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To My Grandson Siddharth

## Preface

Homological algebra arose from many sources in Algebra and Topology. However, the subject appeared as a full-fledged subject in its own right in 1956 when the first book on the subject and still a masterpiece by H. Cartan and S. Eilenberg appeared. More books have appeared on the subject since then, notably by D.G. Northcott, S. MacLane, P.J. Hilton and U. Stammbach, J.J. Rotman, C. A. Wiebel. However, none of these could be adopted as a textbook for a student coming across the subject for the first time. The author felt this difficulty while teaching a one semester course on the subject at Kurukshetra University during the last few years. The students found it hard in the absence of a suitable textbook. The present text is a result of the lectures given at Kurukshetra during which time books by Northcott, Rotman and Hilton and Stammbach were freely used while lecturing. The material covered in the book may be adopted for a two semester course, while a one semester course could be based on the first seven chapters. The book shall also be useful for researchers who like to use the subject in their study. Complete detailed proofs are given to make the book easy for self study.

The book aims at giving just a basic course on the subject and is by no means exhaustive. Several important areas in the subject have not even been touched upon.

We now briefly describe the contents of the book. The book starts with a brief account of modules, homomorphisms of modules and elementary properties of tensor products of modules. Direct and inverse limits of families of modules and pull back and push out diagrams are also introduced. The concept of categories and functors is introduced in Chapter 2. Although homomorphisms and tensor products of modules are studied in Chapter 1, functorial properties of $Hom$ and Tensor Product are discussed here.

Homological algebra may be aptly described as a study of derived functors of (additive) functors, in particular, the functors $Hom$ and Tensor product. Derived functors of additive functors are defined in Chapter 5. For defining these the existence of projective and injective resolutions for every module is needed and the same is also established in this chapter. Chapters 3 and 4 are preparatory for defining and studying derived functors. Derived functors of tensor product are called torsion functors ($Tor_n^R$) while those of $Hom$ are called extension functors ($Ext_R^n$). Some special properties of the functors ($Tor_n^R$) and ($Ext_R^n$) are studied in Chapters 6 and 7. Torsion and extension functors can also be defined for categories not having enough projectives or enough injectives (contrary to the category of modules) and also derived functors of non-additive functors can be defined but we do not discuss these here.

Chapter 8 gives a connection between the ring of scalars and the modules over that ring. Among other things, it is proved that (i) over hereditary rings the distinction between projective and injective modules disappears while (ii) over Dedekind domains the same happens for divisible and injective modules. For studying the (co) homology of the direct sum of two groups, it is necessary to know the connection between the homologies of two complexes $X$, $Y$ and that of the complex $X \otimes Y$. This relationship is given by the Kunneth formula a special case of which as needed later is considered in Chapter 9. Chapter 10 studies projective and injective dimensions of modules and left and right global dimensions of rings. Only simple characterizations of these are given. However, the equality of left and right global dimensions of a ring which is both right and left Noetherian is proved. A characterization of left global dimension of a left Artin ring is also given.

A study of a special case of torsion and extension functors, i.e., the case when the ring of scalars is the integral group ring $ZG$ of a group $G$ is taken up in Chapter 11. These special cases are $H_n(G, A)$ and $H^n(G, A)$ the $n^{\text{th}}$ homology group of $G$ and the $n^{\text{th}}$ cohomology group of $G$ with coefficients in a G-module A. Homology and cohomology groups can also be defined through (co)cycles and (co)boundaries. That the two approaches lead to the same objects, up to isomorphism, is established by introducing the Bar resolution. The last three sections of the chapter are developed to mainly obtain information about the second cohomology group. The connection between the study of $H^2(G, A)$ and the study of group extensions of $A$ by $G$ is discussed. The 5-term exact sequence of Hochschild and Serre connecting the cohomology of a group $G$ to those of a normal subgroup $H$ of $G$ and the quotient group $G/H$ and some extensions of this sequence are obtained.

The last chapter, as applications of homological methods, gives two purely group theoretical problems. One of these is a result of Gasch*ü*tz that every non-Abelian finite $p$-group has an outer automorphism of $p$-power order and the other result as shown by Magnus is that a group having a free presentation with $n + r$ generators and $r$ relations which can also be generated by $n$ elements is a free group of rank $n$.

I would like to express my sincere thanks to my teachers (i) I.B.S. Passi who introduced me to the subject and (ii) D. Rees. I am thankful to my research student Manoj Kumar for his help in transferring the manuscript from M.S.Word to LaTex. Without his help, it would, perhaps, not have been possible to give the manuscript the present shape. I like to place on record my appreciation for (i) my colleagues Vivek Sharma and Pradeep Kumar for their help, (ii) my student Suman Choudhary and (iii) the authorities of JMIT, Radaur for providing facilities during the last stage of the project .

# Contents

# Chapter 1

# Modules

This chapter is preparatory in nature and we give some results on modules. We define a free module and prove that every (left) $R$-module is homomorphic image of a free $R$-module. When $A$, $B$ are left $R$-modules and $C$ is a right $R$-module, the Abelian groups $Hom_R(A, B)$ and $C \otimes_R B$ are defined and some properties of these are obtained. The concepts of direct limit, inverse limit, pull back and push out are introduced and some properties of these are obtained.

## 1.1 Modules

**Definition 1.1.1** Let $R$ be a ring with identity. An additive Abelian group $M$ is called a **left $R$-module** if there exists for every element $r \in R$, and every element $a \in M$, a uniquely determined element $ra$ of $M$ such that the following hold :

(i) $(r + s)a = ra + sa$ for every $r, s \in R, a \in M$;
(ii) $(rs)a = r(sa)$ for every $r, s \in R, a \in M$;
(iii) $r(a + b) = ra + rb$ for every $r \in R, a, b \in M$;
(iv) $1a = a$ for every $a \in M$, where 1 denotes the identity of the ring $R$.

**1.1.2** Similarly, an additive Abelian group $M$ is called a **right $R$-module** if for every $r \in R$ and $a \in M$ there exists a unique element $ar$ of $M$ such that the following hold :

(i)$'$ $a(r + s) = ar + as$ for every $r, s \in R, a \in M$;
(ii)$'$ $(ar)s = a(rs)$ for every $r, s \in R, a \in M$;
(iii)$'$ $(a + b)r = ar + br$ for every $r \in R, a, b \in M$;
(iv)$'$ $a1 = a$ for every $a \in M$, where 1 denotes the identity of the ring $R$.

**1.1.3** If $R$ is a commutative ring with identity and $M$ is a left $R$-module, let us define $a.r$ for $a \in M, r \in R$ by $a.r = ra$. Then, for $a \in M$, $r, s \in R$,

$$a.(rs) = (rs)a = (sr)a = s(ra) = s(a.r) = (a.r).s.$$

Properties (i)$'$, (iii)$'$ and (iv)$'$ of a right $R$-module can be checked even more easily. Thus $M$ has been converted into a right $R$-module. On the other hand, a right $R$-module can similarly be made into a left $R$-module. In view of this when the ring $R$ is commutative, we talk of only an $R$-module rather than a right $R$-module or a left $R$-module.

(i) Let $S$ be a ring and $R$ be a subring of $S$. If $r \in R, s \in S$, then $r, s \in S$ and, therefore, $rs \in S$. Using the associative law for multiplication in $S$ and the distributive laws we find that $S$ becomes a left $R$-module. We can also similarly regard $S$ as a right $R$-module. In particular, we find that every ring $R$ can be regarded as a right as well as a left $R$-module.

(ii) Let $A$ be an Abelian group written additively. For an integer $n$ and $a \in A$ define $na$ to be 0 if $n = 0$, $a + a + \cdots + a(n$ times) if $n$ is positive and $na = (-n)(-a)$ if $n$ is negative. $A$ then becomes a $Z$-module.

(iii) Let $R$ be a ring and $G$ be a group written multiplicatively. Let $RG$ denote the set of all finite formal sums $\sum_{g \in G} r_g g, r_g \in R$ and $r_g = 0$ except for finite number of elements $g \in G$. For $\sum_{g \in G} r_g g, \sum_{g \in G} s_g g \in RG$, say that $\sum_{g \in G} r_g g = \sum_{g \in G} s_g g$ if and only if $r_g = s_g$ for every $g \in G$. For $\sum_{g \in G} r_g g, \sum_{g \in G} s_g g \in RG$, define

$$\sum_{g \in G} r_g g + \sum_{g \in G} s_g g = \sum (r_g + s_g) g$$

and

$$(\sum r_g g)(\sum s_g g) = (\sum r_g g)(\sum s_h h) = \sum (r_g s_h) gh.$$

With these compositions $RG$ becomes a ring. Identifying $r \in R$ with the element $r1$ of $RG$, where 1 denotes the identity of the group $G$, the ring $R$ becomes a subring of $RG$. The ring $RG$ is called the **group ring** of the group $G$ over the ring $R$. By (i) $RG$ becomes a left $R$-module as well as a right $R$-module.

(iv) Let $R$ be a ring, $G$ be a group and $H$ be a subgroup of $G$. The group ring $RH$ is a subring of the group ring $RG$ and, so, $RG$ is left (as well as a right) $RH$-module.

**Definition 1.1.4** If $M$ is a left $R$-module, a subgroup $N$ of the additive group $M$ is called a **submodule** of $M$ if for every $a \in N$, $r \in R$, the element $ra \in N$.

Observe that a nonempty subset $N$ of a left $R$-module $M$ is a submodule of $M$ if and only if (i) for every $a, b \in N$, the element $a - b \in N$; and (ii) $ra \in N$ for every $r \in R, a \in N$.

**1.1.5** Let $M$ be a left $R$-module and $N$ be a submodule of $M$. Then $N$ being a subgroup of the additive Abelian group $M$, we have the Abelian group $M/N = \{a + N | a \in M\}$, the quotient group of $M$ modulo the subgroup $N$. For $a \in M, r \in R$, define

$$r(a + N) = ra + N. \tag{1.1}$$

If $a + N = b + N$, then $b = a + c$ for some $c \in N$ and $r(b + N) = rb + N = r(a+c) + N = ra + rc + N = ra + N = r(a+N)$, as $rc \in N$. Thus the scalar product as in (1.1) is well defined and $M/N$ becomes a left $R$-module and is called the **quotient module** of $M$ modulo the submodule $N$.

**1.1.6** Let $M$, $N$ be two left $R$-modules. A map $f : M \to N$ is called an **$R$-homomorphism** or **module homomorphism** if $f(a+b) = f(a) + f(b)$ for all $a, b \in M$; and $f(ra) = rf(a)$ for all $r \in R, a \in M$.

An $R$-homomorphism $f : M \to N$ is called a **monomorphism** if the map $f$ is one-one; it is called an **epimorphism** if the map $f$ is onto; and it is called an **isomorphism** if the map $f$ is both one-one and onto. Two left $R$-modules $M$ and $N$ are said to be **isomorphic** if there exists an $R$-isomorphism from $M$ to $N$ or $N$ to $M$.

Let $M$, $N$ be left $R$-modules and $f : M \to N$ be an $R$-homomorphism. Then **kernel** and **image** of $f$ are, respectively, defined by

$$\begin{aligned} Ker f &= \{a \in M | f(a) = 0\}; \\ Im f &= \{x \in N | x = f(a) \text{ for some } a \in M\} \\ &= \{f(a) | a \in M\}. \end{aligned}$$

It is fairly easy to see that $Ker\ f$ is a submodule of $M$ and $Im\ f$ is a submodule of $N$. Then **cokernel** and **coimage** of $f$ are, respectively, defined by

$$Coker\ f = N/Im\ f \text{ and } Coim\ f = M/Ker f.$$

**1.1.7 Exercises**

1. If $f : M \to N$ is an $R$-homomorphism of left $R$-modules, prove that $Ker\ f$ is a submodule of $M$ and $Im\ f$ is a submodule of $N$.
2. Prove that an $R$-homomorphism $f : M \to N$ is a monomorphism if and only if $Ker\ f = 0$.
3. If $M$ is a left $R$-module and $N$ is a submodule of $M$, prove that any submodule of $M/N$ is of the form $K/N$ where $K$ is a submodule of $M$ with $N \subset K$.
4. If $f : M \to N$ is an $R$-homomorphism of left $R$-modules, then $M/Ker\ f \cong Im\ f$.

5. If $A, B$ are submodules of a left $R$-module $M$, prove that $A \cap B$ is a submodule of $M$.

6. For submodules $A, B$ of a left $R$-module $M$, define $A+B = \{a+b | a \in A, b \in B\}$. Prove that $A + B$ is a submodule of $M$ containing both $A$ and $B$.

7. For submodules $A, B$ of a left $R$-module $M$, prove that $(A+B)/B \cong A/A \cap B$.

8. If $I$ is a left ideal of $R$, show that $R/I = \{r + I | r \in R\}$ is a left $R$-module.

9. If $S$ is another ring with identity, an Abelian group $M$ is called an $(R, S)$-**bimodule** if $M$ is a left $R$-module, a right $S$-module and for every $r \in R, s \in S, a \in M, (ra)s = r(as)$.

The ring $R$ is an $(R, R)$-bimodule.

*All R-modules considered shall be left R-modules unless mentioned explicitly to the contrary.*

## 1.2 Free Modules

**1.2.1** Let $\{M_i\}_{i \in I}$ be a family of left $R$-modules. Let $\Pi_{i \in I} M_i$ denote the set of all sequences $(x_i)_{i \in I}$, $x_i \in M_i$. If $(x_i)_{i \in I}, (y_i)_{i \in I} \in \Pi_{i \in I} M_i$, say that $(x_i) = (y_i)$ if and only if $x_i = y_i$ for every $i \in I$. For $(x_i)_{i \in I}$, $(y_i)_{i \in I} \in \Pi_{i \in I} M_i$, and $r \in R$, define

$$(x_i) + (y_i) = (x_i + y_i) \text{ and } r(x_i) = (rx_i).$$

With these compositions $\Pi_{i \in I} M_i$ becomes a left $R$-module. Observe that the additive identity of $\Pi_{i \in I} M_i$ is the sequence $(x_i)_{i \in I}$ where $x_i = 0$ for every $i \in I$. We denote this element by $(0)$ or simply $0$. Also, for any $(x_i) \in \Pi_{i \in I} M_i$, its additive inverse is the element $(y_i)$, where $y_i = -x_i$ for every $i \in I$ . We write the additive inverse of $(x_i)$ as $(-x_i)_{i \in I}$.

The left R-module $\Pi_{i \in I} M_i$ is called the **direct product** of the family $\{M_i\}_{i \in I}$ of left $R$-modules.

Let $\oplus \sum_{i \in I} M_i$ denote the subset of $\Pi_{i \in I} M_i$ consisting of those sequences $(x_i)_{i \in I}$ in which $x_i = 0$ except for a finite number of $i \in I$ . It is easy to see that $\oplus \sum_{i \in I} M_i$ is a submodule of the $R$-module $\Pi_{i \in I} M_i$. The left $R$-module $\oplus \sum_{i \in I} M_i$ is called the **external direct sum** of the family $\{M_i\}_{i \in I}$ of left R-modules.

On the other hand, given a left $R$-module $M$ and a family of submodules $\{M_i\}_{i \in I}$ of $M$, $M$ is called the **(internal) direct sum** of the family of submodules if every $a \in M$ can be uniquely written as $\sum_{i=1}^{n} a_{ij}$, where $a_{ij} \in M_{ij}$, $1 \leq i \leq n$. Also each $M_i$ is then called a **direct summand** of $M$.

### 1.2.2 Exercises

1. Let $\{M_i\}_{i\in I}$ be a family of left R-modules and let $M = \oplus \sum_{i\in I} M_i$. Prove that, for every $i \in I$, there exists a submodule $M_i'$ of $M$ such that
(a) $M_i' \cong M_i$ for every $i \in I$;
(b) $M$ is the (internal) direct sum of the family $\{M_i'\}$ of submodules of $M$.

2. Prove that a left $R$-module $M$ is the direct sum of its submodules $M_i$, $1 \leq i \leq n$, if and only if $M = \sum_{i=1}^{n} M_i$ and $M_i \cap \sum_{j=1,\, j\neq i}^{n} M_j = \{0\}$ for every $i, 1 \leq i \leq n$.

**1.2.3** Let $M$ be a left $R$-module and $X$ be a subset of $M$. If every element of $M$ can be written as a finite sum $\sum r_i x_i, r_i \in R, x_i \in X$, then $M$ is said to be generated by $X$ or that $X$ generates the left $R$-module $M$. The module $M$ is said to be **finitely generated** if $X$ is a finite subset of $M$. If the subset $X$ consists of a single element $x$ (say), then $M$ is called a **cyclic module** generated by $x$.

Again consider a family $\{M_i\}_{i\in I}$ of $R$-modules. For any $j \in I$, define $\alpha_j : M_j \to \Pi_{i\in I} M_i$ and $\pi_j : \Pi_{i\in I} M_i \to M_j$ by $\alpha_j(x_j) = (y_k)$,
where $y_k = 0$ for $k \neq j$ and $y_j = x_j \in M_j$; $\pi_j((x_i)) = x_j$,
where $x_i \in M_i$ for $i \in I$.
The maps $\alpha_j, \pi_j$ are $R$-homomorphisms and

$$\pi_k\, \alpha_j = \begin{cases} \text{identity} & \text{if } k = j \\ 0 & \text{if } k \neq j. \end{cases}$$

Also every $\alpha_j$ is a monomorphism while every $\pi_j$ is an epimorphism. It is clear that every $\alpha_j$ takes values in $\oplus \sum_{i\in I} M_i$ so that we have monomorphisms $\alpha_j : M_j \to \oplus \sum_{i\in I} M_i$. Restriction of $\pi_j$ to the submodule $\oplus \sum_{i\in I} M_i$ of $\Pi_{i\in I} M_i$ again yields epimorphisms $\pi_j : \oplus \sum_{i\in I} M_i \to M_j$.

Let $x \in \oplus \sum M_i$. Suppose that in $x = (x_i)$ the non-zero components are $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$. Then $\pi_i(x) = 0$ for $i \notin \{i_1, \ldots, i_k\}$ and $\pi_i(x) = x_i$ for $i \in \{i_1, \ldots, i_k\}$. Therefore $\sum_i \alpha_i \pi_i(x) = x$. Hence $\sum_i \alpha_i \pi_i =$ identity map of $\oplus \sum M_i$.

We next consider the universal property of direct sum and direct product. We consider the family $\{M_i\}_{i\in I}$ of left $R$-modules, $\oplus \sum_{i\in I} M_i$ the direct sum and $\Pi_{i\in I} M_i$ the direct product of this family with monomorphism $\alpha_j : M_j \to \Pi_{i\in I} M_i$ for every $j \in I$ and epimorphism $\pi_j : \Pi_{i\in I} M_i \to M_j$ for every $j$.

**Theorem 1.2.4** *Given any left $R$-module $M$ and monorphisms $f_j : M_j \to M$ for every $j \in I$, then there exists a unique $R$-homomorphism $f : \oplus \sum_{i\in I} M_i \to M$ such that $f\, \alpha_j = f_j$ for every $j \in I$.*

*Proof.* Let $x = (x_i) \in \oplus \sum_{i\in I} M_i$. Since $x_i \neq 0$ for only finitely many values of $i$, we can define $f : \oplus \sum_{i\in I} M_i \to M$ by

$$f((x_i)) = \sum_{i\in I} f_i(x_i).$$

The map $f$ is an $R$-homomorphism. Consider $x_j \in M_j$ for a $j \in I$. Then $\alpha_j(x_j) = (y_i)$, where $y_i = 0$ for $i \neq j$ and $y_j = x_j$. Therefore $(f\alpha_j)(x_j) = f(\alpha_j(x_j)) = f(y_i) = \sum_{i\in I} f_i(y_i) = f_j(x_j)$. Hence $f\ \alpha_j = f_j$ for every $j \in I$.

Let $g : \oplus\sum M_i \to M$ be another $R$-homomorphism such that $g\ \alpha_j = f_j$ for every $j \in I$. An $x \in \oplus\sum M_i$ can be written as $x = \sum \alpha_j(x_j)$, where $j$ runs over a finite subset of $I$. Therefore

$$\begin{aligned} g(x) &= g(\sum \alpha_j(x_j)) = \sum (g\ \alpha_j)(x_j) \\ &= \sum f_j(x_j) = \sum f\ \alpha_j(x_j) \\ &= f(\sum \alpha_j(x_j)) = f(x) \end{aligned}$$

showing that $g = f$. $\square$

Let $A$ be another left $R$-module with monomorphisms $\beta_j : M_j \to A$ for every $j \in I$. Suppose that given any left $R$-module $M$ and $R$-monomorphism $g_j : M_j \to M$ for every $j \in I$, there exists a unique homomorphism $g : A \to M$ such that $g\,\beta_j = g_j$ for every $j \in I$. Taking $M = \oplus\sum M_i$ and $g_j = \alpha_j$, we find that there exists a unique homomorphism $\alpha : A \to \oplus\sum M_i$ such that

$$\alpha\beta_j = \alpha_j \text{ for every } j. \tag{1.2}$$

On the other hand, taking $M = A$ and $f_j = \beta_j$ in Theorem 1.2.4, there exists a unique $R$-homomorphism $\beta : \oplus\sum M_i \to A$ such that

$$\beta\alpha_j = \beta_j \text{ for every } j. \tag{1.3}$$

together (1.2) and (1.3) imply $(\beta\alpha)\beta_j = \beta_j$ and $(\alpha\beta)\alpha_j = \alpha_j$ for every $j \in I$.

Since $1_A\beta_j = \beta_j$ and $1_{\oplus\sum M_i}\alpha_j = \alpha_j$ for every $j \in I$, it follows by uniqueness of the homomorphism which exists as in Theorem 1.2.4 and of the homomorphism which exists as in the case of property of $A$, we get

$$\beta\alpha = 1_A \text{ and } \alpha\beta = 1_{\oplus\sum M_i}.$$

Therefore $\alpha : A \to \oplus\sum M_i$ is an isomorphism with $\beta : \oplus\sum M_i \to A$ as its inverse. This proves that direct sum of modules is determined uniquely up to isomorphism by the **universal property** as in Theorem 1.2.4.

**Theorem 1.2.5** *Given any left $R$-module $M$ and $R$-epimorphisms $g_j : M \to M_j$, for every $j \in I$, then there exists a unique homomorphism $g : M \to \Pi_{i\in I} M_i$ such that $\pi_j g = g_j$ for every $j \in I$.*

*Proof.* Define a map $g : M \to \Pi_{i\in I}M_i$ by $g(x) = (g_i(x)), x \in M$. Then $g$ is an $R$-homomorphism and $\pi_j g = g_j$.

Let $f : M \to \Pi_{i\in I}M_i$ be another $R$-homomorphism such that $\pi_j f = g_j$ for every $j \in I$. For $x \in M$, let $f(x) = (x_i)$. Then $g_j(x) = \pi_j f(x) = \pi_j(x_i) = x_j$ for every $j \in J$. Therefore $f(x) = (x_i) = (g_i(x)) = g(x)$ and we have $g = f$. □

We next prove that direct product of left $R$-modules is determined uniquely up to isomorphism by the universal property as mentioned in Theorem 1.2.5.

Let $A$ be a left $R$-module with epimorphisms $\lambda_j : A \to M_j$. Suppose that for every left $R$-module $M$ and epimorphisms $f_j : M \to M_j$ there exists a unique $R$-homomorphism $f : M \to A$ such that $\lambda_j f = f_j$ for every $j$. By taking $M = \Pi_{i\in I}M_i$ and $f_j = \pi_j$ for every $j$ in the above property of $A$, let $f : \Pi_{i\in I}M_i \to A$ be the unique $R$-homomorphism such that

$$\lambda_j f = \pi_j \text{ for every } j \in I. \tag{1.4}$$

On the other hand, taking $M = A$ and $g_j = \lambda_j$ in Theorem 1.2.5, let $g : A \to \Pi_{i\in I}M_i$ be the unique $R$-homomorphism such that

$$\pi_j g = \lambda_j \text{ for every } j \in I. \tag{1.5}$$

Now (1.4) and (1.5) imply that $g\,f : \Pi_{i\in I}M_i \to \Pi_{i\in I}M_i$ and $f\,g : A \to A$ are homomorphisms such that

$$\pi_j(g\,f) = \pi_j \text{ and } \lambda_j(f\,g) = \lambda_j \text{ for every } j \in I. \tag{1.6}$$

Also $1_A : A \to A$ and $1_{\Pi M_i} : \Pi M_i \to \Pi M_i$ are homomorphisms such that $\lambda_j 1_A = \lambda_j$ and $\pi_j 1_{\Pi M_i} = \pi_j$ for every $j \in I$.

By the uniqueness of the homomorphism in the universal property of $\Pi_{i\in I}M_i$ and of the homomorphism as in the property of $A$, we get $g\,f = 1_{\Pi M_i}, f\,g = 1_A$ which show that $f : \Pi_{i\in I}M_i \to A$ is an isomorphism with $g : A \to \Pi_{i\in I}M_i$ as its inverse.

**Definition 1.2.6** A left $R$-module $F$ is called a **free left $R$-module** on a **basis** $X \neq \phi$, if there is a map $\alpha : X \to F$ such that given any map $f : X \to A$, where $A$ is any left $R$-module, there exists a unique $R$-homomorphism $g : F \to A$ such that $f = g\alpha$.

The unique homomorphism $g : F \to A$ is said to extend the map $f : X \to A$.

We observe that the map $\alpha : X \to F$ is necessarily one-one. If not, suppose that $x_1, x_2 \in X$ such that $\alpha(x_1) = \alpha(x_2)$. Take $A = R \times R = \{(r, s)|r, s \in R\}$ converted into a left $R$-module by defining

$$(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2);\ \ r_1, r_2, s_1, s_2 \in R$$

$$\text{and} \quad r(r_1, s_1) = (rr_1, rs_2), \quad r, r_1, s_1 \in R.$$

Take $f : X \to A$ the map such that $f(x_1) = (1, 0), f(x_2) = (0, 1)$ and $f(x) = 0$ for every $x \in X, x \neq x_1, x \neq x_2$. Let $g : F \to A$ be the unique $R$-homomorphism such that $g\alpha = f$. Now

$$(1, 0) = f(x_1) = g\alpha(x_1) = g\alpha(x_2) = f(x_2) = (0, 1)$$

which is a contradiction.

**Theorem 1.2.7** *For every nonempty set $X$, there exists a free left $R$-module $F$ with $X$ as a basis.*

*Proof.* Consider the family $\{R_x\}_{x \in X}$ of left $R$-modules where $R_x = R$ for every $x \in X$. Let $F = \oplus \sum_{x \in X} R_x$. In view of Exercise 1.2.2 (1), we can regard $F$ as the internal direct sum of the family of submodules $\{R_x\}_{x \in X}$. For $x \in X, R_x$ being equal to $R$, we denote an element $r \in R$ when considered as an element of $R_x$ by $rx$. Under this assumption, every element of $F$ can be uniquely written as a finite sum $\sum_{i=1}^{n} r_i x_i$, where $r_i \in R, x_i \in X$. Let $\alpha : X \to F$ be the map $\alpha(x) = 1.x$. Let $A$ be any left $R$-module and $f : X \to A$ be any map . Define $g : F \to A$ by

$$g(\sum r_i x_i) = \sum r_i f(x_i).$$

Clearly $g$ is a well-defined $R$-homomorphism and $f = g\,\alpha$. That $g$ is unique with $f = g\,\alpha$ is clear. Hence $F$ is a free left $R$-module with $X$ as a basis. □

Observe that the argument in the proof of the above theorem leads to an alternative definition of a free module. In view of the map $\alpha : X \to F$ in the definition of a free module being one-one, we may regard $X$ as a subset of $F$. Let $A = \oplus \sum_{x \in X} R_x$, where $R_x = R$ for every $x \in X$. Let $g : F \to A$ be the unique extension of the inclusion map $X \to A$ to an $R$-homomorphism. Since $X$ may also be regarded as a subset of $A$, $g$ maps $\sum r_i x_i$ onto $\sum r_i x_i$. Also $A$ having been proved to be free, let $h : A \to F$ be the unique $R$-homomorphism which extends the inclusion map $X \to F$. Now $h\,g : F \to F$ is a homomorphism which extends the inclusion map $X \to F$ and identity map $F \to F$ also extends this map. Therefore $h\,g = 1$ which implies that $g$ is a monomorphism and, hence, an isomorphism. Thus $F$ is a free left $R$-module with basis $X$ if and only if $F \cong \oplus \sum_{x \in X} R_x, R_x = R$ for every $x \in X$. Moreover, $F$ is free with basis $X$ if and only if every element of $F$ can be uniquely expressed as $\sum r_i x_i, r_i \in R$.

**Theorem 1.2.8** *Every left $R$-module is homomorphic image of a free left $R$-module.*

*Proof.* Let $M$ be a left $R$-module. Let $X$ be a set the elements of which are in one to one correspondence with the elements of $M$. Let the element of $X$ corresponding to the element $a \in M$ be denoted by $x_a$. Let $F$ be the free left $R$-module with $X$ as a basis. Let $f : X \to M$ be the map given by $f(x_a) = a$. Let $g : F \to M$ be the unique homomorphism which satisfies $a = f(x_a) = g(x_a)$. The homomorphism $g$ is clearly an epimorphism. $\square$

**Theorem 1.2.9** *Let $F$ be a free left $R$-module with basis $X$, $\alpha : A \to B$ an epimorphism of left $R$-modules and $f : F \to B$ an $R$-homomorphism. Then there exists an $R$-homomorphism $g : F \to A$ such that $\alpha g = f$.*

*Proof.* For every $x \in X$, choose an element $a_x \in A$ such that $f(x) = \alpha(a_x)$. Define a map $\beta : X \to A$ by $\beta(x) = a_x, x \in X$. Let $g : F \to A$ be the unique $R$-homomorphism which extends the map $\beta$. Let $\lambda \in F$. Then $\lambda = \sum_i r_i x_i$, where $r_i \in R, x_i \in X$. Therefore

$$\alpha g(\lambda) = \alpha g(\sum r_i x_i) = \alpha(\sum r_i g(x_i)) = \alpha(\sum r_i a_{x_i}) = \sum r_i \alpha(a_{x_i})$$

$$= \sum r_i f(x_i) = f(\sum r_i x_i) = f(\lambda)$$

which proves that $\alpha g = f$. $\square$

### 1.2.10 Examples

1. Observe that $R$ is always a free left $R$-module with a basis consisting of a single element. It is also a free right $R$-module with a basis consisting of a single element.

2. Every submodule of a free left $R$-module need not be free. Consider $R = Z/6Z$-the ring of integers modulo 6. $2Z/6Z = \{0+6Z, 2+6Z, 4+6Z\}$ is a submodule of the free $R$-module $R$. The ring $R$ is of order 6 while the order of the module $2Z/6Z$ is 3. Therefore the module $2Z/6Z$ cannot be direct sum of any copies of $R$. Hence $2Z/6Z$ is not a free $R = Z/6Z$-module. However, every submodule of an $R$-module when $R$ is a principal ideal domain is free (we shall come back to it later).

3. Let $R$ be a commutative ring and $R[X]$ be the polynomial ring in the variable $X$ over $R$. Then $R[X]$ is a free $R$-module with basis $\{X^i\}_{i \geq 0}$.

**Proposition 1.2.11** *Let $G$ be a group, $H$ a subgroup of $G$ and $X$ be a right transversal, i.e., set of right coset representatives, of $H$ in $G$. Then $RG$ is a free left $RH$-module.*

*Proof.* Every element $g$ of $G$ can be uniquely written as $hx, h \in H, x \in X$. Therefore any element $\sum r_i g_i \in RG$ can be written as $\sum r_i g_i = \sum_i r_i h_i x_i$ and $r_i h_i \in RH$. This shows that $RG$ is generated as an $RH$-module by $X$.

Suppose that $\sum_{i=1}^{n} a_i x_i = 0$, where $a_i \in RH$. Let $a_i = \sum_{j=1}^{m} r_{ij} h_{ij}$. Then we have $0 = \sum_i (\sum_j r_{ij} h_{ij}) x_i$.

Consider the elements $\{h_{ij} x_i\}_{i,j}$ of $G$ occurring in the above linear combination. Since $h_{ij} \neq h_{ik}$ for $j \neq k$, therefore $h_{ij} x_i \neq h_{ik} x_i$ for $j \neq k$. Also $x_i, x_l$ for $i \neq l$ being in distinct right cosets of $H$ in $G$, $h_{ij} x_i \neq h_{lk} x_l$ for $i \neq l$. Therefore all the elements in the set $\{h_{ij} x_i\}$ are distinct. Therefore $\sum_{i,j} r_{ij} h_{ij} x_i = 0$ shows that $r_{ij} = 0$ for all $i, j$ which implies that $\sum_j r_{ij} h_{ij} = 0$ for all $i$ or $a_i = 0$ for all $i$. This proves that every element of $RG$ can be uniquely written as $\sum a_i x_i, a_i \in RH$. Hence $RG$ is a free left $RH$-module. □

**1.2.12 Exercise**

Prove that the additive group $Q$ of rational numbers is not a free $Z$-module.

## 1.3 Exact Sequences

Consider a sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{1.7}$$

where $A, B, C$ are $R$-modules and $f : A \to B$, $g : B \to C$ are $R$-homomorphisms. We say that sequence (1.7) is a 0-sequence if $Im\, f \subseteq Ker\, g$ while it is said to be exact if $Im\, f = Ker\, g$.

A sequence

$$\cdots \to A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \to \cdots \tag{1.8}$$

which may extend to infinity, where $A_n$ are $R$-modules and every $f_n$ is an $R$-homomorphism is called a **0-sequence** or a **complex** if every sequence $A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1}$ of three consecutive terms is a 0-sequence while it is said to be an **exact sequence** or an **acylic complex** if every such triplet is an exact sequence.

We write the zero module simply as 0 and $0 \to A$, $A \to 0$ for any $R$-module $A$ are the obvious maps or morphisms.

**Lemma 1.3.1** *For $R$-modules $A, B$ and $R$-homomorphism $f : A \to B$*

*(i) $0 \to A \xrightarrow{f} B$ is exact if and only if $f$ is a monomorphism;*

*(ii) $A \xrightarrow{f} B \to 0$ is exact if and only if $f$ is an epimorphism;*

*(iii) $0 \to A \xrightarrow{f} B \to 0$ is exact if and only if $f$ is an isomorphism.*

*Proof.* Exercise.

Let $A$ be an $R$-module, $B$ be a submodule of $A$, $\alpha : B \to A$ the inclusion map and $\beta : A \to A/B$ be the **natural projection**, i.e., $\beta(a) = a + B$, $a \in$

$A$. Then $\alpha$ is a monomorphism, $\beta$ an epimorphism and $Im\,\alpha = B = Ker\,\beta$. Thus the sequence $0 \to B \xrightarrow{\alpha} A \xrightarrow{\beta} A/B \to 0$ is exact.

An exact sequence of the form $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is called a **short exact sequence.**

Let $p$ be any prime and $Z_{p^2}$ denote the cyclic group of order $p^2$. Let $p : Z_{p^2} \to Z_{p^2}$ be the multiplication by $p$. This is a homomorphism and the long sequence

$$\cdots \to Z_{p^2} \xrightarrow{p} Z_{p^2} \xrightarrow{p} Z_{p^2} \xrightarrow{p} Z_{p^2} \to \cdots$$

is exact. However, the sequence

$$0 \to Z_{p^2} \xrightarrow{p} Z_{p^2} \xrightarrow{p} Z_{p^2} \to 0$$

is not exact but the sequence

$$0 \to Z_p \to Z_{p^2} \xrightarrow{p} Z_{p^2} \to Z_p \to 0$$

is exact. Decide the unmarked maps in this sequence.

**Proposition 1.3.2** *For a short exact sequence*

(1.9) $$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

*of $R$-modules and homomorphisms, the following are equivalent :*
*(a) There exists a homomorphism $\alpha : B \to A$ such that $\alpha\, f = 1_A$;*
*(b) There exists a homomorphism $\beta : C \to B$ such that $g\,\beta = 1_C$;*
*(c) $Im\, f$ is a direct summand of $B$.*

*Proof.* We give a circular proof of this result.
(a) $\Rightarrow$ (c). Suppose that there exists a homomorphism $\alpha : B \to A$ such that $\alpha\, f = 1_A$ – the identity map from $A$ to $A$.

Let $b \in B$. Then $\alpha\, f\, \alpha(b) = \alpha(b)$ so that $\alpha(b - f\alpha(b)) = 0$. Then $b - f\alpha(b) \in Ker\,\alpha = K$ (say) and we have $b = k + f\alpha(b)$ for some $k \in K$. Thus $B = K + Im(f)$.

Suppose that there is also an element $k' \in K$ and an element $a \in A$ such that $b = k + f\alpha(b) = k' + f(a)$. Applying $\alpha$ to both sides of this relation we get

$$\alpha(k) + \alpha\, f\, \alpha(b) = \alpha(k') + \alpha\, f(a)$$

$$or\ \ \alpha(b) = \alpha\, f(a) = a.$$

But then $k' = k$. Hence every element of $B$ can be uniquely written as $k + f(a)$ for some $k \in K$, $a \in A$. Therefore $B = K \oplus Im\, f$ and (c) holds.
(c) $\Rightarrow$ (b). Suppose that there exists a submodule $K$ of $B$ such that $B = K \oplus Im\, f$.

Let $c \in C$. Then there exists $b \in B$ such that $c = g(b)$. Let $b = k + f(a)$ for some $k \in K$, $a \in A$. Then $c = g(k)$. If $k' \in K$ is another element such that $c = g(k')$, then $g(k' - k) = 0$ which shows that $k' - k = f(a')$ for some $a' \in A$. The direct sum property of $B$ shows that $k' - k = f(a') = 0$ or that $k' = k$. Hence there exists a unique $k \in K$ such that $c = g(k)$. Define $\beta : C \to B$ by $\beta(c) = k$, where $k \in K$ is the unique element such that $g(k) = c$. The map $\beta$ is an $R$-homomorphism and we have

$$g\beta(c) = g(k) = c \text{ for every } c \in C.$$

Hence $g\beta = 1_C$ – the identity map of $C$.
(b) $\Rightarrow$ (a) Suppose that there exists an $R$-homomorphism $\beta : C \to B$ such that $g\beta = 1_C$.

Let $b \in B$ and suppose that $g(b) = c$. Then $g(b) = c = g\beta(c)$ so that $g(b - \beta(c)) = 0$. Then there exists $a \in A$ such that $b = \beta(c) + f(a)$. If we also have $\beta(c) + f(a) = \beta(c') + f(a')$, then $\beta(c' - c) = f(a - a')$.

Therefore $0 = gf(a - a') = g\beta(c' - c) = c' - c$, i.e., $c' = c$. But then $a' = a$ also. Hence every element of $B$ can be uniquely written as $\beta(c) + f(a)$, $c \in C$, $a \in A$. Define a map $\alpha : B \to A$ by

$$\alpha(\beta(c) + f(a)) = a,\ c \in C,\ a \in A.$$

$\alpha$ is a well-defined homomorphism and $\alpha f = 1_A$. This proves (a). □

**Definition 1.3.3** If any one of the three equivalent conditions of Proposition 1.3.2 is sastisfied, then (1.9) is called a **split exact sequence** or that the sequence (1.9) is said **to split**.

**Corollary 1.3.4** *If the exact sequence (1.9) splits and* $\alpha, \beta$ *are as in Proposition 1.3.2, then (i)* $B = f(A) \oplus \beta(C) = f(A) \oplus Ker\, g \cong A \oplus C$ *and (ii)* $\alpha\beta = 0$.

### 1.3.5 Exercises

1. Prove that the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of $R$-modules and homomorphisms is split exact if and only if there exist homomorphisms $\alpha : B \to A$, $\beta : C \to B$ satisfying $\alpha f = 1_A$, $g f = 0$, $g \beta = 1_C$, $f\alpha + \beta g = 1_B$ and $\alpha\beta = 0$.

2. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of $R$-modules and homomorphisms.
(a) If $A$, $C$ are finitely generated, then prove that so is $B$.
(b) If $B$ is finitely generated, prove that $C$ is finitely generated. Is $A$ also finitely generated ? Justify.

3. If $0 \to A \xrightarrow{f} B \xrightarrow{g} F \to 0$ is an exact sequence and $F$ is a free $R$-module, prove that the sequence splits. Is the condition of $F$ being free necessary?

4. If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact sequence of finite Abelian groups and orders of $A$ and $C$ are coprime, prove that the sequence splits.
(Hint: Let O(A) = m, O(C) = n, where O(K) (as also $|K|$) denotes for a group $K$ the order of $K$. Let $r, s \in Z$ such that $mr + ns = 1$. Define a map $\gamma : C \to B$ by $\gamma(c) = mr\, b$, where $g(b) = c$.)

One of the central ideas prevalent in homological algebra is that of diagram chasing. We next consider a couple of simple results of this nature one of which is embodied in the Five Lemma.

A diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

of $R$-modules and homomorphisms is said to be **commutative** if $g\alpha = \beta f$ while the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, "\alpha"] \\ C & \end{tikzcd}$$

is said to be commutative if $\alpha f = g$.

The idea of commutativity of larger diagrams is understood in an obovious way.

For example: (i) if $f : A \to B$ is a homomorphism of $R$-modules and $L = Ker\, f$, $\overline{f} : A/L \to B$ is the induced homomorphism, i.e., $\overline{f}(a + L) = f(a)$, $a \in A$, then $f = \overline{f}\pi$, where $\pi : A \to A/L$ is the natural projection. Thus the following diagram is commutative

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[dr, "f"'] & A/L \arrow[d, "\overline{f}"] \\ & B \end{tikzcd}$$

.

(ii) If $M$ is an $R$-module and $A, B$ are submodules of $M$ with $A \subset B$, let $\pi : B \to B/A$, $\pi' : M \to M/A$ be the natural projections and $i : B/A \to M/A$, $j : B \to M$ be the inclusion maps. Then $i\pi = \pi' j$ and the following diagram is commutative

$$\begin{tikzcd} B \arrow[r, "\pi"] \arrow[d, "j"'] & B/A \arrow[d, "i"] \\ M \arrow[r, "\pi'"'] & M/A \end{tikzcd}$$

**Lemma 1.3.6 (Five Lemma)** *Consider a commutative diagram*

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "t_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "t_2"] & A_3 \arrow[r, "f_3"] \arrow[d, "t_3"] & A_4 \arrow[r, "f_4"] \arrow[d, "t_4"] & A_5 \arrow[d, "t_5"] \\ B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3 \arrow[r, "g_3"'] & B_4 \arrow[r, "g_4"'] & B_5 \end{tikzcd}$$

*of $R$-modules and homomorphisms with exact rows. (i) If $t_2$ and $t_4$ are epimorphisms and $t_5$ is a monomorphism, then $t_3$ is an epimorphism.*
*(ii) If $t_2$ and $t_4$ are monomorphisms and $t_1$ is an epimorphism, then $t_3$ is a monomorphism.*

*Proof.* (i) Suppose that $t_2, t_4$ are epimorphisms and $t_5$ is a monomorphism. Let $b_3 \in B_3$. Then $g_3(b_3) \in B_4$ and since $t_4$ is an epimorphism, there exists an element $a_4 \in A_4$ such that $g_3(b_3) = t_4(a_4)$. Now $t_5 f_4(a_4) = g_4 t_4(a_4) = g_4 g_3(b_3) = 0$ and $t_5$ is a monomorphism. Therefore $f_4(a_4) = 0$. The upper row being exact, there exists an element $a_3 \in A_3$ such that $f_3(a_3) = a_4$. Then $g_3(b_3) = t_4(a_4) = t_4 f_3(a_3) = g_3 t_3(a_3)$ and, so, $g_3(b_3 - t_3(a_3)) = 0$. Therefore there exists an element

$b_2 \in B_2$ such that $b_3 - t_3(a_3) = g_2(b_2)$. The homomorphism $t_2$ being an epimorphism, there exists an $a_2 \in A_2$ such that $b_2 = t_2(a_2)$. But then $b_3 - t_3(a_3) = g_2(b_2) = g_2(t_2(a_2)) = (g_2t_2)(a_2) = t_3f_2(a_2)$ or $b_3 = t_3f_2(a_2) + t_3(a_3) = t_3(a_3 + f_2(a_2))$. Hence $t_3$ is an epimorphism.
(ii) Now suppose that $t_2, t_4$ are monomorphisms and $t_1$ is an epimorphism. Let $a_3 \in A_3$ such that $t_3(a_3) = 0$. Then

$$0 = g_3t_3(a_3) = t_4f_3(a_3).$$

$t_4$ being a monomorphism, we have $f_3(a_3) = 0$ and, therefore, there exists an element $a_2 \in A_2$ such that $a_3 = f_2(a_2)$. But then $g_2t_2(a_2) = t_3f_2(a_2) = t_3(a_3) = 0$ and, therefore, there exists an element $b_1 \in B_1$ such that $t_2(a_2) = g_1(b_1)$. Since $t_1$ is an epimorphism, there exists $a_1 \in A_1$, such that $t_1(a_1) = b_1$. Then $t_2(a_2) = g_1(b_1) = g_1t_1(a_1) = t_2f_1(a_1)$.

Now $t_2$ being a monomorphism, we have $a_2 = f_1(a_1)$ and, hence, $a_3 = f_2(f_1(a_1)) = 0$. This completes the proof that $t_3$ is a monomorphism. □

Combining (i) and (ii) of the lemma, we have the following.

**Corollary 1.3.7** *If $t_2$ and $t_4$ are isomorphisms, $t_1$ an epimorphism and $t_5$ a monomorphism, then $t_3$ is an isomorphism.*

**Remark 1.3.8** Observe that in the proof of (i) of the five lemma, the maps $t_1, f_1, g_1$ do not play any part while in the proof of (ii) of the lemma $t_5, f_4, g_4$ do not play any part.

Making use of the above observation, we can have as an immediate consequence of the lemma.

**Corollary 1.3.9** *Consider a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r, "f_1"] \arrow[d, "t_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "t_2"] & A_3 \arrow[r] \arrow[d, "t_3"] & 0 \\ 0 \arrow[r] & B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3 \arrow[r] & 0 \end{tikzcd}$$

*with exact rows. If any two of $t_1, t_2, t_3$ are isomorphisms, then so is the third.*

For the proof of this we can think of maps $t_0 : 0 \to 0$ and $t_4 : 0 \to 0$ given making the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "t_0"] & A_1 \arrow[r, "f_1"] \arrow[d, "t_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "t_2"] & A_3 \arrow[r] \arrow[d, "t_3"] & 0 \arrow[d, "t_4"] \\ 0 \arrow[r] & B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3 \arrow[r] & 0 \end{tikzcd}$$

commutative. Both $t_0, t_4$ are isomorphisms. When $t_1, t_3$ are isomorphisms, then so is $t_2$ is the result of Corollary 1.3.7. If $t_2, t_3$ are isomorphisms, it follows from Lemma 1.3.6 (i) that $t_1$ is an epimorphism. If $a_1 \in A_1$ such that $t_1(a_1) = 0$, then $0 = g_1 t_1(a_1) = t_2 f_1(a)$ and both $f_1, t_2$ being monomorphisms $a_1 = 0$. Hence $t_1$ is a monomorphism and so an isomorphism.

If $t_1, t_2$ are isomosphisms, then Lemma 1.3.6 (ii) shows that $t_3$ is a monomorphism. Also, for any $b_3 \in B_3$, there exists $b_2 \in B_2$ such that $g_2(b_2) = b_3$. Then there exists an $a_2 \in A_2$ such that $b_2 = t_2(a_2)$. Therefore

$$b_3 = g_2(b_2) = g_2(t_2(a_2)) = (g_2 t_2)(a_2) = (t_3 f_2)(a_2) = t_3(f_2(a_2))$$

showing that $t_3$ is an epimorphism. Hence $t_3$ is an isomorphism.

**Proposition 1.3.10** *Consider a diagram of $R$-modules and homomorphisms*

$$\begin{tikzcd} & & M \arrow[d, "f"] & \\ 0 \arrow[r] & A \arrow[r, "\alpha"'] & B \arrow[r, "\beta"'] & C \end{tikzcd}$$

*with exact row such that $\beta f = 0$. Then there exists a unique homomorphism $g : M \to A$ such that $\alpha g = f$.*

*Proof.* Exercise.

**Corollary 1.3.11** *Given a commutative diagram*

$$\begin{tikzcd} & A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\ 0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \end{tikzcd}$$

*of $R$-modules and homomorphisms with $g f = 0$ and lower row exact, there exists a unique homomorphism $\alpha : A \to A'$ such that $f'\alpha = \beta f$.*

*Proof.* Here $\beta f : A \to B'$ is a homomorphism such that $g'(\beta f) = (g'\beta) f = (\gamma g) f = \gamma(g f) = 0$. Then, by the proposition there exists a unique homomorphism $\alpha : A \to A'$ such that $f'\alpha = \beta f$. $\square$

**Proposition 1.3.12** *Given a diagram*

$$\begin{tikzcd} A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] \arrow[d, "f"] & C \arrow[r] & 0 \\ & M & & \end{tikzcd}$$

*of $R$-modules and homomorphisms with exact row such that $f\alpha = 0$, then there exists a unique homomorphism $g : C \to M$ such that $f = g\beta$.*

*Proof.* Let $c \in C$. The map $\beta$ being an epimorphism, there exists a $b \in B$ such that $\beta(b) = c$. If $b' \in B$ is another element such that $\beta(b') = c$, then $\beta(b' - b) = 0$ and there exists an element $a \in A$ such that $b' - b = \alpha(a)$. But then $f(b') - f(b) = f(b' - b) = f\alpha(a) = 0$ showing that the element $f(b) \in M$ is independent of the choice of the element $b \in B$ such that $\beta(b) = c$. Define $g : C \to M$ by $g(c) = f(b)$, where $b \in B$ is such that $\beta(b) = c$.

Let $c_1, c_2 \in C, r \in R$. Choose $b_1, b_2 \in B$ such that $\beta(b_1) = c_1, \beta(b_2) = c_2$ so that $g(c_1) = f(b_1), g(c_2) = f(b_2)$. Then

$$\beta(b_1 + b_2) = \beta(b_1) + \beta(b_2) = c_1 + c_2; \quad \beta(r\, b_1) = r\beta(b_1) = r\, c_1$$

which imply that

$$g(c_1 + c_2) = f(b_1 + b_2) = f(b_1) + f(b_2) = g(c_1) + g(c_2)$$

$$g(r\, c_1) = f(r\, b_1) = rf(b_1) = rg(c_1).$$

Therefore $g$ is an $R$-homomorphism. That $g\,\beta = f$ is clear from the definition of $g$.

Let $h : C \to M$ be another homomorphism such that $h\beta = f$.

Let $c \in C$ and choose $b \in B$ such that $\beta(b) = c$. Then $g(c) = f(b) = (h\beta)(b) = h(\beta(b)) = h(c)$ which shows that $g = h$. □

**Corollary 1.3.13** *Given a commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] & 0 \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' &
\end{tikzcd}$$

*of $R$-modules and homomorphisms with $g'f' = 0$ and the upper row exact, then there exists a unique $R$-homomorphism $\gamma : C \to C'$ such that $\gamma\, g = g'\beta$.*

*Proof.* $g'\beta : B \to C'$ is a homomorphism such that $(g'\beta)f = g'(\beta\, f) = g'(f'\alpha) = (g'f')\alpha = 0$. Therefore by the proposition there exists a unique homomorphism $\gamma : C \to C'$ such that $g'\beta = \gamma g$. □

### 1.3.14 Exercises

Let

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "t_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "t_2"] & A_3 \arrow[d, "t_3"] \\
B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3
\end{tikzcd}$$

be a commutative diagram of $R$-modules and homomorphisms with exact rows.
1. If $t_1, t_3, g_1$ are monomorphisms, prove that $t_2$ is a monomorphism.
2. If $t_1, t_3, f_2$ are epimorphisms, prove that $t_2$ is also an epimorphism.
3. If $t_1, t_3$ are isomorphisms, $g_1$ is a monomorphism and $f_2$ is an epimorphism, prove that $t_2$ is an isomorphism.

## 1.4 Homomorphisms

**1.4.1** Let $A, B$ be $R$-modules. Let $Hom_R(A, B)$ denote the set of all $R$-homomorphisms from $A$ to $B$. For $f, g \in Hom_R(A, B)$ define $f+g : A \to B$ by

$$(f+g)(a) = f(a) + g(a), \quad a \in A.$$

For $a, a_1, a_2 \in A, r \in R$, $(f+g)(a_1+a_2) = f(a_1+a_2) + g(a_1+a_2)$
$= f(a_1) + f(a_2) + g(a_1) + g(a_2) = (f(a_1) + g(a_1)) + (f(a_2) + g(a_2))$
$= (f+g)(a_1) + (f+g)(a_2)$
and $(f+g)(ra) = f(ra) + g(ra) = rf(a) + rg(a) = r(f(a) + g(a))$
$= r(f+g)(a)$.
Hence $f + g \in Hom_R(A, B)$.

The map $0 : A \to B$ given by $0(a) = 0$ for every $a \in A$ is an $R$-homomorphism and for every $f \in Hom_R(A, B)$, $f + 0 = f = 0 + f$.
For $f \in Hom_R(A, B)$ define $g : A \to B$ by

$$g(a) = -f(a), \; a \in A.$$

If $a, a_1, a_2 \in A$, $r \in R$,

$$g(a_1+a_2) = -f(a_1+a_2) = -(f(a_1)+f(a_2)) = -f(a_1)-f(a_2) = g(a_1)+g(a_2)$$

and

$$g(ra) = -f(ra) = -rf(a) = r(-f(a)) = rg(a).$$

Thus $g \in Hom_R(A, B)$. That $f + g = 0 = g + f$ is clear.

If $f, g, h \in Hom_R(A, B)$ and $a \in A$, then $((f+g)+h)(a) = (f+g)(a) + h(a) = (f(a) + g(a)) + h(a) = f(a) + (g(a) + h(a)) = f(a) + (g+h)(a)$
$= (f + (g+h))(a)$

and we have $(f+g)+h = f+(g+h)$, i.e., the additive composition in $Hom_R(A,B)$ satisfies the associative law.

Again , for $f, g \in Hom_R(A,B)$, $a \in A$,

$$(f+g)(a) = f(a)+g(a) = g(a)+f(a) = (g+f)(a).$$

Therefore $f+g = g+f$. Hence $Hom_R(A,B)$ is an Abelian group.

In general $Hom_R(A,B)$ is not an $R$-module . However, we have the following.

**Proposition 1.4.2** *If $R$ is a commutative ring and $A$, $B$ are $R$-modules, then $Hom_R(A,B)$ is an $R$-module.*

*Proof.* For $r \in R, f \in Hom_R(A,B)$ define $r\,f : A \to B$ by

$$(r\,f)(a) = r\,f(a), \quad a \in A.$$

For $a, a_1, a_2 \in A$, $s \in R$, $(rf)(a_1+a_2) = rf(a_1+a_2)$
$= r(f(a_1)+f(a_2)) = rf(a_1)+rf(a_2) = (rf)(a_1)+(rf)(a_2)$ and

$$(rf)(sa) = rf(sa) = r(sf(a)) = (rs)f(a) = (sr)f(a) = s(rf(a)) = s(rf)(a).$$

Therefore $r\,f \in Hom_R(A,B)$.

It is fairly easy to check that then the Abelian group $Hom_R(A,B)$ becomes an $R$-module. □

In the case when the ring $R$ is not necessarily commutative, we can have the following.

**Proposition 1.4.3** *For any left $R$-module $A$, $Hom_R(R,A)$ is again a left $R$-module.*

*Proof.* For $r \in R$, $f \in Hom_R(R,A)$ define $r\,f : R \to A$ by $(r\,f)(s) = f(s\,r)$, $s \in R$. It is almost trivial that $(rf)(s_1+s_2) = (rf)(s_1)+(rf)(s_2)$ for $s_1, s_2 \in R$. Let $s, s_1 \in R$. Then

$$(r\,f)(ss_1) = f((s\,s_1)r) = f(s(s_1r)) = s\,f(s_1r) = s(r\,f)(s_1).$$

Thus $r\,f$ is an $R$-homomorphism.

That $r(f_1+f_2) = r\,f_1+r\,f_2$, $r_1(r_2f) = (r_1r_2)f$ and $1\,f = f$ for $r, r_1, r_2 \in R$, $f_1, f_2, f \in Hom_R(R,A)$ are easy to check. This completes the proof that $Hom_R(R,A)$ is a left $R$-module. □

**Theorem 1.4.4** *For any $R$-module $A$, $Hom_R(R,A) \cong A$ as $R$-modules.*

*Proof.* Define a map $\theta : Hom_R(R,A) \to A$ by $\theta(f) = f(1)$, 1 the identity of $R$, $f \in Hom_R(R,A)$.

The map $\theta$ is a homomorphism of $R$-modules. If $\theta(f) = 0$, for an $f \in Hom_R(R, A)$, then $f(1) = 0$ and for any $r \in R$, $f(r) = f(r.1) = rf(1) = r.0 = 0$, i.e., $f = 0$. Hence $\theta$ is one-one.

Let $a \in A$. Define $f : R \to A$ by $f(r) = r\,a$, $r \in R$. The map $f$ is an $R$-homomorphism and $\theta(f) = f(1) = a$. Thus $\theta$ is an epimorphism and hence an isomorphism. □

Let $\{M_i\}_{i\in I}$ be a family of $R$-modules; $\oplus \sum_{i\in I} M_i$, the direct sum and $\Pi_{i\in I} M_i$, the direct product of the family with $\alpha_j : M_j \to \oplus \sum_{i\in I} M_i$; $j \in I$, the natural injection and $\pi_j : \Pi_i M_i \to M_j$, $j \in I$, the natural projection.

**Theorem 1.4.5** *For any $R$-module $A$,*
*(i)* $\Pi_i Hom_R(M_i, A) \cong Hom_R(\oplus \sum M_i, A)$
*(ii)* $\Pi_i Hom_R(A, M_i) \cong Hom_R(A, \Pi_i M_i)$
*When $R$ is commutative, the isomorphisms in (i) and (ii) above are $R$-isomorphisms.*

*Proof.* (i) Define maps $\theta : Hom_R(\oplus \sum M_i, A) \to \Pi_i Hom_R(M_i, A)$ and $\phi : \Pi_i Hom_R(M_i, A) \to Hom_R(\oplus \sum M_i, A)$ by

$$\theta(f) = (f\alpha_i),\ f \in Hom_R(\oplus \sum M_i, A),$$

and

$$\phi((g_i)) = g,\ g_i \in Hom_R(M_i, A),\ i \in I,$$

where $g((x_i)) = \sum_i g_i(x_i)$, $(x_i) \in \oplus \sum_{i\in I} M_i$.

The maps $\theta$ and $\phi$ are homomorphisms of Abelian groups. Let $f \in Hom_R(\oplus \sum M_i, A)$. Then

$$(\phi\theta)(f) = \phi(\theta(f)) = \phi((f\,\alpha_i)) = \phi((\overline{f_i})) = \overline{f}\ \text{ (say)},$$

where $\overline{f_i} = f\alpha_i$, $i \in I$. For any $(x_i) \in \oplus \sum M_i$,

$$\overline{f}((x_i)) = \sum \overline{f_i}(x_i) = \sum_i f\alpha_i(x_i) = f(\sum_i \alpha_i(x_i)) = f((x_i))$$

showing that $\overline{f} = f$. Therefore $\phi\theta(f) = f$ for every $f \in Hom_R(\oplus \sum M_i, A)$. Hence $\phi\theta =$ identity and, so, $\theta$ is a monomorphism.

Let $f_i \in Hom_R(M_i, A)$, $i \in I$. Define $g : \oplus \sum M_i \to A$ by

$$g((x_i)) = \sum_{i\in I} f_i(x_i).$$

For $x_i \in M_i$, $i \in I$, let $\alpha_i(x_i) = (y_j)$, so that $y_i = x_i$ and $y_j = 0$ for $j \neq i$ and we have

$$(g\alpha_i)(x_i) = g(\alpha_i(x_i)) = g((y_j)) = \sum_j f_j(y_j) = f_i(x_i).$$

Therefore, $g\alpha_i = f_i$ for every $i \in I$ so that $\theta(g) = (g\alpha_i) = (f_i)$ showing that $\theta$ is an epimorphism and, hence, an isomorphism . If the ring $R$ is commutative, both $Hom_R(\oplus \sum M_i, A)$ and $\Pi Hom_R(M_i, A)$ are $R$-modules and it is clear that $\theta$ is an $R$-homomorphism and, hence, an $R$-isomorphism.
(ii) Define maps $\theta : Hom_R(A, \Pi_i M_i) \to \Pi_i Hom_R(A, M_i)$, $\phi : \Pi_i Hom_R (A, M_i) \to Hom_R(A, \Pi_i M_i)$ by $\theta(f) = (\pi_i f)$, $f \in Hom_R(A, \Pi_i M_i)$, $\phi((f_i)) = f$, $f_i \in Hom_R(A, M_i)$, where $f(a) = (f_i(a))$, $a \in A$.

$\theta$ and $\phi$ are homomorphisms of Abelian groups. If $f \in Hom_R(A, \Pi_i M_i)$, $\theta(f) = (\pi_i f)$ and $(\phi\theta)(f) = \phi(\theta(f)) = \phi(\pi_i f) = \overline{f}$, where $\overline{f}(a) = ((\pi_i f)(a)) = ((\pi_i f(a))) = f(a)$ for every $a \in A$. Therefore $\phi\theta =$ identity.

Consider $(f_i) \in \Pi_i Hom_R(A, M_i)$ with $f_i \in Hom_R(A, M_i)$, $i \in I$. By definition $\phi((f_i)) = f$, where

$$f(a) = (f_i(a)),\ a \in A.$$

Then $\theta(\phi((f_i))) = \theta(f) = (\pi_i f)$ and for any $a \in A$, $i \in I$, $(\pi_i f)(a) = \pi_i(f(a)) = f_i(a)$ so that $\pi_i f = f_i$ and $\theta\phi((f_i)) = (\pi_i f) = (f_i)$. Hence $\theta\phi =$ *identity*. Therefore $\theta$ is an isomorphism with $\phi$ as its inverse isomorphism.

When the ring $R$ is commutative, both $\Pi_i Hom_R(A, M_i)$ and $Hom_R(A, \Pi_i M_i)$ are $R$-modules and $\theta$ is trivally an $R$-homomorphism. □

We need a lot more properties of $Hom_R(-,-)$ and we shall come back to these after we have introduced what we call functors.

*When $A$, $B$ are Abelian groups (and so $Z$-modules), we write $Hom(A, B)$ for $Hom_Z(A, B)$.*

### 1.4.6 Exercises

1. If $Q$ is the additive group of rational numbers and $Z$ the additive group of integers, describe
(a) $Hom(Q, Q/Z)$;
(b) $Hom(Q/Z, Q/Z)$;
(c) $Hom(Q/Z, Q)$.

2. If $A$ is a torsion-free, divisible Abelian group and $B$ any Abelian group, prove that $Hom(A, B)$ is a torsion-free, divisible Abelian group.

3. If $A$ is a torsion Abelian group and $B$ a torsion-free Abelian group, prove that $Hom(A, B) = 0$.

4. If $A$ is a left $R$-module, define $rf : R \to A$, for $r \in R$, $f \in Hom_Z(R, A)$, by $(rf)(s) = f(sr)$, $s \in R$. Prove that under this action $Hom_Z(R, A)$ is a left $R$-module.

## 1.5 Tensor Product of Modules

In this section we consider the construction of an Abelian group from a given pair of modules, one of which is a right $R$-module and the other, a left $R$-module.

**Definition 1.5.1** Let $M$ be a right $R$-module, $N$ a left $R$-module and $G$ an Abelian group. A map $f : M \times N \to G$ is called $R$-**biadditive** if
(i) $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$;
(ii) $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$;
(iii) $f(mr, n) = f(m, rn)$
for all $m, m_1, m_2 \in M, n, n_1, n_2 \in N, r \in R$.

**Definition 1.5.2** Let $M$ be a right $R$-module, $N$ a left $R$-module. By **tensor product** of $M$ by $N$ over $R$ we mean an Abelian group $M \otimes_R N$ and $R$-biadditive map $h : M \times N \to M \otimes_R N$ such that for any Abelian group $A$ and any $R$-biadditive map $f : M \times N \to A$, there exists a unique homomorphism $g : M \otimes_R N \to A$ of Abelian groups which makes the diagram

$$\begin{tikzcd} M \times N \arrow[rr, "h"] \arrow[dr, "f"'] & & M \otimes_R N \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

commutative, i.e., $gh = f$.

**Proposition 1.5.3** *Any two tensor products of $M$ by $N$ over $R$ are isomorphic.*

*Proof.* Let $A$ and $B$ be two tensor products of a right $R$-module $M$ by a left $R$-module $N$. Then there also exist $R$-biadditive maps $h_1 : M \times N \to A$ and $h_2 : M \times N \to B$. Since $(A, h_1)$ is a tensor product of $M$ by $N$ and $h_2 : M \times N \to B$ is an $R$-biadditive map, there exists a (unique) homomorphism $f : A \to B$ such that $f h_1 = h_2$. Again $(B, h_2)$ being a tensor product of $M$ by $N$, there exists a (unique) homomorphism $g : B \to A$ such that $g h_2 = h_1$. But then we have $h_1 = g(f h_1) = (g f)h_1$ and $h_2 = (f g)h_1$.
Also the identity homomorphism $1_A : A \to A$ and $1_B : B \to B$ have the property that $h_1 = 1_A h_1$ and $h_2 = 1_B h_2$.

The uniqueness of the homomorphism in the definition of tensor product then shows that $g f = 1_A$ and $f g = 1_B$ showing that $f$ is an isomorphism with $g$ as its inverse. □

**1.5.4** We next consider the question of existence of tensor product of a right $R$-module by a left R-module.

Let $M$ be a right $R$-module and $N$ be a left $R$-module. Let $Z(M, N)$ be the free $Z$-module freely generated by the set $M \times N = \{(m, n) | m \in M, n \in N\}$. Let $B(M, N)$ be the submodule of $Z(M, N)$ generated by all elements of the form