Structural Mechanics Fundamentals

Alberto Carpinteri

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To my family

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Preface

This book intends to provide a complete and uniform treatment of the most fundamental and traditional topics in structural mechanics. It represents the second edition of a substantial part (12 chapters over 20) of my previous book *Structural Mechanics: A Unified Approach*, published in 1997 by E & FN SPON, an imprint of Chapman & Hall.

After introducing the basic topics of the geometry of areas and of kinematics and statics of rigid body systems, the mechanics of linear elastic solids (beams, plates and 3-D solids) is presented, adopting a matrix formulation which is particularly useful for numerical applications. The analysis of strain and stress around a material point is carried out considering the tensorial character of these physical quantities. The linear elastic constitutive law is then introduced, with the related Clapeyron's and Betti's theorems. The kinematic, static and constitutive equations, once composed within the elastic problem, provide an operator equation which has as its unknown the generalized displacement vector. Moreover, constant reference is made to duality, that is to the strict correspondence between statics and kinematics that emerges as soon as the corresponding operators are rendered explicit, and it is at once seen how each of these is the adjoint of the other. The implication of the principle of virtual work by the static–kinematic duality is emphasized, as well as the inverse implication. Once introduced the Saint Venant problem with all the six elementary loading characteristics, the theory of beam systems (statically determinate or indeterminate) is presented, with the solution of numerous examples and the plotting of the corresponding diagrams of axial force, shearing force and bending moment obtained both analytically and graphically. For the examination of indeterminate beam systems, both the methods of forces and energy are applied.

This book is the fruit of many years of teaching in Italian universities, formerly at the University of Bologna and currently at the Politecnico di Torino, where I have been professor of structural mechanics since 1986. It has been written to be used as a text for graduate or undergraduate students of either architecture or engineering, as well as to serve as a useful reference for research workers and practising engineers. It has been my endeavour to update and modernize a basic, and in some respects dated, discipline by merging classical topics with ones that have taken shape in more recent times.

Finally, I wish to express my most sincere gratitude to all those colleagues, collaborators and students, who, having attended my lectures or having read the original manuscript, have, with their suggestions and comments, contributed to the text as it appears in its definitive form. I further wish to thank my master's student, Francesco Armenti, for helping me with the proof corrections and Dr. Amedeo Manuello for his precious advice in realizing the front cover.

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Torino, Italy

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Author

Alberto Carpinteri received his doctoral degrees in nuclear engineering cum laude (1976) and mathematics cum laude (1981) from the University of Bologna (Italy). After two years at the Consiglio Nazionale delle Ricerche, he was appointed assistant professor at the University of Bologna in 1980.

Carpinteri moved to the Politecnico di Torino in 1986 as professor and became the chair of solid and structural mechanics and the director of the Fracture Mechanics Laboratory. During this period, he held different positions of responsibility, including head of the Department of Structural Engineering (1989–1995) and founding member and director of the Post-graduate School of Structural Engineering (1990–).

Prof. Carpinteri was a visiting scientist at Lehigh University, Bethlehem, Pennsylvania (1982–1983), and was appointed a fellow of several academies and professional institutions, including the European Academy of Sciences (2009–), the International Academy of Engineering (2010–), the Turin Academy of Sciences (2005–) and the American Society of Civil Engineers (1996–).

Prof. Carpinteri was the president of various scientific associations and research institutions, as follows: the International Congress on Fracture, ICF (2009–2013); the European Structural Integrity Society, ESIS (2002–2006); the International Association of Fracture Mechanics for Concrete and Concrete Structures, IA-FraMCoS (2004–2007); the Italian Group of Fracture, IGF (1998–2005); and the National Research Institute of Metrology, INRIM (2011–2013). He was appointed a member of the Congress Committee of the International Union of Theoretical and Applied Mechanics, IUTAM (2004–2012); a member of the executive board of the Society for Experimental Mechanics, SEM (2012–2014); a member of the editorial board of 13 international journals; and the editor in chief of the journal *Meccanica* (Springer, IF = 1.568). He is also the author or editor of over 750 publications, of which more than 300 are papers in refereed international journals (ISI h-Index = 31, more than 3800 citations) and 43 are books.

Prof. Carpinteri has received numerous honours and awards, as follows: the Robert L'Hermite Medal from RILEM (1982), the Griffith Medal from ESIS (2008), the Swedlow Memorial Lecture Award from ASTM (2011) and the Inaugural Paul Paris Gold Medal from ICF (2013), among others.

Chapter 1

Introduction

1.1 PRELIMINARY REMARKS

Structural mechanics is the science that studies the **structural response** of solid bodies subjected to external loading. The structural response takes the form of **strains and internal stresses**.

The variation of shape generally involves relative and absolute displacements of the points of the body. The simplest case that can be envisaged is that of a string, one end of which is held firm while a tensile load is applied to the other end. The percentage lengthening or stretching of the string naturally implies a displacement, albeit small, of the end where the force is exerted. Likewise, a membrane, stretched by a system of balanced forces, will dilate in two dimensions, and its points will undergo relative and absolute displacements. Also three-dimensional bodies, when subjected to stress by a system of balanced forces, undergo, point by point and direction by direction, a dilation or a contraction, as well as an angular distortion. Similarly, beams and horizontal plates bend, imposing a certain curvature, respectively, to their axes and to their middle planes, and differentiated deflections to their points.

As regards internal stresses, these can be considered as exchanged between the single (even infinitesimal) parts which make up the body. In the case of the string, the tension is transmitted continuously from the end on which the force is applied right up to the point of constraint. Each elementary segment is thus subject to two equal and opposite forces exerted by the contiguous segments. Likewise, each elementary part of a membrane will be subject to four mutually perpendicular forces, two equal and opposite pairs. In three-dimensional bodies, each elementary part is subject to normal and tangential forces. The former generate dilations and contractions, whilst the latter produce angular distortions. Finally, each element of beam or plate that is bent is subject to self-balanced pairs of moments.

In addition to the shape and properties of the body, it is the external loading applied and the constraints imposed that determine the structural response. The constraints react to the external loads, exerting on the body additional loads called **constraint reactions.** These reactions are *a priori* unknown. In the case where the constraints are not redundant from the kinematic point of view, the calculation of the constraint reactions can be made considering the body as being perfectly rigid and applying only the cardinal equations of statics. In the alternative case where the constraints are redundant, the calculation of the constraint reactions requires, in addition to **equations of equilibrium**, the so-called **equations of congruence.** These equations are obtained by eliminating the redundant constraints, replacing them with the constraint reactions exerted by them and imposing the abeyance of the constraints that have been eliminated. The procedure presupposes that the strains and displacements, produced both by the external loading and by the reactions of the constraints that have been eliminated, are known. A simple example may suffice to illustrate these concepts.

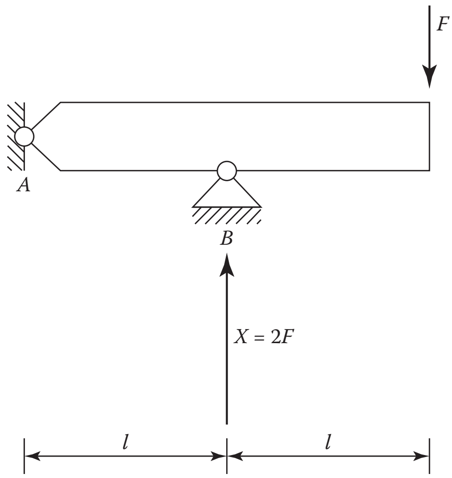


Figure 1.1

Let us consider a bar hinged at point A and supported at point B, subjected to the end force F (Figure 1.1). The reaction X produced by the support B is obtained by imposing equilibrium with regard to rotation about hinge A:

$$F(2l) = Xl \Rightarrow X = 2F \tag{1.1}$$

The equation of equilibrium with regard to vertical translation provides, on the other hand, the reaction of hinge A. The problem is thus **statically determinate** or **isostatic**.

Let us now consider the same bar hinged, not only at A but also at two points B_1 and B_2, distant $\frac{2}{3}l$ and $\frac{4}{3}l$, respectively, from point A (Figure 1.2a). The condition of equilibrium with regard to rotation gives us an equation with two unknowns:

$$F(2l) + X_1 \frac{2}{3} l = X_2 \frac{4}{3} l \tag{1.2}$$

Thus, the pairs of reactions X_1 and X_2 which ensure rotational equilibrium are infinite, but only one of these also ensures congruence, i.e. abeyance of the conditions of constraint. The vertical displacement in both B_1 and B_2 must in fact be zero.

To determine the constraint reactions, we thus proceed to eliminate one of the two hinges B_1 or B_2, for example, B_1, and we find out how much point B_1 rises owing to the external force F (Figure 1.2b) and how much it drops owing to the unknown reaction X_1 (Figure 1.2c). The condition of congruence consists of putting the total displacement of B_1 equal to zero:

$$\upsilon(F) = \upsilon(X_1) \tag{1.3}$$

The equation of equilibrium (1.2) and the equation of congruence (1.3) together solve the problem, which is said to be **statically indeterminate** or **hyperstatic**.

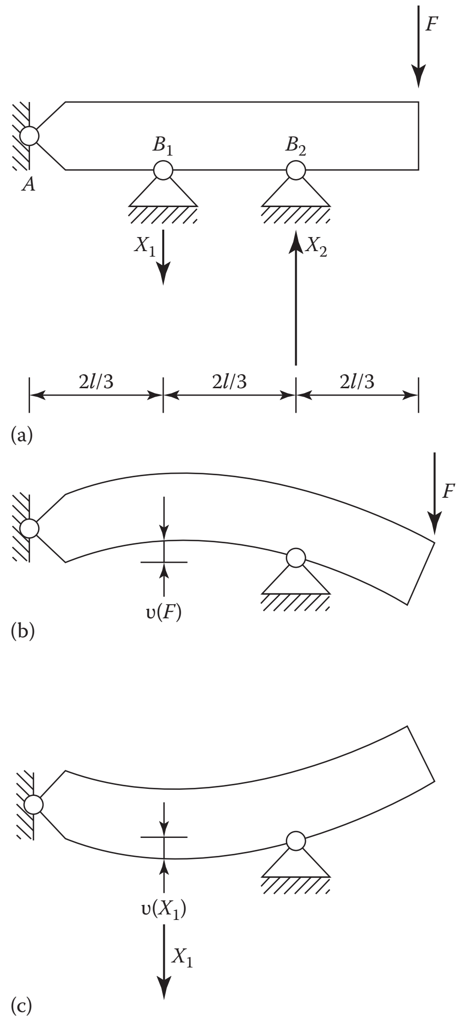


Figure 1.2

1.2 CLASSIFICATION OF STRUCTURAL ELEMENTS

As has already been mentioned in the preliminary remarks, the structural elements which combine to make up the load-bearing structures of civil and industrial constructions, as well as any naturally occurring structure such as rock masses, plants or skeletons, can fit into one of three distinct categories:

1. One-dimensional elements (e.g. ropes, struts, beams, arches)
2. Two-dimensional elements (e.g. membranes, plates, slabs, vaults, shells)
3. Three-dimensional elements (stubby solids)

In the case of one-dimensional elements, for example, beams (Figure 1.3), one of the three dimensions, the length, is much larger than the other two which compose the cross section. Hence, it is possible to neglect the latter two dimensions and consider the entire element as concentrated along the line forming its centroidal axis. In our calculations, features which

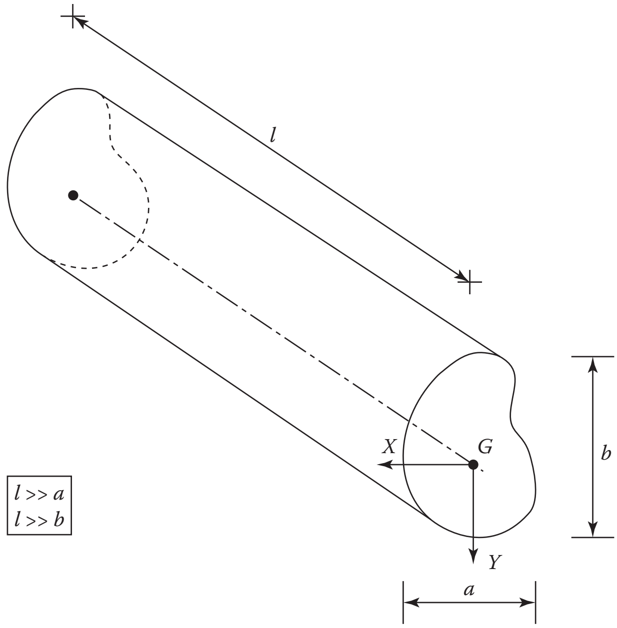


Figure 1.3

represent the geometry of the cross section and, consequently, the three dimensionality of the element will thus be used. Ropes are elements devoid of flexural and compressive stiffness and are able only to bear states of tensile stress. Bars, however, present a high axial stiffness, both in compression (struts) and in tension (tie rods), whilst their flexural stiffness is poor. Beams and, more generally, arches (or curvilinear beams) also present a high degree of flexural stiffness, provided that materials having particularly high tensile strength are used. In the case of stone materials and concrete, which present very low tensile strength, straight beams are reinforced to stand up to bending stresses, whilst arches are traditionally shaped so that only internal compressive stresses are produced.

When, in the cross section of a beam, one dimension is clearly smaller than the others (Figure 1.4), the beam is said to be **thin walled.** Beams of this sort can be easily produced by rolling or welding metal plate and prove to be extremely efficient from the point of view of the ratio of flexural strength to the amount of material employed.

In the case of two-dimensional elements, for example, flat plates (Figure 1.5a) or plates with double curvature (Figure 1.5b), one of the three dimensions, the thickness, is much smaller than the other two, which compose the middle surface. It is thus possible to neglect the thickness and to consider the entire element as being concentrated in its middle surface. Membranes are elements devoid of flexural and compressive stiffness and are able to withstand only states of biaxial traction. Also plates that are of a small thickness present a low flexural stiffness and are able to bear loads only in their middle plane. Thick plates (also referred to as slabs), instead, also withstand bending stresses, provided that materials having particularly high tensile strength are used. In the case of stone materials and concrete, flat plates are, on the other hand, ribbed and reinforced, while vaults and domes are traditionally shaped so that only internal compressive stresses are produced (for instance, in arched dams).

Finally, in the case of so-called **stubby solids**, the three dimensions are all comparable to one another and hence the analysis of the state of strain and internal stress must be three dimensional, without any particular simplifications or approximations.

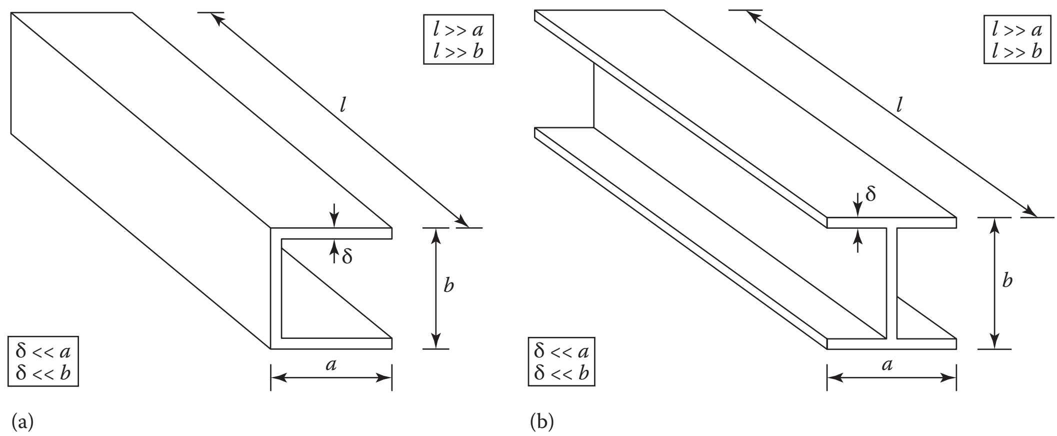


Figure 1.4

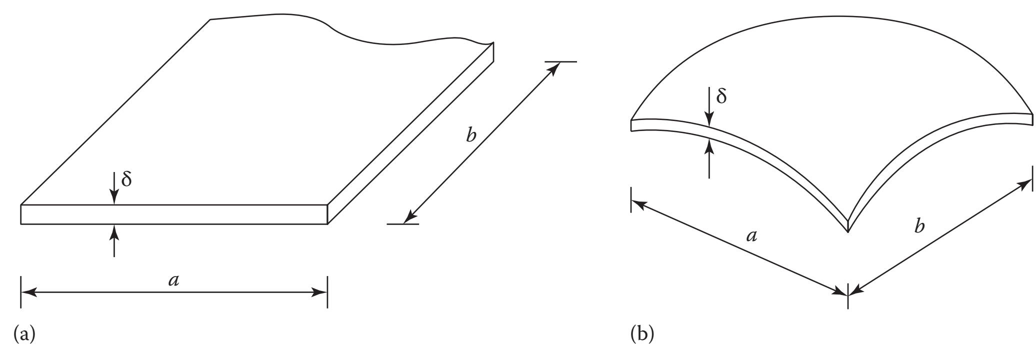


Figure 1.5

1.3 STRUCTURAL TYPES

The single structural elements, introduced in the previous section, are combined to form load-bearing structures. Usually, for buildings of a civil type, one-dimensional and two-dimensional elements are connected together. The characteristics of the individual elements and the way in which they are connected one to another and to the ground together define the structural type, which can be extremely varied, according to the purposes for which the building is designed.

In many cases, the two-dimensional elements do not have a load-bearing function (e.g. the walls of buildings in reinforced concrete), and hence it is necessary to highlight graphically and calculate only the so-called **framework**, made up exclusively of one-dimensional elements. This framework, according to the type of constraint which links together the various beams, will then be said to be **trussed** or **framed.** In the former case, the calculation is made by inserting hinges which connect the beams together, whereas in the latter case the beams are considered as built into one another. In real situations, however, beams are never connected by frictionless hinges or with perfectly rigid joints. Figures 1.6 through 1.11 show some examples of load-bearing frameworks: a timber-beam bridge, a truss in reinforced concrete, an arch centre, a plane steel frame, a grid and a three-dimensional frame.

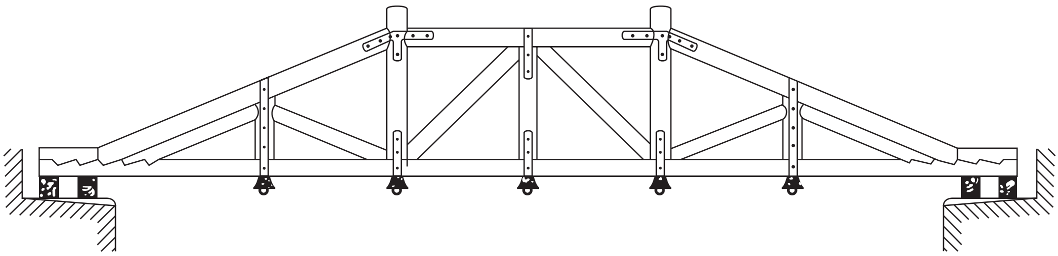


Figure 1.6

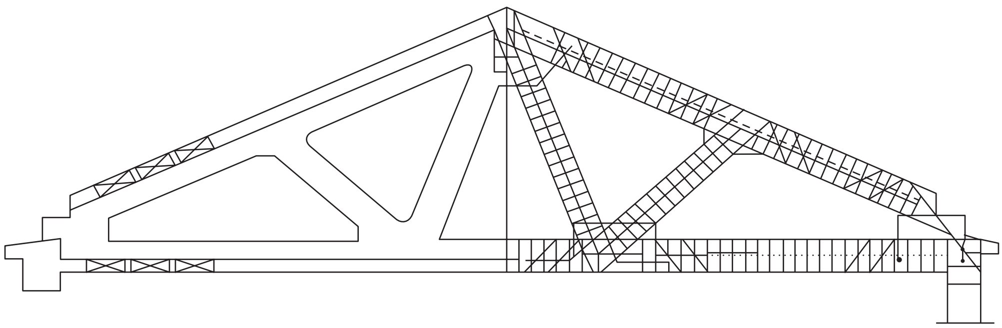


Figure 1.7

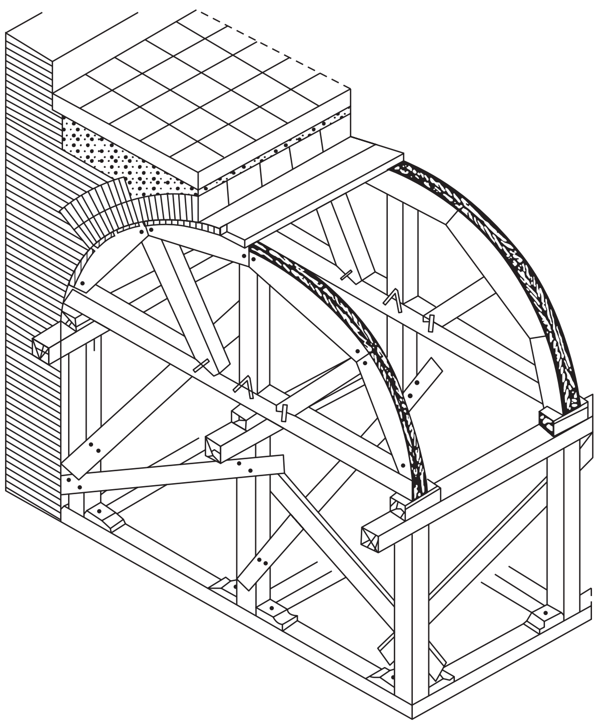
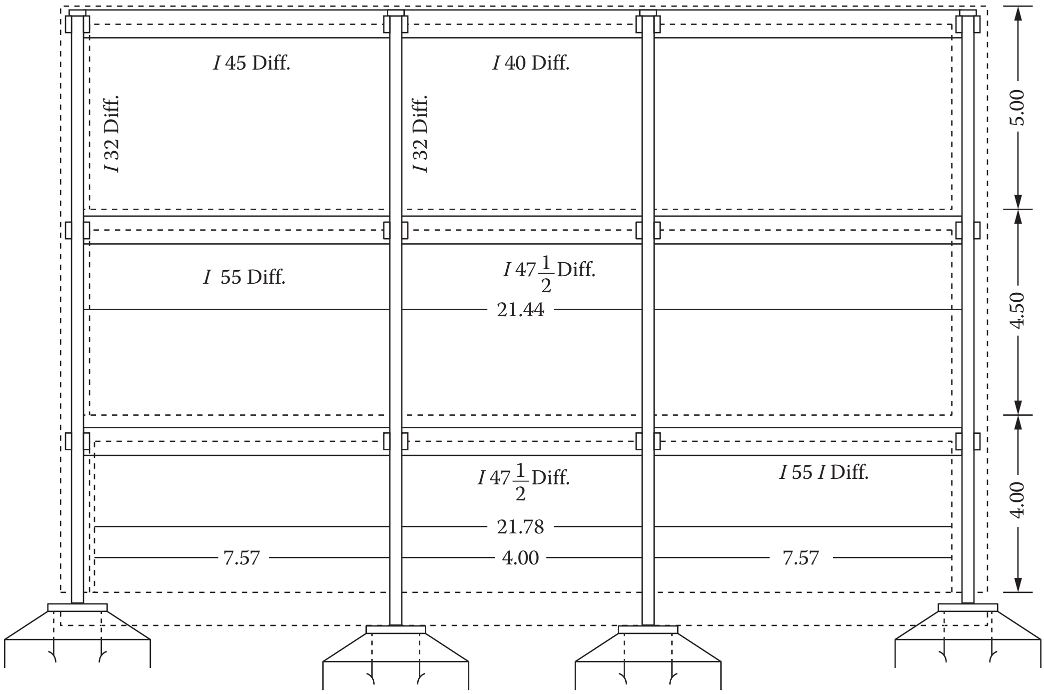


Figure 1.8



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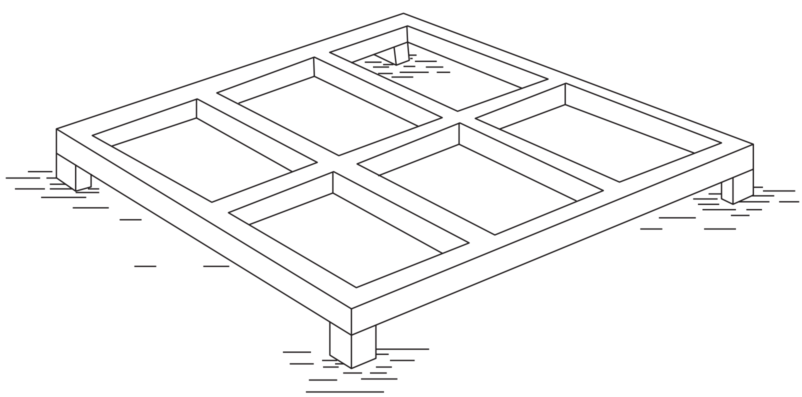
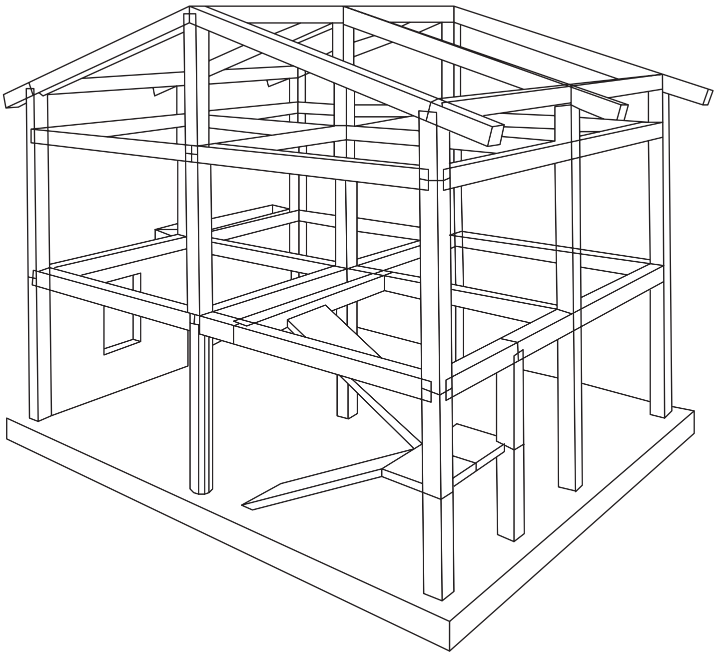


Figure 1.10

Also in the case of **bridges**, it is usually possible to identify a load-bearing structure consisting of one-dimensional elements. The road surface of an **arch bridge** is supported by a parabolic beam which is subject to compression and, if well designed, is devoid of dangerous internal flexural stresses. The road surface can be built to rest above the arch by means of struts (Figure 1.12) or can be suspended beneath the arch by means of tie rods (Figure 1.13). Inverting the static scheme and using a primary load-bearing element subject exclusively to tensile stress, we arrive at the structure of **suspension bridges** (Figure 1.14). In these, the road surface hangs from a parabolic cable by means of tie rods. The cable is, of course, able to withstand only tensile stresses, which are, however, transmitted onto two compressed piers.



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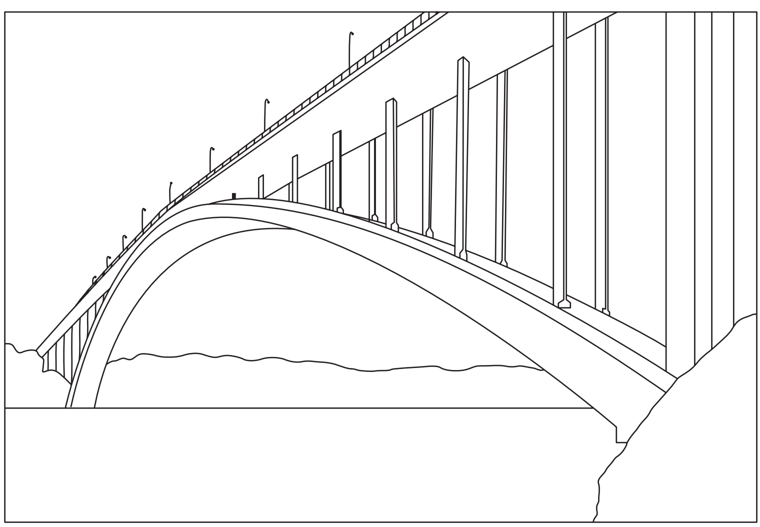
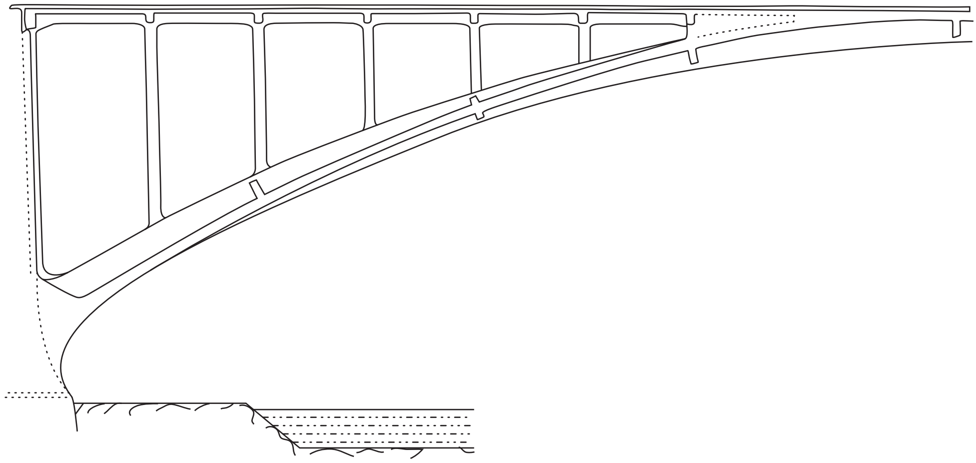


(a)

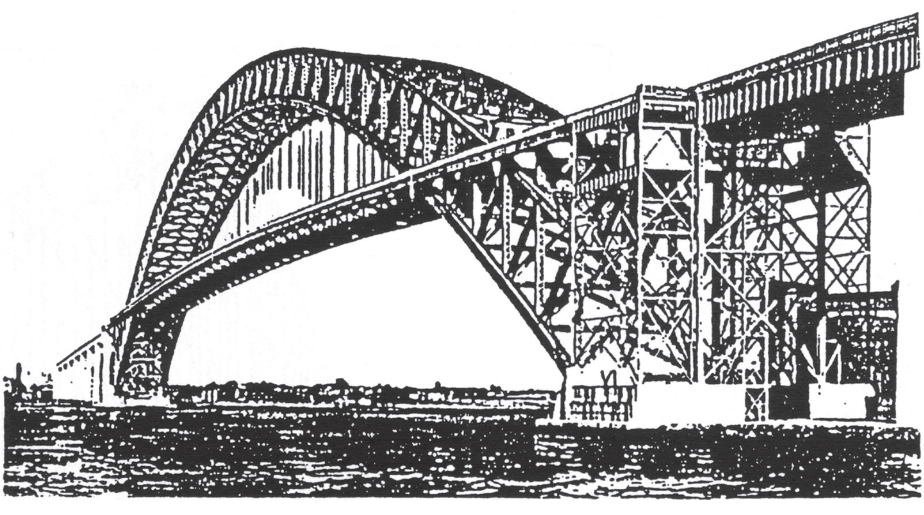
Figure 1.12



(b)

Figure 1.12 (continued)

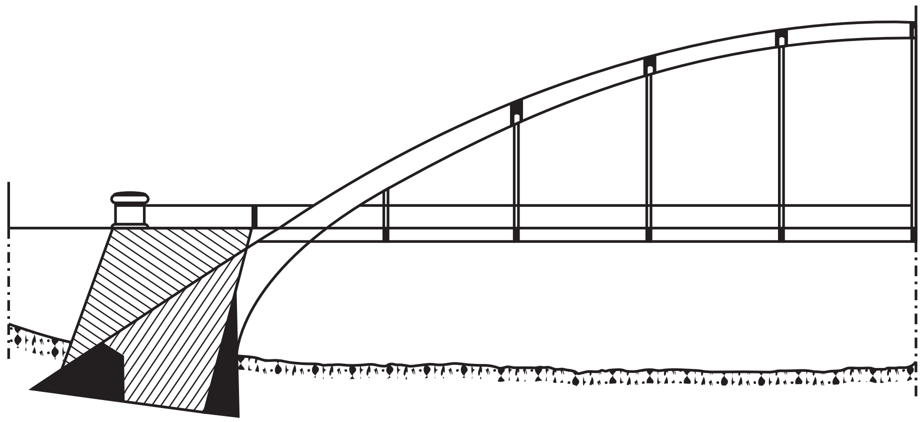
As regards two-dimensional structural elements, it is advantageous to exploit the same static principles already met with in the case of bridges. To avoid, for example, dangerous internal stresses of a flexural nature, the usual approach is to use **vaults** or **domes** having double curvature, which present parabolic sections in both of the principal directions (Figure 1.15a). A variant is provided by the so-called **cross vault** (Figure 1.15b), consisting of two mutually intersecting cylindrical vaults. Membranes, on the other hand, can assume the form of hyperbolic paraboloids, with saddle points and curvatures of opposite sign (Figure 1.15c). In the so-called **prestressed membranes**, both those cables with the concavity facing upwards and those with the concavity facing downwards are subject to tensile stress.



(a)

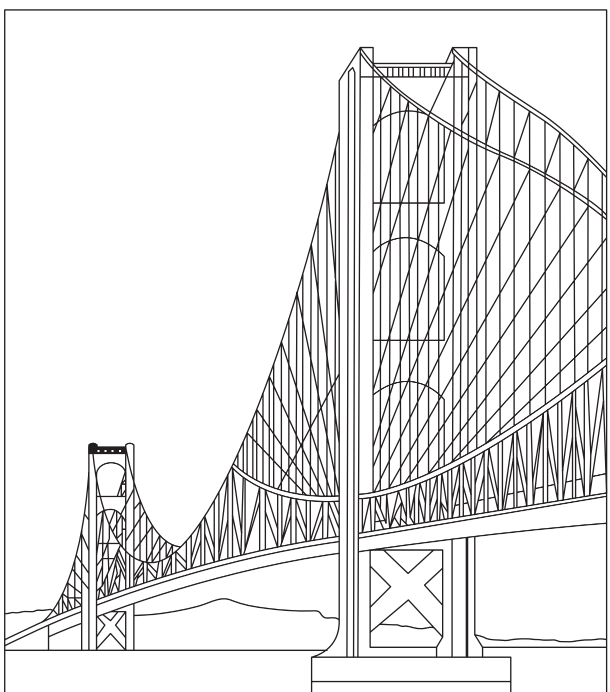
Figure 1.13

(*continued*)



(b)

Figure 1.13 (continued)



(a)

Figure 1.14

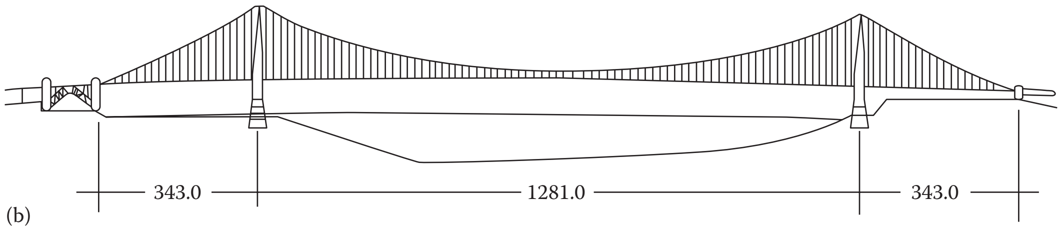


Figure 1.14 (continued)

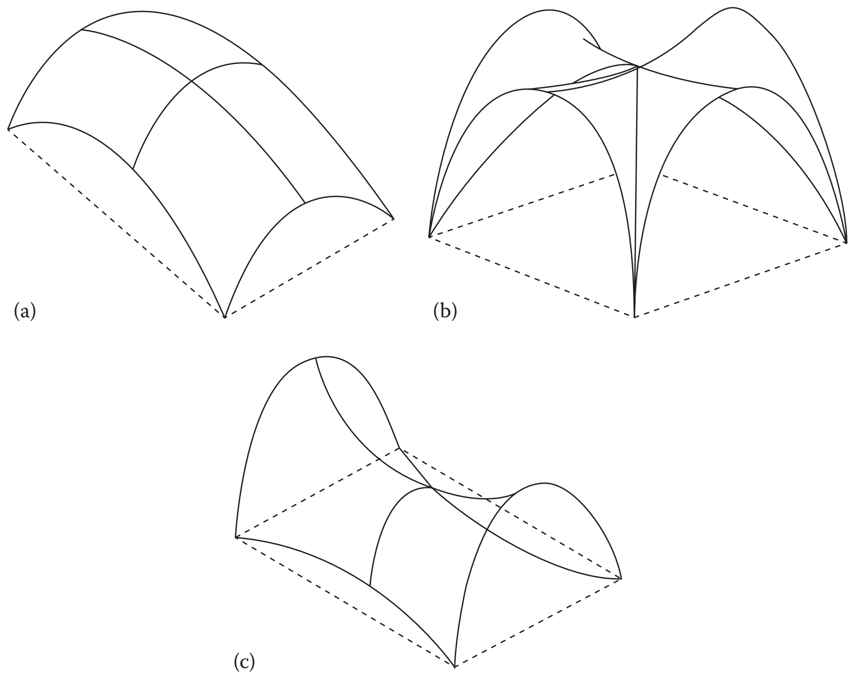


Figure 1.15

1.4 EXTERNAL LOADING AND CONSTRAINT REACTIONS

The strains and internal stresses of a structure obviously depend on the external loads applied to it. These can be of varying nature according to the structure under consideration. In the civil engineering field, the loads are usually represented by the **weight load**, both of the structural elements themselves (**permanent loads**) and of persons, vehicles or objects (**live loads**).

Figure 1.16 represents two load diagrams, used in the early decades of the last century, of horse-drawn carts and carriages. The forces are considered as concentrated and, of course, proceeding over the road surface. Figure 1.17 illustrates the load diagram of a roller, and Figure 1.18 that of a hoisting device. Figure 1.19 compares the permanent load diagrams of two beams, one with constant cross section and the other with linearly variable cross section.

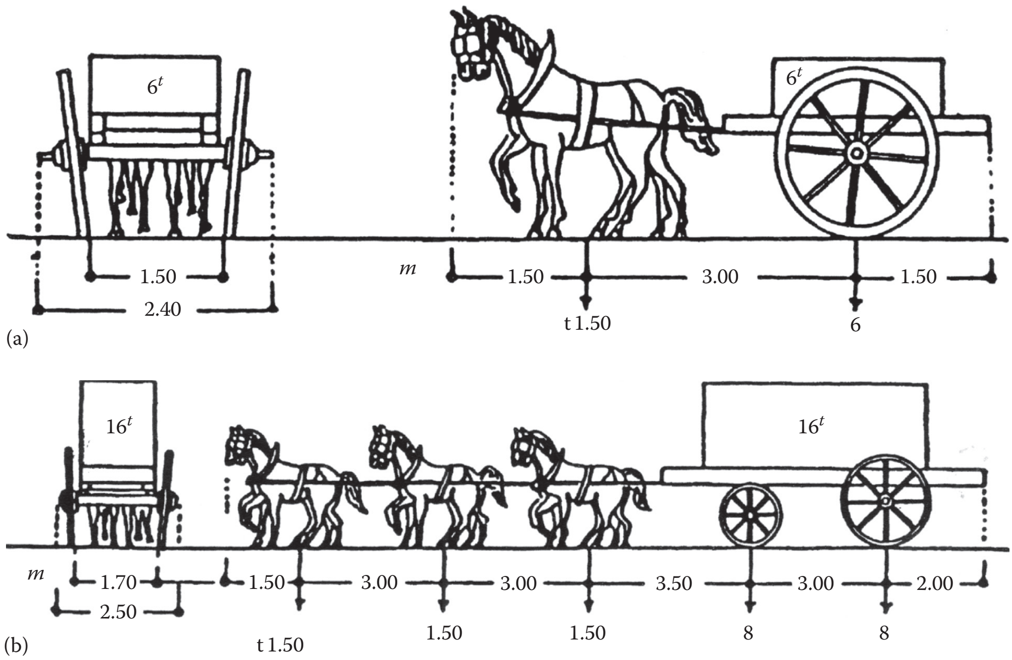


Figure 1.16

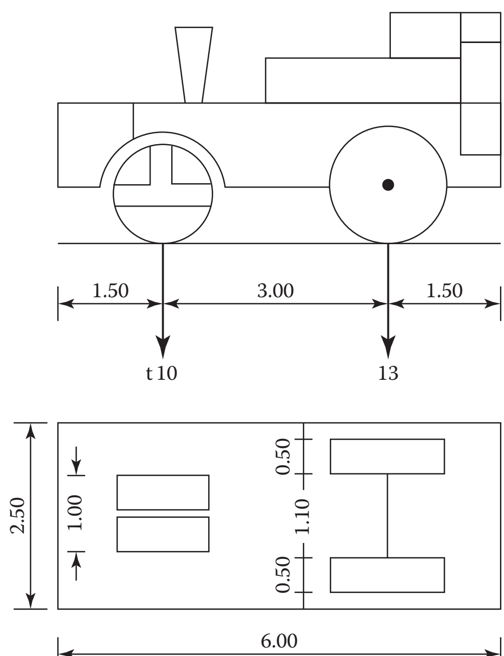
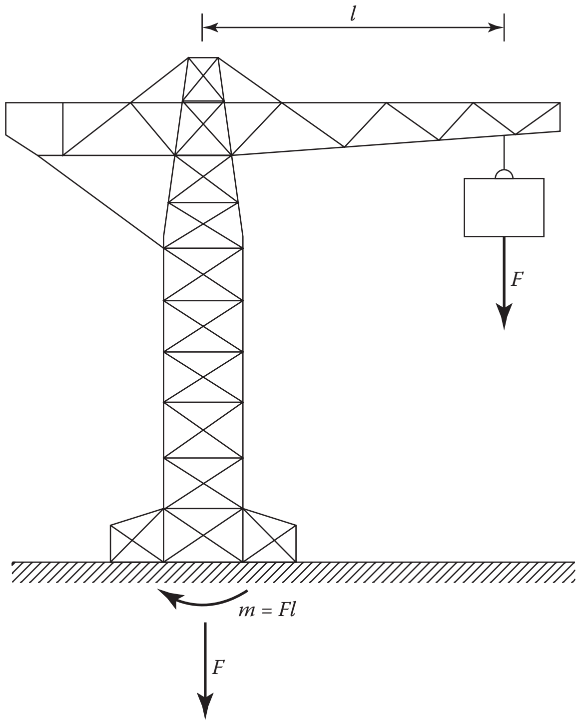


Figure 1.17



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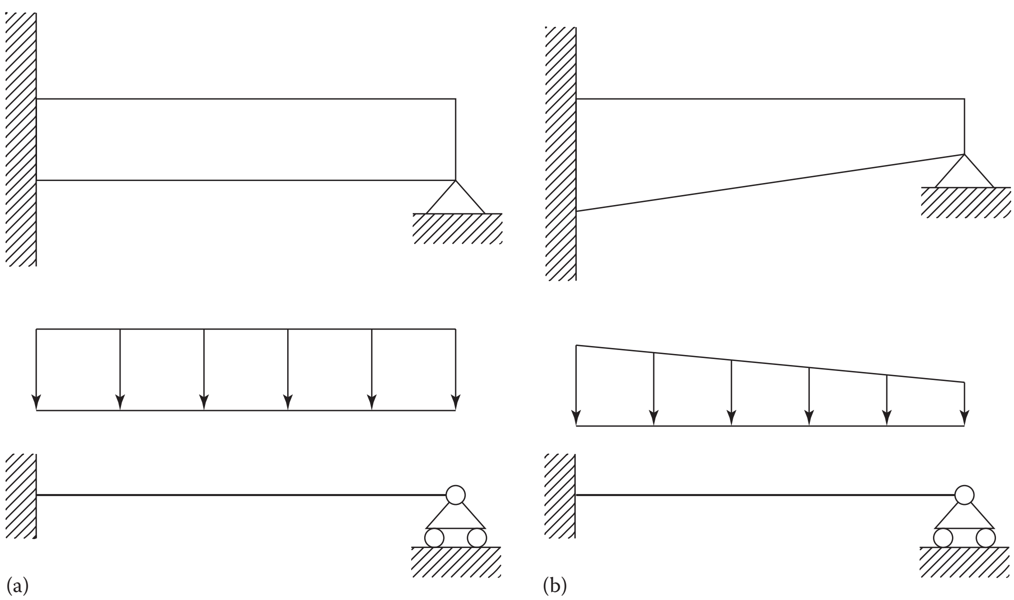

Figure 1.19

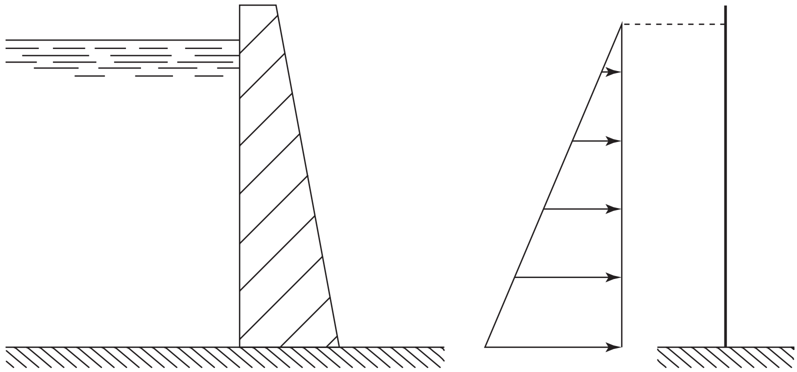

Figure 1.20

Other loads of a mechanical nature are **hydraulic loads** and **pneumatic loads.** Figure 1.20 shows how the thrust of water against a dam can be represented with a triangular distributed load. Then there are **inertial forces**, which act on rotating mechanical components, such as the blades of a turbine, or on the floors of a storeyed building, following ground vibration caused by an earthquake (Figure 1.21). A similar system of horizontal forces can represent the action of the wind on the same building.

In addition to external loading, the structural elements undergo the action of the other structural elements connected to them, including the action of the foundation. These kinds of action are more correctly termed **constraint reactions**, those exchanged between elements being **internal** and those exchanged with the foundation being **external.** The nature of the constraint reaction depends on the conformation and mode of operation of the constraint which connects the two parts.

Figure 1.22 gives examples of some types of **beam support** to the foundation. In the case of Figure 1.22a, we have a pillar in reinforced concrete; in Figure 1.22b, we have joints that are used in bridges, and in Figure 1.22c, a roller support. In all cases, the constraint reaction

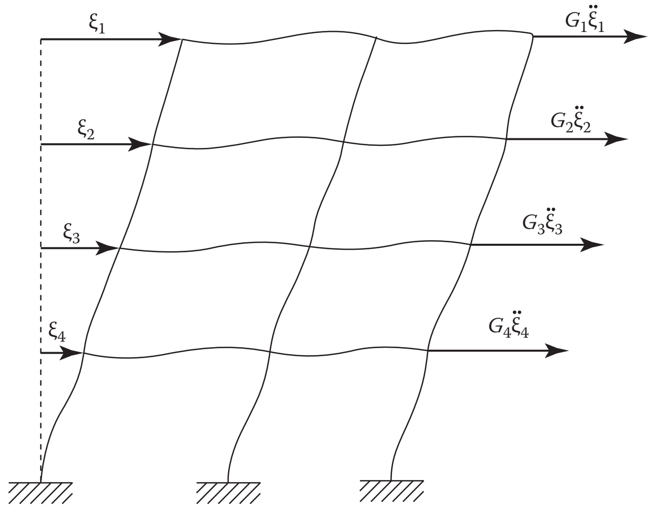

Figure 1.21

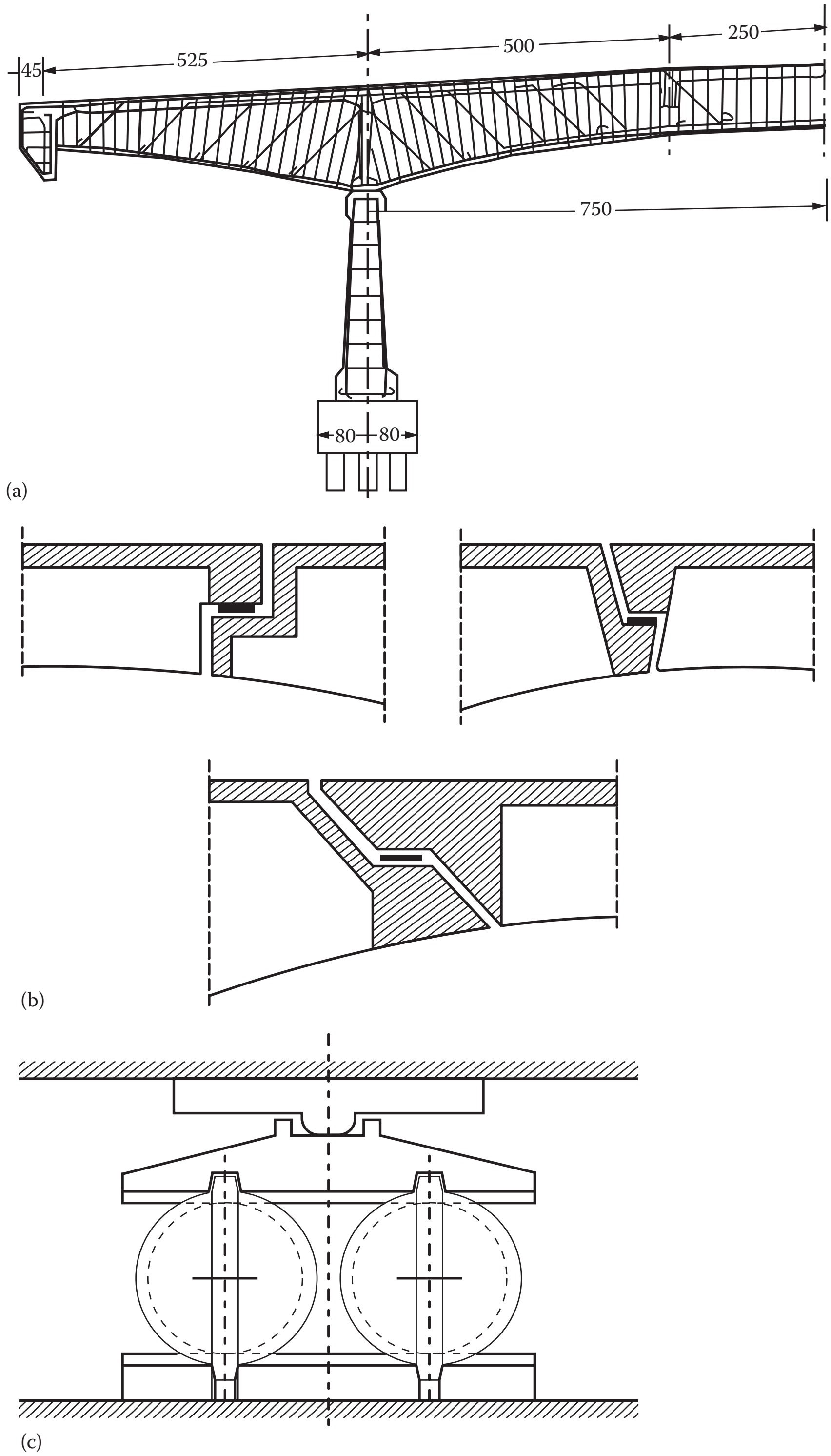

Figure 1.22

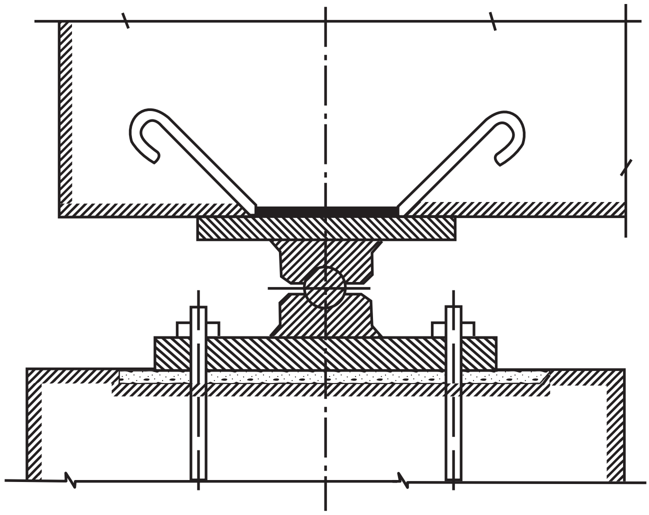

Figure 1.23

exchanged between the foundation and the structural part is constituted by a vertical force, no constraint being exerted horizontally, except for friction.

Figure 1.23 shows the detailed scheme of a hinge connecting a part in reinforced concrete to the foundation. The hinge allows only relative rotations between the two connected parts and hence reacts with a force that passes through its centre. In the case illustrated, there will thus be the possibility of a horizontal reaction, as well as a vertical one.

Figure 1.24 illustrates the joint between two timber beams, built with joining plates and riveting. Similar joints are made for steel girders by means of bolting or welding. This constraint is naturally more severe than a simple hinge, and yet in practice it proves to be much less rigid than a perfectly fixed joint. In the designing of trusses, it is customary to model the joint with a hinge, neglecting the exchange of moment between the two parts. The effect of making such an assumption is, in fact, that of guaranteeing a greater margin of safety.

1.5 STRUCTURAL COLLAPSE

If the loading exerted on a structure exceeds a certain limit, the consequence is the complete collapse or, at any rate, the failure of the structure itself. The loss of stability can occur in different ways depending on the shape and dimensions of the structural elements, as well as on the material of which these are made. In some cases, the constraints and joints can fail, with the result that rigid mechanisms are created, with consequent large displacements, toppling over, etc. In other cases, the structural elements themselves can give way; the mechanisms of structural collapse can be divided schematically into three distinct categories:

1. Buckling
2. Yielding
3. Brittle fracturing

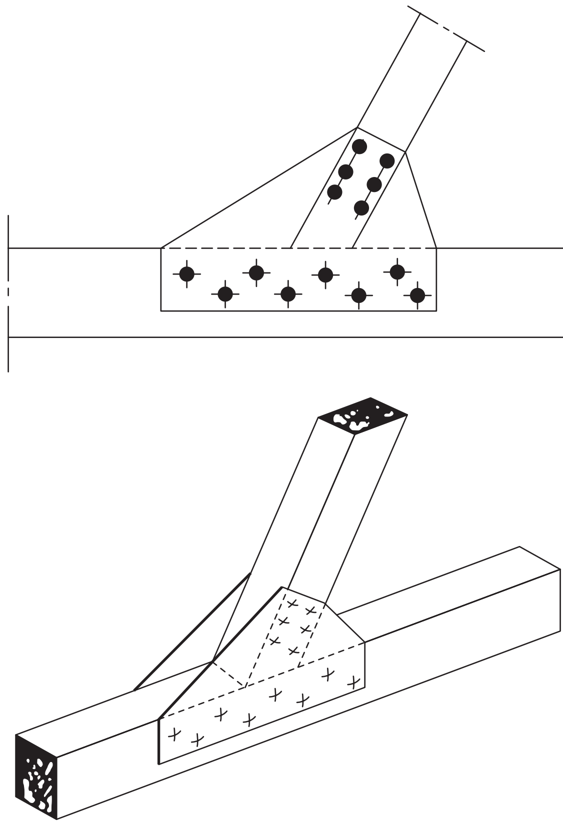

Figure 1.24

In real situations, however, many cases of structural collapse occur in such a way as to involve two of these mechanisms, if not all three.

Buckling, or instability of elastic equilibrium, is the type of structural collapse which involves slender structural elements, subject prevalently to compression, such as struts of trusses, columns of frameworks, piers and arches of bridges, valve stems, crankshafts, ceiling shells, submarine hulls, etc. This kind of collapse often occurs even before the material of which the element is made has broken or yielded.

Unlike buckling, **yielding**, or plastic deformation, involves also the material itself and occurs in a localized manner in one or more points of the structure. When, with the increase in load, plastic deformation has taken place in a sufficient number of points, the structure can give way altogether since it has become hypostatic, i.e. it has become a mechanism. This type of generalized structural collapse usually involves structures built of rather ductile material, such as metal frames and plates, which are mainly prone to bending.

Finally, **brittle fracturing** is of a localized origin, as is plastic deformation, but spreads throughout the structure and hence constitutes a structural collapse of a generalized nature. This type of collapse affects prevalently one- and two-dimensional structural elements of considerable thickness (bridges, dams, ships, large ceilings and vessels, etc.), large three-dimensional elements (rock masses, the Earth's crust, etc.), brittle materials (high-strength steel and concrete, rocks, ceramics, glass, etc.) and tensile conditions.

As, with the decrease in their degree of slenderness, certain structures, subject prevalently to compression and bending, very gradually pass from a collapse due to buckling to one due to plastic deformation, likewise, as we move down the size scale, other structures, prone to tension and bending, gradually pass from a collapse due to brittle fracturing to one due to plastic deformation.

1.6 NUMERICAL MODELS

With the development of electronics technology and the production of computers of ever-increasing power and capacity, structural analysis has undergone, in the last four or five decades, a remarkable metamorphosis. Calculations which were carried out manually by individual engineers, with at most the help of the traditional graphical methods, can now be performed using computer software.

Up to a few years ago, since the calculation of strains and internal stresses of complex structures could not be handled in such a way as to obtain an exact result, such calculations were carried out using a procedure of approximation. These approximations, at times, were somewhat crude and, in certain cases, far from being altogether realistic. Today numerical models allow us to consider enormous numbers of points, or nodes, with their corresponding displacements and corresponding strains and internal stresses. The so-called **finite-element method** is both a discretization method, since it considers a finite number, albeit a very large one, of structural nodes, and an interpolation method, since it allows us to estimate the static and kinematic quantities even outside the nodes.

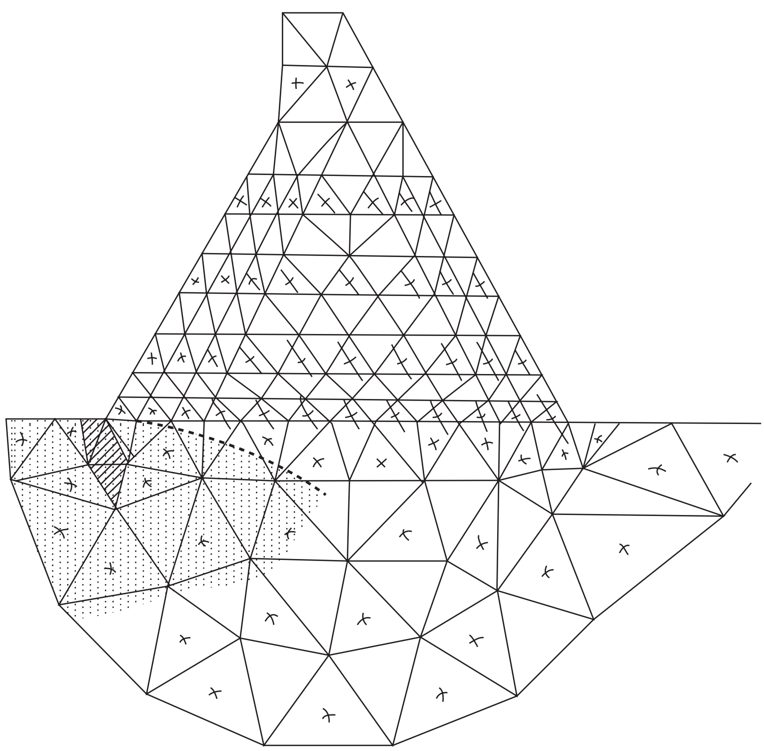

Figure 1.25

The enormous amount of information to be handled is organized and ordered in a matrix form by the computer. In this way, the language itself of structural analysis has taken on a different appearance, undoubtedly more synthetic and homogeneous. This means that, for every type of structural element, it is possible to write static, kinematic and constitutive equations having the same form. Once discretized, these provide a matrix of global stiffness which presents a dimension equal to the number of degrees of freedom considered. This matrix, multiplied by the vector of the nodal displacements, which constitutes the primary unknown of the problem, provides the vector of the external forces applied to the nodes; this represents the known term of the problem. Once this matrix equation has been resolved, taking into account any boundary conditions, it is then possible to arrive at the nodal strains and nodal internal stresses.

As an illustration of these mathematical techniques, a number of **finite-element meshes** are presented. They correspond to a buttress dam (Figure 1.25), a rock mass with a tunnel system (Figure 1.26), an eye hook (Figure 1.27), two mechanical components having supporting functions (Figure 1.28), a concrete vessel for a nuclear reactor (Figure 1.29) and an arch dam (Figure 1.30).

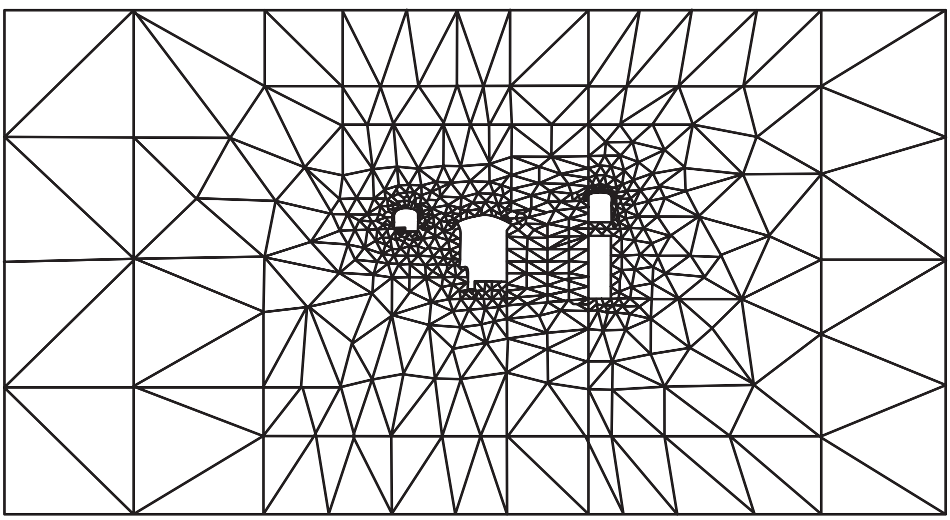

Figure 1.26

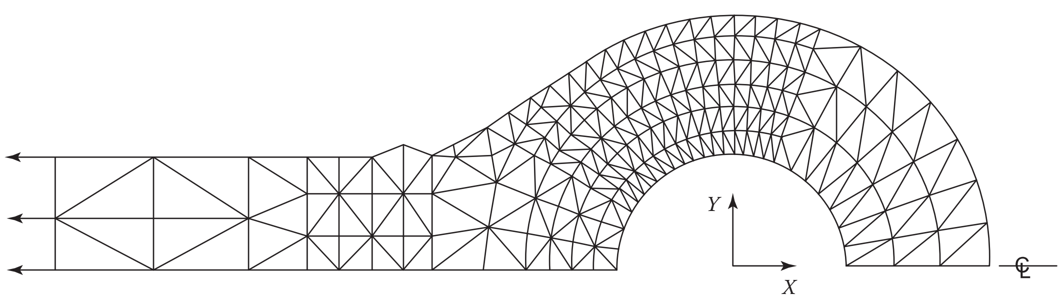

Figure 1.27

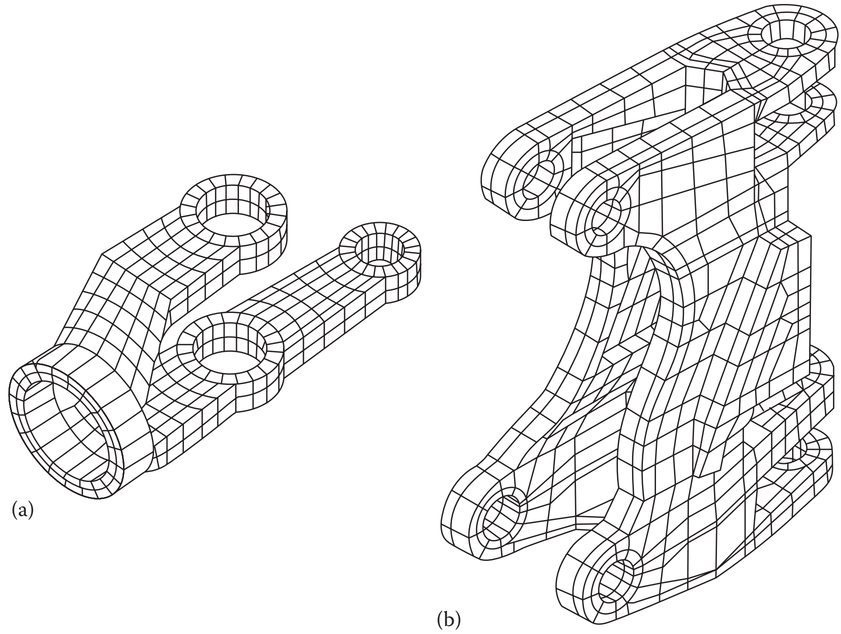

Figure 1.28

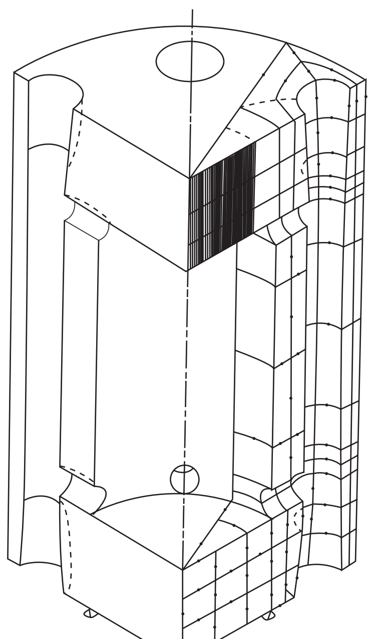

Figure 1.29

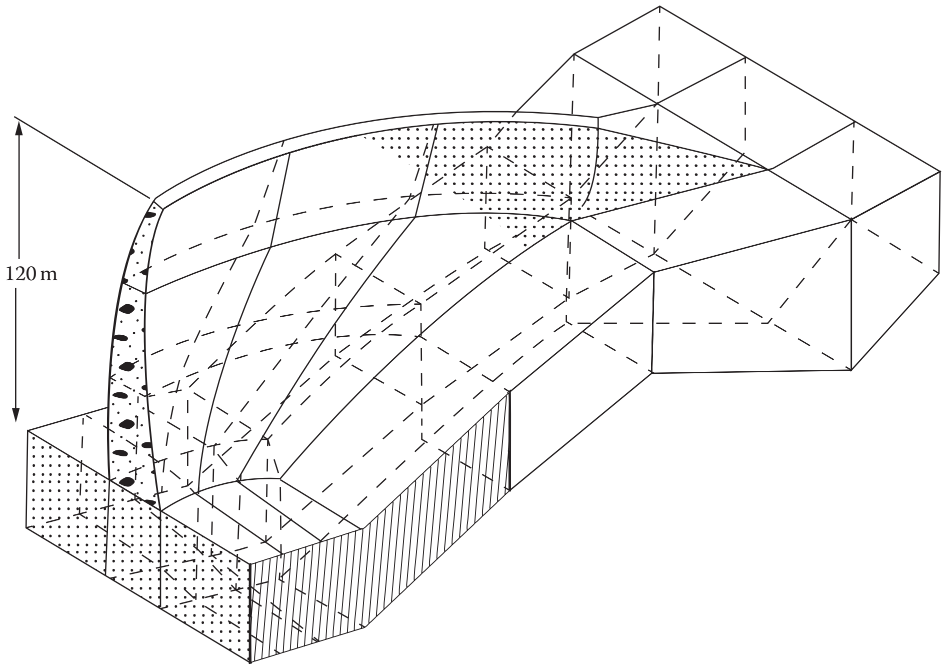

Figure 1.30

Chapter 2

Geometry of areas

2.1 INTRODUCTION

When analysing beam resistance, it is necessary to consider the geometrical features of the corresponding right sections. These features, as will emerge more clearly hereafter, amount to a scalar quantity, the **area**; a vector quantity, the position of the **centroid**; and a tensor quantity, consisting of the **central directions** and the **central moments of inertia**.

The laws of transformation, by translation and rotation of the reference system, both of the vector of static moments and of the tensor of moments of inertia, will be considered. It will thus become possible also to calculate composite sections, consisting of the combination of a number of elementary parts, and the graphical interpretation (due to Mohr) of this calculation will be given.

Particular attention will be paid to the cases of sections presenting symmetry, whether axial or polar, and of thin-walled beam sections, which have already been mentioned in the introductory chapter and for which a simplified calculation is possible. A number of examples will close the chapter.

2.2 LAWS OF TRANSFORMATION OF THE POSITION VECTOR

The coordinates x, y of a point of the plane in the XY reference system are linked to the coordinates $\bar{x}$, $\bar{y}$ of the same point in the **translated reference system** $\bar{X}\bar{Y}$ (Figure 2.1) by the following relations:

$$\bar{x} = x - x_0 \tag{2.1a}$$

$$\bar{y} = y - y_0 \tag{2.1b}$$

where x_0, y_0 are the coordinates of the origin $\bar{O}$ of the translated system, with respect to the original XY axes.

The laws of transformation (2.1) can be reproposed in a vector form as follows:

$$\{\bar{r}\} = \{r\} - \{r_0\} \tag{2.2}$$

where $\{r\}$ indicates the position vector $[x, y]^T$ of the generic point in the original reference system, with $\{\bar{r}\}$ being the position vector $[\bar{x}, \bar{y}]^T$ in the translated system and with $\{r_0\}$ being the position vector $[x_0, y_0]^T$ of the origin $\bar{O}$ of the translated system in the original reference system.

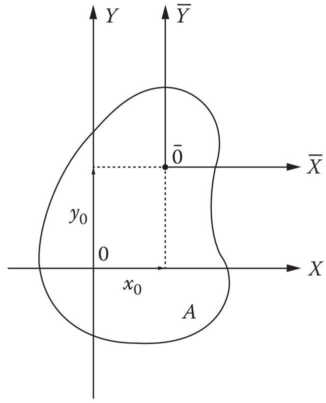

Figure 2.1

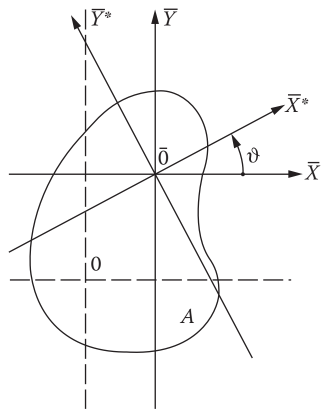

Figure 2.2

The coordinates $\bar{x}$, $\bar{y}$ of a point of the plane $\overline{XY}$ are linked to the coordinates $\bar{x}^*$, $\bar{y}^*$ of the same point in the rotated reference system $\bar{X}^*\bar{Y}^*$ (Figure 2.2) *via* the following relations:

$$\bar{x}^* = \bar{x}\cos\vartheta + \bar{y}\sin\vartheta \tag{2.3a}$$

$$\bar{y}^* = -\bar{x}\sin\vartheta + \bar{y}\cos\vartheta \tag{2.3b}$$

where ϑ indicates the angle of rotation of the second reference system with respect to the first (positive if the rotation is counterclockwise).

These transformation laws can be reproposed in a matrix form as follows:

$$\{\bar{r}^*\} = [N]\{\bar{r}\} \tag{2.4}$$

where

$$[N] = \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix} \tag{2.5}$$

is the orthogonal matrix of rotation.

2.3 LAWS OF TRANSFORMATION OF THE STATIC MOMENT VECTOR

Consider the area A in the XY reference system (Figure 2.1). The definition of **static moment vector**, relative to the area A and calculated in the XY reference system, is given by the following two-component vector:

$$\{S\} = \begin{bmatrix} S_y \\ S_x \end{bmatrix} = \begin{bmatrix} \int_A x\,dA \\ \int_A y\,dA \end{bmatrix} = \int_A \{r\}dA \tag{2.6}$$

The static moment vector, again referred to the area A, calculated in the translated $\overline{XY}$ system, can be expressed in the following way:

$$\{\bar{S}\} = \begin{bmatrix} S_{\bar{y}} \\ S_{\bar{x}} \end{bmatrix} = \begin{bmatrix} \int_A \bar{x}\,dA \\ \int_A \bar{y}\,dA \end{bmatrix} = \int_A \{\bar{r}\}dA \tag{2.7}$$

Applying the transformation law (2.2), Equation 2.7 becomes

$$\{\bar{S}\} = \int_A \{r\}dA - \{r_0\}\int_A dA \tag{2.8}$$

since $\{r_0\}$ is a constant vector. Recalling definition (2.6), we obtain finally the static moment vector transformation law for translations of the reference system:

$$\{\bar{S}\} = \{S\} - A\{r_0\} \tag{2.9}$$

Vector relation (2.9) is equivalent to the following two scalar relations:

$$S_{\bar{y}} = S_y - A x_0 \tag{2.10a}$$

$$S_{\bar{x}} = S_x - A y_0 \tag{2.10b}$$

The reference system, translated with respect to the original one, for which both static moments vanish, is determined by the following position vector:

$$x_G = \frac{S_y}{A} \tag{2.11a}$$

$$y_G = \frac{S_x}{A} \tag{2.11b}$$

The origin G of this particular reference system is termed the **centroid** of area A and is a characteristic point of the area itself, in the sense that it is altogether independent of the choice of the original XY system.

Now consider the reference system $\bar{X}^*\bar{Y}^*$, rotated with respect to the $\overline{XY}$ system (Figure 2.2). The static moment vector, relative to area A and calculated in the rotated system $\bar{X}^*\bar{Y}^*$, may be expressed using the law (2.4):

$$\{\bar{S}^*\} = \int_A \{\bar{r}^*\} dA = [N] \int_A \{\bar{r}\} dA \tag{2.12}$$

where $[N]$ is the constant matrix (2.5). Finally, recalling definition (2.7), the static moment vector transformation law for rotations of the reference system is obtained as follows:

$$\{\bar{S}^*\} = [N]\{\bar{S}\} \tag{2.13}$$

The matrix relation (2.13) is equivalent to the following two scalar relations:

$$S_{\bar{y}^*} = S_{\bar{y}} \cos\vartheta + S_{\bar{x}} \sin\vartheta \tag{2.14a}$$

$$S_{\bar{x}^*} = -S_{\bar{y}} \sin\vartheta + S_{\bar{x}} \cos\vartheta \tag{2.14b}$$

From Equations 2.14, two important conclusions may be drawn.

1. The static moments are zero with respect to any pair of centroidal orthogonal axes.
2. If the origin $\bar{O}$ of the reference system does not coincide with the centroid G of area A, there exists no angle of rotation ϑ of the reference system for which the static moments both vanish. In fact, from Equations 2.14, we obtain

$$S_{\bar{y}^*} = 0 \quad \text{for } \vartheta = \arctan\left(-\frac{S_{\bar{y}}}{S_{\bar{x}}}\right) \tag{2.15a}$$

$$S_{\bar{x}^*} = 0 \quad \text{for } \vartheta = \arctan\left(+\frac{S_{\bar{x}}}{S_{\bar{y}}}\right) \tag{2.15b}$$

The conditions (2.15) are not, however, compatible.

If we consider a reference system $\bar{X}^*\bar{Y}^*$ obtained by translating and then rotating the original XY system (Figures 2.1 and 2.2), it is possible to formulate the general static moment vector transformation law for rototranslations of the reference system, combining the foregoing partial laws (2.9) and (2.13):

$$\{\bar{S}^*\} = [N]\left(\{S\} - A\{r_0\}\right) \tag{2.16}$$

The inverse rototranslation formula may be obtained from the previous one by premultiplying both members by $[N]^T = [N]^{-1}$

$$\{S\} = [N]^T \{\bar{S}^*\} + A\{r_0\} \tag{2.17}$$

2.4 LAWS OF TRANSFORMATION OF THE MOMENT OF INERTIA TENSOR

Consider the following matrix product (referred to as the **dyadic product**):

$$\{r\}\{r\}^T = \begin{bmatrix} x \\ y \end{bmatrix} [x\ y] = \begin{bmatrix} x^2 & xy \\ yx & y^2 \end{bmatrix} \tag{2.18}$$

The definition of the **moment of inertia tensor**, relative to area A and calculated in the XY reference system, is given by the following symmetric (2 × 2) tensor:

$$[I] = \begin{bmatrix} I_{yy} & I_{xy} \\ I_{yx} & I_{xx} \end{bmatrix} = \begin{bmatrix} \int_A x^2\, dA & \int_A xy\, dA \\ \int_A yx\, dA & \int_A y^2\, dA \end{bmatrix} \tag{2.19}$$

Taking into account relation (2.18), definition (2.19) can be expressed in the following compact form:

$$[I] = \int_A \{r\}\{r\}^T\, dA \tag{2.20}$$

The moment of inertia tensor, relative again to area A and calculated in the translated reference system $\overline{X\,Y}$ (Figure 2.1), can be expressed as follows:

$$[\bar{I}] = \int_A \{\bar{r}\}\{\bar{r}\}^T\, dA \tag{2.21}$$

And, thus, applying the position vector transformation law for the translations of the reference system (Equation 2.2), we obtain

$$[\bar{I}] = \int_A (\{r\} - \{r_G\})(\{r\} - \{r_G\})^T\, dA \tag{2.22}$$

Since the transpose of the sum of two matrices is equal to the sum of the transposes, we have

$$[\bar{I}] = \int_A (\{r\} - \{r_G\})(\{r\}^T - \{r_G\}^T)\, dA$$

$$= \int_A \{r\}\{r\}^T\, dA - \int_A \{r\} dA\, \{r_G\}^T - \{r_G\} \int_A \{r\}^T\, dA + \{r_G\}\{r_G\}^T \int_A dA \tag{2.23}$$

Finally, recalling definitions (2.6) and (2.20), we obtain the law of transformation of the moment of inertia tensor for translations of the reference system

$$[\bar{I}] = [I] + A\{r_G\}\{r_G\}^T - \{r_G\}\{S\}^T - \{S\}\{r_G\}^T \tag{2.24}$$

The matrix relation (2.24) can be rendered explicit as follows:

$$I_{\bar{x}\bar{x}} = I_{xx} + A y_0^2 - 2 y_0 S_x \tag{2.25a}$$

$$I_{\bar{y}\bar{y}} = I_{yy} + A x_0^2 - 2 x_0 S_y \tag{2.25b}$$

$$I_{\bar{x}\bar{y}} = I_{\bar{y}\bar{x}} = I_{xy} + A x_0 y_0 - x_0 S_x - y_0 S_y \tag{2.25c}$$

The earlier relations simplify in the case where the origin of the primitive *XY* reference system coincides with the centroid *G* of area A. In this case, we have

$$S_x = S_y = 0 \tag{2.26}$$

and Equations 2.25 assume the form of well-known **Huygens' laws**:

$$I_{\bar{x}\bar{x}} = I_{x_G x_G} + A y_0^2 \tag{2.27a}$$

$$I_{\bar{y}\bar{y}} = I_{y_G y_G} + A x_0^2 \tag{2.27b}$$

$$I_{\bar{x}\bar{y}} = I_{x_G y_G} + A x_0 y_0 \tag{2.27c}$$

As regards relations (2.27a) and (2.27b), it may be noted how the centroidal moment of inertia is the minimum of all those corresponding to an infinite number of parallel straight lines.

Now consider the moment of inertia tensor, relative to area *A* and calculated in the rotated reference system $\bar{X}^*\bar{Y}^*$ (Figure 2.2):

$$[\bar{I}^*] = \int_A \{\bar{r}^*\}\{\bar{r}^*\}^T \, dA \tag{2.28}$$

Using the law (2.4) of transformation of the position vector for rotations of the reference system, we have

$$[\bar{I}^*] = \int_A ([N]\{\bar{r}\})([N]\{\bar{r}\})^T \, dA \tag{2.29}$$

Now applying the law by which the transpose of the product of two matrices is equal to the inverse product of the transposes, we have

$$[\bar{I}^*] = \int_A ([N]\{\bar{r}\})(\{\bar{r}\}^T [N]^T) \, dA \tag{2.30}$$

Exploiting the associative law and carrying the constant matrices $[N]$ and $[N]^T$ outside the integral sign, Equation 2.30 becomes

$$[\bar{I}^*] = [N]\int_A \{\bar{r}\}\{\bar{r}\}^T \, dA\,[N]^T \tag{2.31}$$

Finally, recalling definition (2.21), we obtain the law of transformation of the moment of inertia tensor for rotations of the reference system

$$[\bar{I}^*] = [N][\bar{I}][N]^T \tag{2.32}$$

Matrix relation (2.32) can be rendered explicit as follows:

$$I_{\bar{x}^*\bar{x}^*} = I_{\bar{x}\bar{x}}\cos^2\vartheta + I_{\bar{y}\bar{y}}\sin^2\vartheta - 2I_{\bar{x}\bar{y}}\sin\vartheta\cos\vartheta \tag{2.33a}$$

$$I_{\bar{y}^*\bar{y}^*} = I_{\bar{x}\bar{x}}\sin^2\vartheta + I_{\bar{y}\bar{y}}\cos^2\vartheta + 2I_{\bar{x}\bar{y}}\sin\vartheta\cos\vartheta \tag{2.33b}$$

$$I_{\bar{x}^*\bar{y}^*} = I_{\bar{y}^*\bar{x}^*} = I_{\bar{x}\bar{y}}\cos 2\vartheta + \frac{1}{2}(I_{\bar{x}\bar{x}} - I_{\bar{y}\bar{y}})\sin 2\vartheta \tag{2.33c}$$

Two important conclusions can be derived from Equations 2.33:

1. The sum of the two moments of inertia I_{xx} and I_{yy} remains constant as the angle of rotation ϑ varies. We have in fact

   $$I_{\bar{x}^*\bar{x}^*} + I_{\bar{y}^*\bar{y}^*} = I_{\bar{x}\bar{x}} + I_{\bar{y}\bar{y}} \tag{2.34}$$

   This sum is the first scalar invariant of the moment of inertia tensor and can be interpreted as the **polar moment of inertia** of area A with respect to the origin of the reference system:

   $$I_p = \int_A r^2 \, dA \tag{2.35}$$

2. Equating to zero the expression of the product of inertia $I_{\bar{x}^*\bar{y}^*}$, it is possible to obtain the angle of rotation ϑ_0 which renders the moment of inertia tensor diagonal:

   $$I_{\bar{x}^*\bar{y}^*} = I_{\bar{y}^*\bar{x}^*} = 0 \quad \text{for}$$

   $$\vartheta_0 = \frac{1}{2}\arctan\left(\frac{2I_{\bar{x}\bar{y}}}{I_{\bar{y}\bar{y}} - I_{\bar{x}\bar{x}}}\right), \quad -\frac{\pi}{4} < \vartheta_0 < \frac{\pi}{4} \tag{2.36}$$

Substituting Equation 2.36 in (2.33a and 2.33b), the so-called **principal moments of inertia** are determined. The two orthogonal directions defined by the angle ϑ_0 are referred to as the **principal directions of inertia**. It can be demonstrated how the principal moments of inertia are, in one case, the minimum and, in the other, the maximum of all the moments of inertia $I_{\bar{x}^*\bar{x}^*}$ and $I_{\bar{y}^*\bar{y}^*}$, which we have as the angle of rotation ϑ varies. When the axes, in addition to being principal, are also centroidal, they are referred to as **central,** as are the corresponding moments of inertia.

The general law of transformation of the moment of inertia tensor for rototranslations of the reference system (Figures 2.1 and 2.2) is obtained by combining the partial laws (2.24) and (2.32):

$$[\bar{I}^*] = [N]([I] + A\{r_0\}\{r_0\}^T - \{r_0\}\{S\}^T - \{S\}\{r_0\}^T)[N]^T \tag{2.37}$$

The inverse rototranslation formula may be obtained from (2.37) by premultiplying both sides of the equation by $[N]^T$ and postmultiplying them by $[N]$ and inserting Equation 2.17:

$$[I] = [N]^T[\bar{I}^*][N] + [N]^T\{\bar{S}^*\}\{r_0\}^T + \{r_0\}\{\bar{S}^*\}^T[N] + A\{r_0\}\{r_0\}^T \tag{2.38}$$

2.5 PRINCIPAL AXES AND MOMENTS OF INERTIA

Using well-known trigonometric formulas, relation (2.33a) becomes

$$I_{\bar{x}^*\bar{x}^*} = I_{\overline{xx}}\frac{1+\cos 2\vartheta}{2} + I_{\overline{yy}}\frac{1-\cos 2\vartheta}{2} - I_{\overline{xy}}\sin 2\vartheta \tag{2.39}$$

Via Equation 2.36, we obtain

$$I_{\bar{x}^*\bar{x}^*}(\vartheta_0) = \frac{I_{\overline{xx}} + I_{\overline{yy}}}{2} + \frac{I_{\overline{xx}} - I_{\overline{yy}}}{2}\cos 2\vartheta_0 + \frac{I_{\overline{xx}} - I_{\overline{yy}}}{2}\tan 2\vartheta_0 \sin 2\vartheta_0 \tag{2.40}$$

and hence

$$I_{\bar{x}^*\bar{x}^*}(\vartheta_0) = \frac{I_{\overline{xx}} + I_{\overline{yy}}}{2} + \frac{I_{\overline{xx}} - I_{\overline{yy}}}{2}\frac{1}{\cos 2\vartheta_0} \tag{2.41}$$

Since we know from trigonometry that

$$\frac{1}{\cos 2\vartheta_0} = (1 + \tan^2 2\vartheta_0)^{\frac{1}{2}} \tag{2.42}$$

it is possible to apply Equation 2.36 once more:

$$\frac{1}{\cos 2\vartheta_0} = \left(1 + \frac{4I^2_{\overline{xy}}}{(I_{\overline{yy}} - I_{\overline{xx}})^2}\right)^{\frac{1}{2}} = \begin{cases} \dfrac{1}{I_{\overline{xx}} - I_{\overline{yy}}}\left((I_{\overline{xx}} - I_{\overline{yy}})^2 + 4I^2_{\overline{xy}}\right)^{\frac{1}{2}} & \text{when } I_{\overline{xx}} > I_{\overline{yy}} \\[2ex] \dfrac{1}{I_{\overline{yy}} - I_{\overline{xx}}}\left((I_{\overline{xx}} - I_{\overline{yy}})^2 + 4I^2_{\overline{xy}}\right)^{\frac{1}{2}} & \text{when } I_{\overline{xx}} < I_{\overline{yy}} \end{cases} \tag{2.43}$$

Then, indicating $I_{\bar{x}^*\bar{x}^*}(\vartheta_0)$ with the simpler notation I_ξ, we have

$$I_\xi = \begin{cases} \dfrac{I_{\bar{x}\bar{x}} + I_{\bar{y}\bar{y}}}{2} + \dfrac{1}{2}\left((I_{\bar{x}\bar{x}} - I_{\bar{y}\bar{y}})^2 + 4I^2_{\bar{x}\bar{y}} \right)^{\frac{1}{2}} & \text{when } I_{\bar{x}\bar{x}} > I_{\bar{y}\bar{y}} \\ \dfrac{I_{\bar{x}\bar{x}} + I_{\bar{y}\bar{y}}}{2} - \dfrac{1}{2}\left((I_{\bar{x}\bar{x}} - I_{\bar{y}\bar{y}})^2 + 4I^2_{\bar{x}\bar{y}} \right)^{\frac{1}{2}} & \text{when } I_{\bar{x}\bar{x}} < I_{\bar{y}\bar{y}} \end{cases} \tag{2.44}$$

Likewise, indicating $I_{\bar{y}^*\bar{y}^*}(\vartheta_0)$ with I_η, we have

$$I_\eta = \begin{cases} \dfrac{I_{\bar{x}\bar{x}} + I_{\bar{y}\bar{y}}}{2} - \dfrac{1}{2}\left((I_{\bar{x}\bar{x}} - I_{\bar{y}\bar{y}})^2 + 4I^2_{\bar{x}\bar{y}} \right)^{\frac{1}{2}} & \text{when } I_{\bar{x}\bar{x}} > I_{\bar{y}\bar{y}} \\ \dfrac{I_{\bar{x}\bar{x}} + I_{\bar{y}\bar{y}}}{2} + \dfrac{1}{2}\left((I_{\bar{x}\bar{x}} - I_{\bar{y}\bar{y}})^2 + 4I^2_{\bar{x}\bar{y}} \right)^{\frac{1}{2}} & \text{when } I_{\bar{x}\bar{x}} < I_{\bar{y}\bar{y}} \end{cases} \tag{2.45}$$

We can thus conclude that, when the $\overline{XY}$ axes, by rotation, become the principal axes, the order relation is conserved:

$$I_{\bar{x}\bar{x}} > I_{\bar{y}\bar{y}} \Rightarrow I_\xi > I_\eta \tag{2.46a}$$

$$I_{\bar{x}\bar{x}} < I_{\bar{y}\bar{y}} \Rightarrow I_\xi < I_\eta \tag{2.46b}$$

When

$$I_{\bar{x}\bar{x}} = I_{\bar{y}\bar{y}}, \quad I_{\bar{x}\bar{y}} \neq 0 \tag{2.47}$$

relation (2.36) is not defined and thus it makes no difference whether the $\overline{XY}$ reference system is rotated by π/4 clockwise or counterclockwise ($\vartheta_0 = \pm\pi/4$) in order to obtain the principal directions.

Moreover, when

$$I_{\bar{x}\bar{x}} = I_{\bar{y}\bar{y}}, \quad I_{\bar{x}\bar{y}} = 0 \tag{2.48}$$

all the rotated reference systems $\bar{X}^*\bar{Y}^*$ are principal systems, for any angle of rotation ϑ_0. The areas that satisfy conditions (2.48) are said to be **gyroscopic**. As will be seen in the next section, it is possible to give a synthetic graphical interpretation of cases (2.47) and (2.48).

2.6 MOHR'S CIRCLE

With the aim of introducing the graphical method of **Mohr's circle**, let us consider the inverse problem of the one previously solved: Given an area A, and its principal axes of inertia ξη and the corresponding principal moments known, with respect to a point O of the plane (Figure 2.3), we intend to express the moments of inertia with respect to a reference system rotated by an angle ϑ, counterclockwise with respect to the principal reference system.

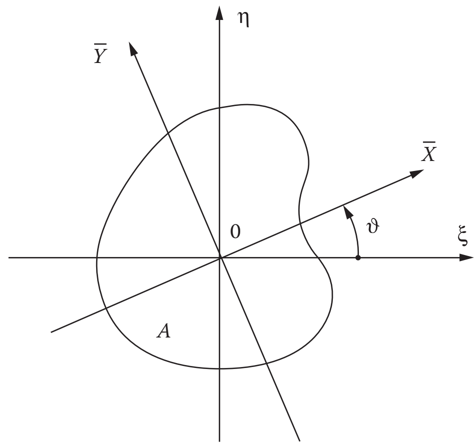

Figure 2.3

Applying Equations 2.33, and since $I_{\xi\eta} = 0$, we have

$$I_{\overline{xx}} = I_\xi \cos^2 \vartheta + I_\eta \sin^2 \vartheta \tag{2.49a}$$

$$I_{\overline{yy}} = I_\xi \sin^2 \vartheta + I_\eta \cos^2 \vartheta \tag{2.49b}$$

$$I_{\overline{xy}} = \frac{I_\xi - I_\eta}{2} \sin 2\vartheta \tag{2.49c}$$

The trigonometry formulas used previously give

$$I_{\overline{xx}} = \frac{I_\xi + I_\eta}{2} + \frac{I_\xi - I_\eta}{2} \cos 2\vartheta \tag{2.50a}$$

$$I_{\overline{yy}} = \frac{I_\xi + I_\eta}{2} - \frac{I_\xi - I_\eta}{2} \cos 2\vartheta \tag{2.50b}$$

$$I_{\overline{xy}} = \frac{I_\xi - I_\eta}{2} \sin 2\vartheta \tag{2.50c}$$

Relations (2.50a and 2.50c) constitute the parametric equations of a circumference having as its centre

$$C\left(\frac{I_\xi + I_\eta}{2}, 0\right) \tag{2.51a}$$

and as its radius

$$R = \frac{I_\xi - I_\eta}{2} \tag{2.51b}$$

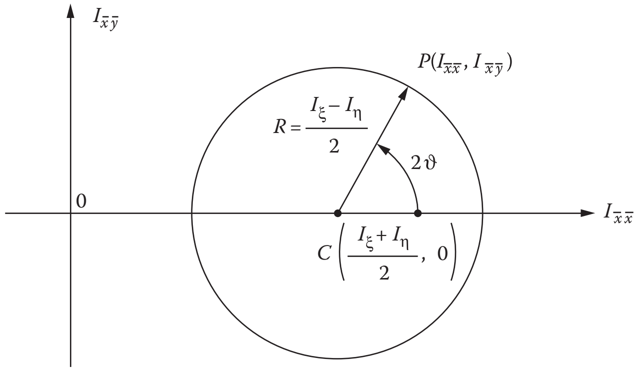

Figure 2.4

in Mohr's plane (Figure 2.4). The earlier circumference represents all the pairs $(I_{\bar{x}\bar{x}}, I_{\bar{x}\bar{y}})$ which succeed one another as the angle ϑ (Figure 2.3) varies. Note that, since $I_{\bar{x}\bar{x}}$ is in any case positive, we have in fact a Mohr's half-plane.

Let us now reconsider the direct problem: Given the moments of inertia with respect to the two generic orthogonal axes $\overline{X}\,\overline{Y}$ (Figure 2.3), determine the principal axes and moments of inertia. This determination has already been made analytically in Section 2.5. We shall now proceed to repropose it graphically using Mohr's circle (Figure 2.5):

1. The first operation to be carried out is to identify the two notable points P and P' on Mohr's plane:

$$P(I_{\bar{x}\bar{x}}, I_{\bar{y}\bar{y}}), \quad P'(I_{\bar{y}\bar{y}}, -I_{\bar{x}\bar{y}}) \tag{2.52}$$

2. The intersection C of the segment PP' with the axis $I_{\bar{x}\bar{x}}$ identifies the centre of Mohr's circle, while the segments CP and CP' represent two radii of that circle.
3. Draw through the point P the line parallel to the axis $I_{\bar{x}\bar{x}}$ and through P' the line parallel to the axis $I_{\bar{x}\bar{y}}$. These two lines meet in point P^*, called the **pole**, again belonging to Mohr's circle.

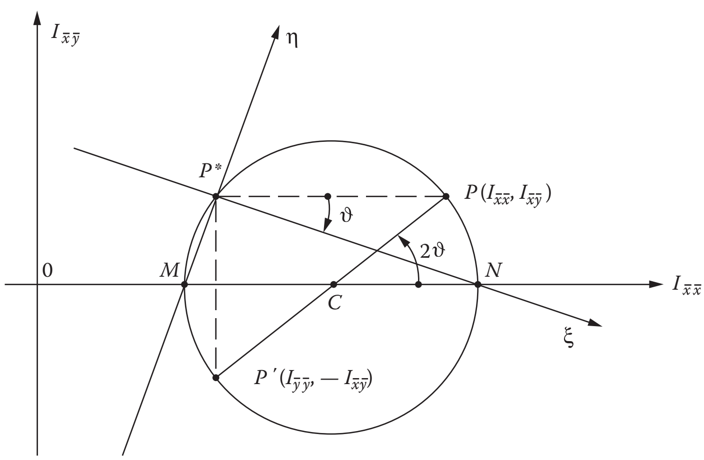

Figure 2.5

4. The lines joining pole P^* with points M and N of the $I_{\overline{xx}}$ axis, which are the intersections of the circumference with the axis, give the directions of the two principal axes of inertia. Naturally, points M and N each have the value of one of the two principal moments of inertia as abscissa. In particular, in Figure 2.5, the abscissa of M is I_η, while the abscissa of N is I_ξ, since we have assumed $I_{\overline{xx}} > I_{\overline{yy}}$. Pole P^* can obviously also fall in one of the three remaining quadrants corresponding to Mohr's circle.

The graphical construction described earlier and shown in Figure 2.5 is justified by noting that the circumferential angle $\widehat{PP^*N}$ is half of the corresponding central angle $\widehat{PCN} = 2\vartheta$ and that thus its amplitude is equal to the angle ϑ.

2.7 AREAS PRESENTING SYMMETRY

An area is said to present **oblique axial symmetry** (Figure 2.6a) when there exists a straight line s which cuts the area into two parts, and a direction s' conjugate with this straight line, such that, if we consider a generic point P, belonging to the area and the line PC, parallel to the direction s' and we draw on that line the segment $\overline{CP'} = \overline{PC}$ on the opposite side of P with respect to s, the point P' still belongs to the area. When the angle α between the directions of the lines s and s' is equal to 90°, then we have **right axial symmetry** (Figure 2.6b).

It is easy to verify that the centroid of a section having axial symmetry lies on the corresponding axis of symmetry. The centroid relative to the pair of symmetrical elementary areas located in P and P' coincides in fact with point C (Figure 2.6). Applying the so-called **distributive law** of the centroid, it is possible to think of concentrating the whole area on the axis of symmetry s, and thus the global centroid is sure to lie on the same line s.

In the case of an area presenting right symmetry (Figure 2.6b), the axis of symmetry is also a central axis of inertia. In fact, it is centroidal and, with respect to it and to any orthogonal axis, the product of inertia $I_{ss'}$ vanishes by symmetry.

When there are two or more axes of symmetry (oblique or right), since the centroid must belong to each axis, it coincides with their intersection (Figure 2.7). In the case of double right symmetry (Figure 2.7a), the axes of symmetry are also central axes of inertia.

An area is said to present **polar symmetry** (Figure 2.8) when there exists a point C such that, if we consider a generic point P belonging to the area and the line PC joining the two

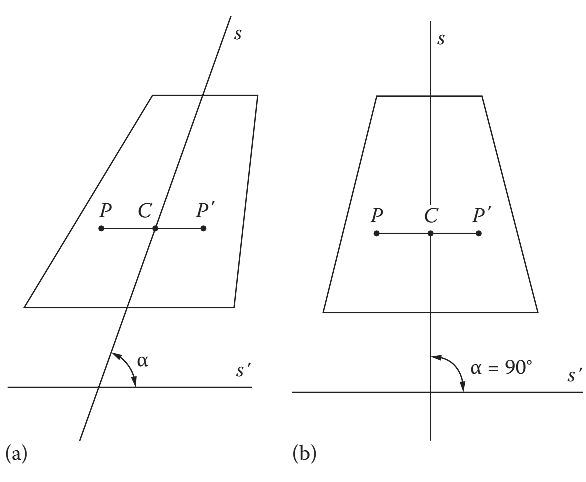

Figure 2.6

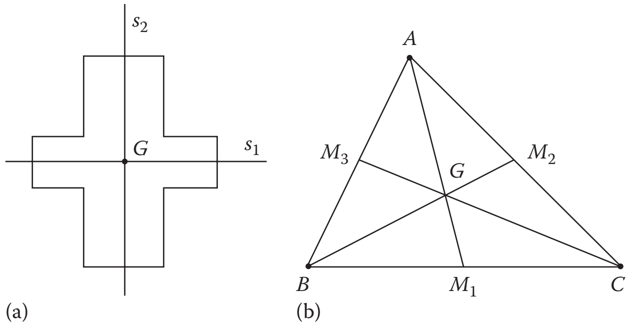

Figure 2.7

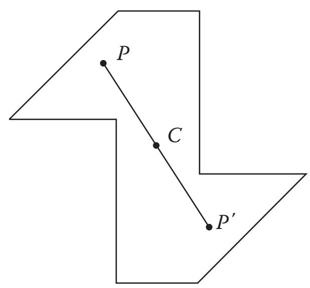

Figure 2.8

points, and we draw on this line the segment $\overline{CP'} = \overline{PC}$ on the side opposite to P with respect to C, the point P' still belongs to the area.

It is immediately verifiable that the centroid of a section having polar symmetry coincides with its geometrical centre C. The centroid corresponding to the pair of symmetrical elementary areas in P and P' coincides, in fact, with point C (Figure 2.8). Applying the distributive law of the centroid, it is possible to think of concentrating the whole area in point C, and thus the global centroid must certainly coincide with the same point C.

It is interesting to note how an n-tuple right symmetry area, with n being an even number ($2 \leq n < \infty$), is also a polar symmetry area (Figure 2.9), whereas a polar symmetry area is not necessarily also an n-tuple right symmetry area.

Areas having n-tuple right symmetry, with n being an odd number ($3 \leq n < \infty$), do not, however, present polar symmetry, even though they are gyroscopic areas, as also are those with n even.

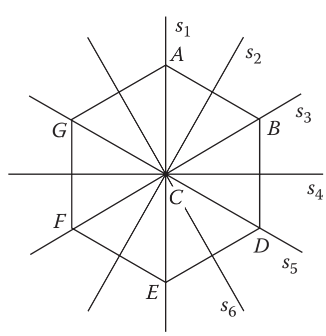

Figure 2.9

2.8 ELEMENTARY AREAS

If, on an XY plane, we assign n disjoint areas, $A_1, A_2,\ldots, A_n$, the distributive law of static moments and, respectively, that of the moments of inertia are defined as follows (Figure 2.10):

$$S\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} S(A_i) \tag{2.53a}$$

$$I\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} I(A_i) \tag{2.53b}$$

where S and I indicate generically a static moment and a moment of inertia, respectively, calculated with respect to the coordinate axes (S_x, S_y, I_{xx}, I_{yy}, I_{xy}).

In determining the static and inertial characteristics of composite areas, it is necessary to exploit the aforementioned laws. These derive from the integral nature of the definitions which have previously been given of first- and second-order moments. The first law expresses the fact that the static moment of a composite area (i.e. of the disjoint union of more than one elementary area) is equal to the sum of the static moments of the single areas. The second law refers to the moments of inertia and is altogether analogous.

Since it is therefore possible to reduce the calculation of composite areas to the calculation of simpler areas, the importance of calculating once and for all the static and inertial features of elementary areas emerges clearly. In the sequel, we shall examine the rectangle, the right triangle and the annulus sector.

Consider the **rectangle** having base b and height h (Figure 2.11). From the definition of centroid, we immediately obtain the static moments in the XY reference system:

$$S_x = A y_G = \frac{1}{2} bh^2 \tag{2.54a}$$

$$S_y = A x_G = \frac{1}{2} hb^2 \tag{2.54b}$$

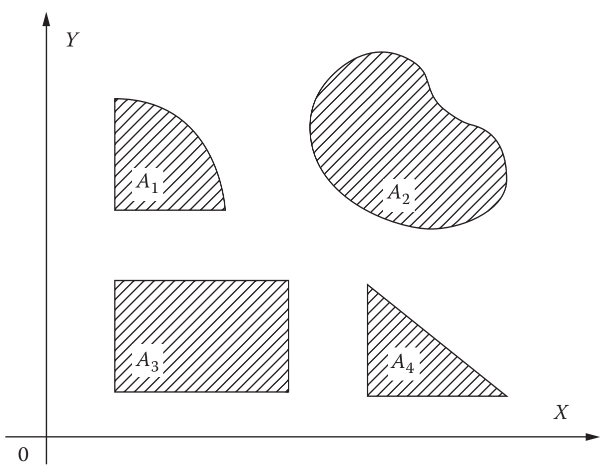

Figure 2.10