2}}.$$

⊙ Reference: P. P. Zabreyko, A. I. Koshelev, et al. (1975).

1.2. Equations Whose Kernels Contain Exponential Functions

1.2-1. Kernels Containing Exponential Functions.

$$1. \int_a^x e^{\lambda(x-t)} y(t) dt = f(x).$$

Solution: $y(x) = f'_x(x) - \lambda f(x)$.

Example. In the special case $a = 0$ and $f(x) = Ax$, the solution has the form $y(x) = A(1 - \lambda x)$.

$$2. \quad \int_a^x e^{\lambda x + \beta t} y(t) dt = f(x).$$

$$\text{Solution: } y(x) = e^{-(\lambda + \beta)x} [f'_x(x) - \lambda f(x)].$$

Example. In the special case $a = 0$ and $f(x) = A \sin(\gamma x)$, the solution has the form $y(x) = A e^{-(\lambda + \beta)x} \times [\gamma \cos(\gamma x) - \lambda \sin(\gamma x)]$.

$$3. \quad \int_a^x [e^{\lambda(x-t)} - 1] y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

$$\text{Solution: } y(x) = \frac{1}{\lambda} f''_{xx}(x) - f'_x(x).$$

$$4. \quad \int_a^x [e^{\lambda(x-t)} + b] y(t) dt = f(x).$$

For $b = -1$, see equation 1.2.3. Differentiating with respect to x yields an equation of the form 2.2.1:

$$y(x) + \frac{\lambda}{b+1} \int_a^x e^{\lambda(x-t)} y(t) dt = \frac{f'_x(x)}{b+1}.$$

Solution:

$$y(x) = \frac{f'_x(x)}{b+1} - \frac{\lambda}{(b+1)^2} \int_a^x \exp\left[\frac{\lambda b}{b+1}(x-t)\right] f'_t(t) dt.$$

$$5. \quad \int_a^x (e^{\lambda x + \beta t} + b) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = e^{\lambda x}$, $h_1(t) = e^{\beta t}$, $g_2(x) = 1$, and $h_2(t) = b$. For $\beta = -\lambda$, see equation 1.2.4.

$$6. \quad \int_a^x (e^{\lambda x} - e^{\lambda t}) y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

This is a special case of equation 1.9.2 with $g(x) = e^{\lambda x}$.

$$\text{Solution: } y(x) = e^{-\lambda x} \left[\frac{1}{\lambda} f''_{xx}(x) - f'_x(x) \right].$$

$$7. \quad \int_a^x (e^{\lambda x} - e^{\lambda t} + b) y(t) dt = f(x).$$

This is a special case of equation 1.9.3 with $g(x) = e^{\lambda x}$. For $b = 0$, see equation 1.2.6.

Solution:

$$y(x) = \frac{1}{b} f'_x(x) - \frac{\lambda}{b^2} e^{\lambda x} \int_a^x \exp\left(\frac{e^{\lambda t} - e^{\lambda x}}{b}\right) f'_t(t) dt.$$

$$8. \quad \int_a^x (A e^{\lambda x} + B e^{\lambda t}) y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = e^{\lambda x}$. For $B = -A$, see equation 1.2.6.

$$\text{Solution: } y(x) = \frac{1}{A+B} \frac{d}{dx} \left[\exp\left(-\frac{A\lambda}{A+B}x\right) \int_a^x \exp\left(-\frac{B\lambda}{A+B}t\right) f'_t(t) dt \right].$$

$$9. \quad \int_a^x (A e^{\lambda x} + B e^{\lambda t} + C) y(t) dt = f(x).$$

This is a special case of equation 1.9.5 with $g(x) = e^{\lambda x}$.

10. $\int_a^x (Ae^{\lambda x} + Be^{\mu t})y(t) dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = Ae^{\lambda x}$ and $h(t) = Be^{\mu t}$. For $\lambda = \mu$, see equation 1.2.8.

11. $\int_a^x [e^{\lambda(x-t)} - e^{\mu(x-t)}]y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$

Solution:

$$y(x) = \frac{1}{\lambda - \mu} [f''_{xx} - (\lambda + \mu)f'_x + \lambda\mu f], \quad f = f(x).$$

12. $\int_a^x [Ae^{\lambda(x-t)} + Be^{\mu(x-t)}]y(t) dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\lambda x}$, $h_1(t) = e^{-\lambda t}$, $g_2(x) = Be^{\mu x}$, and $h_2(t) = e^{-\mu t}$. For $B = -A$, see equation 1.2.11.

Solution:

$$y(x) = \frac{e^{\lambda x}}{A+B} \frac{d}{dx} \left\{ e^{(\mu-\lambda)x} \Phi(x) \int_a^x \left[\frac{f(t)}{e^{\mu t}} \right]'_t \frac{dt}{\Phi(t)} \right\}, \quad \Phi(x) = \exp \left[\frac{B(\lambda - \mu)}{A+B} x \right].$$

13. $\int_a^x [Ae^{\lambda(x-t)} + Be^{\mu(x-t)} + C]y(t) dt = f(x).$

This is a special case of equation 1.2.14 with $\beta = 0$.

14. $\int_a^x [Ae^{\lambda(x-t)} + Be^{\mu(x-t)} + Ce^{\beta(x-t)}]y(t) dt = f(x).$

Differentiating the equation with respect to x yields

$$(A+B+C)y(x) + \int_a^x [A\lambda e^{\lambda(x-t)} + B\mu e^{\mu(x-t)} + C\beta e^{\beta(x-t)}]y(t) dt = f'_x(x).$$

Eliminating the term with $e^{\beta(x-t)}$ with the aid of the original equation, we arrive at an equation of the form 2.2.10:

$$(A+B+C)y(x) + \int_a^x [A(\lambda - \beta)e^{\lambda(x-t)} + B(\mu - \beta)e^{\mu(x-t)}]y(t) dt = f'_x(x) - \beta f(x).$$

In the special case $A+B+C=0$, this is an equation of the form 1.2.12.

15. $\int_a^x [Ae^{\lambda(x-t)} + Be^{\mu(x-t)} + Ce^{\beta(x-t)} - A - B - C]y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$

Differentiating with respect to x , we arrive at an equation of the form 1.2.14:

$$\int_a^x [A\lambda e^{\lambda(x-t)} + B\mu e^{\mu(x-t)} + C\beta e^{\beta(x-t)}]y(t) dt = f'_x(x).$$

16. $\int_a^x (e^{\lambda x + \mu t} - e^{\mu x + \lambda t})y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$

This is a special case of equation 1.9.11 with $g(x) = e^{\lambda x}$ and $h(t) = e^{\mu t}$.

Solution:

$$y(x) = \frac{f''_{xx} - (\lambda + \mu)f'_x(x) + \lambda\mu f(x)}{(\lambda - \mu) \exp[(\lambda + \mu)x]}.$$

$$17. \int_a^x (Ae^{\lambda x + \mu t} + Be^{\mu x + \lambda t})y(t) dt = f(x).$$

This is a special case of equation 1.9.12 with $g(x) = e^{\lambda x}$ and $h(t) = e^{\mu t}$. For $B = -A$, see equation 1.2.16.

Solution:

$$y(x) = \frac{1}{(A+B)e^{\mu x}} \frac{d}{dx} \left\{ \Phi^A(x) \int_a^x \Phi^B(t) \frac{d}{dt} \left[\frac{f(t)}{e^{\mu t}} \right] dt \right\}, \quad \Phi(x) = \exp\left(\frac{\mu - \lambda}{A+B}x\right).$$

$$18. \int_a^x (Ae^{\lambda x + \mu t} + Be^{\beta x + \gamma t})y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\lambda x}$, $h_1(t) = e^{\mu t}$, $g_2(x) = Be^{\beta x}$, and $h_2(t) = e^{\gamma t}$.

$$19. \int_a^x (Ae^{2\lambda x} + Be^{2\beta t} + Ce^{\lambda x} + De^{\beta t} + E)y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = Ae^{2\lambda x} + Ce^{\lambda x}$ and $h(t) = Be^{2\beta t} + De^{\beta t} + E$.

$$20. \int_a^x (Ae^{\lambda x + \beta t} + Be^{2\beta t} + Ce^{\lambda x} + De^{\beta t} + E)y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = e^{\lambda x}$, $h_1(t) = Ae^{\beta t} + C$, and $g_2(x) = 1$, $h_2(t) = Be^{2\beta t} + De^{\beta t} + E$.

$$21. \int_a^x (Ae^{2\lambda x} + Be^{\lambda x + \beta t} + Ce^{\lambda x} + De^{\beta t} + E)y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Be^{\lambda x} + D$, $h_1(t) = e^{\beta t}$, and $g_2(x) = Ae^{2\lambda x} + Ce^{\lambda x} + E$, $h_2(t) = 1$.

$$22. \int_a^x [1 + Ae^{\lambda x}(e^{\mu t} - e^{\mu x})]y(t) dt = f(x).$$

This is a special case of equation 1.9.13 with $g(x) = e^{\mu x}$ and $h(x) = Ae^{\lambda x}$.

Solution:

$$y(x) = \frac{d}{dx} \left\{ e^{\lambda x} \Phi(x) \int_a^x \left[\frac{f(t)}{e^{\lambda t}} \right]'_t \frac{dt}{\Phi(t)} \right\}, \quad \Phi(x) = \exp \left[\frac{A\mu}{\lambda + \mu} e^{(\lambda + \mu)x} \right].$$

$$23. \int_a^x [Ae^{\lambda x}(e^{\mu x} - e^{\mu t}) + Be^{\beta x}(e^{\gamma x} - e^{\gamma t})]y(t) dt = f(x).$$

This is a special case of equation 1.9.47 with $g_1(x) = Ae^{\lambda x}$, $h_1(t) = -e^{\mu t}$, $g_2(x) = Be^{\beta x}$, and $h_2(t) = -e^{\gamma t}$.

$$24. \int_a^x \{ A \exp(\lambda x + \mu t) + B \exp[(\lambda + \beta)x + (\mu - \beta)t] \\ - (A + B) \exp[(\lambda + \gamma)x + (\mu - \gamma)t] \} y(t) dt = f(x).$$

This is a special case of equation 1.9.49 with $g_1(x) = e^x$.

$$25. \int_a^x (e^{\lambda x} - e^{\lambda t})^n y(t) dt = f(x), \quad n = 1, 2, \dots$$

Solution:

$$y(x) = \frac{1}{\lambda^n n!} e^{\lambda x} \left(\frac{1}{e^{\lambda x}} \frac{d}{dx} \right)^{n+1} f(x).$$

$$26. \int_a^x \sqrt{e^{\lambda x} - e^{\lambda t}} y(t) dt = f(x), \quad \lambda > 0.$$

Solution:

$$y(x) = \frac{2}{\pi} e^{\lambda x} \left(e^{-\lambda x} \frac{d}{dx} \right)^2 \int_a^x \frac{e^{\lambda t} f(t) dt}{\sqrt{e^{\lambda x} - e^{\lambda t}}}.$$

$$27. \int_a^x \frac{y(t) dt}{\sqrt{e^{\lambda x} - e^{\lambda t}}} = f(x), \quad \lambda > 0.$$

Solution:

$$y(x) = \frac{\lambda}{\pi} \frac{d}{dx} \int_a^x \frac{e^{\lambda t} f(t) dt}{\sqrt{e^{\lambda x} - e^{\lambda t}}}.$$

$$28. \int_a^x (e^{\lambda x} - e^{\lambda t})^\mu y(t) dt = f(x), \quad \lambda > 0, \quad 0 < \mu < 1.$$

Solution:

$$y(x) = k e^{\lambda x} \left(e^{-\lambda x} \frac{d}{dx} \right)^2 \int_a^x \frac{e^{\lambda t} f(t) dt}{(e^{\lambda x} - e^{\lambda t})^\mu}, \quad k = \frac{\sin(\pi\mu)}{\pi\mu}.$$

$$29. \int_a^x \frac{y(t) dt}{(e^{\lambda x} - e^{\lambda t})^\mu} = f(x), \quad \lambda > 0, \quad 0 < \mu < 1.$$

Solution:

$$y(x) = \frac{\lambda \sin(\pi\mu)}{\pi} \frac{d}{dx} \int_a^x \frac{e^{\lambda t} f(t) dt}{(e^{\lambda x} - e^{\lambda t})^{1-\mu}}.$$

1.2-2. Kernels Containing Power-Law and Exponential Functions.

$$30. \int_a^x [A(x-t) + B e^{\lambda(x-t)}] y(t) dt = f(x).$$

Differentiating with respect to x , we arrive at an equation of the form 2.2.4:

$$B y(x) + \int_a^x [A + B \lambda e^{\lambda(x-t)}] y(t) dt = f'_x(x).$$

$$31. \int_a^x (x-t) e^{\lambda(x-t)} y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

Solution: $y(x) = f''_{xx}(x) - 2\lambda f'_x(x) + \lambda^2 f(x).$

$$32. \int_a^x (Ax + Bt + C) e^{\lambda(x-t)} y(t) dt = f(x).$$

The substitution $u(x) = e^{-\lambda x} y(x)$ leads to an equation of the form 1.1.3:

$$\int_a^x (Ax + Bt + C) u(t) dt = e^{-\lambda x} f(x).$$

$$33. \int_a^x (Axe^{\lambda t} + Bte^{\mu x})y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ax$, $h_1(t) = e^{\lambda t}$, and $g_2(x) = Be^{\mu x}$, $h_2(t) = t$.

$$34. \int_a^x [Axe^{\lambda(x-t)} + Bte^{\mu(x-t)}]y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Axe^{\lambda x}$, $h_1(t) = e^{-\lambda t}$, $g_2(x) = Be^{\mu x}$, and $h_2(t) = te^{-\mu t}$.

$$35. \int_a^x (x-t)^2 e^{\lambda(x-t)} y(t) dt = f(x), \quad f(a) = f'_x(a) = f''_{xx}(a) = 0.$$

Solution: $y(x) = \frac{1}{2} [f'''_{xxx}(x) - 3\lambda f''_{xx}(x) + 3\lambda^2 f'_x(x) - \lambda^3 f(x)]$.

$$36. \int_a^x (x-t)^n e^{\lambda(x-t)} y(t) dt = f(x), \quad n = 1, 2, \dots$$

It is assumed that $f(a) = f'_x(a) = \dots = f^{(n)}_x(a) = 0$.

Solution: $y(x) = \frac{1}{n!} e^{\lambda x} \frac{d^{n+1}}{dx^{n+1}} [e^{-\lambda x} f(x)]$.

$$37. \int_a^x (Ax^\beta + Be^{\lambda t})y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = Ax^\beta$ and $h(t) = Be^{\lambda t}$.

$$38. \int_a^x (Ae^{\lambda x} + Bt^\beta)y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = Ae^{\lambda x}$ and $h(t) = Bt^\beta$.

$$39. \int_a^x (Ax^\beta e^{\lambda t} + Bt^\gamma e^{\mu x})y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\beta$, $h_1(t) = e^{\lambda t}$, $g_2(x) = Be^{\mu x}$, and $h_2(t) = t^\gamma$.

$$40. \int_a^x e^{\lambda(x-t)} \sqrt{x-t} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{2}{\pi} e^{\lambda x} \frac{d^2}{dx^2} \int_a^x \frac{e^{-\lambda t} f(t) dt}{\sqrt{x-t}}.$$

$$41. \int_a^x \frac{e^{\lambda(x-t)}}{\sqrt{x-t}} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{1}{\pi} e^{\lambda x} \frac{d}{dx} \int_a^x \frac{e^{-\lambda t} f(t) dt}{\sqrt{x-t}}.$$

$$42. \int_a^x (x-t)^\lambda e^{\mu(x-t)} y(t) dt = f(x), \quad 0 < \lambda < 1.$$

Solution:

$$y(x) = k e^{\mu x} \frac{d^2}{dx^2} \int_a^x \frac{e^{-\mu t} f(t) dt}{(x-t)^\lambda}, \quad k = \frac{\sin(\pi \lambda)}{\pi \lambda}.$$

$$43. \int_a^x \frac{e^{\lambda(x-t)}}{(x-t)^\mu} y(t) dt = f(x), \quad 0 < \mu < 1.$$

Solution:

$$y(x) = \frac{\sin(\pi \mu)}{\pi} e^{\lambda x} \frac{d}{dx} \int_a^x \frac{e^{-\lambda t} f(t)}{(x-t)^{1-\mu}} dt.$$

$$44. \int_a^x (\sqrt{x} - \sqrt{t})^\lambda e^{\mu(x-t)} y(t) dt = f(x), \quad 0 < \lambda < 1.$$

The substitution $u(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.1.49:

$$\int_a^x (\sqrt{x} - \sqrt{t})^\lambda u(t) dt = e^{-\mu x} f(x).$$

$$45. \int_a^x \frac{e^{\mu(x-t)} y(t) dt}{(\sqrt{x} - \sqrt{t})^\lambda} = f(x), \quad 0 < \lambda < 1.$$

The substitution $u(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.1.50:

$$\int_a^x \frac{u(t) dt}{(\sqrt{x} - \sqrt{t})^\lambda} = e^{-\mu x} f(x).$$

$$46. \int_a^x \frac{e^{\lambda(x-t)}}{\sqrt{x^2 - t^2}} y(t) dt = f(x).$$

$$\text{Solution: } y = \frac{2}{\pi} e^{\lambda x} \frac{d}{dx} \int_a^x \frac{t e^{-\lambda t}}{\sqrt{x^2 - t^2}} f(t) dt.$$

$$47. \int_a^x \exp[\lambda(x^2 - t^2)] y(t) dt = f(x).$$

$$\text{Solution: } y(x) = f'_x(x) - 2\lambda x f(x).$$

$$48. \int_a^x [\exp(\lambda x^2) - \exp(\lambda t^2)] y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \exp(\lambda x^2)$.

$$\text{Solution: } y(x) = \frac{1}{2\lambda} \frac{d}{dx} \left[\frac{f'_x(x)}{x \exp(\lambda x^2)} \right].$$

$$49. \int_a^x [A \exp(\lambda x^2) + B \exp(\lambda t^2) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.5 with $g(x) = \exp(\lambda x^2)$.

$$50. \int_a^x [A \exp(\lambda x^2) + B \exp(\mu t^2)] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \exp(\lambda x^2)$ and $h(t) = B \exp(\mu t^2)$.

$$51. \int_a^x \sqrt{x-t} \exp[\lambda(x^2 - t^2)] y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{2}{\pi} \exp(\lambda x^2) \frac{d^2}{dx^2} \int_a^x \frac{\exp(-\lambda t^2)}{\sqrt{x-t}} f(t) dt.$$

$$52. \int_a^x \frac{\exp[\lambda(x^2 - t^2)]}{\sqrt{x-t}} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{1}{\pi} \exp(\lambda x^2) \frac{d}{dx} \int_a^x \frac{\exp(-\lambda t^2)}{\sqrt{x-t}} f(t) dt.$$

$$53. \int_a^x (x-t)^\lambda \exp[\mu(x^2 - t^2)] y(t) dt = f(x), \quad 0 < \lambda < 1.$$

Solution:

$$y(x) = k \exp(\mu x^2) \frac{d^2}{dx^2} \int_a^x \frac{\exp(-\mu t^2)}{(x-t)^\lambda} f(t) dt, \quad k = \frac{\sin(\pi\lambda)}{\pi\lambda}.$$

$$54. \int_a^x \exp[\lambda(x^\beta - t^\beta)] y(t) dt = f(x).$$

Solution: $y(x) = f'_x(x) - \lambda\beta x^{\beta-1} f(x).$

$$55. \int_0^x (-1)^{[(x-t)/b]} y(t) dt = f(x), \quad f(0) = f'_x(0) = 0.$$

Here $b = \text{const}$ and $[A]$ stands for the integer part of the number A .

Solution:

$$y(x) = \frac{1}{2} \int_0^x \left(2 \left[\frac{x-t}{b} \right] + 1 \right) f''_{tt}(t) dt.$$

⊙ References: H. M. Srivastava and R. G. Buschman (1977), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 434).

1.3. Equations Whose Kernels Contain Hyperbolic Functions

1.3-1. Kernels Containing Hyperbolic Cosine.

$$1. \int_a^x \cosh[\lambda(x-t)] y(t) dt = f(x).$$

Solution: $y(x) = f'_x(x) - \lambda^2 \int_a^x f(x) dx.$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 435).

$$2. \int_a^x \{ \cosh[\lambda(x-t)] - 1 \} y(t) dt = f(x), \quad f(a) = f'_x(a) = f''_{xx}(x) = 0.$$

Solution: $y(x) = \frac{1}{\lambda^2} f'''_{xxx}(x) - f'_x(x).$

$$3. \quad \int_a^x \{\cosh[\lambda(x-t)] + b\} y(t) dt = f(x).$$

For $b = 0$, see equation 1.3.1. For $b = -1$, see equation 1.3.2. For $\lambda = 0$, see equation 1.1.1. Differentiating the equation with respect to x , we arrive at an equation of the form 2.3.16:

$$y(x) + \frac{\lambda}{b+1} \int_a^x \sinh[\lambda(x-t)] y(t) dt = \frac{f'_x(x)}{b+1}.$$

1°. Solution with $b(b+1) < 0$:

$$y(x) = \frac{f'_x(x)}{b+1} - \frac{\lambda^2}{k(b+1)^2} \int_a^x \sin[k(x-t)] f'_t(t) dt, \quad \text{where } k = \lambda \sqrt{\frac{-b}{b+1}}.$$

2°. Solution with $b(b+1) > 0$:

$$y(x) = \frac{f'_x(x)}{b+1} - \frac{\lambda^2}{k(b+1)^2} \int_a^x \sinh[k(x-t)] f'_t(t) dt, \quad \text{where } k = \lambda \sqrt{\frac{b}{b+1}}.$$

$$4. \quad \int_a^x \cosh(\lambda x + \beta t) y(t) dt = f(x).$$

For $\beta = -\lambda$, see equation 1.3.1.

Differentiating the equation with respect to x twice, we obtain

$$\cosh[(\lambda + \beta)x] y(x) + \lambda \int_a^x \sinh(\lambda x + \beta t) y(t) dt = f'_x(x), \quad (1)$$

$$\{\cosh[(\lambda + \beta)x] y(x)\}'_x + \lambda \sinh[(\lambda + \beta)x] y(x) + \lambda^2 \int_a^x \cosh(\lambda x + \beta t) y(t) dt = f''_{xx}(x). \quad (2)$$

Eliminating the integral term from (2) with the aid of the original equation, we arrive at the first-order linear ordinary differential equation

$$w'_x + \lambda \tanh[(\lambda + \beta)x] w = f''_{xx}(x) - \lambda^2 f(x), \quad w = \cosh[(\lambda + \beta)x] y(x). \quad (3)$$

Setting $x = a$ in (1) yields the initial condition $w(a) = f'_x(a)$. On solving equation (3) with this condition, after some manipulations we obtain the solution of the original integral equation in the form

$$y(x) = \frac{1}{\cosh[(\lambda + \beta)x]} f'_x(x) - \frac{\lambda \sinh[(\lambda + \beta)x]}{\cosh^2[(\lambda + \beta)x]} f(x) + \frac{\lambda \beta}{\cosh^{k+1}[(\lambda + \beta)x]} \int_a^x f(t) \cosh^{k-2}[(\lambda + \beta)t] dt, \quad k = \frac{\lambda}{\lambda + \beta}.$$

$$5. \quad \int_a^x [\cosh(\lambda x) - \cosh(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \cosh(\lambda x)$.

$$\text{Solution: } y(x) = \frac{1}{\lambda} \frac{d}{dx} \left[\frac{f'_x(x)}{\sinh(\lambda x)} \right].$$

$$6. \quad \int_a^x [A \cosh(\lambda x) + B \cosh(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \cosh(\lambda x)$. For $B = -A$, see equation 1.3.5.

$$\text{Solution: } y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ [\cosh(\lambda x)]^{-\frac{A}{A+B}} \int_a^x [\cosh(\lambda t)]^{-\frac{B}{A+B}} f'_t(t) dt \right\}.$$

7. $\int_a^x [A \cosh(\lambda x) + B \cosh(\mu t) + C] y(t) dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A \cosh(\lambda x)$ and $h(t) = B \cosh(\mu t) + C$.

8. $\int_a^x \{A_1 \cosh[\lambda_1(x-t)] + A_2 \cosh[\lambda_2(x-t)]\} y(t) dt = f(x).$

The equation is equivalent to the equation

$$\int_a^x \{B_1 \sinh[\lambda_1(x-t)] + B_2 \sinh[\lambda_2(x-t)]\} y(t) dt = F(x),$$

$$B_1 = \frac{A_1}{\lambda_1}, \quad B_2 = \frac{A_2}{\lambda_2}, \quad F(x) = \int_a^x f(t) dt,$$

of the form 1.3.49. (Differentiating this equation yields the original equation.)

9. $\int_a^x \cosh^2[\lambda(x-t)] y(t) dt = f(x).$

Differentiation yields an equation of the form 2.3.16:

$$y(x) + \lambda \int_a^x \sinh[2\lambda(x-t)] y(t) dt = f'_x(x).$$

Solution:

$$y(x) = f'_x(x) - \frac{2\lambda^2}{k} \int_a^x \sinh[k(x-t)] f'_t(t) dt, \quad \text{where } k = \lambda\sqrt{2}.$$

10. $\int_a^x [\cosh^2(\lambda x) - \cosh^2(\lambda t)] y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$

Solution: $y(x) = \frac{1}{\lambda} \frac{d}{dx} \left[\frac{f'_x(x)}{\sinh(2\lambda x)} \right].$

11. $\int_a^x [A \cosh^2(\lambda x) + B \cosh^2(\lambda t)] y(t) dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \cosh^2(\lambda x)$. For $B = -A$, see equation 1.3.10.

Solution:

$$y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ [\cosh(\lambda x)]^{-\frac{2A}{A+B}} \int_a^x [\cosh(\lambda t)]^{-\frac{2B}{A+B}} f'_t(t) dt \right\}.$$

12. $\int_a^x [A \cosh^2(\lambda x) + B \cosh^2(\mu t) + C] y(t) dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A \cosh^2(\lambda x)$, and $h(t) = B \cosh^2(\mu t) + C$.

13. $\int_a^x \cosh[\lambda(x-t)] \cosh[\lambda(x+t)] y(t) dt = f(x).$

Using the formula

$$\cosh(\alpha - \beta) \cosh(\alpha + \beta) = \frac{1}{2} [\cosh(2\alpha) + \cosh(2\beta)], \quad \alpha = \lambda x, \quad \beta = \lambda t,$$

we transform the original equation to an equation of the form 1.3.6 with $A = B = 1$:

$$\int_a^x [\cosh(2\lambda x) + \cosh(2\lambda t)] y(t) dt = 2f(x).$$

Solution:

$$y(x) = \frac{d}{dx} \left[\frac{1}{\sqrt{\cosh(2\lambda x)}} \int_a^x \frac{f'_t(t) dt}{\sqrt{\cosh(2\lambda t)}} \right].$$

14. $\int_a^x [\cosh(\lambda x) \cosh(\mu t) + \cosh(\beta x) \cosh(\gamma t)] y(t) dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = \cosh(\lambda x)$, $h_1(t) = \cosh(\mu t)$, $g_2(x) = \cosh(\beta x)$, and $h_2(t) = \cosh(\gamma t)$.

15. $\int_a^x \cosh^3[\lambda(x-t)] y(t) dt = f(x).$

Using the formula $\cosh^3 \beta = \frac{1}{4} \cosh 3\beta + \frac{3}{4} \cosh \beta$, we arrive at an equation of the form 1.3.8:

$$\int_a^x \left\{ \frac{1}{4} \cosh[3\lambda(x-t)] + \frac{3}{4} \cosh[\lambda(x-t)] \right\} y(t) dt = f(x).$$

16. $\int_a^x [\cosh^3(\lambda x) - \cosh^3(\lambda t)] y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$

Solution: $y(x) = \frac{1}{3\lambda} \frac{d}{dx} \left[\frac{f'_x(x)}{\sinh(\lambda x) \cosh^2(\lambda x)} \right].$

17. $\int_a^x [A \cosh^3(\lambda x) + B \cosh^3(\lambda t)] y(t) dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \cosh^3(\lambda x)$. For $B = -A$, see equation 1.3.16.

Solution:

$$y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ [\cosh(\lambda x)]^{-\frac{3A}{A+B}} \int_a^x [\cosh(\lambda t)]^{-\frac{3B}{A+B}} f'_t(t) dt \right\}.$$

18. $\int_a^x [A \cosh^2(\lambda x) \cosh(\mu t) + B \cosh(\beta x) \cosh^2(\gamma t)] y(t) dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = A \cosh^2(\lambda x)$, $h_1(t) = \cosh(\mu t)$, $g_2(x) = B \cosh(\beta x)$, and $h_2(t) = \cosh^2(\gamma t)$.

19. $\int_a^x \cosh^4[\lambda(x-t)] y(t) dt = f(x).$

Let us transform the kernel of the integral equation using the formula

$$\cosh^4 \beta = \frac{1}{8} \cosh 4\beta + \frac{1}{2} \cosh 2\beta + \frac{3}{8}, \quad \text{where } \beta = \lambda(x-t),$$

and differentiate the resulting equation with respect to x . Then we obtain an equation of the form 2.3.18:

$$y(x) + \lambda \int_a^x \left\{ \frac{1}{2} \sinh[4\lambda(x-t)] + \sinh[2\lambda(x-t)] \right\} y(t) dt = f'_x(x).$$

20. $\int_a^x [\cosh(\lambda x) - \cosh(\lambda t)]^n y(t) dt = f(x), \quad n = 1, 2, \dots$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \dots = f_x^{(n)}(a) = 0$.

Solution: $y(x) = \frac{\sinh(\lambda x)}{\lambda^n n!} \left[\frac{1}{\sinh(\lambda x)} \frac{d}{dx} \right]^{n+1} f(x).$

$$21. \int_a^x \sqrt{\cosh x - \cosh t} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{2}{\pi} \sinh x \left(\frac{1}{\sinh x} \frac{d}{dx} \right)^2 \int_a^x \frac{\sinh t f(t) dt}{\sqrt{\cosh x - \cosh t}}.$$

$$22. \int_a^x \frac{y(t) dt}{\sqrt{\cosh x - \cosh t}} = f(x).$$

Solution:

$$y(x) = \frac{1}{\pi} \frac{d}{dx} \int_a^x \frac{\sinh t f(t) dt}{\sqrt{\cosh x - \cosh t}}.$$

$$23. \int_a^x (\cosh x - \cosh t)^\lambda y(t) dt = f(x), \quad 0 < \lambda < 1.$$

Solution:

$$y(x) = k \sinh x \left(\frac{1}{\sinh x} \frac{d}{dx} \right)^2 \int_a^x \frac{\sinh t f(t) dt}{(\cosh x - \cosh t)^\lambda}, \quad k = \frac{\sin(\pi\lambda)}{\pi\lambda}.$$

$$24. \int_a^x (\cosh^\mu x - \cosh^\mu t) y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \cosh^\mu x$.

$$\text{Solution: } y(x) = \frac{1}{\mu} \frac{d}{dx} \left[\frac{f'_x(x)}{\sinh x \cosh^{\mu-1} x} \right].$$

$$25. \int_a^x (A \cosh^\mu x + B \cosh^\mu t) y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \cosh^\mu x$. For $B = -A$, see equation 1.3.24.

Solution:

$$y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ [\cosh(\lambda x)]^{-\frac{A\mu}{A+B}} \int_a^x [\cosh(\lambda t)]^{-\frac{B\mu}{A+B}} f'_t(t) dt \right\}.$$

$$26. \int_a^x \frac{y(t) dt}{(\cosh x - \cosh t)^\lambda} = f(x), \quad 0 < \lambda < 1.$$

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi} \frac{d}{dx} \int_a^x \frac{\sinh t f(t) dt}{(\cosh x - \cosh t)^{1-\lambda}}.$$

$$27. \int_a^x (x-t) \cosh[\lambda(x-t)] y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

Differentiating the equation twice yields

$$y(x) + 2\lambda \int_a^x \sinh[\lambda(x-t)] y(t) dt + \lambda^2 \int_a^x (x-t) \cosh[\lambda(x-t)] y(t) dt = f''_{xx}(x).$$

Eliminating the third term on the right-hand side with the aid of the original equation, we arrive at an equation of the form 2.3.16:

$$y(x) + 2\lambda \int_a^x \sinh[\lambda(x-t)] y(t) dt = f''_{xx}(x) - \lambda^2 f(x).$$

$$28. \int_a^x \frac{\cosh[\lambda(x-t)]}{\sqrt{x-t}} y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

Solution:

$$y(x) = \frac{2}{\pi\lambda} \int_a^x \frac{\cosh[\lambda(x-t)]}{\sqrt{x-t}} [f''_{tt}(t) - \lambda^2 f(t)] dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 436).

$$29. \int_a^x \sqrt{x-t} \cosh(\lambda\sqrt{x-t}) y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{1}{\pi} \int_a^x \frac{\cos(\lambda\sqrt{x-t})}{\sqrt{x-t}} f'_t(t) dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 437), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

$$30. \int_a^x \frac{\cosh(\lambda\sqrt{x-t})}{\sqrt{x-t}} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{1}{\pi} \frac{d}{dx} \int_a^x \frac{\cos(\lambda\sqrt{x-t})}{\sqrt{x-t}} f(t) dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 437), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

$$31. \int_x^\infty \frac{\cosh(\lambda\sqrt{t-x})}{\sqrt{t-x}} y(t) dt = f(x).$$

Solution:

$$y(x) = -\frac{1}{\pi} \frac{d}{dx} \int_x^\infty \frac{\cos(\lambda\sqrt{t-x})}{\sqrt{t-x}} f(t) dt.$$

⊙ References: H. M. Srivastava and R. G. Buschman (1977), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 439), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

$$32. \int_0^x \frac{\cosh(\lambda\sqrt{x^2-t^2})}{\sqrt{x^2-t^2}} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{2}{\pi} \frac{d}{dx} \int_0^x t \frac{\cos(\lambda\sqrt{x^2-t^2})}{\sqrt{x^2-t^2}} f(t) dt.$$

$$33. \int_x^\infty \frac{\cosh(\lambda\sqrt{t^2-x^2})}{\sqrt{t^2-x^2}} y(t) dt = f(x).$$

Solution:

$$y(x) = -\frac{2}{\pi} \frac{d}{dx} \int_x^\infty t \frac{\cos(\lambda\sqrt{t^2-x^2})}{\sqrt{t^2-x^2}} f(t) dt.$$

$$34. \int_0^x \frac{\cosh(\lambda\sqrt{xt-t^2})}{\sqrt{x-t}} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{1}{\pi x} \int_0^x \frac{\cos(\lambda\sqrt{x^2-xt})}{\sqrt{x-t}} [f(t)/2 + t f'_t(t)] dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 438), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

$$35. \int_0^x \frac{\cosh(\lambda\sqrt{x^2-xt})}{\sqrt{x-t}} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{\sqrt{x}}{\pi} \frac{d}{dx} \left[\sqrt{x} \int_0^x \frac{\cos(\lambda\sqrt{xt-t^2})}{\sqrt{x-t}} f(t) dt \right].$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 438), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

$$36. \int_a^x \frac{\cosh[\lambda\sqrt{(x-t)(x-t+\gamma)}]}{\sqrt{x-t}} y(t) dt = f(x).$$

It is assumed that $f(a) = f'_x(a) = f''_{xx}(a) = f'''_{xxx}(a) = 0$.

Solution:

$$y(x) = \frac{2}{\pi\lambda^2} \int_a^x \frac{\sinh[\lambda\sqrt{(x-t)(x-t-\gamma)}]}{\sqrt{x-t-\gamma}} \int_a^t \sinh[\lambda(t-s)] \left(\frac{d^2}{ds^2} - \lambda^2 \right)^2 f(s) ds dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 438), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

$$37. \int_a^x [Ax^\beta + B \cosh^\gamma(\lambda t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = Ax^\beta$ and $h(t) = B \cosh^\gamma(\lambda t) + C$.

$$38. \int_a^x [A \cosh^\gamma(\lambda x) + Bt^\beta + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \cosh^\gamma(\lambda x)$ and $h(t) = Bt^\beta + C$.

$$39. \int_a^x (Ax^\lambda \cosh^\mu t + Bt^\beta \cosh^\gamma x) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \cosh^\mu t$, $g_2(x) = B \cosh^\gamma x$, and $h_2(t) = t^\beta$.

1.3-2. Kernels Containing Hyperbolic Sine.

$$40. \int_a^x \sinh[\lambda(x-t)] y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

Solution: $y(x) = \frac{1}{\lambda} f''_{xx}(x) - \lambda f(x)$.

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 435).

$$41. \int_a^x \frac{\sinh[\lambda(x-t)]}{\sqrt{x-t}} y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

Solution:

$$y(x) = \frac{2}{\pi\lambda} \int_a^x \frac{\sinh[\lambda(x-t)]}{\sqrt{x-t}} [f''_{tt}(t) - \lambda^2 f(t)] dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 436).

$$42. \int_a^x \frac{\sinh[\lambda(x-t)]}{(x-t)^{3/2}} y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

Solution:

$$y(x) = \frac{2}{\pi\lambda} \int_a^x \frac{\sinh[\lambda(x-t)]}{\sqrt{x-t}} \left[f''_{tt}(t) - \lambda^2 f(t) + \frac{f'(t)}{x-t} \right] dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 437).

$$43. \int_a^x \{ \sinh[\lambda(x-t)] + b \} y(t) dt = f(x).$$

For $b = 0$, see equation 1.3.40. Assume that $b \neq 0$.

Differentiating the equation with respect to x , we arrive at an equation of the form 2.3.3:

$$y(x) + \frac{\lambda}{b} \int_a^x \cosh[\lambda(x-t)] y(t) dt = \frac{1}{b} f'_x(x).$$

Solution:

$$y(x) = \frac{1}{b} f'_x(x) + \int_a^x R(x-t) f'_t(t) dt,$$

$$R(x) = \frac{\lambda}{b^2} \exp\left(-\frac{\lambda x}{2b}\right) \left[\frac{\lambda}{2bk} \sinh(kx) - \cosh(kx) \right], \quad k = \frac{\lambda\sqrt{1+4b^2}}{2b}.$$

$$44. \int_a^x \sinh(\lambda x + \beta t) y(t) dt = f(x).$$

For $\beta = -\lambda$, see equation 1.3.40. Assume that $\beta \neq -\lambda$.

Differentiating the equation with respect to x twice yields

$$\sinh[(\lambda + \beta)x] y(x) + \lambda \int_a^x \cosh(\lambda x + \beta t) y(t) dt = f'_x(x), \quad (1)$$

$$\{ \sinh[(\lambda + \beta)x] y(x) \}'_x + \lambda \cosh[(\lambda + \beta)x] y(x) + \lambda^2 \int_a^x \sinh(\lambda x + \beta t) y(t) dt = f''_{xx}(x). \quad (2)$$

Eliminating the integral term from (2) with the aid of the original equation, we arrive at the first-order linear ordinary differential equation

$$w'_x + \lambda \coth[(\lambda + \beta)x] w = f''_{xx}(x) - \lambda^2 f(x), \quad w = \sinh[(\lambda + \beta)x] y(x). \quad (3)$$

Setting $x = a$ in (1) yields the initial condition $w(a) = f'_x(a)$. On solving equation (3) with this condition, after some manipulations we obtain the solution of the original integral equation in the form

$$y(x) = \frac{1}{\sinh[(\lambda + \beta)x]} f'_x(x) - \frac{\lambda \cosh[(\lambda + \beta)x]}{\sinh^2[(\lambda + \beta)x]} f(x) - \frac{\lambda\beta}{\sinh^{k+1}[(\lambda + \beta)x]} \int_a^x f(t) \sinh^{k-2}[(\lambda + \beta)t] dt, \quad k = \frac{\lambda}{\lambda + \beta}.$$

45. $\int_a^x [\sinh(\lambda x) - \sinh(\lambda t)]y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$

This is a special case of equation 1.9.2 with $g(x) = \sinh(\lambda x)$.

Solution: $y(x) = \frac{1}{\lambda} \frac{d}{dx} \left[\frac{f'_x(x)}{\cosh(\lambda x)} \right].$

46. $\int_a^x [A \sinh(\lambda x) + B \sinh(\lambda t)]y(t) dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \sinh(\lambda x)$. For $B = -A$, see equation 1.3.45.

Solution: $y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ [\sinh(\lambda x)]^{-\frac{A}{A+B}} \int_a^x [\sinh(\lambda t)]^{-\frac{B}{A+B}} f'_t(t) dt \right\}.$

47. $\int_a^x [A \sinh(\lambda x) + B \sinh(\mu t)]y(t) dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A \sinh(\lambda x)$ and $h(t) = B \sinh(\mu t)$.

48. $\int_a^x \{ \mu \sinh[\lambda(x-t)] - \lambda \sinh[\mu(x-t)] \} y(t) dt = f(x).$

It is assumed that $f(a) = f'_x(a) = f''_{xx}(a) = f'''_{xxx}(a) = 0$.

Solution:

$$y(x) = \frac{f'''_{xxx} - (\lambda^2 + \mu^2)f''_{xx} + \lambda^2\mu^2 f}{\mu\lambda^3 - \lambda\mu^3}, \quad f = f(x).$$

49. $\int_a^x \{ A_1 \sinh[\lambda_1(x-t)] + A_2 \sinh[\lambda_2(x-t)] \} y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$

1°. Introduce the notation

$$I_1 = \int_a^x \sinh[\lambda_1(x-t)]y(t) dt, \quad I_2 = \int_a^x \sinh[\lambda_2(x-t)]y(t) dt,$$

$$J_1 = \int_a^x \cosh[\lambda_1(x-t)]y(t) dt, \quad J_2 = \int_a^x \cosh[\lambda_2(x-t)]y(t) dt.$$

Let us successively differentiate the integral equation four times. As a result, we have (the first line is the original equation):

$$A_1 I_1 + A_2 I_2 = f, \quad f = f(x), \quad (1)$$

$$A_1 \lambda_1 J_1 + A_2 \lambda_2 J_2 = f'_x, \quad (2)$$

$$(A_1 \lambda_1 + A_2 \lambda_2)y + A_1 \lambda_1^2 I_1 + A_2 \lambda_2^2 I_2 = f''_{xx}, \quad (3)$$

$$(A_1 \lambda_1 + A_2 \lambda_2)y'_x + A_1 \lambda_1^3 J_1 + A_2 \lambda_2^3 J_2 = f'''_{xxx}, \quad (4)$$

$$(A_1 \lambda_1 + A_2 \lambda_2)y''_{xx} + (A_1 \lambda_1^3 + A_2 \lambda_2^3)y + A_1 \lambda_1^4 I_1 + A_2 \lambda_2^4 I_2 = f''''_{xxxx}. \quad (5)$$

Eliminating I_1 and I_2 from (1), (3), and (5), we arrive at the following second-order linear ordinary differential equation with constant coefficients:

$$(A_1 \lambda_1 + A_2 \lambda_2)y''_{xx} - \lambda_1 \lambda_2 (A_1 \lambda_2 + A_2 \lambda_1)y = f''''_{xxxx} - (\lambda_1^2 + \lambda_2^2)f''_{xx} + \lambda_1^2 \lambda_2^2 f. \quad (6)$$

The initial conditions can be obtained by substituting $x = a$ into (3) and (4):

$$(A_1 \lambda_1 + A_2 \lambda_2)y(a) = f''_{xx}(a), \quad (A_1 \lambda_1 + A_2 \lambda_2)y'_x(a) = f'''_{xxx}(a). \quad (7)$$

Solving the differential equation (6) under conditions (7) allows us to find the solution of the integral equation.

2°. Denote

$$\Delta = \lambda_1 \lambda_2 \frac{A_1 \lambda_2 + A_2 \lambda_1}{A_1 \lambda_1 + A_2 \lambda_2}.$$

2.1. Solution for $\Delta > 0$:

$$(A_1 \lambda_1 + A_2 \lambda_2)y(x) = f''_{xx}(x) + Bf(x) + C \int_a^x \sinh[k(x-t)]f(t) dt,$$

$$k = \sqrt{\Delta}, \quad B = \Delta - \lambda_1^2 - \lambda_2^2, \quad C = \frac{1}{\sqrt{\Delta}} [\Delta^2 - (\lambda_1^2 + \lambda_2^2)\Delta + \lambda_1^2 \lambda_2^2].$$

2.2. Solution for $\Delta < 0$:

$$(A_1 \lambda_1 + A_2 \lambda_2)y(x) = f''_{xx}(x) + Bf(x) + C \int_a^x \sin[k(x-t)]f(t) dt,$$

$$k = \sqrt{-\Delta}, \quad B = \Delta - \lambda_1^2 - \lambda_2^2, \quad C = \frac{1}{\sqrt{-\Delta}} [\Delta^2 - (\lambda_1^2 + \lambda_2^2)\Delta + \lambda_1^2 \lambda_2^2].$$

2.3. Solution for $\Delta = 0$:

$$(A_1 \lambda_1 + A_2 \lambda_2)y(x) = f''_{xx}(x) - (\lambda_1^2 + \lambda_2^2)f(x) + \lambda_1^2 \lambda_2^2 \int_a^x (x-t)f(t) dt.$$

2.4. Solution for $\Delta = \infty$:

$$y(x) = \frac{f''''_{xxxx} - (\lambda_1^2 + \lambda_2^2)f''_{xx} + \lambda_1^2 \lambda_2^2 f}{A_1 \lambda_1^3 + A_2 \lambda_2^3}, \quad f = f(x).$$

In the last case, the relation $A_1 \lambda_1 + A_2 \lambda_2 = 0$ is valid, and the right-hand side of the integral equation is assumed to satisfy the conditions $f(a) = f'_x(a) = f''_{xx}(a) = f'''_{xxx}(a) = 0$.

50. $\int_a^x \{ A \sinh[\lambda(x-t)] + B \sinh[\mu(x-t)] + C \sinh[\beta(x-t)] \} y(t) dt = f(x).$

It is assumed that $f(a) = f'_x(a) = 0$. Differentiating the integral equation twice yields

$$(A\lambda + B\mu + C\beta)y(x) + \int_a^x \{ A\lambda^2 \sinh[\lambda(x-t)] + B\mu^2 \sinh[\mu(x-t)] \} y(t) dt$$

$$+ C\beta^2 \int_a^x \sinh[\beta(x-t)]y(t) dt = f''_{xx}(x).$$

Eliminating the last integral with the aid of the original equation, we arrive at an equation of the form 2.3.18:

$$(A\lambda + B\mu + C\beta)y(x)$$

$$+ \int_a^x \{ A(\lambda^2 - \beta^2) \sinh[\lambda(x-t)] + B(\mu^2 - \beta^2) \sinh[\mu(x-t)] \} y(t) dt = f''_{xx}(x) - \beta^2 f(x).$$

In the special case $A\lambda + B\mu + C\beta = 0$, this is an equation of the form 1.3.49.

51. $\int_a^x \sinh^2[\lambda(x-t)]y(t) dt = f(x), \quad f(a) = f'_x(a) = f''_{xx}(a) = 0.$

Differentiating yields an equation of the form 1.3.40:

$$\int_a^x \sinh[2\lambda(x-t)]y(t) dt = \frac{1}{\lambda} f'_x(x).$$

Solution: $y(x) = \frac{1}{2} \lambda^{-2} f'''_{xxx}(x) - 2 f'_x(x).$

$$52. \int_a^x [\sinh^2(\lambda x) - \sinh^2(\lambda t)] y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

$$\text{Solution: } y(x) = \frac{1}{\lambda} \frac{d}{dx} \left[\frac{f'_x(x)}{\sinh(2\lambda x)} \right].$$

$$53. \int_a^x [A \sinh^2(\lambda x) + B \sinh^2(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \sinh^2(\lambda x)$. For $B = -A$, see equation 1.3.52.

Solution:

$$y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ [\sinh(\lambda x)]^{-\frac{2A}{A+B}} \int_a^x [\sinh(\lambda t)]^{-\frac{2B}{A+B}} f'_t(t) dt \right\}.$$

$$54. \int_a^x [A \sinh^2(\lambda x) + B \sinh^2(\mu t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \sinh^2(\lambda x)$ and $h(t) = B \sinh^2(\mu t)$.

$$55. \int_a^x \sinh[\lambda(x-t)] \sinh[\lambda(x+t)] y(t) dt = f(x).$$

Using the formula

$$\sinh(\alpha - \beta) \sinh(\alpha + \beta) = \frac{1}{2} [\cosh(2\alpha) - \cosh(2\beta)], \quad \alpha = \lambda x, \quad \beta = \lambda t,$$

we reduce the original equation to an equation of the form 1.3.5:

$$\int_a^x [\cosh(2\lambda x) - \cosh(2\lambda t)] y(t) dt = 2f(x).$$

$$\text{Solution: } y(x) = \frac{1}{\lambda} \frac{d}{dx} \left[\frac{f'_x(x)}{\sinh(2\lambda x)} \right].$$

$$56. \int_a^x [A \sinh(\lambda x) \sinh(\mu t) + B \sinh(\beta x) \sinh(\gamma t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = A \sinh(\lambda x)$, $h_1(t) = \sinh(\mu t)$, $g_2(x) = B \sinh(\beta x)$, and $h_2(t) = \sinh(\gamma t)$.

$$57. \int_a^x \sinh^3[\lambda(x-t)] y(t) dt = f(x), \quad f(a) = f'_x(a) = f''_{xx}(a) = f'''_{xxx}(a) = 0.$$

Using the formula $\sinh^3 \beta = \frac{1}{4} \sinh 3\beta - \frac{3}{4} \sinh \beta$, we arrive at an equation of the form 1.3.49:

$$\int_a^x \left\{ \frac{1}{4} \sinh[3\lambda(x-t)] - \frac{3}{4} \sinh[\lambda(x-t)] \right\} y(t) dt = f(x).$$

$$58. \int_a^x [\sinh^3(\lambda x) - \sinh^3(\lambda t)] y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

This is a special case of equation 1.9.2 with $g(x) = \sinh^3(\lambda x)$.

59.
$$\int_a^x [A \sinh^3(\lambda x) + B \sinh^3(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \sinh^3(\lambda x)$.

Solution:

$$y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ [\sinh(\lambda x)]^{-\frac{3A}{A+B}} \int_a^x [\sinh(\lambda t)]^{-\frac{3B}{A+B}} f'_t(t) dt \right\}.$$

60.
$$\int_a^x [A \sinh^2(\lambda x) \sinh(\mu t) + B \sinh(\beta x) \sinh^2(\gamma t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = A \sinh^2(\lambda x)$, $h_1(t) = \sinh(\mu t)$, $g_2(x) = B \sinh(\beta x)$, and $h_2(t) = \sinh^2(\gamma t)$.

61.
$$\int_a^x \sinh^4[\lambda(x-t)] y(t) dt = f(x).$$

It is assumed that $f(a) = f'_x(a) = \dots = f''''_{xxx}(a) = 0$.

Let us transform the kernel of the integral equation using the formula

$$\sinh^4 \beta = \frac{1}{8} \cosh 4\beta - \frac{1}{2} \cosh 2\beta + \frac{3}{8}, \quad \text{where } \beta = \lambda(x-t),$$

and differentiate the resulting equation with respect to x . Then we arrive at an equation of the form 1.3.49:

$$\lambda \int_a^x \left\{ \frac{1}{2} \sinh[4\lambda(x-t)] - \sinh[2\lambda(x-t)] \right\} y(t) dt = f'_x(x).$$

62.
$$\int_a^x \sinh^n[\lambda(x-t)] y(t) dt = f(x), \quad n = 2, 3, \dots$$

It is assumed that $f(a) = f'_x(a) = \dots = f^{(n)}_x(a) = 0$.

1°. Let us differentiate the equation with respect to x twice and transform the kernel of the resulting integral equation using the formula $\cosh^2 \beta = 1 + \sinh^2 \beta$, where $\beta = \lambda(x-t)$. Then we have

$$\lambda^2 n^2 \int_a^x \sinh^n[\lambda(x-t)] y(t) dt + \lambda^2 n(n-1) \int_a^x \sinh^{n-2}[\lambda(x-t)] y(t) dt = f''_{xx}(x).$$

Eliminating the first term on the left-hand side with the aid of the original equation, we obtain

$$\int_a^x \sinh^{n-2}[\lambda(x-t)] y(t) dt = \frac{1}{\lambda^2 n(n-1)} [f''_{xx}(x) - \lambda^2 n^2 f(x)].$$

This equation has the same form as the original equation, but the exponent of the kernel has been reduced by two.

By applying this technique sufficiently many times, we finally arrive at simple integral equations of the form 1.1.1 (for even n) or 1.3.40 (for odd n).

2°. Solution:

$$y(x) = \frac{1}{\lambda^n n!} \left(\frac{d}{dx} + n\lambda \right) \left(\frac{d}{dx} + (n-2)\lambda \right) \dots \left(\frac{d}{dx} - n\lambda \right) f(x).$$

63. $\int_a^x \sinh(\lambda\sqrt{x-t})y(t) dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi\lambda} \frac{d^2}{dx^2} \int_a^x \frac{\cos(\lambda\sqrt{x-t})}{\sqrt{x-t}} f(t) dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 437), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

64. $\int_x^\infty \sinh(\lambda\sqrt{t-x})y(t) dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi\lambda} \frac{d^2}{dx^2} \int_x^\infty \frac{\cos(\lambda\sqrt{t-x})}{\sqrt{t-x}} f(t) dt.$$

⊙ References: H. M. Srivastava and R. G. Buschman (1977), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 439), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

65. $\int_a^x \sqrt{\sinh x - \sinh t} y(t) dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi} \cosh x \left(\frac{1}{\cosh x} \frac{d}{dx} \right)^2 \int_a^x \frac{\cosh t f(t) dt}{\sqrt{\sinh x - \sinh t}}.$$

66. $\int_a^x \frac{y(t) dt}{\sqrt{\sinh x - \sinh t}} = f(x).$

Solution:

$$y(x) = \frac{1}{\pi} \frac{d}{dx} \int_a^x \frac{\cosh t f(t) dt}{\sqrt{\sinh x - \sinh t}}.$$

67. $\int_a^x (\sinh x - \sinh t)^\lambda y(t) dt = f(x), \quad 0 < \lambda < 1.$

Solution:

$$y(x) = k \cosh x \left(\frac{1}{\cosh x} \frac{d}{dx} \right)^2 \int_a^x \frac{\cosh t f(t) dt}{(\sinh x - \sinh t)^\lambda}, \quad k = \frac{\sin(\pi\lambda)}{\pi\lambda}.$$

68. $\int_a^x (\sinh^\mu x - \sinh^\mu t) y(t) dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \sinh^\mu x$.

Solution: $y(x) = \frac{1}{\mu} \frac{d}{dx} \left[\frac{f'_x(x)}{\cosh x \sinh^{\mu-1} x} \right].$

69. $\int_a^x [A \sinh^\mu(\lambda x) + B \sinh^\mu(\lambda t)] y(t) dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \sinh^\mu(\lambda x)$.

Solution with $B \neq -A$:

$$y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ [\sinh(\lambda x)]^{-\frac{A\mu}{A+B}} \int_a^x [\sinh(\lambda t)]^{-\frac{B\mu}{A+B}} f'_t(t) dt \right\}.$$

70.
$$\int_a^x \frac{y(t) dt}{(\sinh x - \sinh t)^\lambda} = f(x), \quad 0 < \lambda < 1.$$

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi} \frac{d}{dx} \int_a^x \frac{\cosh t f(t) dt}{(\sinh x - \sinh t)^{1-\lambda}}.$$

71.
$$\int_a^x (x-t) \sinh[\lambda(x-t)] y(t) dt = f(x), \quad f(a) = f'_x(a) = f''_{xx}(a) = 0.$$

Double differentiation yields

$$2\lambda \int_a^x \cosh[\lambda(x-t)] y(t) dt + \lambda^2 \int_a^x (x-t) \sinh[\lambda(x-t)] y(t) dt = f''_{xx}(x).$$

Eliminating the second term on the left-hand side with the aid of the original equation, we arrive at an equation of the form 1.3.1:

$$\int_a^x \cosh[\lambda(x-t)] y(t) dt = \frac{1}{2\lambda} [f''_{xx}(x) - \lambda^2 f(x)].$$

Solution:

$$y(x) = \frac{1}{2\lambda} f'''_{xxx}(x) - \lambda f'_x(x) + \frac{1}{2} \lambda^3 \int_a^x f(t) dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 436).

72.
$$\int_a^x \frac{\sinh[\lambda(x-t)]}{\sqrt{x-t}} y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

Solution:

$$y(x) = \frac{2}{\pi\lambda} \int_a^x \frac{\sinh[\lambda(x-t)]}{\sqrt{x-t}} [f''_{tt}(t) - \lambda^2 f(t)] dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 436).

73.
$$\int_a^x \frac{\sinh[\lambda(x-t)]}{(x-t)^{3/2}} y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

Solution:

$$y(x) = \frac{2}{\pi\lambda} \int_a^x \frac{\sinh[\lambda(x-t)]}{\sqrt{x-t}} \left[f''_{tt}(t) - \lambda^2 f(t) + \frac{f'(t)}{x-t} \right] dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 437).

74.
$$\int_a^x \frac{\sinh[\lambda\sqrt{(x-t)(x-t+\gamma)}]}{\sqrt{x-t+\gamma}} y(t) dt = f(x).$$

It is assumed that $f(a) = f'_x(a) = f''_{xx}(a) = f'''_{xxx}(a) = 0$.

Solution:

$$y(x) = \frac{2}{\pi\lambda^2} \int_a^x \frac{\cosh[\lambda\sqrt{(x-t)(x-t+\gamma)}]}{\sqrt{x-t}} \int_a^t \sinh[\lambda(t-s)] \left(\frac{d^2}{ds^2} + \lambda^2 \right)^2 f(s) ds dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 438), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

$$75. \int_a^x [Ax^\beta + B \sinh^\gamma(\lambda t) + C]y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = Ax^\beta$ and $h(t) = B \sinh^\gamma(\lambda t) + C$.

$$76. \int_a^x [A \sinh^\gamma(\lambda x) + Bt^\beta + C]y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \sinh^\gamma(\lambda x)$ and $h(t) = Bt^\beta + C$.

$$77. \int_a^x (Ax^\lambda \sinh^\mu t + Bt^\beta \sinh^\gamma x)y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \sinh^\mu t$, $g_2(x) = B \sinh^\gamma x$, and $h_2(t) = t^\beta$.

1.3-3. Kernels Containing Hyperbolic Tangent.

$$78. \int_a^x [\tanh(\lambda x) - \tanh(\lambda t)]y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \tanh(\lambda x)$.

$$\text{Solution: } y(x) = \frac{1}{\lambda} [\cosh^2(\lambda x) f'_x(x)]'_x.$$

$$79. \int_a^x [A \tanh(\lambda x) + B \tanh(\lambda t)]y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \tanh(\lambda x)$. For $B = -A$, see equation 1.3.78.

$$\text{Solution: } y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ [\tanh(\lambda x)]^{-\frac{A}{A+B}} \int_a^x [\tanh(\lambda t)]^{-\frac{B}{A+B}} f'_t(t) dt \right\}.$$

$$80. \int_a^x [A \tanh(\lambda x) + B \tanh(\mu t) + C]y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \tanh(\lambda x)$ and $h(t) = B \tanh(\mu t) + C$.

$$81. \int_a^x [\tanh^2(\lambda x) - \tanh^2(\lambda t)]y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \tanh^2(\lambda x)$.

$$\text{Solution: } y(x) = \frac{d}{dx} \left[\frac{\cosh^3(\lambda x) f'_x(x)}{2\lambda \sinh(\lambda x)} \right].$$

$$82. \int_a^x [A \tanh^2(\lambda x) + B \tanh^2(\lambda t)]y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \tanh^2(\lambda x)$. For $B = -A$, see equation 1.3.81.

$$\text{Solution: } y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ [\tanh(\lambda x)]^{-\frac{2A}{A+B}} \int_a^x [\tanh(\lambda t)]^{-\frac{2B}{A+B}} f'_t(t) dt \right\}.$$

$$83. \int_a^x [A \tanh^2(\lambda x) + B \tanh^2(\mu t) + C]y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \tanh^2(\lambda x)$ and $h(t) = B \tanh^2(\mu t) + C$.

$$84. \int_a^x [\tanh(\lambda x) - \tanh(\lambda t)]^n y(t) dt = f(x), \quad n = 1, 2, \dots$$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \dots = f_x^{(n)}(a) = 0$.

$$\text{Solution: } y(x) = \frac{1}{\lambda^n n! \cosh^2(\lambda x)} \left[\cosh^2(\lambda x) \frac{d}{dx} \right]^{n+1} f(x).$$

$$85. \int_a^x \sqrt{\tanh x - \tanh t} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{2}{\pi \cosh^2 x} \left(\cosh^2 x \frac{d}{dx} \right)^2 \int_a^x \frac{f(t) dt}{\cosh^2 t \sqrt{\tanh x - \tanh t}}.$$

$$86. \int_a^x \frac{y(t) dt}{\sqrt{\tanh x - \tanh t}} = f(x).$$

Solution:

$$y(x) = \frac{1}{\pi} \frac{d}{dx} \int_a^x \frac{f(t) dt}{\cosh^2 t \sqrt{\tanh x - \tanh t}}.$$

$$87. \int_a^x (\tanh x - \tanh t)^\lambda y(t) dt = f(x), \quad 0 < \lambda < 1.$$

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi \lambda \cosh^2 x} \left(\cosh^2 x \frac{d}{dx} \right)^2 \int_a^x \frac{f(t) dt}{\cosh^2 t (\tanh x - \tanh t)^\lambda}.$$

$$88. \int_a^x (\tanh^\mu x - \tanh^\mu t) y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \tanh^\mu x$.

$$\text{Solution: } y(x) = \frac{1}{\mu} \frac{d}{dx} \left[\frac{\cosh^{\mu+1} x f'_x(x)}{\sinh^{\mu-1} x} \right].$$

$$89. \int_a^x (A \tanh^\mu x + B \tanh^\mu t) y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \tanh^\mu x$. For $B = -A$, see equation 1.3.88.

Solution:

$$y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ [\tanh(\lambda x)]^{-\frac{A\mu}{A+B}} \int_a^x [\tanh(\lambda t)]^{-\frac{B\mu}{A+B}} f'_t(t) dt \right\}.$$

$$90. \int_a^x \frac{y(t) dt}{[\tanh(\lambda x) - \tanh(\lambda t)]^\mu} = f(x), \quad 0 < \mu < 1.$$

This is a special case of equation 1.9.44 with $g(x) = \tanh(\lambda x)$ and $h(x) \equiv 1$.

Solution:

$$y(x) = \frac{\lambda \sin(\pi\mu)}{\pi} \frac{d}{dx} \int_a^x \frac{f(t) dt}{\cosh^2(\lambda t) [\tanh(\lambda x) - \tanh(\lambda t)]^{1-\mu}}.$$

$$91. \int_a^x [Ax^\beta + B \tanh^\gamma(\lambda t) + C]y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = Ax^\beta$ and $h(t) = B \tanh^\gamma(\lambda t) + C$.

$$92. \int_a^x [A \tanh^\gamma(\lambda x) + Bt^\beta + C]y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \tanh^\gamma(\lambda x)$ and $h(t) = Bt^\beta + C$.

$$93. \int_a^x (Ax^\lambda \tanh^\mu t + Bt^\beta \tanh^\gamma x)y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \tanh^\mu t$, $g_2(x) = B \tanh^\gamma x$, and $h_2(t) = t^\beta$.

1.3-4. Kernels Containing Hyperbolic Cotangent.

$$94. \int_a^x [\coth(\lambda x) - \coth(\lambda t)]y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \coth(\lambda x)$.

$$\text{Solution: } y(x) = -\frac{1}{\lambda} \frac{d}{dx} [\sinh^2(\lambda x) f'_x(x)].$$

$$95. \int_a^x [A \coth(\lambda x) + B \coth(\lambda t)]y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \coth(\lambda x)$. For $B = -A$, see equation 1.3.94.

$$\text{Solution: } y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ [\tanh(\lambda x)]^{\frac{A}{A+B}} \int_a^x [\tanh(\lambda t)]^{\frac{B}{A+B}} f'_t(t) dt \right\}.$$

$$96. \int_a^x [A \coth(\lambda x) + B \coth(\mu t) + C]y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \coth(\lambda x)$ and $h(t) = B \coth(\mu t) + C$.

$$97. \int_a^x [\coth^2(\lambda x) - \coth^2(\lambda t)]y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \coth^2(\lambda x)$.

$$\text{Solution: } y(x) = -\frac{d}{dx} \left[\frac{\sinh^3(\lambda x) f'_x(x)}{2\lambda \cosh(\lambda x)} \right].$$

$$98. \int_a^x [A \coth^2(\lambda x) + B \coth^2(\lambda t)]y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \coth^2(\lambda x)$. For $B = -A$, see equation 1.3.97.

$$\text{Solution: } y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ [\tanh(\lambda x)]^{\frac{2A}{A+B}} \int_a^x [\tanh(\lambda t)]^{\frac{2B}{A+B}} f'_t(t) dt \right\}.$$

$$99. \int_a^x [A \coth^2(\lambda x) + B \coth^2(\mu t) + C]y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \coth^2(\lambda x)$ and $h(t) = B \coth^2(\mu t) + C$.

$$100. \int_a^x [\coth(\lambda x) - \coth(\lambda t)]^n y(t) dt = f(x), \quad n = 1, 2, \dots$$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \dots = f_x^{(n)}(a) = 0$.

$$\text{Solution: } y(x) = \frac{(-1)^n}{\lambda^n n! \sinh^2(\lambda x)} \left[\sinh^2(\lambda x) \frac{d}{dx} \right]^{n+1} f(x).$$

$$101. \int_a^x (\coth^\mu x - \coth^\mu t) y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \coth^\mu x$.

$$\text{Solution: } y(x) = -\frac{1}{\mu} \frac{d}{dx} \left[\frac{\sinh^{\mu+1} x f'_x(x)}{\cosh^{\mu-1} x} \right].$$

$$102. \int_a^x (A \coth^\mu x + B \coth^\mu t) y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \coth^\mu x$. For $B = -A$, see equation 1.3.101.

Solution:

$$y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ \left| \tanh x \right|^{\frac{A\mu}{A+B}} \int_a^x \left| \tanh t \right|^{\frac{B\mu}{A+B}} f'_t(t) dt \right\}.$$

$$103. \int_a^x [Ax^\beta + B \coth^\gamma(\lambda t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = Ax^\beta$ and $h(t) = B \coth^\gamma(\lambda t) + C$.

$$104. \int_a^x [A \coth^\gamma(\lambda x) + Bt^\beta + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \coth^\gamma(\lambda x)$ and $h(t) = Bt^\beta + C$.

$$105. \int_a^x (Ax^\lambda \coth^\mu t + Bt^\beta \coth^\gamma x) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \coth^\mu t$, $g_2(x) = B \coth^\gamma x$, and $h_2(t) = t^\beta$.

1.3-5. Kernels Containing Combinations of Hyperbolic Functions.

$$106. \int_a^x \{ \cosh[\lambda(x-t)] + A \sinh[\mu(x-t)] \} y(t) dt = f(x).$$

Let us differentiate the equation with respect to x and then eliminate the integral with the hyperbolic cosine. As a result, we arrive at an equation of the form 2.3.16:

$$y(x) + (\lambda - A^2\mu) \int_a^x \sinh[\mu(x-t)] y(t) dt = f'_x(x) - A\mu f(x).$$

$$107. \int_a^x [A \cosh(\lambda x) + B \sinh(\mu t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \cosh(\lambda x)$ and $h(t) = B \sinh(\mu t) + C$.

108. $\int_a^x [A \cosh^2(\lambda x) + B \sinh^2(\mu t) + C] y(t) dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A \cosh^2(\lambda x)$ and $h(t) = B \sinh^2(\mu t) + C$.

109. $\int_a^x \sinh[\lambda(x-t)] \cosh[\lambda(x+t)] y(t) dt = f(x).$

Using the formula

$$\sinh(\alpha - \beta) \cosh(\alpha + \beta) = \frac{1}{2} [\sinh(2\alpha) - \sinh(2\beta)], \quad \alpha = \lambda x, \quad \beta = \lambda t,$$

we reduce the original equation to an equation of the form 1.3.45:

$$\int_a^x [\sinh(2\lambda x) - \sinh(2\lambda t)] y(t) dt = 2f(x).$$

Solution: $y(x) = \frac{1}{\lambda} \frac{d}{dx} \left[\frac{f'_x(x)}{\cosh(2\lambda x)} \right].$

110. $\int_a^x \cosh[\lambda(x-t)] \sinh[\lambda(x+t)] y(t) dt = f(x).$

Using the formula

$$\cosh(\alpha - \beta) \sinh(\alpha + \beta) = \frac{1}{2} [\sinh(2\alpha) + \sinh(2\beta)], \quad \alpha = \lambda x, \quad \beta = \lambda t,$$

we reduce the original equation to an equation of the form 1.3.46 with $A = B = 1$:

$$\int_a^x [\sinh(2\lambda x) + \sinh(2\lambda t)] y(t) dt = 2f(x).$$

111. $\int_a^x [A \cosh(\lambda x) \sinh(\mu t) + B \cosh(\beta x) \sinh(\gamma t)] y(t) dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = A \cosh(\lambda x)$, $h_1(t) = \sinh(\mu t)$, $g_2(x) = B \cosh(\beta x)$, and $h_2(t) = \sinh(\gamma t)$.

112. $\int_a^x [\sinh(\lambda x) \cosh(\mu t) + \sinh(\beta x) \cosh(\gamma t)] y(t) dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = \sinh(\lambda x)$, $h_1(t) = \cosh(\mu t)$, $g_2(x) = \sinh(\beta x)$, and $h_2(t) = \cosh(\gamma t)$.

113. $\int_a^x [\cosh(\lambda x) \cosh(\mu t) + \sinh(\beta x) \sinh(\gamma t)] y(t) dt = f(x).$

This is a special case of equation 1.9.15 with $g_1(x) = \cosh(\lambda x)$, $h_1(t) = \cosh(\mu t)$, $g_2(x) = \sinh(\beta x)$, and $h_2(t) = \sinh(\gamma t)$.

114. $\int_a^x [A \cosh^\beta(\lambda x) + B \sinh^\gamma(\mu t)] y(t) dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A \cosh^\beta(\lambda x)$ and $h(t) = B \sinh^\gamma(\mu t)$.

$$115. \int_a^x [A \sinh^\beta(\lambda x) + B \cosh^\gamma(\mu t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \sinh^\beta(\lambda x)$ and $h(t) = B \cosh^\gamma(\mu t)$.

$$116. \int_a^x (Ax^\lambda \cosh^\mu t + Bt^\beta \sinh^\gamma x) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \cosh^\mu t$, $g_2(x) = B \sinh^\gamma x$, and $h_2(t) = t^\beta$.

$$117. \int_a^x \{ (x-t) \sinh[\lambda(x-t)] - \lambda(x-t)^2 \cosh[\lambda(x-t)] \} y(t) dt = f(x).$$

Solution:

$$y(x) = \int_a^x g(t) dt,$$

where

$$g(t) = \sqrt{\frac{\pi}{2\lambda}} \frac{1}{64\lambda^5} \left(\frac{d^2}{dt^2} - \lambda^2 \right)^6 \int_a^t (t-\tau)^{\frac{5}{2}} I_{\frac{5}{2}}[\lambda(t-\tau)] f(\tau) d\tau.$$

$$118. \int_a^x \left\{ \frac{\sinh[\lambda(x-t)]}{x-t} - \lambda \cosh[\lambda(x-t)] \right\} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{1}{2\lambda^4} \left(\frac{d^2}{dx^2} - \lambda^2 \right)^3 \int_a^x \sinh[\lambda(x-t)] f(t) dt.$$

$$119. \int_a^x [\sinh(\lambda\sqrt{x-t}) - \lambda\sqrt{x-t} \cosh(\lambda\sqrt{x-t})] y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

Solution:

$$y(x) = -\frac{4}{\pi\lambda^3} \frac{d^3}{dx^3} \int_a^x \frac{\cos(\lambda\sqrt{x-t})}{\sqrt{x-t}} f(t) dt.$$

$$120. \int_a^x (Ax^\lambda \sinh^\mu t + Bt^\beta \cosh^\gamma x) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \sinh^\mu t$, $g_2(x) = B \cosh^\gamma x$, and $h_2(t) = t^\beta$.

$$121. \int_a^x [A \tanh(\lambda x) + B \coth(\mu t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \tanh(\lambda x)$ and $h(t) = B \coth(\mu t) + C$.

$$122. \int_a^x [A \tanh^2(\lambda x) + B \coth^2(\mu t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \tanh^2(\lambda x)$ and $h(t) = B \coth^2(\mu t)$.

$$123. \int_a^x [\tanh(\lambda x) \coth(\mu t) + \tanh(\beta x) \coth(\gamma t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = \tanh(\lambda x)$, $h_1(t) = \coth(\mu t)$, $g_2(x) = \tanh(\beta x)$, and $h_2(t) = \coth(\gamma t)$.

$$124. \int_a^x [\coth(\lambda x) \tanh(\mu t) + \coth(\beta x) \tanh(\gamma t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = \coth(\lambda x)$, $h_1(t) = \tanh(\mu t)$, $g_2(x) = \coth(\beta x)$, and $h_2(t) = \tanh(\gamma t)$.

$$125. \int_a^x [\tanh(\lambda x) \tanh(\mu t) + \coth(\beta x) \coth(\gamma t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = \tanh(\lambda x)$, $h_1(t) = \tanh(\mu t)$, $g_2(x) = \coth(\beta x)$, and $h_2(t) = \coth(\gamma t)$.

$$126. \int_a^x [A \tanh^\beta(\lambda x) + B \coth^\gamma(\mu t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \tanh^\beta(\lambda x)$ and $h(t) = B \coth^\gamma(\mu t)$.

$$127. \int_a^x [A \coth^\beta(\lambda x) + B \tanh^\gamma(\mu t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \coth^\beta(\lambda x)$ and $h(t) = B \tanh^\gamma(\mu t)$.

$$128. \int_a^x (Ax^\lambda \tanh^\mu t + Bt^\beta \coth^\gamma x) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \tanh^\mu t$, $g_2(x) = B \coth^\gamma x$, and $h_2(t) = t^\beta$.

$$129. \int_a^x (Ax^\lambda \coth^\mu t + Bt^\beta \tanh^\gamma x) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \coth^\mu t$, $g_2(x) = B \tanh^\gamma x$, and $h_2(t) = t^\beta$.

1.4. Equations Whose Kernels Contain Logarithmic Functions

1.4-1. Kernels Containing Logarithmic Functions.

$$1. \int_a^x (\ln x - \ln t) y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \ln x$.

Solution: $y(x) = x f''_{xx}(x) + f'_x(x)$.

$$2. \int_0^x \ln(x-t) y(t) dt = f(x).$$

Solution:

$$y(x) = - \int_0^x f''_{tt}(t) dt \int_0^\infty \frac{(x-t)^z e^{-Cz}}{\Gamma(z+1)} dz - f'_x(0) \int_0^\infty \frac{x^z e^{-Cz}}{\Gamma(z+1)} dz,$$

where $C = \lim_{k \rightarrow \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k+1} - \ln k\right) = 0.5772 \dots$ is the Euler constant and $\Gamma(z)$ is the gamma function.

⊙ References: M. L. Krasnov, A. I. Kisilev, and G. I. Makarenko (1971), A. G. Butkovskii (1979).

$$3. \quad \int_a^x [\ln(x-t) + A]y(t) dt = f(x).$$

Solution:

$$y(x) = -\frac{d}{dx} \int_a^x \nu_A(x-t)f(t) dt, \quad \nu_A(x) = \frac{d}{dx} \int_0^\infty \frac{x^z e^{(A-C)z}}{\Gamma(z+1)} dz,$$

where $C = 0.5772 \dots$ is the Euler constant and $\Gamma(z)$ is the gamma function.

For $a = 0$, the solution can be written in the form

$$y(x) = -\int_0^x f_{tt}''(t) dt \int_0^\infty \frac{(x-t)^z e^{(A-C)z}}{\Gamma(z+1)} dz - f'_x(0) \int_0^\infty \frac{x^z e^{(A-C)z}}{\Gamma(z+1)} dz.$$

⊙ Reference: S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

$$4. \quad \int_a^x (A \ln x + B \ln t)y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \ln x$. For $B = -A$, see equation 1.4.1.

Solution:

$$y(x) = \frac{\text{sign}(\ln x)}{A+B} \frac{d}{dx} \left\{ |\ln x|^{-\frac{A}{A+B}} \int_a^x |\ln t|^{-\frac{B}{A+B}} f'_t(t) dt \right\}.$$

$$5. \quad \int_a^x (A \ln x + B \ln t + C)y(t) dt = f(x).$$

This is a special case of equation 1.9.5 with $g(x) = x$.

$$6. \quad \int_a^x [\ln^2(\lambda x) - \ln^2(\lambda t)]y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

$$\text{Solution: } y(x) = \frac{d}{dx} \left[\frac{x f'_x(x)}{2 \ln(\lambda x)} \right].$$

$$7. \quad \int_a^x [A \ln^2(\lambda x) + B \ln^2(\lambda t)]y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \ln^2(\lambda x)$. For $B = -A$, see equation 1.4.6.

Solution:

$$y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ |\ln(\lambda x)|^{-\frac{2A}{A+B}} \int_a^x |\ln(\lambda t)|^{-\frac{2B}{A+B}} f'_t(t) dt \right\}.$$

$$8. \quad \int_a^x [A \ln^2(\lambda x) + B \ln^2(\mu t) + C]y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \ln^2(\lambda x)$ and $h(t) = B \ln^2(\mu t) + C$.

$$9. \quad \int_a^x [\ln(x/t)]^n y(t) dt = f(x), \quad n = 1, 2, \dots$$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \dots = f_x^{(n)}(a) = 0$.

$$\text{Solution: } y(x) = \frac{1}{n! x} \left(x \frac{d}{dx} \right)^{n+1} f(x).$$

$$10. \quad \int_a^x (\ln^2 x - \ln^2 t)^n y(t) dt = f(x), \quad n = 1, 2, \dots$$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \dots = f_x^{(n)}(a) = 0$.

$$\text{Solution: } y(x) = \frac{\ln x}{2^n n! x} \left(\frac{x}{\ln x} \frac{d}{dx} \right)^{n+1} f(x).$$

$$11. \quad \int_a^x \ln \left(\frac{x+b}{t+b} \right) y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \ln(x+b)$.

$$\text{Solution: } y(x) = (x+b) f''_{xx}(x) + f'_x(x).$$

$$12. \quad \int_a^x \sqrt{\ln(x/t)} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{2}{\pi x} \left(x \frac{d}{dx} \right)^2 \int_a^x \frac{f(t) dt}{t \sqrt{\ln(x/t)}}.$$

$$13. \quad \int_a^x \frac{y(t) dt}{\sqrt{\ln(x/t)}} = f(x).$$

Solution:

$$y(x) = \frac{1}{\pi} \frac{d}{dx} \int_a^x \frac{f(t) dt}{t \sqrt{\ln(x/t)}}.$$

$$14. \quad \int_a^x [\ln^\mu(\lambda x) - \ln^\mu(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \ln^\mu(\lambda x)$.

$$\text{Solution: } y(x) = \frac{1}{\mu} \frac{d}{dx} [x \ln^{1-\mu}(\lambda x) f'_x(x)].$$

$$15. \quad \int_a^x [A \ln^\beta(\lambda x) + B \ln^\gamma(\mu t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \ln^\beta(\lambda x)$ and $h(t) = B \ln^\gamma(\mu t) + C$.

$$16. \quad \int_a^x [\ln(x/t)]^\lambda y(t) dt = f(x), \quad 0 < \lambda < 1.$$

Solution:

$$y(x) = \frac{k}{x} \left(x \frac{d}{dx} \right)^2 \int_a^x \frac{f(t) dt}{t [\ln(x/t)]^\lambda}, \quad k = \frac{\sin(\pi \lambda)}{\pi \lambda}.$$

$$17. \quad \int_a^x \frac{y(t) dt}{[\ln(x/t)]^\lambda} = f(x), \quad 0 < \lambda < 1.$$

This is a special case of equation 1.9.44 with $g(x) = \ln x$ and $h(x) \equiv 1$.

Solution:

$$y(x) = \frac{\sin(\pi \lambda)}{\pi} \frac{d}{dx} \int_a^x \frac{f(t) dt}{t [\ln(x/t)]^{1-\lambda}}.$$

18. $\int_0^x \ln \frac{\sqrt{x} + \sqrt{x-t}}{\sqrt{x} - \sqrt{x-t}} y(t) dt = f(x).$

Solution:

$$y(x) = \frac{1}{\pi} \frac{d}{dx} \int_0^x \frac{\sqrt{t}}{\sqrt{x-t}} \frac{d}{dt} f(t) dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 451).

19. $\int_x^\infty \ln \frac{\sqrt{t} + \sqrt{t-x}}{\sqrt{t} - \sqrt{t-x}} y(t) dt = f(x).$

Solution:

$$y(x) = \frac{1}{\pi} \frac{1}{\sqrt{x}} \frac{d}{dx} \int_x^\infty \frac{t}{\sqrt{t-x}} \frac{d}{dt} f(t) dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 452).

1.4-2. Kernels Containing Power-Law and Logarithmic Functions.

20. $\int_a^x (x-t) [\ln(x-t) + A] y(t) dt = f(x).$

Solution:

$$y(x) = -\frac{d^2}{dx^2} \int_a^x \nu_A(x-t) f(t) dt, \quad \nu_A(x) = \frac{d}{dx} \int_0^\infty \frac{x^z e^{(A-C)z}}{\Gamma(z+1)} dz,$$

where $C = 0.5772 \dots$ is the Euler constant and $\Gamma(z)$ is the gamma function.

⊙ Reference: S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

21. $\int_a^x \frac{\ln(x-t) + A}{(x-t)^\lambda} y(t) dt = f(x), \quad 0 < \lambda < 1.$

Solution:

$$y(x) = -\frac{\sin(\pi\lambda)}{\pi} \frac{d}{dx} \int_a^x \frac{F(t) dt}{(x-t)^{1-\lambda}}, \quad F(x) = \int_a^x \nu_h(x-t) f(t) dt,$$

$$\nu_h(x) = \frac{d}{dx} \int_0^\infty \frac{x^z e^{hz}}{\Gamma(z+1)} dz, \quad h = A + \psi(1-\lambda),$$

where $\Gamma(z)$ is the gamma function and $\psi(z) = [\Gamma(z)]'_z$ is the logarithmic derivative of the gamma function.

⊙ Reference: S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

22. $\int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} [\ln(x-t) + A] y(t) dt = f(x), \quad \alpha > 0.$

Solution:

$$y(x) = -\frac{1}{\Gamma([\alpha] - \alpha + 1)} \left(\frac{d}{dx} \right)^{[\alpha]+1} \int_a^x \frac{F(t) dt}{(x-t)^{\alpha-[\alpha]}}, \quad F(x) = \int_a^x \nu_h(x-t) f(t) dt,$$

$$\nu_h(x) = \frac{d}{dx} \int_0^\infty \frac{x^z e^{hz}}{\Gamma(z+1)} dz, \quad h = A + \psi(\alpha),$$

where $\Gamma(z)$ is the gamma function and $\psi(z) = [\Gamma(z)]'_z$ is the logarithmic derivative of the gamma function.

⊙ Reference: S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993, p. 483).

$$23. \int_a^x (t^\beta \ln^\lambda x - x^\beta \ln^\lambda t) y(t) dt = f(x).$$

This is a special case of equation 1.9.11 with $g(x) = \ln^\lambda x$ and $h(t) = t^\beta$.

$$24. \int_a^x (At^\beta \ln^\lambda x + Bx^\mu \ln^\gamma t) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = A \ln^\lambda x$, $h_1(t) = t^\beta$, $g_2(x) = Bx^\mu$, and $h_2(t) = \ln^\gamma t$.

$$25. \int_a^x \ln \left(\frac{x^\mu + b}{ct^\lambda + s} \right) y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = \ln(x^\mu + b)$ and $h(t) = -\ln(ct^\lambda + s)$.

1.5. Equations Whose Kernels Contain Trigonometric Functions

1.5-1. Kernels Containing Cosine.

$$1. \int_a^x \cos[\lambda(x-t)] y(t) dt = f(x).$$

Solution: $y(x) = f'_x(x) + \lambda^2 \int_a^x f(x) dx$.

⊙ References: H. M. Srivastava and R. G. Buschman (1977), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 442).

$$2. \int_a^x \{ \cos[\lambda(x-t)] - 1 \} y(t) dt = f(x), \quad f(a) = f'_x(a) = f''_{xx}(a) = 0.$$

Solution: $y(x) = -\frac{1}{\lambda^2} f'''_{xxx}(x) - f'_x(x)$.

$$3. \int_a^x \{ \cos[\lambda(x-t)] + b \} y(t) dt = f(x).$$

For $b = 0$, see equation 1.5.1. For $b = -1$, see equation 1.5.2. For $\lambda = 0$, see equation 1.1.1. Differentiating the equation with respect to x , we arrive at an equation of the form 2.5.16:

$$y(x) - \frac{\lambda}{b+1} \int_a^x \sin[\lambda(x-t)] y(t) dt = \frac{f'_x(x)}{b+1}.$$

1°. Solution with $b(b+1) > 0$:

$$y(x) = \frac{f'_x(x)}{b+1} + \frac{\lambda^2}{k(b+1)^2} \int_a^x \sin[k(x-t)] f'_t(t) dt, \quad \text{where } k = \lambda \sqrt{\frac{b}{b+1}}.$$

2°. Solution with $b(b+1) < 0$:

$$y(x) = \frac{f'_x(x)}{b+1} + \frac{\lambda^2}{k(b+1)^2} \int_a^x \sinh[k(x-t)] f'_t(t) dt, \quad \text{where } k = \lambda \sqrt{\frac{-b}{b+1}}.$$

4. $\int_a^x \cos(\lambda x + \beta t) y(t) dt = f(x).$

Differentiating the equation with respect to x twice yields

$$\cos[(\lambda + \beta)x]y(x) - \lambda \int_a^x \sin(\lambda x + \beta t)y(t) dt = f'_x(x), \quad (1)$$

$$\left\{ \cos[(\lambda + \beta)x]y(x) \right\}'_x - \lambda \sin[(\lambda + \beta)x]y(x) - \lambda^2 \int_a^x \cos(\lambda x + \beta t)y(t) dt = f''_{xx}(x). \quad (2)$$

Eliminating the integral term from (2) with the aid of the original equation, we arrive at the first-order linear ordinary differential equation

$$w'_x - \lambda \tan[(\lambda + \beta)x]w = f''_{xx}(x) + \lambda^2 f(x), \quad w = \cos[(\lambda + \beta)x]y(x). \quad (3)$$

Setting $x = a$ in (1) yields the initial condition $w(a) = f'_x(a)$. On solving equation (3) under this condition, after some transformations we obtain the solution of the original integral equation in the form

$$y(x) = \frac{1}{\cos[(\lambda + \beta)x]} f'_x(x) + \frac{\lambda \sin[(\lambda + \beta)x]}{\cos^2[(\lambda + \beta)x]} f(x) - \frac{\lambda \beta}{\cos^{k+1}[(\lambda + \beta)x]} \int_a^x f(t) \cos^{k-2}[(\lambda + \beta)t] dt, \quad k = \frac{\lambda}{\lambda + \beta}.$$

5. $\int_a^x [\cos(\lambda x) - \cos(\lambda t)] y(t) dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \cos(\lambda x)$.

Solution: $y(x) = -\frac{1}{\lambda} \frac{d}{dx} \left[\frac{f'_x(x)}{\sin(\lambda x)} \right].$

6. $\int_a^x [A \cos(\lambda x) + B \cos(\lambda t)] y(t) dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \cos(\lambda x)$. For $B = -A$, see equation 1.5.5.

Solution with $B \neq -A$:

$$y(x) = \frac{\text{sign } \cos(\lambda x)}{A + B} \frac{d}{dx} \left\{ |\cos(\lambda x)|^{-\frac{A}{A+B}} \int_a^x |\cos(\lambda t)|^{-\frac{B}{A+B}} f'_t(t) dt \right\}.$$

7. $\int_a^x [A \cos(\lambda x) + B \cos(\mu t) + C] y(t) dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A \cos(\lambda x)$ and $h(t) = B \cos(\mu t) + C$.

8. $\int_a^x \{A_1 \cos[\lambda_1(x - t)] + A_2 \cos[\lambda_2(x - t)]\} y(t) dt = f(x).$

The equation is equivalent to the equation

$$\int_a^x \{B_1 \sin[\lambda_1(x - t)] + B_2 \sin[\lambda_2(x - t)]\} y(t) dt = F(x),$$

$$B_1 = \frac{A_1}{\lambda_1}, \quad B_2 = \frac{A_2}{\lambda_2}, \quad F(x) = \int_a^x f(t) dt,$$

which has the form 1.5.41. (Differentiation of this equation yields the original integral equation.)

9. $\int_a^x \cos^2[\lambda(x-t)]y(t) dt = f(x).$

Differentiating yields an equation of the form 2.5.16:

$$y(x) - \lambda \int_a^x \sin[2\lambda(x-t)]y(t) dt = f'_x(x).$$

Solution:

$$y(x) = f'_x(x) + \frac{2\lambda^2}{k} \int_a^x \sin[k(x-t)]f'_t(t) dt, \quad \text{where } k = \lambda\sqrt{2}.$$

10. $\int_a^x [\cos^2(\lambda x) - \cos^2(\lambda t)]y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$

Solution: $y(x) = -\frac{1}{\lambda} \frac{d}{dx} \left[\frac{f'_x(x)}{\sin(2\lambda x)} \right].$

11. $\int_a^x [A \cos^2(\lambda x) + B \cos^2(\lambda t)]y(t) dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \cos^2(\lambda x)$. For $B = -A$, see equation 1.5.10.

Solution:

$$y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ [\cos(\lambda x)]^{-\frac{2A}{A+B}} \int_a^x [\cos(\lambda t)]^{-\frac{2B}{A+B}} f'_t(t) dt \right\}.$$

12. $\int_a^x [A \cos^2(\lambda x) + B \cos^2(\mu t) + C]y(t) dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A \cos^2(\lambda x)$ and $h(t) = B \cos^2(\mu t) + C$.

13. $\int_a^x \cos[\lambda(x-t)] \cos[\lambda(x+t)]y(t) dt = f(x).$

Using the trigonometric formula

$$\cos(\alpha - \beta) \cos(\alpha + \beta) = \frac{1}{2} [\cos(2\alpha) + \cos(2\beta)], \quad \alpha = \lambda x, \quad \beta = \lambda t,$$

we reduce the original equation to an equation of the form 1.5.6 with $A = B = 1$:

$$\int_a^x [\cos(2\lambda x) + \cos(2\lambda t)]y(t) dt = 2f(x).$$

Solution with $\cos(2\lambda x) > 0$:

$$y(x) = \frac{d}{dx} \left[\frac{1}{\sqrt{\cos(2\lambda x)}} \int_a^x \frac{f'_t(t) dt}{\sqrt{\cos(2\lambda t)}} \right].$$

14. $\int_a^x \cos[\lambda(x-t)] \cos[\mu(x-t)]y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$

Solution:

$$y(x) = \frac{1}{\sqrt{\lambda^2 - \mu^2}} \left[\frac{d^2}{dx^2} + (\lambda + \mu)^2 \right] \left[\frac{d^2}{dx^2} + (\lambda - \mu)^2 \right] \int_a^x \int_a^t \sin[\sqrt{\lambda^2 + \mu^2} (t-s)] f(s) ds dt.$$

$$15. \int_a^x [A \cos(\lambda x) \cos(\mu t) + B \cos(\beta x) \cos(\gamma t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = A \cos(\lambda x)$, $h_1(t) = \cos(\mu t)$, $g_2(x) = B \cos(\beta x)$, and $h_2(t) = \cos(\gamma t)$.

$$16. \int_a^x \cos^3[\lambda(x-t)] y(t) dt = f(x).$$

Using the formula $\cos^3 \beta = \frac{1}{4} \cos 3\beta + \frac{3}{4} \cos \beta$, we arrive at an equation of the form 1.5.8:

$$\int_a^x \left\{ \frac{1}{4} \cos[3\lambda(x-t)] + \frac{3}{4} \cos[\lambda(x-t)] \right\} y(t) dt = f(x).$$

$$17. \int_a^x [\cos^3(\lambda x) - \cos^3(\lambda t)] y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

$$\text{Solution: } y(x) = -\frac{1}{3\lambda} \frac{d}{dx} \left[\frac{f'_x(x)}{\sin(\lambda x) \cos^2(\lambda x)} \right].$$

$$18. \int_a^x [A \cos^3(\lambda x) + B \cos^3(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \cos^3(\lambda x)$. For $B = -A$, see equation 1.5.17.

Solution:

$$y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ [\cos(\lambda x)]^{-\frac{3A}{A+B}} \int_a^x [\cos(\lambda t)]^{-\frac{3B}{A+B}} f'_t(t) dt \right\}.$$

$$19. \int_a^x [\cos^2(\lambda x) \cos(\mu t) + \cos(\beta x) \cos^2(\gamma t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = \cos^2(\lambda x)$, $h_1(t) = \cos(\mu t)$, $g_2(x) = \cos(\beta x)$, and $h_2(t) = \cos^2(\gamma t)$.

$$20. \int_a^x \cos^4[\lambda(x-t)] y(t) dt = f(x).$$

Let us transform the kernel of the integral equation using the trigonometric formula $\cos^4 \beta = \frac{1}{8} \cos 4\beta + \frac{1}{2} \cos 2\beta + \frac{3}{8}$, where $\beta = \lambda(x-t)$, and differentiate the resulting equation with respect to x . Then we arrive at an equation of the form 2.5.18:

$$y(x) - \lambda \int_a^x \left\{ \frac{1}{2} \sin[4\lambda(x-t)] + \sin[2\lambda(x-t)] \right\} y(t) dt = f'_x(x).$$

$$21. \int_a^x [\cos(\lambda x) - \cos(\lambda t)]^n y(t) dt = f(x), \quad n = 1, 2, \dots$$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \dots = f_x^{(n)}(a) = 0$.

$$\text{Solution: } y(x) = \frac{(-1)^n}{\lambda^n n!} \sin(\lambda x) \left[\frac{1}{\sin(\lambda x)} \frac{d}{dx} \right]^{n+1} f(x).$$

$$22. \int_a^x \sqrt{\cos t - \cos x} y(t) dt = f(x).$$

This is a special case of equation 1.9.40 with $g(x) = 1 - \cos x$.

Solution:

$$y(x) = \frac{2}{\pi} \sin x \left(\frac{1}{\sin x} \frac{d}{dx} \right)^2 \int_a^x \frac{\sin t f(t) dt}{\sqrt{\cos t - \cos x}}.$$

$$23. \int_a^x \frac{y(t) dt}{\sqrt{\cos t - \cos x}} = f(x).$$

Solution:

$$y(x) = \frac{1}{\pi} \frac{d}{dx} \int_a^x \frac{\sin t f(t) dt}{\sqrt{\cos t - \cos x}}.$$

$$24. \int_a^x (\cos t - \cos x)^\lambda y(t) dt = f(x), \quad 0 < \lambda < 1.$$

Solution:

$$y(x) = k \sin x \left(\frac{1}{\sin x} \frac{d}{dx} \right)^2 \int_a^x \frac{\sin t f(t) dt}{(\cos t - \cos x)^\lambda}, \quad k = \frac{\sin(\pi\lambda)}{\pi\lambda}.$$

$$25. \int_a^x (\cos^\mu x - \cos^\mu t) y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \cos^\mu x$.

$$\text{Solution: } y(x) = -\frac{1}{\mu} \frac{d}{dx} \left[\frac{f'_x(x)}{\sin x \cos^{\mu-1} x} \right].$$

$$26. \int_a^x (A \cos^\mu x + B \cos^\mu t) y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \cos^\mu x$. For $B = -A$, see equation 1.5.25.

Solution:

$$y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ |\cos x|^{-\frac{A\mu}{A+B}} \int_a^x |\cos t|^{-\frac{B\mu}{A+B}} f'_t(t) dt \right\}.$$

$$27. \int_a^x \frac{y(t) dt}{(\cos t - \cos x)^\lambda} = f(x), \quad 0 < \lambda < 1.$$

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi} \frac{d}{dx} \int_a^x \frac{\sin t f(t) dt}{(\cos t - \cos x)^{1-\lambda}}.$$

$$28. \int_a^x (x-t) \cos[\lambda(x-t)] y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

Differentiating the equation twice yields

$$y(x) - 2\lambda \int_a^x \sin[\lambda(x-t)] y(t) dt - \lambda^2 \int_a^x (x-t) \cos[\lambda(x-t)] y(t) dt = f''_{xx}(x).$$

Eliminating the third term on the left-hand side with the aid of the original equation, we arrive at an equation of the form 2.5.16:

$$y(x) - 2\lambda \int_a^x \sin[\lambda(x-t)] y(t) dt = f''_{xx}(x) + \lambda^2 f(x).$$

$$29. \int_a^x \frac{\cos[\lambda(x-t)]}{\sqrt{x-t}} y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

Solution:

$$y(x) = \frac{2}{\pi\lambda} \int_a^x \frac{\sin[\lambda(x-t)]}{\sqrt{x-t}} [f''_{tt}(t) + \lambda^2 f(t)] dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 445).

$$30. \int_a^x \sqrt{x-t} \cos(\lambda\sqrt{x-t}) y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{1}{\pi} \int_a^x \frac{\cosh(\lambda\sqrt{x-t})}{\sqrt{x-t}} f'_t(t) dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, pp. 445–446), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

$$31. \int_a^x \frac{\cos(\lambda\sqrt{x-t})}{\sqrt{x-t}} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{1}{\pi} \frac{d}{dx} \int_a^x \frac{\cosh(\lambda\sqrt{x-t})}{\sqrt{x-t}} f(t) dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 446), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

$$32. \int_x^\infty \frac{\cos(\lambda\sqrt{t-x})}{\sqrt{t-x}} y(t) dt = f(x).$$

Solution:

$$y(x) = -\frac{1}{\pi} \frac{d}{dx} \int_x^\infty \frac{\cosh(\lambda\sqrt{t-x})}{\sqrt{t-x}} f(t) dt.$$

⊙ References: H. M. Srivastava and R. G. Buschman (1977), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 448), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

$$33. \int_0^x \frac{\cos(\lambda\sqrt{x^2-t^2})}{\sqrt{x^2-t^2}} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{2}{\pi} \frac{d}{dx} \int_0^x t \frac{\cosh(\lambda\sqrt{x^2-t^2})}{\sqrt{x^2-t^2}} f(t) dt.$$

$$34. \int_x^\infty \frac{\cos(\lambda\sqrt{t^2-x^2})}{\sqrt{t^2-x^2}} y(t) dt = f(x).$$

Solution:

$$y(x) = -\frac{2}{\pi} \frac{d}{dx} \int_x^\infty t \frac{\cosh(\lambda\sqrt{t^2-x^2})}{\sqrt{t^2-x^2}} f(t) dt.$$

$$35. \int_0^x \frac{\cos(\lambda\sqrt{xt-t^2})}{\sqrt{x-t}} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{1}{\pi x} \int_0^x \frac{\cosh(\lambda\sqrt{x^2-xt})}{\sqrt{x-t}} [f(t)/2 + t f'_t(t)] dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 446).

$$36. \int_0^x \frac{\cos(\lambda\sqrt{x^2 - xt})}{\sqrt{x-t}} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{\sqrt{x}}{\pi} \frac{d}{dx} \left[\sqrt{x} \int_0^x \frac{\cosh(\lambda\sqrt{xt - t^2})}{\sqrt{x-t}} f(t) dt \right].$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 446).

$$37. \int_a^x \frac{\cos[\lambda\sqrt{(x-t)(x-t+\gamma)}]}{\sqrt{x-t}} y(t) dt = f(x).$$

It is assumed that $f(a) = f'_x(a) = f''_{xx}(a) = f'''_{xxx}(a) = 0$.

Solution:

$$y(x) = \frac{2}{\pi\lambda^2} \int_a^x \frac{\sin[\lambda\sqrt{(x-t)(x-t-\gamma)}]}{\sqrt{x-t-\gamma}} \int_a^t \sin[\lambda(t-s)] \left(\frac{d^2}{ds^2} + \lambda^2 \right)^2 f(s) ds dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 447).

$$38. \int_a^x [Ax^\beta + B \cos^\gamma(\lambda t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = Ax^\beta$ and $h(t) = B \cos^\gamma(\lambda t) + C$.

$$39. \int_a^x [A \cos^\gamma(\lambda x) + Bt^\beta + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \cos^\gamma(\lambda x)$ and $h(t) = Bt^\beta + C$.

$$40. \int_a^x (Ax^\lambda \cos^\mu t + Bt^\beta \cos^\gamma x) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \cos^\mu t$, $g_2(x) = B \cos^\gamma x$, and $h_2(t) = t^\beta$.

1.5-2. Kernels Containing Sine.

$$41. \int_a^x \sin[\lambda(x-t)] y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

Solution: $y(x) = \frac{1}{\lambda} f''_{xx}(x) + \lambda f(x)$.

⊙ References: H. M. Srivastava and R. G. Buschman (1977), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 442).

$$42. \int_a^x \{ \sin[\lambda(x-t)] + b \} y(t) dt = f(x).$$

For $b = 0$, see equation 1.5.41. Assume that $b \neq 0$.

Differentiating the equation with respect to x yields an equation of the form 2.5.3:

$$y(x) + \frac{\lambda}{b} \int_a^x \cos[\lambda(x-t)] y(t) dt = \frac{1}{b} f'_x(x).$$

43. $\int_a^x \sin(\lambda x + \beta t) y(t) dt = f(x).$

For $\beta = -\lambda$, see equation 1.5.41. Assume that $\beta \neq -\lambda$.

Differentiating the equation with respect to x twice yields

$$\sin[(\lambda + \beta)x]y(x) + \lambda \int_a^x \cos(\lambda x + \beta t)y(t) dt = f'_x(x), \quad (1)$$

$$\{\sin[(\lambda + \beta)x]y(x)\}'_x + \lambda \cos[(\lambda + \beta)x]y(x) - \lambda^2 \int_a^x \sin(\lambda x + \beta t)y(t) dt = f''_{xx}(x). \quad (2)$$

Eliminating the integral term from (2) with the aid of the original equation, we arrive at the first-order linear ordinary differential equation

$$w'_x + \lambda \cot[(\lambda + \beta)x]w = f''_{xx}(x) + \lambda^2 f(x), \quad w = \sin[(\lambda + \beta)x]y(x). \quad (3)$$

Setting $x = a$ in (1) yields the initial condition $w(a) = f'_x(a)$. On solving equation (3) under this condition, after some transformation we obtain the solution of the original integral equation in the form

$$y(x) = \frac{1}{\sin[(\lambda + \beta)x]} f'_x(x) - \frac{\lambda \cos[(\lambda + \beta)x]}{\sin^2[(\lambda + \beta)x]} f(x) - \frac{\lambda \beta}{\sin^{k+1}[(\lambda + \beta)x]} \int_a^x f(t) \sin^{k-2}[(\lambda + \beta)t] dt, \quad k = \frac{\lambda}{\lambda + \beta}.$$

44. $\int_a^x [\sin(\lambda x) - \sin(\lambda t)] y(t) dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \sin(\lambda x)$.

Solution: $y(x) = \frac{1}{\lambda} \frac{d}{dx} \left[\frac{f'_x(x)}{\cos(\lambda x)} \right].$

45. $\int_a^x [A \sin(\lambda x) + B \sin(\lambda t)] y(t) dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \sin(\lambda x)$. For $B = -A$, see equation 1.5.44.

Solution with $B \neq -A$:

$$y(x) = \frac{\text{sign } \sin(\lambda x)}{A + B} \frac{d}{dx} \left\{ |\sin(\lambda x)|^{-\frac{A}{A+B}} \int_a^x |\sin(\lambda t)|^{-\frac{B}{A+B}} f'_t(t) dt \right\}.$$

46. $\int_a^x [A \sin(\lambda x) + B \sin(\mu t) + C] y(t) dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A \sin(\lambda x)$ and $h(t) = B \sin(\mu t) + C$.

47. $\int_a^x \{\mu \sin[\lambda(x - t)] - \lambda \sin[\mu(x - t)]\} y(t) dt = f(x).$

It is assumed that $f(a) = f'_x(a) = f''_{xx}(a) = f'''_{xxx}(a) = 0$.

Solution:

$$y(x) = \frac{f'''_{xxx} + (\lambda^2 + \mu^2)f''_{xx} + \lambda^2\mu^2 f}{\lambda\mu^3 - \lambda^3\mu}, \quad f = f(x).$$

48. $\int_a^x \{A_1 \sin[\lambda_1(x-t)] + A_2 \sin[\lambda_2(x-t)]\} y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$

This equation can be solved in the same manner as equation 1.3.49, i.e., by reducing it to a second-order linear ordinary differential equation with constant coefficients.

Let

$$\Delta = -\lambda_1 \lambda_2 \frac{A_1 \lambda_2 + A_2 \lambda_1}{A_1 \lambda_1 + A_2 \lambda_2}.$$

1°. Solution for $\Delta > 0$:

$$(A_1 \lambda_1 + A_2 \lambda_2) y(x) = f''_{xx}(x) + B f(x) + C \int_a^x \sinh[k(x-t)] f(t) dt,$$

$$k = \sqrt{\Delta}, \quad B = \Delta + \lambda_1^2 + \lambda_2^2, \quad C = \frac{1}{\sqrt{\Delta}} [\Delta^2 + (\lambda_1^2 + \lambda_2^2) \Delta + \lambda_1^2 \lambda_2^2].$$

2°. Solution for $\Delta < 0$:

$$(A_1 \lambda_1 + A_2 \lambda_2) y(x) = f''_{xx}(x) + B f(x) + C \int_a^x \sin[k(x-t)] f(t) dt,$$

$$k = \sqrt{-\Delta}, \quad B = \Delta + \lambda_1^2 + \lambda_2^2, \quad C = \frac{1}{\sqrt{-\Delta}} [\Delta^2 + (\lambda_1^2 + \lambda_2^2) \Delta + \lambda_1^2 \lambda_2^2].$$

3°. Solution for $\Delta = 0$:

$$(A_1 \lambda_1 + A_2 \lambda_2) y(x) = f''_{xx}(x) + (\lambda_1^2 + \lambda_2^2) f(x) + \lambda_1^2 \lambda_2^2 \int_a^x (x-t) f(t) dt.$$

4°. Solution for $\Delta = \infty$:

$$y(x) = -\frac{f''''_{xxxx} + (\lambda_1^2 + \lambda_2^2) f''_{xx} + \lambda_1^2 \lambda_2^2 f}{A_1 \lambda_1^3 + A_2 \lambda_2^3}, \quad f = f(x).$$

In the last case, the relation $A_1 \lambda_1 + A_2 \lambda_2 = 0$ holds and the right-hand side of the integral equation is assumed to satisfy the conditions $f(a) = f'_x(a) = f''_{xx}(a) = f'''_{xxx}(a) = 0$.

Remark. The solution can be obtained from the solution of equation 1.3.49 in which the change of variables $\lambda_k \rightarrow i\lambda_k$, $A_k \rightarrow -iA_k$, $i^2 = -1$ ($k = 1, 2$), should be made.

49. $\int_a^x \{A \sin[\lambda(x-t)] + B \sin[\mu(x-t)] + C \sin[\beta(x-t)]\} y(t) dt = f(x).$

It is assumed that $f(a) = f'_x(a) = 0$. Differentiating the integral equation twice yields

$$(A\lambda + B\mu + C\beta)y(x) - \int_a^x \{A\lambda^2 \sin[\lambda(x-t)] + B\mu^2 \sin[\mu(x-t)]\} y(t) dt$$

$$- C\beta^2 \int_a^x \sin[\beta(x-t)] y(t) dt = f''_{xx}(x).$$

Eliminating the last integral with the aid of the original equation, we arrive at an equation of the form 2.5.18:

$$(A\lambda + B\mu + C\beta)y(x) + \int_a^x \{A(\beta^2 - \lambda^2) \sin[\lambda(x-t)]$$

$$+ B(\beta^2 - \mu^2) \sin[\mu(x-t)]\} y(t) dt = f''_{xx}(x) + \beta^2 f(x).$$

In the special case $A\lambda + B\mu + C\beta = 0$, this is an equation of the form 1.5.41.

$$50. \quad \int_a^x \sin^2[\lambda(x-t)]y(t) dt = f(x), \quad f(a) = f'_x(a) = f''_{xx}(a) = 0.$$

Differentiation yields an equation of the form 1.5.41:

$$\int_a^x \sin[2\lambda(x-t)]y(t) dt = \frac{1}{\lambda} f'_x(x).$$

$$\text{Solution: } y(x) = \frac{1}{2} \lambda^{-2} f'''_{xxx}(x) + 2f'_x(x).$$

$$51. \quad \int_a^x [\sin^2(\lambda x) - \sin^2(\lambda t)] y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

$$\text{Solution: } y(x) = \frac{1}{\lambda} \frac{d}{dx} \left[\frac{f'_x(x)}{\sin(2\lambda x)} \right].$$

$$52. \quad \int_a^x [A \sin^2(\lambda x) + B \sin^2(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \sin^2(\lambda x)$. For $B = -A$, see equation 1.5.51.

Solution:

$$y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ |\sin(\lambda x)|^{-\frac{2A}{A+B}} \int_a^x |\sin(\lambda t)|^{-\frac{2B}{A+B}} f'_t(t) dt \right\}.$$

$$53. \quad \int_a^x [A \sin^2(\lambda x) + B \sin^2(\mu t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \sin^2(\lambda x)$ and $h(t) = B \sin^2(\mu t) + C$.

$$54. \quad \int_a^x \sin[\lambda(x-t)] \sin[\lambda(x+t)] y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

Using the trigonometric formula

$$\sin(\alpha - \beta) \sin(\alpha + \beta) = \frac{1}{2} [\cos(2\beta) - \cos(2\alpha)], \quad \alpha = \lambda x, \quad \beta = \lambda t,$$

we reduce the original equation to an equation of the form 1.5.5:

$$\int_a^x [\cos(2\lambda x) - \cos(2\lambda t)] y(t) dt = -2f(x).$$

$$\text{Solution: } y(x) = \frac{1}{\lambda} \frac{d}{dx} \left[\frac{f'_x(x)}{\sin(2\lambda x)} \right].$$

$$55. \quad \int_a^x \sin[\lambda(x-t)] \sin[\mu(x-t)] y(t) dt = f(x), \quad f(a) = f'_x(a) = f''_{xx}(a) = 0.$$

Solution:

$$y(x) = \left[\frac{d^2}{dx^2} + (\lambda + \mu)^2 \right] \left[\frac{d^2}{dx^2} + (\lambda - \mu)^2 \right] \frac{1}{2\lambda\mu} \int_a^x f(t) dt.$$

$$56. \int_a^x [\sin(\lambda x) \sin(\mu t) + \sin(\beta x) \sin(\gamma t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = \sin(\lambda x)$, $h_1(t) = \sin(\mu t)$, $g_2(x) = \sin(\beta x)$, and $h_2(t) = \sin(\gamma t)$.

$$57. \int_a^x \sin^3[\lambda(x-t)] y(t) dt = f(x).$$

It is assumed that $f(a) = f'_x(a) = f''_{xx}(a) = f'''_{xxx}(a) = 0$.

Using the formula $\sin^3 \beta = -\frac{1}{4} \sin 3\beta + \frac{3}{4} \sin \beta$, we arrive at an equation of the form 1.5.48:

$$\int_a^x \left\{ -\frac{1}{4} \sin[3\lambda(x-t)] + \frac{3}{4} \sin[\lambda(x-t)] \right\} y(t) dt = f(x).$$

$$58. \int_a^x [\sin^3(\lambda x) - \sin^3(\lambda t)] y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

This is a special case of equation 1.9.2 with $g(x) = \sin^3(\lambda x)$.

$$59. \int_a^x [A \sin^3(\lambda x) + B \sin^3(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \sin^3(\lambda x)$. For $B = -A$, see equation 1.5.58.

Solution:

$$y(x) = \frac{\text{sign} \sin(\lambda x)}{A+B} \frac{d}{dx} \left\{ |\sin(\lambda x)|^{-\frac{3A}{A+B}} \int_a^x |\sin(\lambda t)|^{-\frac{3B}{A+B}} f'_t(t) dt \right\}.$$

$$60. \int_a^x [\sin^2(\lambda x) \sin(\mu t) + \sin(\beta x) \sin^2(\gamma t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = \sin^2(\lambda x)$, $h_1(t) = \sin(\mu t)$, $g_2(x) = \sin(\beta x)$, and $h_2(t) = \sin^2(\gamma t)$.

$$61. \int_a^x \sin^4[\lambda(x-t)] y(t) dt = f(x).$$

It is assumed that $f(a) = f'_x(a) = \dots = f''''_{xxxx}(a) = 0$.

Let us transform the kernel of the integral equation using the trigonometric formula $\sin^4 \beta = \frac{1}{8} \cos 4\beta - \frac{1}{2} \cos 2\beta + \frac{3}{8}$, where $\beta = \lambda(x-t)$, and differentiate the resulting equation with respect to x . Then we obtain an equation of the form 1.5.48:

$$\lambda \int_a^x \left\{ -\frac{1}{2} \sin[4\lambda(x-t)] + \sin[2\lambda(x-t)] \right\} y(t) dt = f'_x(x).$$

$$62. \int_a^x \sin^n[\lambda(x-t)] y(t) dt = f(x), \quad n = 2, 3, \dots$$

It is assumed that $f(a) = f'_x(a) = \dots = f^{(n)}_x(a) = 0$.

1°. Let us differentiate the equation with respect to x twice and transform the kernel of the resulting integral equation using the formula $\cos^2 \beta = 1 - \sin^2 \beta$, where $\beta = \lambda(x-t)$. We have

$$-\lambda^2 n^2 \int_a^x \sin^n[\lambda(x-t)] y(t) dt + \lambda^2 n(n-1) \int_a^x \sin^{n-2}[\lambda(x-t)] y(t) dt = f''_{xx}(x).$$

Eliminating the first term on the left-hand side with the aid of the original equation, we obtain

$$\int_a^x \sin^{n-2}[\lambda(x-t)]y(t) dt = \frac{1}{\lambda^2 n(n-1)} [f''_{xx}(x) + \lambda^2 n^2 f(x)].$$

This equation has the same form as the original equation, but the degree characterizing the kernel has been reduced by two.

By applying this technique sufficiently many times, we finally arrive at simple integral equations of the form 1.1.1 (for even n) or 1.5.41 (for odd n).

2°. Solution:

$$y(x) = \frac{1}{\lambda^n n!} \left(\frac{d}{dx} \right)^{1-\alpha} \prod_{k=1}^{\beta} \left[\frac{d^2}{dx^2} + (2k + \alpha)\lambda^2 \right] f(x),$$

where $\alpha = n - 2[n/2]$, $\beta = [(n+1)/2]$, $[A]$ denotes the integer part of number A .

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 443).

$$63. \quad \int_a^x (x-t) \sin[\lambda(x-t)]y(t) dt = f(x), \quad f(a) = f'_x(a) = f''_{xx}(a) = 0.$$

Solution:

$$y(x) = \frac{1}{2\lambda} \left(\frac{d^2}{dx^2} + \lambda^2 \right)^2 \int_a^x f(t) dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 444), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

$$64. \quad \int_a^x \sin(\lambda\sqrt{x-t})y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{2}{\pi\lambda} \frac{d^2}{dx^2} \int_a^x \frac{\cosh(\lambda\sqrt{x-t})}{\sqrt{x-t}} f(t) dt.$$

See also Example 2 in Section 10.4.

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 445), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

$$65. \quad \int_x^\infty \sin(\lambda\sqrt{t-x})y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{2}{\pi\lambda} \frac{d^2}{dx^2} \int_x^\infty \frac{\cos(\lambda\sqrt{t-x})}{\sqrt{t-x}} f(t) dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 447).

$$66. \quad \int_a^x \frac{\sin[\lambda(x-t)]}{\sqrt{x-t}} y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

Solution:

$$y(x) = \frac{2}{\pi\lambda} \int_a^x \frac{\cos[\lambda(x-t)]}{\sqrt{x-t}} [f''_{tt}(t) + \lambda^2 f(t)] dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 445), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

67. $\int_a^x \frac{\sin[\lambda(x-t)]}{(x-t)^{3/2}} y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$

Solution:

$$y(x) = \frac{2}{\pi\lambda^2} \int_a^x \frac{\sin[\lambda(x-t)]}{\sqrt{x-t}} \left[f''_{tt}(t) + \lambda^2 f(t) + \frac{f'(t)}{x-t} \right] dt.$$

⊙ References: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 445), S. G. Samko, A. A. Kilbas, and O. I. Marichev (1993).

68. $\int_a^x \frac{\sin [\lambda \sqrt{(x-t)(x-t+\gamma)}]}{\sqrt{x-t+\gamma}} y(t) dt = f(x).$

It is assumed that $f(a) = f'_x(a) = f''_{xx}(a) = f'''_{xxx}(a) = 0.$

Solution:

$$y(x) = \frac{2}{\pi\lambda^2} \int_a^x \frac{\cos[\lambda\sqrt{(x-t)(x-t-\gamma)}]}{\sqrt{x-t}} \int_a^t \sin[\lambda(t-s)] \left(\frac{d^2}{ds^2} + \lambda^2 \right)^2 f(s) ds dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 447).

69. $\int_a^x \sqrt{\sin x - \sin t} y(t) dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi} \cos x \left(\frac{1}{\cos x} \frac{d}{dx} \right)^2 \int_a^x \frac{\cos t f(t) dt}{\sqrt{\sin x - \sin t}}.$$

70. $\int_a^x \frac{y(t) dt}{\sqrt{\sin x - \sin t}} = f(x).$

Solution:

$$y(x) = \frac{1}{\pi} \frac{d}{dx} \int_a^x \frac{\cos t f(t) dt}{\sqrt{\sin x - \sin t}}.$$

71. $\int_a^x (\sin x - \sin t)^\lambda y(t) dt = f(x), \quad 0 < \lambda < 1.$

Solution:

$$y(x) = k \cos x \left(\frac{1}{\cos x} \frac{d}{dx} \right)^2 \int_a^x \frac{\cos t f(t) dt}{(\sin x - \sin t)^\lambda}, \quad k = \frac{\sin(\pi\lambda)}{\pi\lambda}.$$

72. $\int_a^x (\sin^\mu x - \sin^\mu t) y(t) dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \sin^\mu x.$

Solution: $y(x) = \frac{1}{\mu} \frac{d}{dx} \left[\frac{f'_x(x)}{\cos x \sin^{\mu-1} x} \right].$

73. $\int_a^x \{ A |\sin(\lambda x)|^\mu + B |\sin(\lambda t)|^\mu \} y(t) dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = |\sin(\lambda x)|^\mu.$

Solution:

$$y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ |\sin(\lambda x)|^{-\frac{A\mu}{A+B}} \int_a^x |\sin(\lambda t)|^{-\frac{B\mu}{A+B}} f'_t(t) dt \right\}.$$

$$74. \int_a^x \frac{y(t) dt}{[\sin(\lambda x) - \sin(\lambda t)]^\mu} = f(x), \quad 0 < \mu < 1.$$

This is a special case of equation 1.9.44 with $g(x) = \sin(\lambda x)$ and $h(x) \equiv 1$.

Solution:

$$y(x) = \frac{\lambda \sin(\pi\mu)}{\pi} \frac{d}{dx} \int_a^x \frac{\cos(\lambda t) f(t) dt}{[\sin(\lambda x) - \sin(\lambda t)]^{1-\mu}}.$$

$$75. \int_a^x (x-t) \sin[\lambda(x-t)] y(t) dt = f(x), \quad f(a) = f'_x(a) = f''_{xx}(a) = 0.$$

Double differentiation yields

$$2\lambda \int_a^x \cos[\lambda(x-t)] y(t) dt - \lambda^2 \int_a^x (x-t) \sin[\lambda(x-t)] y(t) dt = f''_{xx}(x).$$

Eliminating the second integral on the left-hand side of this equation with the aid of the original equation, we arrive at an equation of the form 1.5.1:

$$\int_a^x \cos[\lambda(x-t)] y(t) dt = \frac{1}{2\lambda} [f''_{xx}(x) + \lambda^2 f(x)].$$

Solution:

$$y(x) = \frac{1}{2\lambda} f'''_{xxx}(x) + \lambda f'_x(x) + \frac{1}{2} \lambda^3 \int_a^x f(t) dt.$$

$$76. \int_a^x |\sin(\lambda(x-t))| y(t) dt = f(x), \quad f(a) = f'_x(a) = f''_{xx}(a) = 0.$$

Solution:

$$y(x) = \frac{1}{\lambda} \int_a^x (-1)^{[\lambda(x-t)/\pi]} (f'''_{ttt}(t) + \lambda^2 f'_t(t)) dt,$$

where $[A]$ denotes the integer part of number A .

⊙ References: H. M. Srivastava and R. G. Buschman (1977), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 443).

$$77. \int_a^x [Ax^\beta + B \sin^\gamma(\lambda t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = Ax^\beta$ and $h(t) = B \sin^\gamma(\lambda t) + C$.

$$78. \int_a^x [A \sin^\gamma(\lambda x) + Bt^\beta + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \sin^\gamma(\lambda x)$ and $h(t) = Bt^\beta + C$.

$$79. \int_a^x (Ax^\lambda \sin^\mu t + Bt^\beta \sin^\gamma x) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \sin^\mu t$, $g_2(x) = B \sin^\gamma x$, and $h_2(t) = t^\beta$.

1.5-3. Kernels Containing Tangent.

$$80. \quad \int_a^x [\tan(\lambda x) - \tan(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \tan(\lambda x)$.

$$\text{Solution: } y(x) = \frac{1}{\lambda} \frac{d}{dx} [\cos^2(\lambda x) f'_x(x)].$$

$$81. \quad \int_a^x [A \tan(\lambda x) + B \tan(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \tan(\lambda x)$. For $B = -A$, see equation 1.5.80.

$$\text{Solution: } y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ [\tan(\lambda x)]^{-\frac{A}{A+B}} \int_a^x [\tan(\lambda t)]^{-\frac{B}{A+B}} f'_t(t) dt \right\}.$$

$$82. \quad \int_a^x [A \tan(\lambda x) + B \tan(\mu t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \tan(\lambda x)$ and $h(t) = B \tan(\mu t) + C$.

$$83. \quad \int_a^x [\tan^2(\lambda x) - \tan^2(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \tan^2(\lambda x)$.

$$\text{Solution: } y(x) = \frac{d}{dx} \left[\frac{\cos^3(\lambda x) f'_x(x)}{2\lambda \sin(\lambda x)} \right].$$

$$84. \quad \int_a^x [A \tan^2(\lambda x) + B \tan^2(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \tan^2(\lambda x)$. For $B = -A$, see equation 1.5.83.

$$\text{Solution: } y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ |\tan(\lambda x)|^{-\frac{2A}{A+B}} \int_a^x |\tan(\lambda t)|^{-\frac{2B}{A+B}} f'_t(t) dt \right\}.$$

$$85. \quad \int_a^x [A \tan^2(\lambda x) + B \tan^2(\mu t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \tan^2(\lambda x)$ and $h(t) = B \tan^2(\mu t) + C$.

$$86. \quad \int_a^x [\tan(\lambda x) - \tan(\lambda t)]^n y(t) dt = f(x), \quad n = 1, 2, \dots$$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \dots = f_x^{(n)}(a) = 0$.

$$\text{Solution: } y(x) = \frac{1}{\lambda^n n! \cos^2(\lambda x)} \left[\cos^2(\lambda x) \frac{d}{dx} \right]^{n+1} f(x).$$

$$87. \quad \int_a^x \sqrt{\tan x - \tan t} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{2}{\pi \cos^2 x} \left(\cos^2 x \frac{d}{dx} \right)^2 \int_a^x \frac{f(t) dt}{\cos^2 t \sqrt{\tan x - \tan t}}.$$

$$88. \int_a^x \frac{y(t) dt}{\sqrt{\tan x - \tan t}} = f(x).$$

Solution:

$$y(x) = \frac{1}{\pi} \frac{d}{dx} \int_a^x \frac{f(t) dt}{\cos^2 t \sqrt{\tan x - \tan t}}.$$

$$89. \int_a^x (\tan x - \tan t)^\lambda y(t) dt = f(x), \quad 0 < \lambda < 1.$$

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi\lambda \cos^2 x} \left(\cos^2 x \frac{d}{dx} \right)^2 \int_a^x \frac{f(t) dt}{\cos^2 t (\tan x - \tan t)^\lambda}.$$

$$90. \int_a^x (\tan^\mu x - \tan^\mu t) y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \tan^\mu x$.

$$\text{Solution: } y(x) = \frac{1}{\mu} \frac{d}{dx} \left[\frac{\cos^{\mu+1} x f'_x(x)}{\sin^{\mu-1} x} \right].$$

$$91. \int_a^x (A \tan^\mu x + B \tan^\mu t) y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \tan^\mu x$. For $B = -A$, see equation 1.5.90.

Solution:

$$y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ [\tan(\lambda x)]^{-\frac{A\mu}{A+B}} \int_a^x [\tan(\lambda t)]^{-\frac{B\mu}{A+B}} f'_t(t) dt \right\}.$$

$$92. \int_a^x \frac{y(t) dt}{[\tan(\lambda x) - \tan(\lambda t)]^\mu} = f(x), \quad 0 < \mu < 1.$$

This is a special case of equation 1.9.44 with $g(x) = \tan(\lambda x)$ and $h(x) \equiv 1$.

Solution:

$$y(x) = \frac{\lambda \sin(\pi\mu)}{\pi} \frac{d}{dx} \int_a^x \frac{f(t) dt}{\cos^2(\lambda t) [\tan(\lambda x) - \tan(\lambda t)]^{1-\mu}}.$$

$$93. \int_a^x [Ax^\beta + B \tan^\gamma(\lambda t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = Ax^\beta$ and $h(t) = B \tan^\gamma(\lambda t) + C$.

$$94. \int_a^x [A \tan^\gamma(\lambda x) + Bt^\beta + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \tan^\gamma(\lambda x)$ and $h(t) = Bt^\beta + C$.

$$95. \int_a^x (Ax^\lambda \tan^\mu t + Bt^\beta \tan^\gamma x) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \tan^\mu t$, $g_2(x) = B \tan^\gamma x$, and $h_2(t) = t^\beta$.

1.5-4. Kernels Containing Cotangent.

96. $\int_a^x [\cot(\lambda x) - \cot(\lambda t)] y(t) dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \cot(\lambda x)$.

Solution: $y(x) = -\frac{1}{\lambda} \frac{d}{dx} [\sin^2(\lambda x) f'_x(x)].$

97. $\int_a^x [A \cot(\lambda x) + B \cot(\lambda t)] y(t) dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \cot(\lambda x)$. For $B = -A$, see equation 1.5.96.

Solution: $y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ [\tan(\lambda x)]^{\frac{A}{A+B}} \int_a^x [\tan(\lambda t)]^{\frac{B}{A+B}} f'_t(t) dt \right\}.$

98. $\int_a^x [A \cot(\lambda x) + B \cot(\mu t) + C] y(t) dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A \cot(\lambda x)$ and $h(t) = B \cot(\mu t) + C$.

99. $\int_a^x [\cot^2(\lambda x) - \cot^2(\lambda t)] y(t) dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \cot^2(\lambda x)$.

Solution: $y(x) = -\frac{d}{dx} \left[\frac{\sin^3(\lambda x) f'_x(x)}{2\lambda \cos(\lambda x)} \right].$

100. $\int_a^x [A \cot^2(\lambda x) + B \cot^2(\lambda t)] y(t) dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \cot^2(\lambda x)$. For $B = -A$, see equation 1.5.99.

Solution: $y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ |\tan(\lambda x)|^{\frac{2A}{A+B}} \int_a^x |\tan(\lambda t)|^{\frac{2B}{A+B}} f'_t(t) dt \right\}.$

101. $\int_a^x [A \cot^2(\lambda x) + B \cot^2(\mu t) + C] y(t) dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A \cot^2(\lambda x)$ and $h(t) = B \cot^2(\mu t) + C$.

102. $\int_a^x [\cot(\lambda x) - \cot(\lambda t)]^n y(t) dt = f(x), \quad n = 1, 2, \dots$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \dots = f_x^{(n)}(a) = 0$.

Solution: $y(x) = \frac{(-1)^n}{\lambda^n n! \sin^2(\lambda x)} \left[\sin^2(\lambda x) \frac{d}{dx} \right]^{n+1} f(x).$

103. $\int_a^x (\cot^\mu x - \cot^\mu t) y(t) dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \cot^\mu x$.

Solution: $y(x) = -\frac{1}{\mu} \frac{d}{dx} \left[\frac{\sin^{\mu+1} x f'_x(x)}{\cos^{\mu-1} x} \right].$

$$104. \int_a^x (A \cot^\mu x + B \cot^\mu t) y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \cot^\mu x$. For $B = -A$, see equation 1.5.103.

Solution:

$$y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ |\tan x|^{\frac{A\mu}{A+B}} \int_a^x |\tan t|^{\frac{B\mu}{A+B}} f'_t(t) dt \right\}.$$

$$105. \int_a^x [Ax^\beta + B \cot^\gamma(\lambda t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = Ax^\beta$ and $h(t) = B \cot^\gamma(\lambda t) + C$.

$$106. \int_a^x [A \cot^\gamma(\lambda x) + Bt^\beta + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \cot^\gamma(\lambda x)$ and $h(t) = Bt^\beta + C$.

$$107. \int_a^x (Ax^\lambda \cot^\mu t + Bt^\beta \cot^\gamma x) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \cot^\mu t$, $g_2(x) = B \cot^\gamma x$, and $h_2(t) = t^\beta$.

1.5-5. Kernels Containing Combinations of Trigonometric Functions.

$$108. \int_a^x \{ \cos[\lambda(x-t)] + A \sin[\mu(x-t)] \} y(t) dt = f(x).$$

Differentiating the equation with respect to x followed by eliminating the integral with the cosine yields an equation of the form 2.3.16:

$$y(x) - (\lambda + A^2\mu) \int_a^x \sin[\mu(x-t)] y(t) dt = f'_x(x) - A\mu f(x).$$

$$109. \int_a^x [A \cos(\lambda x) + B \sin(\mu t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \cos(\lambda x)$ and $h(t) = B \sin(\mu t) + C$.

$$110. \int_a^x [A \sin(\lambda x) + B \cos(\mu t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \sin(\lambda x)$ and $h(t) = B \cos(\mu t) + C$.

$$111. \int_a^x [A \cos^2(\lambda x) + B \sin^2(\mu t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \cos^2(\lambda x)$ and $h(t) = B \sin^2(\mu t)$.

$$112. \int_a^x \sin[\lambda(x-t)] \cos[\lambda(x+t)] y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

Using the trigonometric formula

$$\sin(\alpha - \beta) \cos(\alpha + \beta) = \frac{1}{2} [\sin(2\alpha) - \sin(2\beta)], \quad \alpha = \lambda x, \quad \beta = \lambda t,$$

we reduce the original equation to an equation of the form 1.5.44:

$$\int_a^x [\sin(2\lambda x) - \sin(2\lambda t)] y(t) dt = 2f(x).$$

$$\text{Solution: } y(x) = \frac{1}{\lambda} \frac{d}{dx} \left[\frac{f'_x(x)}{\cos(2\lambda x)} \right].$$

$$113. \int_a^x \cos[\lambda(x-t)] \sin[\lambda(x+t)] y(t) dt = f(x).$$

Using the trigonometric formula

$$\cos(\alpha - \beta) \sin(\alpha + \beta) = \frac{1}{2} [\sin(2\alpha) + \sin(2\beta)], \quad \alpha = \lambda x, \quad \beta = \lambda t,$$

we reduce the original equation to an equation of the form 1.5.45 with $A = B = 1$:

$$\int_a^x [\sin(2\lambda x) + \sin(2\lambda t)] y(t) dt = 2f(x).$$

Solution with $\sin(2\lambda x) > 0$:

$$y(x) = \frac{d}{dx} \left[\frac{1}{\sqrt{\sin(2\lambda x)}} \int_a^x \frac{f'_t(t) dt}{\sqrt{\sin(2\lambda t)}} \right].$$

$$114. \int_a^x \sin[\lambda(x-t)] \cos[\mu(x-t)] y(t) dt = f(x), \quad f(a) = f'_x(a) = f''_{xx}(a) = 0.$$

Solution with $\mu < \lambda$:

$$y(x) = \frac{1}{\lambda \sqrt{\lambda^2 - \mu^2}} \left[\frac{d^2}{dx^2} + (\lambda + \mu)^2 \right] \left[\frac{d^2}{dx^2} + (\lambda - \mu)^2 \right] \int_a^x \sin[\sqrt{\lambda^2 - \mu^2}(x-t)] f(t) dt.$$

Solution with $\mu > \lambda$:

$$y(x) = \frac{1}{\lambda \sqrt{\lambda^2 - \mu^2}} \left[\frac{d^2}{dx^2} + (\lambda + \mu)^2 \right] \left[\frac{d^2}{dx^2} + (\lambda - \mu)^2 \right] \int_a^x \sinh[\sqrt{\mu^2 - \lambda^2}(x-t)] f(t) dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 444).

$$115. \int_a^x [A \cos(\lambda x) \sin(\mu t) + B \cos(\beta x) \sin(\gamma t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = A \cos(\lambda x)$, $h_1(t) = \sin(\mu t)$, $g_2(x) = B \cos(\beta x)$, and $h_2(t) = \sin(\gamma t)$.

$$116. \int_a^x [A \sin(\lambda x) \cos(\mu t) + B \sin(\beta x) \cos(\gamma t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = A \sin(\lambda x)$, $h_1(t) = \cos(\mu t)$, $g_2(x) = B \sin(\beta x)$, and $h_2(t) = \cos(\gamma t)$.

$$117. \int_a^x [A \cos(\lambda x) \cos(\mu t) + B \sin(\beta x) \sin(\gamma t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = A \cos(\lambda x)$, $h_1(t) = \cos(\mu t)$, $g_2(x) = B \sin(\beta x)$, and $h_2(t) = \sin(\gamma t)$.

$$118. \int_a^x [A \cos^\beta(\lambda x) + B \sin^\gamma(\mu t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \cos^\beta(\lambda x)$ and $h(t) = B \sin^\gamma(\mu t)$.

$$119. \int_a^x [A \sin^\beta(\lambda x) + B \cos^\gamma(\mu t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \sin^\beta(\lambda x)$ and $h(t) = B \cos^\gamma(\mu t)$.

$$120. \int_a^x (Ax^\lambda \cos^\mu t + Bt^\beta \sin^\gamma x) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \cos^\mu t$, $g_2(x) = B \sin^\gamma x$, and $h_2(t) = t^\beta$.

$$121. \int_a^x (Ax^\lambda \sin^\mu t + Bt^\beta \cos^\gamma x) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \sin^\mu t$, $g_2(x) = B \cos^\gamma x$, and $h_2(t) = t^\beta$.

$$122. \int_a^x \{ (x-t) \sin[\lambda(x-t)] - \lambda(x-t)^2 \cos[\lambda(x-t)] \} y(t) dt = f(x).$$

Solution:

$$y(x) = \int_a^x g(t) dt,$$

where

$$g(t) = \sqrt{\frac{\pi}{2\lambda}} \frac{1}{64\lambda^5} \left(\frac{d^2}{dt^2} + \lambda^2 \right)^6 \int_a^t (t-\tau)^{5/2} J_{5/2}[\lambda(t-\tau)] f(\tau) d\tau.$$

$$123. \int_a^x \left\{ \frac{\sin[\lambda(x-t)]}{x-t} - \lambda \cos[\lambda(x-t)] \right\} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{1}{2\lambda^4} \left(\frac{d^2}{dx^2} + \lambda^2 \right)^3 \int_a^x \sin[\lambda(x-t)] f(t) dt.$$

$$124. \int_a^x [\sin(\lambda\sqrt{x-t}) - \lambda\sqrt{x-t} \cos(\lambda\sqrt{x-t})] y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

Solution:

$$y(x) = \frac{4}{\pi\lambda^3} \frac{d^3}{dx^3} \int_a^x \frac{\cosh(\lambda\sqrt{x-t})}{\sqrt{x-t}} f(t) dt.$$

$$125. \int_a^x [A \tan(\lambda x) + B \cot(\mu t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \tan(\lambda x)$ and $h(t) = B \cot(\mu t) + C$.

$$126. \int_a^x [A \tan^2(\lambda x) + B \cot^2(\mu t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \tan^2(\lambda x)$ and $h(t) = B \cot^2(\mu t)$.

$$127. \int_a^x [\tan(\lambda x) \cot(\mu t) + \tan(\beta x) \cot(\gamma t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = \tan(\lambda x)$, $h_1(t) = \cot(\mu t)$, $g_2(x) = \tan(\beta x)$, and $h_2(t) = \cot(\gamma t)$.

$$128. \int_a^x [\cot(\lambda x) \tan(\mu t) + \cot(\beta x) \tan(\gamma t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = \cot(\lambda x)$, $h_1(t) = \tan(\mu t)$, $g_2(x) = \cot(\beta x)$, and $h_2(t) = \tan(\gamma t)$.

$$129. \int_a^x [\tan(\lambda x) \tan(\mu t) + \cot(\beta x) \cot(\gamma t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = \tan(\lambda x)$, $h_1(t) = \tan(\mu t)$, $g_2(x) = \cot(\beta x)$, and $h_2(t) = \cot(\gamma t)$.

$$130. \int_a^x [A \tan^\beta(\lambda x) + B \cot^\gamma(\mu t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \tan^\beta(\lambda x)$ and $h(t) = B \cot^\gamma(\mu t)$.

$$131. \int_a^x [A \cot^\beta(\lambda x) + B \tan^\gamma(\mu t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \cot^\beta(\lambda x)$ and $h(t) = B \tan^\gamma(\mu t)$.

$$132. \int_a^x (Ax^\lambda \tan^\mu t + Bt^\beta \cot^\gamma x) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \tan^\mu t$, $g_2(x) = B \cot^\gamma x$, and $h_2(t) = t^\beta$.

$$133. \int_a^x (Ax^\lambda \cot^\mu t + Bt^\beta \tan^\gamma x) y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ax^\lambda$, $h_1(t) = \cot^\mu t$, $g_2(x) = B \tan^\gamma x$, and $h_2(t) = t^\beta$.

1.6. Equations Whose Kernels Contain Inverse Trigonometric Functions

1.6-1. Kernels Containing Arccosine.

$$1. \int_a^x [\arccos(\lambda x) - \arccos(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \arccos(\lambda x)$.

$$\text{Solution: } y(x) = -\frac{1}{\lambda} \frac{d}{dx} \left[\sqrt{1 - \lambda^2 x^2} f'_x(x) \right].$$

$$2. \quad \int_a^x [A \arccos(\lambda x) + B \arccos(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \arccos(\lambda x)$. For $B = -A$, see equation 1.6.1.

Solution:

$$y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ [\arccos(\lambda x)]^{-\frac{A}{A+B}} \int_a^x [\arccos(\lambda t)]^{-\frac{B}{A+B}} f'_t(t) dt \right\}.$$

$$3. \quad \int_a^x [A \arccos(\lambda x) + B \arccos(\mu t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \arccos(\lambda x)$ and $h(t) = B \arccos(\mu t) + C$.

$$4. \quad \int_a^x [\arccos(\lambda x) - \arccos(\lambda t)]^n y(t) dt = f(x), \quad n = 1, 2, \dots$$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \dots = f_x^{(n)}(a) = 0$.

Solution:

$$y(x) = \frac{(-1)^n}{\lambda^n n! \sqrt{1-\lambda^2 x^2}} \left(\sqrt{1-\lambda^2 x^2} \frac{d}{dx} \right)^{n+1} f(x).$$

$$5. \quad \int_a^x \sqrt{\arccos(\lambda t) - \arccos(\lambda x)} y(t) dt = f(x).$$

This is a special case of equation 1.9.40 with $g(x) = 1 - \arccos(\lambda x)$.

Solution:

$$y(x) = \frac{2}{\pi} \varphi(x) \left(\frac{1}{\varphi(x)} \frac{d}{dx} \right)^2 \int_a^x \frac{\varphi(t) f(t) dt}{\sqrt{\arccos(\lambda t) - \arccos(\lambda x)}}, \quad \varphi(x) = \frac{1}{\sqrt{1-\lambda^2 x^2}}.$$

$$6. \quad \int_a^x \frac{y(t) dt}{\sqrt{\arccos(\lambda t) - \arccos(\lambda x)}} = f(x).$$

Solution:

$$y(x) = \frac{\lambda}{\pi} \frac{d}{dx} \int_a^x \frac{\varphi(t) f(t) dt}{\sqrt{\arccos(\lambda t) - \arccos(\lambda x)}}, \quad \varphi(x) = \frac{1}{\sqrt{1-\lambda^2 x^2}}.$$

$$7. \quad \int_a^x [\arccos(\lambda t) - \arccos(\lambda x)]^\mu y(t) dt = f(x), \quad 0 < \mu < 1.$$

Solution:

$$y(x) = k \varphi(x) \left(\frac{1}{\varphi(x)} \frac{d}{dx} \right)^2 \int_a^x \frac{\varphi(t) f(t) dt}{[\arccos(\lambda t) - \arccos(\lambda x)]^\mu},$$

$$\varphi(x) = \frac{1}{\sqrt{1-\lambda^2 x^2}}, \quad k = \frac{\sin(\pi \mu)}{\pi \mu}.$$

$$8. \quad \int_a^x [\arccos^\mu(\lambda x) - \arccos^\mu(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \arccos^\mu(\lambda x)$.

$$\text{Solution: } y(x) = -\frac{1}{\lambda \mu} \frac{d}{dx} \left[\frac{f'_x(x) \sqrt{1-\lambda^2 x^2}}{\arccos^{\mu-1}(\lambda x)} \right].$$

$$9. \quad \int_a^x \frac{y(t) dt}{[\arccos(\lambda t) - \arccos(\lambda x)]^\mu} = f(x), \quad 0 < \mu < 1.$$

Solution:

$$y(x) = \frac{\lambda \sin(\pi\mu)}{\pi} \frac{d}{dx} \int_a^x \frac{\varphi(t)f(t) dt}{[\arccos(\lambda t) - \arccos(\lambda x)]^{1-\mu}}, \quad \varphi(x) = \frac{1}{\sqrt{1-\lambda^2 x^2}}.$$

$$10. \quad \int_a^x [A \arccos^\beta(\lambda x) + B \arccos^\gamma(\mu t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \arccos^\beta(\lambda x)$ and $h(t) = B \arccos^\gamma(\mu t) + C$.

1.6-2. Kernels Containing Arcsine.

$$11. \quad \int_a^x [\arcsin(\lambda x) - \arcsin(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \arcsin(\lambda x)$.

$$\text{Solution: } y(x) = \frac{1}{\lambda} \frac{d}{dx} \left[\sqrt{1-\lambda^2 x^2} f'_x(x) \right].$$

$$12. \quad \int_a^x [A \arcsin(\lambda x) + B \arcsin(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \arcsin(\lambda x)$. For $B = -A$, see equation 1.6.11.

Solution:

$$y(x) = \frac{\text{sign } x}{A+B} \frac{d}{dx} \left\{ |\arcsin(\lambda x)|^{-\frac{A}{A+B}} \int_a^x |\arcsin(\lambda t)|^{-\frac{B}{A+B}} f'_t(t) dt \right\}.$$

$$13. \quad \int_a^x [A \arcsin(\lambda x) + B \arcsin(\mu t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \arcsin(\lambda x)$ and $h(t) = B \arcsin(\mu t) + C$.

$$14. \quad \int_a^x [\arcsin(\lambda x) - \arcsin(\lambda t)]^n y(t) dt = f(x), \quad n = 1, 2, \dots$$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \dots = f_x^{(n)}(a) = 0$.

Solution:

$$y(x) = \frac{1}{\lambda^n n! \sqrt{1-\lambda^2 x^2}} \left(\sqrt{1-\lambda^2 x^2} \frac{d}{dx} \right)^{n+1} f(x).$$

$$15. \quad \int_a^x \sqrt{\arcsin(\lambda x) - \arcsin(\lambda t)} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{2}{\pi} \varphi(x) \left(\frac{1}{\varphi(x)} \frac{d}{dx} \right)^2 \int_a^x \frac{\varphi(t)f(t) dt}{\sqrt{\arcsin(\lambda x) - \arcsin(\lambda t)}}, \quad \varphi(x) = \frac{1}{\sqrt{1-\lambda^2 x^2}}.$$

$$16. \int_a^x \frac{y(t) dt}{\sqrt{\arcsin(\lambda x) - \arcsin(\lambda t)}} = f(x).$$

Solution:

$$y(x) = \frac{\lambda}{\pi} \frac{d}{dx} \int_a^x \frac{\varphi(t) f(t) dt}{\sqrt{\arcsin(\lambda x) - \arcsin(\lambda t)}}, \quad \varphi(x) = \frac{1}{\sqrt{1 - \lambda^2 x^2}}.$$

$$17. \int_a^x [\arcsin(\lambda x) - \arcsin(\lambda t)]^\mu y(t) dt = f(x), \quad 0 < \mu < 1.$$

Solution:

$$y(x) = k \varphi(x) \left(\frac{1}{\varphi(x)} \frac{d}{dx} \right)^2 \int_a^x \frac{\varphi(t) f(t) dt}{[\arcsin(\lambda x) - \arcsin(\lambda t)]^\mu},$$

$$\varphi(x) = \frac{1}{\sqrt{1 - \lambda^2 x^2}}, \quad k = \frac{\sin(\pi \mu)}{\pi \mu}.$$

$$18. \int_a^x [\arcsin^\mu(\lambda x) - \arcsin^\mu(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \arcsin^\mu(\lambda x)$.

$$\text{Solution: } y(x) = \frac{1}{\lambda \mu} \frac{d}{dx} \left[\frac{f'_x(x) \sqrt{1 - \lambda^2 x^2}}{\arcsin^{\mu-1}(\lambda x)} \right].$$

$$19. \int_a^x \frac{y(t) dt}{[\arcsin(\lambda x) - \arcsin(\lambda t)]^\mu} = f(x), \quad 0 < \mu < 1.$$

Solution:

$$y(x) = \frac{\lambda \sin(\pi \mu)}{\pi} \frac{d}{dx} \int_a^x \frac{\varphi(t) f(t) dt}{[\arcsin(\lambda x) - \arcsin(\lambda t)]^{1-\mu}}, \quad \varphi(x) = \frac{1}{\sqrt{1 - \lambda^2 x^2}}.$$

$$20. \int_a^x [A \arcsin^\beta(\lambda x) + B \arcsin^\gamma(\mu t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \arcsin^\beta(\lambda x)$ and $h(t) = B \arcsin^\gamma(\mu t) + C$.

$$21. \int_0^x \arcsin \sqrt{1 - \frac{t}{x}} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{2}{\pi} \frac{1}{\sqrt{x}} \frac{d}{dx} \int_0^x \frac{t}{\sqrt{x-t}} \frac{d}{dt} f(t) dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 452).

$$22. \int_x^\infty \arcsin \sqrt{1 - \frac{x}{t}} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{2}{\pi} \frac{d}{dx} \int_x^\infty \frac{\sqrt{t}}{\sqrt{t-x}} \frac{d}{dt} f(t) dt.$$

⊙ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 453).

1.6-3. Kernels Containing Arctangent.

23. $\int_a^x [\arctan(\lambda x) - \arctan(\lambda t)] y(t) dt = f(x).$

This is a special case of equation 1.9.2 with $g(x) = \arctan(\lambda x)$.

Solution: $y(x) = \frac{1}{\lambda} \frac{d}{dx} [(1 + \lambda^2 x^2) f'_x(x)].$

24. $\int_a^x [A \arctan(\lambda x) + B \arctan(\lambda t)] y(t) dt = f(x).$

This is a special case of equation 1.9.4 with $g(x) = \arctan(\lambda x)$. For $B = -A$, see equation 1.6.21.

Solution:

$$y(x) = \frac{\text{sign } x}{A+B} \frac{d}{dx} \left\{ |\arctan(\lambda x)|^{-\frac{A}{A+B}} \int_a^x |\arctan(\lambda t)|^{-\frac{B}{A+B}} f'_t(t) dt \right\}.$$

25. $\int_a^x [A \arctan(\lambda x) + B \arctan(\mu t) + C] y(t) dt = f(x).$

This is a special case of equation 1.9.6 with $g(x) = A \arctan(\lambda x)$ and $h(t) = B \arctan(\mu t) + C$.

26. $\int_a^x [\arctan(\lambda x) - \arctan(\lambda t)]^n y(t) dt = f(x), \quad n = 1, 2, \dots$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \dots = f_x^{(n)}(a) = 0$.

Solution:

$$y(x) = \frac{1}{\lambda^n n! (1 + \lambda^2 x^2)} \left((1 + \lambda^2 x^2) \frac{d}{dx} \right)^{n+1} f(x).$$

27. $\int_a^x \sqrt{\arctan(\lambda x) - \arctan(\lambda t)} y(t) dt = f(x).$

Solution:

$$y(x) = \frac{2}{\pi} \varphi(x) \left(\frac{1}{\varphi(x)} \frac{d}{dx} \right)^2 \int_a^x \frac{\varphi(t) f(t) dt}{\sqrt{\arctan(\lambda x) - \arctan(\lambda t)}}, \quad \varphi(x) = \frac{1}{1 + \lambda^2 x^2}.$$

28. $\int_a^x \frac{y(t) dt}{\sqrt{\arctan(\lambda x) - \arctan(\lambda t)}} = f(x).$

Solution:

$$y(x) = \frac{\lambda}{\pi} \frac{d}{dx} \int_a^x \frac{\varphi(t) f(t) dt}{\sqrt{\arctan(\lambda x) - \arctan(\lambda t)}}, \quad \varphi(x) = \frac{1}{1 + \lambda^2 x^2}.$$

29. $\int_a^x \sqrt{t} \arctan \left(\sqrt{\frac{x-t}{t}} \right) y(t) dt = f(x).$

The equation can be rewritten in terms of the Gaussian hypergeometric function in the form

$$\int_a^x (x-t)^{\gamma-1} F \left(\alpha, \beta, \gamma; 1 - \frac{x}{t} \right) y(t) dt = f(x), \quad \text{where } \alpha = \frac{1}{2}, \quad \beta = 1, \quad \gamma = \frac{3}{2}.$$

See 1.8.135 for the solution of this equation.

$$30. \int_a^x [\arctan(\lambda x) - \arctan(\lambda t)]^\mu y(t) dt = f(x), \quad 0 < \mu < 1.$$

Solution:

$$y(x) = k\varphi(x) \left(\frac{1}{\varphi(x)} \frac{d}{dx} \right)^2 \int_a^x \frac{\varphi(t)f(t) dt}{[\arctan(\lambda x) - \arctan(\lambda t)]^\mu},$$

$$\varphi(x) = \frac{1}{1 + \lambda^2 x^2}, \quad k = \frac{\sin(\pi\mu)}{\pi\mu}.$$

$$31. \int_a^x [\arctan^\mu(\lambda x) - \arctan^\mu(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \arctan^\mu(\lambda x)$.

$$\text{Solution: } y(x) = \frac{1}{\lambda\mu} \frac{d}{dx} \left[\frac{(1 + \lambda^2 x^2) f'_x(x)}{\arctan^{\mu-1}(\lambda x)} \right].$$

$$32. \int_a^x \frac{y(t) dt}{[\arctan(\lambda x) - \arctan(\lambda t)]^\mu} = f(x), \quad 0 < \mu < 1.$$

Solution:

$$y(x) = \frac{\lambda \sin(\pi\mu)}{\pi} \frac{d}{dx} \int_a^x \frac{\varphi(t)f(t) dt}{[\arctan(\lambda x) - \arctan(\lambda t)]^{1-\mu}}, \quad \varphi(x) = \frac{1}{1 + \lambda^2 x^2}.$$

$$33. \int_a^x [A \arctan^\beta(\lambda x) + B \arctan^\gamma(\mu t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \arctan^\beta(\lambda x)$ and $h(t) = B \arctan^\gamma(\mu t) + C$.

1.6-4. Kernels Containing Arccotangent.

$$34. \int_a^x [\operatorname{arccot}(\lambda x) - \operatorname{arccot}(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \operatorname{arccot}(\lambda x)$.

$$\text{Solution: } y(x) = -\frac{1}{\lambda} \frac{d}{dx} [(1 + \lambda^2 x^2) f'_x(x)].$$

$$35. \int_a^x [A \operatorname{arccot}(\lambda x) + B \operatorname{arccot}(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.4 with $g(x) = \operatorname{arccot}(\lambda x)$. For $B = -A$, see equation 1.6.34.

Solution:

$$y(x) = \frac{1}{A+B} \frac{d}{dx} \left\{ [\operatorname{arccot}(\lambda x)]^{-\frac{A}{A+B}} \int_a^x [\operatorname{arccot}(\lambda t)]^{-\frac{B}{A+B}} f'_t(t) dt \right\}.$$

$$36. \int_a^x [A \operatorname{arccot}(\lambda x) + B \operatorname{arccot}(\mu t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \operatorname{arccot}(\lambda x)$ and $h(t) = B \operatorname{arccot}(\mu t) + C$.

$$37. \int_a^x [\operatorname{arccot}(\lambda x) - \operatorname{arccot}(\lambda t)]^n y(t) dt = f(x), \quad n = 1, 2, \dots$$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \dots = f_x^{(n)}(a) = 0$.

Solution:

$$y(x) = \frac{(-1)^n}{\lambda^n n! (1 + \lambda^2 x^2)} \left((1 + \lambda^2 x^2) \frac{d}{dx} \right)^{n+1} f(x).$$

$$38. \int_a^x \sqrt{\operatorname{arccot}(\lambda t) - \operatorname{arccot}(\lambda x)} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{2}{\pi} \varphi(x) \left(\frac{1}{\varphi(x)} \frac{d}{dx} \right)^2 \int_a^x \frac{\varphi(t) f(t) dt}{\sqrt{\operatorname{arccot}(\lambda t) - \operatorname{arccot}(\lambda x)}}, \quad \varphi(x) = \frac{1}{1 + \lambda^2 x^2}.$$

$$39. \int_a^x \frac{y(t) dt}{\sqrt{\operatorname{arccot}(\lambda t) - \operatorname{arccot}(\lambda x)}} = f(x).$$

Solution:

$$y(x) = \frac{\lambda}{\pi} \frac{d}{dx} \int_a^x \frac{\varphi(t) f(t) dt}{\sqrt{\operatorname{arccot}(\lambda t) - \operatorname{arccot}(\lambda x)}}, \quad \varphi(x) = \frac{1}{1 + \lambda^2 x^2}.$$

$$40. \int_a^x [\operatorname{arccot}(\lambda t) - \operatorname{arccot}(\lambda x)]^\mu y(t) dt = f(x), \quad 0 < \mu < 1.$$

Solution:

$$y(x) = k \varphi(x) \left(\frac{1}{\varphi(x)} \frac{d}{dx} \right)^2 \int_a^x \frac{\varphi(t) f(t) dt}{[\operatorname{arccot}(\lambda t) - \operatorname{arccot}(\lambda x)]^\mu},$$

$$\varphi(x) = \frac{1}{1 + \lambda^2 x^2}, \quad k = \frac{\sin(\pi \mu)}{\pi \mu}.$$

$$41. \int_a^x [\operatorname{arccot}^\mu(\lambda x) - \operatorname{arccot}^\mu(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.2 with $g(x) = \operatorname{arccot}^\mu(\lambda x)$.

$$\text{Solution: } y(x) = -\frac{1}{\lambda \mu} \frac{d}{dx} \left[\frac{(1 + \lambda^2 x^2) f'_x(x)}{\operatorname{arccot}^{\mu-1}(\lambda x)} \right].$$

$$42. \int_a^x \frac{y(t) dt}{[\operatorname{arccot}(\lambda t) - \operatorname{arccot}(\lambda x)]^\mu} = f(x), \quad 0 < \mu < 1.$$

Solution:

$$y(x) = \frac{\lambda \sin(\pi \mu)}{\pi} \frac{d}{dx} \int_a^x \frac{\varphi(t) f(t) dt}{[\operatorname{arccot}(\lambda t) - \operatorname{arccot}(\lambda x)]^{1-\mu}}, \quad \varphi(x) = \frac{1}{1 + \lambda^2 x^2}.$$

$$43. \int_a^x [A \operatorname{arccot}^\beta(\lambda x) + B \operatorname{arccot}^\gamma(\mu t) + C] y(t) dt = f(x).$$

This is a special case of equation 1.9.6 with $g(x) = A \operatorname{arccot}^\beta(\lambda x)$ and $h(t) = B \operatorname{arccot}^\gamma(\mu t) + C$.

1.7. Equations Whose Kernels Contain Combinations of Elementary Functions

1.7-1. Kernels Containing Exponential and Hyperbolic Functions.

$$1. \quad \int_a^x e^{\mu(x-t)} \{A_1 \cosh[\lambda_1(x-t)] + A_2 \cosh[\lambda_2(x-t)]\} y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.3.8:

$$\int_a^x \{A_1 \cosh[\lambda_1(x-t)] + A_2 \cosh[\lambda_2(x-t)]\} w(t) dt = e^{-\mu x} f(x).$$

$$2. \quad \int_a^x e^{\mu(x-t)} \cosh^2[\lambda(x-t)] y(t) dt = f(x).$$

Solution:

$$y(x) = \varphi(x) - \frac{2\lambda^2}{k} \int_a^x e^{\mu(x-t)} \sinh[k(x-t)] \varphi(x) dt, \quad k = \lambda\sqrt{2}, \quad \varphi(x) = f'_x(x) - \mu f(x).$$

$$3. \quad \int_a^x e^{\mu(x-t)} \cosh^3[\lambda(x-t)] y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.3.15:

$$\int_a^x \cosh^3[\lambda(x-t)] w(t) dt = e^{-\mu x} f(x).$$

$$4. \quad \int_a^x e^{\mu(x-t)} \cosh^4[\lambda(x-t)] y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.3.19:

$$\int_a^x \cosh^4[\lambda(x-t)] w(t) dt = e^{-\mu x} f(x).$$

$$5. \quad \int_a^x e^{\mu(x-t)} [\cosh(\lambda x) - \cosh(\lambda t)]^n y(t) dt = f(x), \quad n = 1, 2, \dots$$

Solution:

$$y(x) = \frac{1}{\lambda^n n!} e^{\mu x} \sinh(\lambda x) \left[\frac{1}{\sinh(\lambda x)} \frac{d}{dx} \right]^{n+1} F_\mu(x), \quad F_\mu(x) = e^{-\mu x} f(x).$$

$$6. \quad \int_a^x e^{\mu(x-t)} \sqrt{\cosh x - \cosh t} y(t) dt = f(x), \quad f(a) = 0.$$

Solution:

$$y(x) = \frac{2}{\pi} e^{\mu x} \sinh x \left(\frac{1}{\sinh x} \frac{d}{dx} \right)^2 \int_a^x \frac{e^{-\mu t} \sinh t f(t) dt}{\sqrt{\cosh x - \cosh t}}.$$

$$7. \quad \int_a^x \frac{e^{\mu(x-t)} y(t) dt}{\sqrt{\cosh x - \cosh t}} = f(x).$$

Solution:

$$y(x) = \frac{1}{\pi} e^{\mu x} \frac{d}{dx} \int_a^x \frac{e^{-\mu t} \sinh t f(t) dt}{\sqrt{\cosh x - \cosh t}}.$$

$$8. \quad \int_a^x e^{\mu(x-t)} (\cosh x - \cosh t)^\lambda y(t) dt = f(x), \quad 0 < \lambda < 1.$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.3.23:

$$\int_a^x (\cosh x - \cosh t)^\lambda w(t) dt = e^{-\mu x} f(x).$$

$$9. \quad \int_a^x [Ae^{\mu(x-t)} + B \cosh^\lambda x] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B \cosh^\lambda x$, and $h_2(t) = 1$.

$$10. \quad \int_a^x [Ae^{\mu(x-t)} + B \cosh^\lambda t] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B$, and $h_2(t) = \cosh^\lambda t$.

$$11. \quad \int_a^x e^{\mu(x-t)} (\cosh^\lambda x - \cosh^\lambda t) y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.3.24:

$$\int_a^x (\cosh^\lambda x - \cosh^\lambda t) w(t) dt = e^{-\mu x} f(x).$$

$$12. \quad \int_a^x e^{\mu(x-t)} (A \cosh^\lambda x + B \cosh^\lambda t) y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.3.25:

$$\int_a^x (A \cosh^\lambda x + B \cosh^\lambda t) w(t) dt = e^{-\mu x} f(x).$$

$$13. \quad \int_a^x \frac{e^{\mu(x-t)} y(t) dt}{(\cosh x - \cosh t)^\lambda} = f(x), \quad 0 < \lambda < 1.$$

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi} e^{\mu x} \frac{d}{dx} \int_a^x \frac{e^{-\mu t} \sinh t f(t) dt}{(\cosh x - \cosh t)^{1-\lambda}}.$$

$$14. \quad \int_a^x e^{\mu(x-t)} \{A_1 \sinh[\lambda_1(x-t)] + A_2 \sinh[\lambda_2(x-t)]\} y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.3.49:

$$\int_a^x \{A_1 \sinh[\lambda_1(x-t)] + A_2 \sinh[\lambda_2(x-t)]\} w(t) dt = e^{-\mu x} f(x).$$

$$15. \quad \int_a^x e^{\mu(x-t)} \sinh^2[\lambda(x-t)] y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.3.51:

$$\int_a^x \sinh^2[\lambda(x-t)] w(t) dt = e^{-\mu x} f(x).$$

$$16. \int_a^x e^{\mu(x-t)} \sinh^3[\lambda(x-t)]y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x}y(x)$ leads to an equation of the form 1.3.57:

$$\int_a^x \sinh^3[\lambda(x-t)]w(t) dt = e^{-\mu x}f(x).$$

$$17. \int_a^x e^{\mu(x-t)} \sinh^n[\lambda(x-t)]y(t) dt = f(x), \quad n = 2, 3, \dots$$

The substitution $w(x) = e^{-\mu x}y(x)$ leads to an equation of the form 1.3.62:

$$\int_a^x \sinh^n[\lambda(x-t)]w(t) dt = e^{-\mu x}f(x).$$

$$18. \int_a^x e^{\mu(x-t)} \sinh(k\sqrt{x-t})y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{2}{\pi k} e^{\mu x} \frac{d^2}{dx^2} \int_a^x \frac{e^{-\mu t} \cos(k\sqrt{x-t})}{\sqrt{x-t}} f(t) dt.$$

$$19. \int_a^x e^{\mu(x-t)} \sqrt{\sinh x - \sinh t} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{2}{\pi} e^{\mu x} \cosh x \left(\frac{1}{\cosh x} \frac{d}{dx} \right)^2 \int_a^x \frac{e^{-\mu t} \cosh t f(t) dt}{\sqrt{\sinh x - \sinh t}}.$$

$$20. \int_a^x \frac{e^{\mu(x-t)} y(t) dt}{\sqrt{\sinh x - \sinh t}} = f(x).$$

Solution:

$$y(x) = \frac{1}{\pi} e^{\mu x} \frac{d}{dx} \int_a^x \frac{e^{-\mu t} \cosh t f(t) dt}{\sqrt{\sinh x - \sinh t}}.$$

$$21. \int_a^x e^{\mu(x-t)} (\sinh x - \sinh t)^\lambda y(t) dt = f(x), \quad 0 < \lambda < 1.$$

The substitution $w(x) = e^{-\mu x}y(x)$ leads to an equation of the form 1.3.67:

$$\int_a^x (\sinh x - \sinh t)^\lambda w(t) dt = e^{-\mu x}f(x).$$

$$22. \int_a^x e^{\mu(x-t)} (\sinh^\lambda x - \sinh^\lambda t) y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x}y(x)$ leads to an equation of the form 1.3.68:

$$\int_a^x (\sinh^\lambda x - \sinh^\lambda t) w(t) dt = e^{-\mu x}f(x).$$

$$23. \int_a^x e^{\mu(x-t)} (A \sinh^\lambda x + B \sinh^\lambda t) y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x}y(x)$ leads to an equation of the form 1.3.69:

$$\int_a^x (A \sinh^\lambda x + B \sinh^\lambda t) w(t) dt = e^{-\mu x}f(x).$$

$$24. \int_a^x [Ae^{\mu(x-t)} + B \sinh^\lambda x] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B \sinh^\lambda x$, and $h_2(t) = 1$.

$$25. \int_a^x [Ae^{\mu(x-t)} + B \sinh^\lambda t] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B$, and $h_2(t) = \sinh^\lambda t$.

$$26. \int_a^x \frac{e^{\mu(x-t)} y(t) dt}{(\sinh x - \sinh t)^\lambda} = f(x), \quad 0 < \lambda < 1.$$

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi} e^{\mu x} \frac{d}{dx} \int_a^x \frac{e^{-\mu t} \cosh t f(t) dt}{(\sinh x - \sinh t)^{1-\lambda}}.$$

$$27. \int_a^x e^{\mu(x-t)} (A \tanh^\lambda x + B \tanh^\lambda t) y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.3.89:

$$\int_a^x (A \tanh^\lambda x + B \tanh^\lambda t) w(t) dt = e^{-\mu x} f(x).$$

$$28. \int_a^x e^{\mu(x-t)} (A \tanh^\lambda x + B \tanh^\beta t + C) y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.9.6 with $g(x) = A \tanh^\lambda x$, $h(t) = B \tanh^\beta t + C$:

$$\int_a^x (A \tanh^\lambda x + B \tanh^\beta t + C) w(t) dt = e^{-\mu x} f(x).$$

$$29. \int_a^x [Ae^{\mu(x-t)} + B \tanh^\lambda x] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B \tanh^\lambda x$, and $h_2(t) = 1$.

$$30. \int_a^x [Ae^{\mu(x-t)} + B \tanh^\lambda t] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B$, and $h_2(t) = \tanh^\lambda t$.

$$31. \int_a^x e^{\mu(x-t)} (A \coth^\lambda x + B \coth^\lambda t) y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.3.102:

$$\int_a^x (A \coth^\lambda x + B \coth^\lambda t) w(t) dt = e^{-\mu x} f(x).$$

$$32. \int_a^x e^{\mu(x-t)} (A \coth^\lambda x + B \coth^\beta t + C) y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.9.6 with $g(x) = A \coth^\lambda x$, $h(t) = B \coth^\beta t + C$:

$$\int_a^x (A \coth^\lambda x + B \coth^\beta t + C) w(t) dt = e^{-\mu x} f(x).$$

$$33. \int_a^x [A e^{\mu(x-t)} + B \coth^\lambda x] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = A e^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B \coth^\lambda x$, and $h_2(t) = 1$.

$$34. \int_a^x [A e^{\mu(x-t)} + B \coth^\lambda t] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = A e^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B$, and $h_2(t) = \coth^\lambda t$.

1.7-2. Kernels Containing Exponential and Logarithmic Functions.

$$35. \int_a^x e^{\lambda(x-t)} (\ln x - \ln t) y(t) dt = f(x).$$

Solution:

$$y(x) = e^{\lambda x} [x \varphi''_{xx}(x) + \varphi'_x(x)], \quad \varphi(x) = e^{-\lambda x} f(x).$$

$$36. \int_0^x e^{\lambda(x-t)} \ln(x-t) y(t) dt = f(x).$$

The substitution $w(x) = e^{-\lambda x} y(x)$ leads to an equation of the form 1.4.2:

$$\int_0^x \ln(x-t) w(t) dt = e^{-\lambda x} f(x).$$

$$37. \int_a^x e^{\lambda(x-t)} (A \ln x + B \ln t) y(t) dt = f(x).$$

The substitution $w(x) = e^{-\lambda x} y(x)$ leads to an equation of the form 1.4.4:

$$\int_a^x (A \ln x + B \ln t) w(t) dt = e^{-\lambda x} f(x).$$

$$38. \int_a^x e^{\mu(x-t)} [A \ln^2(\lambda x) + B \ln^2(\lambda t)] y(t) dt = f(x).$$

The substitution $w(x) = e^{-\lambda x} y(x)$ leads to an equation of the form 1.4.7:

$$\int_a^x [A \ln^2(\lambda x) + B \ln^2(\lambda t)] w(t) dt = e^{-\lambda x} f(x).$$

$$39. \int_a^x e^{\lambda(x-t)} [\ln(x/t)]^n y(t) dt = f(x), \quad n = 1, 2, \dots$$

Solution:

$$y(x) = \frac{1}{n!x} e^{\lambda x} \left(x \frac{d}{dx} \right)^{n+1} F_\lambda(x), \quad F_\lambda(x) = e^{-\lambda x} f(x).$$

$$40. \int_a^x e^{\lambda(x-t)} \sqrt{\ln(x/t)} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{2e^{\lambda x}}{\pi x} \left(x \frac{d}{dx} \right)^2 \int_a^x \frac{e^{-\lambda t} f(t) dt}{t \sqrt{\ln(x/t)}}.$$

$$41. \int_a^x \frac{e^{\lambda(x-t)}}{\sqrt{\ln(x/t)}} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{1}{\pi} e^{\lambda x} \frac{d}{dx} \int_a^x \frac{e^{-\lambda t} f(t) dt}{t \sqrt{\ln(x/t)}}.$$

$$42. \int_a^x [Ae^{\mu(x-t)} + B \ln^\nu(\lambda x)] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B \ln^\nu(\lambda x)$, and $h_2(t) = 1$.

$$43. \int_a^x [Ae^{\mu(x-t)} + B \ln^\nu(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = Ae^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B$, and $h_2(t) = \ln^\nu(\lambda t)$.

$$44. \int_a^x e^{\mu(x-t)} [\ln(x/t)]^\lambda y(t) dt = f(x), \quad 0 < \lambda < 1.$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.4.16:

$$\int_a^x [\ln(x/t)]^\lambda w(t) dt = e^{-\mu x} f(x).$$

$$45. \int_a^x \frac{e^{\mu(x-t)}}{[\ln(x/t)]^\lambda} y(t) dt = f(x), \quad 0 < \lambda < 1.$$

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi} e^{\mu x} \frac{d}{dx} \int_a^x \frac{f(t) dt}{t e^{\mu t} [\ln(x/t)]^{1-\lambda}}.$$

1.7-3. Kernels Containing Exponential and Trigonometric Functions.

$$46. \int_a^x e^{\mu(x-t)} \cos[\lambda(x-t)] y(t) dt = f(x).$$

Solution: $y(x) = f'_x(x) - \mu f(x) + \lambda^2 \int_a^x e^{\mu(x-t)} f(t) dt.$

$$47. \int_a^x e^{\mu(x-t)} \{A_1 \cos[\lambda_1(x-t)] + A_2 \cos[\lambda_2(x-t)]\} y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.5.8:

$$\int_a^x \{A_1 \cos[\lambda_1(x-t)] + A_2 \cos[\lambda_2(x-t)]\} w(t) dt = e^{-\mu x} f(x).$$

$$48. \int_a^x e^{\mu(x-t)} \cos^2[\lambda(x-t)] y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.5.9.

Solution:

$$y(x) = \varphi(x) + \frac{2\lambda^2}{k} \int_a^x e^{\mu(x-t)} \sin[k(x-t)] \varphi(t) dt, \quad k = \lambda\sqrt{2}, \quad \varphi(x) = f'_x(x) - \mu f(x).$$

$$49. \int_a^x e^{\mu(x-t)} \cos^3[\lambda(x-t)] y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.5.16:

$$\int_a^x \cos^3[\lambda(x-t)] w(t) dt = e^{-\mu x} f(x).$$

$$50. \int_a^x e^{\mu(x-t)} \cos^4[\lambda(x-t)] y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.5.20:

$$\int_a^x \cos^4[\lambda(x-t)] w(t) dt = e^{-\mu x} f(x).$$

$$51. \int_a^x e^{\mu(x-t)} [\cos(\lambda x) - \cos(\lambda t)]^n y(t) dt = f(x), \quad n = 1, 2, \dots$$

The right-hand side of the equation is assumed to satisfy the conditions $f(a) = f'_x(a) = \dots = f_x^{(n)}(a) = 0$.

Solution:

$$y(x) = \frac{(-1)^n}{\lambda^n n!} e^{\mu x} \sin(\lambda x) \left[\frac{1}{\sin(\lambda x)} \frac{d}{dx} \right]^{n+1} F_\mu(x), \quad F_\mu(x) = e^{-\mu x} f(x).$$

$$52. \int_a^x e^{\mu(x-t)} \sqrt{\cos t - \cos x} y(t) dt = f(x).$$

Solution:

$$y(x) = \frac{2}{\pi} e^{\mu x} \sin x \left(\frac{1}{\sin x} \frac{d}{dx} \right)^2 \int_a^x \frac{e^{-\mu t} \sin t f(t) dt}{\sqrt{\cos t - \cos x}}.$$

$$53. \int_a^x \frac{e^{\mu(x-t)} y(t) dt}{\sqrt{\cos t - \cos x}} = f(x).$$

Solution:

$$y(x) = \frac{1}{\pi} e^{\mu x} \frac{d}{dx} \int_a^x \frac{e^{-\mu t} \sin t f(t) dt}{\sqrt{\cos t - \cos x}}.$$

$$54. \int_a^x e^{\mu(x-t)} (\cos t - \cos x)^\lambda y(t) dt = f(x), \quad 0 < \lambda < 1.$$

Solution:

$$y(x) = k e^{\mu x} \sin x \left(\frac{1}{\sin x} \frac{d}{dx} \right)^2 \int_a^x \frac{e^{-\mu t} \sin t f(t) dt}{(\cos t - \cos x)^\lambda}, \quad k = \frac{\sin(\pi \lambda)}{\pi \lambda}.$$

$$55. \int_a^x e^{\mu(x-t)} (\cos^\lambda x - \cos^\lambda t) y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.5.25:

$$\int_a^x (\cos^\lambda x - \cos^\lambda t) w(t) dt = e^{-\mu x} f(x).$$

$$56. \int_a^x e^{\mu(x-t)} (A \cos^\lambda x + B \cos^\lambda t) y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.5.26:

$$\int_a^x (A \cos^\lambda x + B \cos^\lambda t) w(t) dt = e^{-\mu x} f(x).$$

$$57. \int_a^x \frac{e^{\mu(x-t)} y(t) dt}{(\cos t - \cos x)^\lambda} = f(x), \quad 0 < \lambda < 1.$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.5.27:

$$\int_a^x \frac{w(t) dt}{(\cos t - \cos x)^\lambda} = e^{-\mu x} f(x).$$

$$58. \int_a^x [A e^{\mu(x-t)} + B \cos^\nu(\lambda x)] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = A e^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B \cos^\nu(\lambda x)$, and $h_2(t) = 1$.

$$59. \int_a^x [A e^{\mu(x-t)} + B \cos^\nu(\lambda t)] y(t) dt = f(x).$$

This is a special case of equation 1.9.15 with $g_1(x) = A e^{\mu x}$, $h_1(t) = e^{-\mu t}$, $g_2(x) = B$, and $h_2(t) = \cos^\nu(\lambda t)$.

$$60. \int_a^x e^{\mu(x-t)} \sin[\lambda(x-t)] y(t) dt = f(x), \quad f(a) = f'_x(a) = 0.$$

Solution: $y(x) = \frac{1}{\lambda} [f''_{xx}(x) - 2\mu f'_x(x) + (\lambda^2 + \mu^2) f(x)]$.

$$61. \int_a^x e^{\mu(x-t)} \{A_1 \sin[\lambda_1(x-t)] + A_2 \sin[\lambda_2(x-t)]\} y(t) dt = f(x).$$

The substitution $w(x) = e^{-\mu x} y(x)$ leads to an equation of the form 1.5.48:

$$\int_a^x \{A_1 \sin[\lambda_1(x-t)] + A_2 \sin[\lambda_2(x-t)]\} w(t) dt = e^{-\mu x} f(x).$$