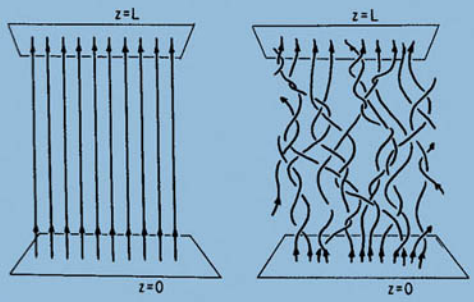
Spontaneous Current Sheets in Magnetic Fields

With Applications to Stellar X-Rays

Eugene N. Parker



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SPONTANEOUS CURRENT SHEETS IN MAGNETIC FIELDS

With Applications to Stellar X-rays

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Preface

The magnetic heating of stellar coronas and galactic halos has led to the realization over the years that the electric currents associated with the magnetic fields are universally partially concentrated into widely separated thin sheets. For otherwise there is insufficient dissipation of magnetic energy to provide the observed heating. The problem has been to understand why the currents should be concentrated, rather than spread entirely smoothly over the field. Various special circumstances, e.g., the collision of two distinct magnetic lobes, have been conceived and described (over the last four decades) with an eye to understanding the flare phenomenon as the most intense magnetic heating of all. However, the less intense and more continuous heating of the X-ray corona of a star has proved more difficult, because the heating appears where observation shows only a continuous field. The continuous form of magnetic fields is taken for granted unless discontinuous fluid motions are specified.

The need for this monograph arises, at least in part, from the widespread habit of thinking that fields are generally continuous. The classical linear Maxwell equations with continuous sources have continuous solutions (excluding such supraluminous phenomena as Čerenkov radiation) and we all learned field theory in that context. But only those fields described by fully elliptic equations, e.g., Laplace's equation or the wave equation $\nabla^2\phi + k^2\phi = 0$, have exclusively continuous solutions. The fact is that the field equations of magnetostatics in an electrically conducting medium have the field lines as a set of real characteristics in addition to the two sets of complex characteristics of the elliptic equation. So one should expect surfaces of tangential discontinuity extending along the field lines unless there is some special circumstance that would provide an entirely continuous field. That is to say, we should expect the electric currents to be concentrated into thin sheets unless conditions conspire to distribute the currents more smoothly.

The situation is summarized by the basic theorem of magnetostatics that, in relaxing to magnetostatic equilibrium in an infinitely conducting fluid, almost all field topologies form internal surfaces of tangential discontinuity (current sheets). The formation of the tangential discontinuities is caused by the balance of the Maxwell stresses, and if the formation of a true mathematical discontinuity is frustrated by the nonvanishing resistivity of real fluids, there can be no complete static equilibrium. The Maxwell stresses drive fluid motions in their constant pursuit of discontinuity. This is, of course, the phenomenon commonly called rapid reconnection, or neutral point reconnection, of the magnetic field across the site of the potential tangential discontinuity.

This monograph is the extension of the theory of the universal suprathermal activity of magnetic fields in both laboratory and astronomical settings, initiated in *Cosmical Magnetic Fields*. Chapter 14 of that writing dealt with the dynamical

nonequilibrium of magnetic fields lacking invariance of one form or another. The nonequilibrium arises because, as already noted, complete magnetostatic equilibrium of a magnetic field in a conducting fluid requires either a simple symmetric or invariant field topology or, lacking the necessary symmetry, it requires the formation of surfaces of tangential discontinuity (current sheets) within the magnetic field. Any slight resistivity in the fluid prevents the full achievement of the necessary mathematical discontinuity, of course, so the absence of static equilibrium, viz. dynamical nonequilibrium, is the result. The present writing approaches the problem from another direction, beginning with the idealized case in which resistivity is identically zero and the field has time to relax into a final asymptotic magnetostatic state. Interest centers on fields with general topologies, lacking the special, and apriori unlikely, topologies that provide a static field that is everywhere continuous.

The length of the monograph arises from the need to understand the basis for the theorem and to understand the implications of the theorem. So the writing "begins at the beginning," with a brief development of the magnetohydrodynamic equations as the proper description of the large-scale properties of a magnetic field in a noninsulating fluid or plasma. There is a curious popular notion to the contrary that has arisen in the past decade.

The basic theorem of magnetostatics then follows from the magnetohydrodynamic equations for static equilibrium. To see the theorem in perspective a number of examples are presented to contrast the special character of the fully continuous field. In particular, it is shown by example that the specification of a magnetostatic field on the boundaries of a region provides a unique determination of a continuous field throughout the region, exercising the elliptic aspect of the field equations when there are no discontinuities along the real characteristics. On the other hand, considering that the field is frozen into the ponderable conducting fluid, the topology of the field in a region could have been manipulated into most any internal form, with no impact on the normal component of the field on the boundaries, etc. The discontinuity along the field lines is the means by which the mathematics accommodates the arbitrary topology. Without it the field equations would contradict the physics, and that would have far-reaching implications indeed!

Then the geometry and topology of the surfaces of discontinuity need to be studied, at least in a preliminary fashion. The magnetic field lines, as characteristics of the magnetostatic equilibrium equations, play a prominent role in the development so that the optical analogy is an appropriate device for understanding the form of the static field.

Finally, we come to the specific application of the basic theorem to the corona of the Sun, suggesting the origin of the X-ray corona of a solitary star like the Sun, so that the necessary observational tests can be described. For it is the observations now of the motion of the footpoints of the magnetic field of active regions and of the detailed space and time behavior of the X-ray emission that must carry on from the formal theoretical principles. For the basic theorem of magnetostatics asserts in effect that the magnetic heating of the solar atmosphere depends only on the rate at which the swirling and intermixing of the photospheric footpoints of the field introduces magnetic free energy into the field. The spontaneous tangential discontinuities automatically take care of the dissipation of that free energy into heat.

The observations have not yet established the necessary swirling and intermixing of the photospheric footpoints of the bipolar magnetic field on the Sun. It is that continuing quasi-static deformation of the footpoints to which the necessary magnetic free energy is attributed. Without such free energy there is nothing to dissipate into coronal heat. The most interesting discovery of all would be the absence of the assumed mixing of the footpoints. In that instance the only available theoretical possibility would seem to be the dissipation of intense high frequency Alfven waves (with periods of the general order of 1 sec or less). Their origin would be mysterious indeed, requiring a wholly new and arbitrary dynamical state beneath the visible surface of the Sun. And if in the Sun, what then in other stars? It follows that the necessary studies of the small-scale dynamics of the photosphere should be undertaken with a full appreciation of the implications of the results, whatever those results might prove to be.

This is perhaps the appropriate place to note that a semantic difficulty has arisen in the past year or so, based on the application of dynamical terminology to magnetostatic phenomena. Specifically, some authors have presented elaborate theories of magnetohydrodynamic "turbulence," with which they propose to describe the small-scale structure of quasi-static magnetic fields in the corona of the Sun, referring to the formation of current sheets as the "cascade of magnetic energy to large wave number k." But the asymptotic relaxation of a magnetic field to static equilibrium is neither "turbulent" nor "cascading," and the use of such terms is a disservice to both the authors and the readers. It is a fact that the formation of a tangential discontinuity represents an extension of a tail on the Fourier spectrum to large wave numbers. But the extension is not a dynamical cascade in any sense. The only dynamical aspect is the *inhibiting* effect of the inertia of the fluid being ejected in the process of forming the discontinuities. Rather the declining thickness of the magnetic shear layers and the associated ejection of fluid jets is in response to the requirement for ultimate *static* balance of the Maxwell stresses.

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Acknowledgments

It is a pleasure to acknowledge the many contributions of colleagues to the development of this monograph. First of all, there is the intellectual stimulus, provided by rational discussion and a general stirring of ideas on the one hand, and enduring disbelief on the other. All of these press the author to provide a clearer exposition. Perhaps the most helpful of all have been the five published papers of which I am aware "proving" the impossibility of the spontaneous development of tangential discontinuities, or current sheets, by continuous deformation of an initially continuous field. For such papers have opened up new dimensions to the theory, showing solutions to the field equations to be added to the existing repertoire and disposing of special situations that are not part of the general picture.

No monograph is complete without figures, and I wish to express my appreciation to Dr. Gerard Van Hoven, Dr. D.D. Schnack, and Dr. Z. Mikic for permission to publish two figures from their important numerical simulation of the rapid formation of current sheets during the continuous deformation of an initially uniform field. Dr. W.H. Matthaeus and Dr. David Montgomery gave permission to publish figures created from their numerical simulation of the fields, flows, and current sheets that develop in 2D magnetohydrodynamic turbulence. Dr. B.C. Low generously agreed to my using several figures from his work, and from his work in collaboration with Dr. Y.Q. Hu, showing the properties of tangential discontinuities that form when a simple poloidal field is deformed. My thanks also to Dr. H.R. Strauss and Dr. N.F. Otani for permission to use their figures showing the formation of discontinuities in the numerical simulation of the ballooning mode. These graphic simulations show more vividly than words and theorems the nature of the spontaneous formation of tangential discontinuities as the continuous deformation of a magnetic field progresses away from the necessary simple symmetry of a continuous magnetostatic field.

Dr. Leon Golub furnished the spectacular high resolution X-ray picture of the Sun that is the frontispiece of this monograph. The detail is essential in evaluating the nature of the coronal heating and has become possible through years of technical development of the interference coating to make the mirror of the Normal Incidence X-Ray Telescope (NIXT). The intrinsic resolution is better than one second of arc. The picture was one of many taken during the few minutes of observing available from a sounding rocket.

Finally, this monograph was created by the nimble fingers, sharp eyes, and agile mind of Ms. Valerie Smith, working from my handwritten manuscript with its many erasures, insertions, deletions, rewritings, and general smudging and illegibility by the time I was through with it. Ms. Smith learned TeX from reading the manual. She honed her skill through endless hours of struggling with manuscripts such as this one. Much the same way as I learned physics, and I hope with the same satisfaction in the accomplishment.

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Contents

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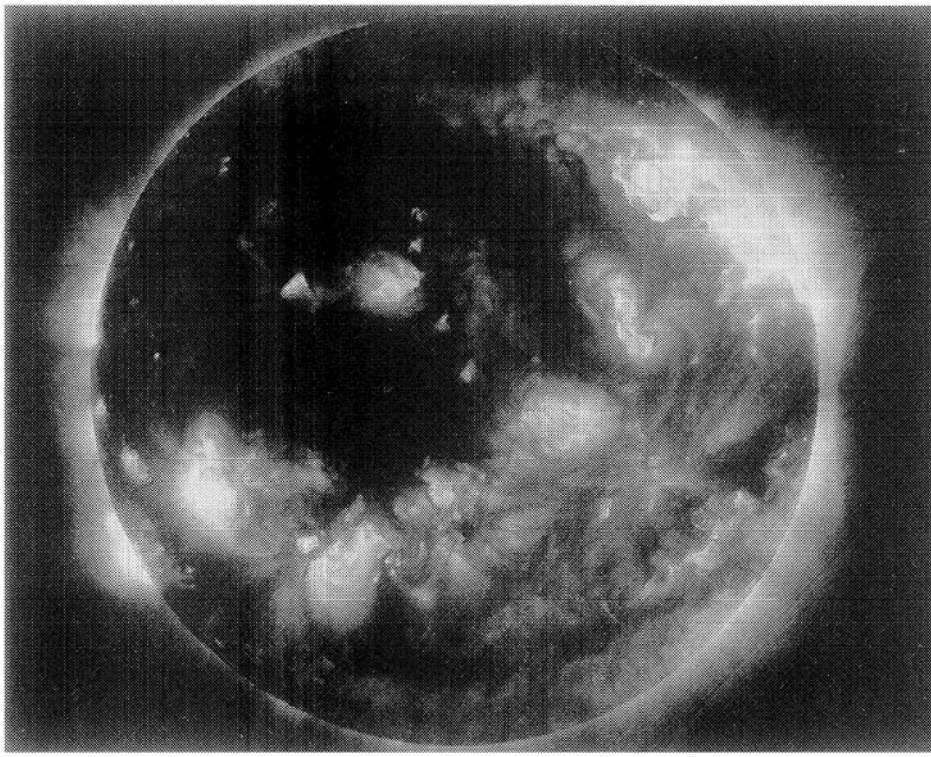

An X-ray photograph of the sun, seen in soft X-rays (17:03 UT, 11 July 1991) showing the denser coronal gas with temperature in the range 1–3 × 10^6 K. The photograph was made with the Normal Incidence X-ray Telescope (NIXT) above the atmosphere of earth (Golub, et al. 1990, Chapter 11) and was kindly furnished by Dr Leon Golub, Harvard Smithsonian Observatory.

1

Introduction

1.1 The General Picture

This monograph treats the basic theorem of magnetostatics, that the lowest available energy state of a magnetic field $\mathbf{B}(\mathbf{r})$ in an infinitely conducting fluid contains surfaces of tangential discontinuity (current sheets, across which the direction of the field changes discontinuously) for all but the most carefully tailored field topologies. That is to say, almost all continuous magnetic field configurations develop internal discontinuities as they relax to equilibrium. The theorem may be stated conversely to the effect that continuous fields are associated only with special topologies. The theorem is a consequence of the basic structure of the Maxwell stress tensor.

The magnetostatic theorem has broad application to the activity of the external magnetic fields of planets, stars, interstellar gas clouds, and galaxies, and to the magnetic fields in laboratory plasmas. In particular the theorem indicates that magnetic fields are highly dissipative, as a consequence of their internal current sheets, providing the principal heat source that creates the flares and X-ray coronas of stars and galaxies, and providing the aurora in the magnetic field of the Earth and other planets.

Observations show the remarkable fact that most stars emit X-rays[1] as thermal bremsstrahlung and line emission, indicating outer atmospheres (coronas) of 10^6–10^7 K. The Sun provides a laboratory to study the structure and the physics of the stellar X-ray corona, which are otherwise lost in the unresolved telescopic images of the more distant stars. Detailed observations of the Sun show that the X-ray emission arises from gas trapped in local bipolar regions of magnetic field and heated by some form of magnetic dissipation in the enclosing field. The theoretical dilemma has been that the very small electrical resistivity of the hot X-ray emitting gas is not conducive to dissipation of magnetic field. However, the ubiquitous tangential discontinuity is unique in that it causes the free energy of the field to dissipate by dynamical neutral-point reconnection at a rate determined more by the Alfven speed than by the slight resistivity of the medium. It appears that the X-ray luminosity of the Sun, and presumably, therefore, the X-ray luminosity of most solitary middle and late main sequence stars, is a consequence of a sea of small

[1]White dwarfs and most solitary red giants provide exceptions. In the opposite extreme, the extraordinary X-ray luminosities of certain special multiple star systems are attributed to the gravitational energy of matter from a giant star falling onto the surface of a compact star (white dwarf or neutron star), and, in somewhat less extreme cases, to the strong tidal churning of close companions.

reconnection events — nanoflares — in the local surfaces of tangential discontinuity throughout the bipolar magnetic fields of active regions. The degree of fluctuation, i.e., the duration and intensity of the individual nanoflare, is not quantitatively defined yet by the theory. The magnetic fields are continually deformed by the underlying convection, so that they continually develop new tangential discontinuities as the old discontinuities are dissipated, thereby providing an ongoing source of heat for the active X-ray corona. Thus the spontaneous discontinuity is the basis for much of X-ray astronomy.

The X-ray luminosity of solitary stars shows occasional transient increases as a result of concentrated outbursts, or flares, at the star. The individual flare can be studied at the Sun where it appears as an intense burst of dissipation of magnetic energy in the corona (Parker, 1957a) as the subphotospheric convection rams together two otherwise separate external magnetic lobes (usually bipoles) to produce a particularly strong magnetic discontinuity. Following the initial burst of dissipation at the discontinuity the flare continues with what appears to be a sea of nanoflares within the colliding bipoles, triggered by the initial burst and by the overall deformation of the colliding bipoles.

Much the same happens in the external magnetic fields of spiral galaxies, which are continually and rapidly (20–100 km/sec) inflated by the powerful relativistic cosmic ray gas generated within the disks of the galaxies. The current sheets produced in the geomagnetic field by the strong deformation of the field by the confining solar wind and by the dynamical reconnection with the field of the solar wind represent another facet of the same general situation, that deformation of magnetic field usually produces internal discontinuities.

In summary, wherever magnetic fields are deformed from the special geometrical form and internal topology of continuous fields, there arise internal surfaces of tangential discontinuity, providing strong dissipation of magnetic energy in an otherwise essentially dissipationless system. This process is manifest throughout the astronomical universe in the exotic phenomena of X-ray emission.

The spontaneous formation of tangential discontinuities in a magnetic field undergoing a simple (or complex) continuous deformation is a basic (but largely unfamiliar) physical phenomenon arising directly from the nonlinear character of the Maxwell stresses in the deformed magnetic field. In view of the unfamiliar character of the special properties of the magnetostatic equation giving rise to the discontinuities, the theoretical development progresses a step at a time, exploring in detail the properties of the field equations for magnetostatic equilibrium to show how the tangential discontinuity is a natural and necessary part of the equilibrium of almost all field topologies.

As we shall see, the equilibrium equations for a magnetic field in an infinitely conducting fluid are qualitatively different from the equilibrium equations for fields in vacuum. The equations for a vacuum field are fully elliptic, with two sets of imaginery characteristics. In a conducting medium the equations possess two sets of imaginery characteristics, but in addition the equations possess a set of real characteristics. The real characteristics are represented by the field lines, thereby providing for the surfaces of tangential discontinuity. As with all physical phenomena, the basic equations, with their stark economy of structure, possess precisely those features that are necessary to reconcile the diverse physical properties of the field. In the present case, it is the arbitrary topology of the field that must somehow

be reconciled to the invariance of fluid pressure and/or the invariance of the torsion along the field lines. The tangential discontinuity is precisely the means by which the reconciliation is achieved. The essential point is that the convective motions in stars and galaxies, and sometimes in laboratory plasma devices, deform magnetic fields without regard for the special topological conditions necessary to avoid the formation of discontinuities. Hence the ubiquitous character of the tangential discontinuity in the astronomical universe with the exotic pyrotechnic consequences already mentioned.

Conventional mathematical methods are not particularly effective in dealing with the nonlinear magnetostatic field equations, so in Chapter 7 the optical analogy is introduced, which greatly facilitates the treatment the field line topology associated with deformation of a magnetic field. The optical analogy takes advantage of the fact that the lines of force of a static magnetic field $\mathbf{B}(\mathbf{r})$ in any isobaric surface follow the same pattern as the optical ray paths in an index of refraction $B(\mathbf{r}) = |\mathbf{B}(\mathbf{r})|$. Indeed, the optical analogy applies to the projection of any vector field $\mathbf{F}(\mathbf{r})$ onto the local flux surfaces of $\nabla \times \mathbf{F}$. Hence a sufficiently concentrated maximum in $B(\mathbf{r})$ causes a bifurcation in the field pattern, as the field lines pass around on either side, rather than over, the maximum. The bifurcation of the field pattern is the singular feature that creates the tangential discontinuity. The gap in the field pattern associated with the bifurcation is centered over the maximum and permits the otherwise separated regions of field on either side to come into contact through the gap. The separate fields create a tangential discontinuity at their contact surface in the gap.

The optical analogy applies to stationary flow of ideal inviscid incompressible fluid in the same special way that it applies to the magnetic field, because the stationary Euler equation and the magnetostatic equation are identical in form. In its general form the optical analogy applies to time-dependent turbulent hydrodynamic flows, showing the relation between local velocity maxima and vortex sheets. A brief discussion is provided in Chapter 7. The essential point is that the dynamical formation of vortex sheets in turbulent flows is a trend that is already conspicuous in the stationary flow, of which the vortex sheet is an intrinsic part.

The general phenomenon of spontaneous internal tangential discontinuities has received only limited attention in the literature, in most cases without appreciating its general occurrence and its importance for astrophysics. In particular, the optical analogy has not been previously recognized or exploited to describe the creation of a discontinuity by the adjoining regions of the field. Hence one of the goals of this monograph is to develop the optical analogy and then to exploit the analogy to extend the general theory of the spontaneous tangential discontinuities in magnetostatic fields. We apply the general theory to the universal magnetic activity of stars, planets, and galaxies and to the magnetic confinement of plasma in the laboratory.

Now the theoretical development is extensive, as is the range of applications. Hence this first chapter establishes a road map, a "cultural history," and a commentary on some of the major points of interest along the way, describing the general ideas involved in both the magnetostatic theorem and the astronomical settings in which the magnetostatic theorem is to be applied. The succeeding chapters provide detailed examination of the many individual aspects of the theory and its applications.

1.2 Activity of Stars and Galaxies

Consider the general nature of the activity of astronomical objects. Observations of the Sun show an active, rather than a placid, object. Observation leaves no alternative to the idea that the activity is a direct consequence of magnetic fields. Where there is magnetic field, there is activity, and vice versa.

Cowling (1958) gives a brief history of the study of magnetic fields in the Sun (see also Cowling, 1953; Kiepenheuer, 1953; Parker, 1979, pp. 739–746; Priest, 1982; Weiss, 1983; Foukal, 1990). The existence of magnetic fields was first inferred in 1889 by Bigelow from the filamentary appearance of the coronal streamers seen during total eclipse. Hale (1908a–d, 1913; see also Hale and Nicholson, 1938) was the first to observe the Zeeman effect, establishing that sunspots represent regions where the field is $2–3 \times 10^3$ gauss. Hale's instrumental noise was evidently about 50 gauss, because he thought (erroneously) that he detected a general dipole magnetic field of about that intensity at the North and South poles.

Detection and study of the magnetic fields outside sunspots had to wait for the development of electronics and the Babcock magnetograph (Babcock and Babcock, 1955; Babcock, 1959) to map the line-of-sight component of magnetic field over the solar photosphere (see also Howard, 1959; Leighton, 1959). The complex nature of the large-scale photospheric magnetic fields throughout three complete 11-year sunspot cycles is now a matter of record.

The outstanding aspects of the magnetic activity (besides the conspicuous sunspots) are the suprathermal effects and the violent mass motions, with the transient solar flare and the coronal mass ejection as the extreme examples, respectively. On a continuing basis there is coronal gas confined in the 100 gauss bipolar magnetic fields of active regions, and heated to $2–3 \times 10^6$ K with densities as large as 10^{10} H atoms/cm^3 so as to emit X-rays at a rate 10^7 ergs/cm^2 sec, to be compared to the photospheric radiation intensity of 6×10^{10} ergs/cm^2 sec (Withbroe and Noyes, 1977). The coronal gas in open field configurations (5–10 gauss) reaches $1.5–2 \times 10^6$ K and expands continually to produce the solar wind, requiring a heat input of about 5×10^5 ergs/cm^2 (Withbroe and Noyes, 1977; Withbroe, 1988). The relatively low density (10^8 atoms/cm^3) precludes significant emission of X-rays.

The coronal mass ejection represents a magnetic catapult that flings matter out into space (Illing and Hundhausen, 1986; Athay and Illing, 1986; Athay, Low, and Rompolt, 1987; Webb and Hundhausen, 1987) with individual ejections estimated to be sometimes as large as 10^{32} ergs (Hundhausen, 1990). The solar flare, which may also be as large as 10^{32} ergs, is an example of extreme intensity, emitting hard X-rays and gamma-rays, and accelerating ions and electrons, sometimes to relativistic energies (Svestka, 1976; Priest, 1981, 1982).

The more closely one looks at the Sun, the more activity there is to see on progressively smaller scales. There is continual microflaring in the small-scale network fields as small magnetic bipoles emerge in supergranule cells and are swept into the cell boundaries, where they accumulate to provide the network fields. It appears that this microflaring may be the principal source of heat in the regions of weak open field (Martin, 1984, 1988, 1990; Porter, et al. 1987; Porter and Moore, 1988; Parker, 1991a), as already noted. Dere, Bartoe, and Brueckner (1991), Brueckner and Bartoe (1983) and Brueckner, et al. (1986) find tiny jets and explo-

sive events in the chromosphere–corona transition layer, evidently associated with the microflaring in the network fields (Porter and Dere, 1991).

The X-ray corona, even with the very high space and time resolution of the recent normal-incidence X-ray telescope (Walker, et al. 1988; Golub, et al. 1990), appears as a filamentary continuum, the individual nanoflares being unresolved. The existence of the nanoflares is indicated by the observed electromagnetic emission spectrum, showing excitation well above the mean temperature (Sturrock, et al. 1990; Laming and Feldman, 1992; Feldman, 1992; Feldman, et al. 1992), and implying that the temperature varies sharply and intermittently through a wide range.

A somewhat similar situation is inferred for the solar flare, where it appears that the principal emission arises from a sea of nanoflares in one or more of the magnetic bipoles whose collision creates the initial impulsive phase of hard radiation (Parker, 1987; Machado, et al. 1988).

The magnetic field of the Sun at the photosphere is composed of tiny, intense, and widely separated magnetic fibrils of $1\text{–}2 \times 10^3$ gauss across diameters as small as 10^7 cm (Beckers and Schröter, 1968; Livingston and Harvey, 1969, 1971; Simon and Noyes, 1971; Howard and Stenflo, 1972; Frazier and Stenflo, 1972; Stenflo, 1973; Chapman, 1973), with the fibrils expanding to fill the entire space in the chromosphere above (Kopp and Kuperus, 1968; Gabriel, 1976; Athay, 1981). The magnetic fibrils are unresolved for the most part (cf. Dunn and Zirker, 1973) and are carried with the photospheric convection (Title, et al. 1989).

Observations of stars and galaxies show universal X-ray emission, flaring, etc., suggesting that the active Sun is a paradigm rather than an anomaly. Evidently magnetic fields and magnetic activity are everywhere (Parker, 1979, p. 6). On the other hand, from the point of view of the physicist, the Sun is unique, being the only star for which the form of the activity can be seen. This is essential because, first, the nature of the magnetic activity is exotic, lying outside the realm of the terrestrial physics laboratory. Second, the basic equations of physics admit of so many different classes of solutions (for which general mathematical descriptions are not available) that the nature of the observed magnetic activity cannot be deduced from first principles. Hence a theoretical understanding can be developed only with quantitative and qualitative guidance from detailed observations.

1.3 The Nature of Active Magnetic Fields

There is an initial puzzle at the simplest theoretical level. For it must be remembered that the general occurrence of magnetic fields in astronomical objects can be understood only from the fact of their long life implied by the relative unimportance of resistive dissipation in the interior of the objects, whereas the observed continuing activity of the external fields of astronomical objects implies bursts of rapid dissipation, converting magnetic energy into heat, fast particles, etc. To elaborate, the existence of magnetic fields in planets, stars, gas clouds, and galaxies can be understood only from the fact of the relatively small effective resistive diffusion coefficient η and the relatively large scale ℓ, so that the characteristic resistive decay time ℓ^2/η is long compared to any convective turn over time ℓ/v in the internal fluid motion v. The appropriate dimensionless number is the magnetic Reynolds number $N_M = \ell v/\eta$, representing the ratio of these two characteristic times. It is sometimes

convenient to define the Lundquist number N_L as $\ell C/\eta$, where C is the characteristic Alfven speed $C = B/(4\pi\rho)^{\frac{1}{2}}$ in the field. A typical value of N_L in the solar corona, where $\ell \cong 10^{10}$ cm, $\eta \cong 10^3$ cm^2/sec, and $C \cong 10^8$ cm/sec, is 10^{15}, indicating the relative smallness of resistive dissipation of the magnetic field.

Note, then, the limitations of the terrestrial plasma laboratory where ℓ may perhaps be as large as a meter, which is 10^{-7} or less of the gross scale of the fields in the solar corona. The resistive decay time ℓ^2/η in a cubic meter of laboratory plasma with a thermal energy of 10^2 eV may be 1 sec where the Lundquist number is generally 10^4 or less, making it impossible to study more than relatively transient dynamical effects. Needless to say, the laboratory experiments that have been performed on plasma confinement, on the formation and coalescence of islands, and on a variety of major instabilities, have been essential in guiding the theoretical development of the basic plasma phenomena. But the quasi-equilibrium magnetic configurations that ultimately ignite into extended sequences of flaring require the enormous Lundquist numbers of the astronomical setting.

Having established the essential resistive longevity of the magnetic fields in astronomical settings, how is it, then, that the fields observed in the Sun are in a state of perpetual internal dissipation (and sometimes explosive dissipation), diverting free magnetic energy to heat the X-ray corona and the solar wind, sometimes accelerating particles to cosmic ray energies, sometimes flinging mass out into space, etc.? These dissipative phenomena occur in seconds or minutes. They involve reconnection of field lines, which is intrinsically a resistive effect. For without resistivity the field lines are permanently connected and can do little or nothing to heat the ambient gases, nor can they cut loose from their moorings to depart into space. So on the one hand there is longevity because of the small resistivity and large scale, while on the other hand there is vigorous dissipation.

The resolution of the contradiction has gradually emerged over the years, beginning with studies of large solar flares, where it is well established that the explosive dissipation is a consequence of rapid neutral point reconnection of magnetic fields. Evidently this provides the conspicuous impulsive onset of a flare, as already noted, when separate lobes (topological regions) of field are rammed together (cf. Parker, 1957b; Sweet, 1958) squashing the X-type neutral point where the fields come into contact. Unless by chance the colliding fields are closely parallel, one field component meets its opposite number in the other bipole and dynamical annihilation occurs. The process is simply that the intervening gas is rapidly squeezed out from between the opposite components until the separation becomes so small that the opposite fields dissipate. No matter how small the effective resistivity the separation is soon sufficiently small, and the electric current density sufficiently large, as to dissipate the opposite components in a short time. The dissipation frees the gas from the field, but the gas thus liberated is continually squeezed out from between the opposite components. So the process of rapid reconnection continues as long as there is free energy available in the colliding fields (Parker, 1957b).

Flaring by rapid reconnection was not initially associated with the general heating of the active X-ray corona because the field within a magnetic bipole was assumed to be continuous throughout. A prescient paper by Gold (1964) noted that the photospheric convective turbulence deforms, wraps, and winds the bipolar magnetic fields above the surface of the Sun, continually increasing the magnetic

free energy. He proposed that the convective turbulence twists the fields so tightly that some form of dissipation (dynamical instability, reconnection, etc.) must occur, continually converting magnetic energy into heat and causing the elevated temperature of the solar atmosphere. Syrovatskii (1971, 1978, 1981) was the first to recognize the universal vulnerability of the X-type neutral point in the projection of the magnetic field onto any plane perpendicular to **B**. He noted that any squeezing of the neutral point creates a current sheet, or pinch sheet as he sometimes referred to it. Parker (1972, 1973, 1979 pp. 511–519) pointed out that special invariance of the magnetic field is necessary to avoid current sheets or tangential discontinuities as an intrinsic part of the equilibrium of a magnetic field. He noted that the magnetic fields created in the convective fluid of a star generally do not have the necessary invariance or symmetry and so they may be expected to contain surfaces of tangential discontinuity, subject to rapid reconnection. He proposed that the otherwise runaway increase in the small-scale components (large wave number k) of a magnetic field in a turbulent fluid may be checked by such rapid reconnection. It was Glencross (1975, 1980) who first suggested explicitly that the general occurrence of current sheets pointed out by Parker provides the heat source for the X-ray corona of the Sun. Parker (1981, 1983a) sketched out some specific circumstances for heating the active corona in that way.

The basic idea behind the application of the magnetostatic theorem to magnetic activity and coronal heating can be expressed in the following way. A simple continuous magnetic field configuration $\mathbf{B}(\mathbf{r})$ is preserved by its large scale $L(N_L \gg 1)$ in the presence of small resistivity. But the large–scale field $\mathbf{B}(\mathbf{r})$ of a convective object, e.g., a star or galaxy, is internally wrapped and interwoven, producing strong local deformation $\Delta\mathbf{B}$ on intermediate scales, ℓ. These intermediate scales are sufficiently large that they too are preserved. However, the topology of $\mathbf{B} + \Delta\mathbf{B}$ is no longer the simple topology of the basic form $\mathbf{B}(\mathbf{r})$. The magnetostatic theorem asserts that the field $\mathbf{B} + \Delta\mathbf{B}$ develops internal discontinuities as it relaxes to equilibrium. The internal tangential discontinuities involve magnetic free energy, and, since η is small but not identically zero in the real physical world, the discontinuities provide rapid reconnection and quick dissipation of the free energy into heat. The dissipation consumes $\Delta\mathbf{B}$ but not $\mathbf{B}$, of course, because the topology of $\mathbf{B}(\mathbf{r})$ is simple enough to permit a continuous equilibrium field, which is preserved by its large-scale L, as remarked in the beginning. Thus the dissipation is active so long as there is enough $\Delta\mathbf{B}(\mathbf{r})$ that the topology requires discontinuities for equilibrium.

This fundamental property of magnetostatic fields merits further elaboration, because there is at least some slight resistive dissipation everywhere in the gases in the astronomical universe or in the terrestrial laboratory. The resistivity converts each ideal surface of tangential discontinuity into a thin transition layer, or current sheet, in which magnetic energy is rapidly dissipated as a consequence of the high current density. In fact the characteristic thickness of the current sheet is prevented from reaching the zero of the true discontinuity in a ideal fluid with zero resistivity. The thickness falls only to a value sufficiently small that the dissipation balances the dynamical trend toward zero thickness. The rapid dissipation and the associated field line reconnection continues only so long as the field topology requires discontinuities for magnetostatic equilibrium. The thin current sheet is not in internal equilibrium, of course. It is the topology of the quasi-static equilibrium fields in the

regions of continuous field, filling the volume between the current sheets, that drives the formation of the discontinuities. The dissipation continually reduces the topology of the continuous field to simpler forms through reconnection of field across the current sheet. So the rapid dissipation continues until the topology is so simple that it no longer requires the current sheets.

This point is often ignored and it deserves an illustration. Consider a tangential discontinuity in a static field in an infinite space, where the discontinuity is not required by the topology, i.e., a passive discontinuity. An example would be an initial equilibrium field extending uniformly in the z-direction ($B_x = B_y = 0$) from $z = -\infty$ to $z = +\infty$, with $B_z = +B_0$ in $y > 0$ and $B_z = -B_0$ in $y < 0$. The field is filled with an infinitely conducting incompressible fluid and is clearly in magnetostatic equilibrium with a surface of discontinuity (current sheet) at $y = 0$. Then suppose that at some time $t = 0$ a small uniform resistivity η is introduced throughout the fluid, so that the field remains in static equilibrium, evolving according to the familiar magnetohydrodynamic diffusion equation

$$\partial B_z/\partial t = \eta\, \partial^2 B_z/\partial y^2.$$

It is readily shown that

$$B_z = B_0 \operatorname{erf}[y/(4\eta t)^{\frac{1}{2}}]$$

for $t > 0$, where erf denotes the error function

$$\operatorname{erf}\chi = \frac{2}{\pi^{\frac{1}{2}}} \int_0^{\chi} ds \exp(-s^2).$$

The characteristic thickness of the current sheet increases from zero with the passage of time, in the form $(4\eta t)^{\frac{1}{2}}$. The dissipation rate per unit volume varies as $(B_0^2/4\pi t)\exp(-y^2/2\eta t)$. The total rate of dissipation of energy (throughout $-\infty < y < +\infty$) per unit area of the current sheet is $(\eta/2\pi t)^{\frac{1}{2}} B_0^2/8\pi$, declining as $t^{-1/2}$ with the passage of time.[2]

The essential point of this example of a passive discontinuity is that the initial rapid dissipation of magnetic energy into heat quickly declines to a low level because the current sheet rapidly thickens when it is not rejuvenated by the Maxwell stresses. It follows that the creation of tangential discontinuities or current sheets is not sufficient to guarantee continuing magnetic dissipation unless the discontinuities are required by the topology of the continuous portions of the field. In that case the Maxwell stresses continually extract the field and fluid from the thickening current sheet, thereby maintaining a thickness δ so small as to continue rapid dissipation and reconnection of field. This is, of course, the basis for rapid reconnection of fields (Sweet, 1958; Parker, 1957b, 1963a) across a scale ℓ in a time $\ell N_L^{1/2}/C$ or less, instead of the passive diffusion time $\ell N_L/C$, where C is the characteristic Alfven speed and N_L ($\gg 1$) is the Lundquist number $\ell C/\eta$.

One may wonder, then, if the resistive dissipation of magnetic energy can be maintained at a substantial level if some discontinuous motion (e.g., $v_x = +vkz$ in $y > 0$ and $v_y = -vkz$ in $y < 0$ in the uniform field $e_z B$) is introduced to regenerate the passive discontinuity at a fixed rate. It is a simple matter to show (in §10.2) that,

[2]The onset of the resistive tearing instability might enhance the dissipation somewhat, but that does not prevent the decline with increasing t.

even in that case, the dissipation is confined to a transition layer with the characteristic diffusion thickness $(4\eta t)^{\frac{1}{2}}$, which grows too slowly to consume a significant fraction of the available free magnetic energy.

In summary, rapid reconnection occurs only where the topology provides Maxwell stresses in a form to drive the current sheet toward vanishing thickness. Other surfaces of discontinuity are passive and do not drive the dynamical reconnection, so they are limited to the characteristic diffusion scale $(4\eta t)^{\frac{1}{2}}$.

If we view from a distance the theoretical problem of dissipation of magnetic free energy with small resistivity, there are two obvious approaches. One, adopted in this monograph, is to consider the ideal case of a magnetic field embedded in a fluid whose resistivity is identically zero, so that strong deformation of the field develops whatever tangential discontinuities are required for equilibrium by the field topology.

The other approach is to start with the nonvanishing resistivity and inquire into its effects in a field containing internal shear. This begins with the universal resistive magnetohydrodynamic instabilities shown by Spicer (1976, 1977, 1982) and Van Hoven (1976, 1979, 1981) many years ago to occur wherever the magnetic field is subject to shear or torsion. The resistive kink and tearing instabilities are the primary candidates. If h is the characteristic scale of the shear, of the order of $|\mathbf{B}|/|\nabla \times \mathbf{B}$, then the characteristic growth time τ for the resistive instability is given by (Furth, Killeen, and Rosenbluth, 1963; Parker, 1979; Van Hoven, 1981)

$$\begin{aligned}\tau &\sim h^2 (k^2/\eta^3 C^2)^{\frac{1}{5}} \\ &= (kh)^{\frac{2}{5}} \tau_R^{\frac{3}{5}} \tau_A^{\frac{2}{5}}\end{aligned}$$

for wavelength $2\pi/k$ along the current sheet, Alfven speed $C = |\mathbf{B}|/(4\pi\rho)^{\frac{1}{2}}$, and resistive diffusion coefficient η. The characteristic diffusion time τ_R is h^2/η and the characteristic Alfven transit time is h/C. Spicer and Van Hoven point out that τ is a relatively long time because of the magnitude of τ_R in the large dimension h of magnetic fields in the typical astronomical setting. The essential point, then, is that the trend toward a surface of tangential discontinuity provides a local shear scale h that tends rapidly toward zero. It follows that the resistive tearing instability arises primarily in the declining thickness of the current sheets when the sheets are well on their way to becoming tangential discontinuities. Thus the resistive approach leads to the same situation as the development that starts with a fluid of zero resistivity and notes that the thickness of the current sheet declines until resistive dissipation — presumably in the form of a resistive instability — prevents further decline (Parker, 1990d). In either case we end up in the same final state, with the dissipation arising at the site of the potential tangential discontinuities. Then since the theory of the spontaneous formation of tangential discontinuities does not depend upon the resistivity, it is easiest to develop the theory before bringing in the final limiting effect of resistivity. The conclusion is simply that any magnetic field in a fluid of small resistivity in a convective astronomical setting is subject to rapid internal dissipation of its magnetic free energy.

It follows that there is continual dissipation of magnetic energy into heat throughout the bipolar fields of magnetic active regions and throughout the strong small-scale bipoles of the network fields on the Sun. Quantitative estimates indicate that the rate of dissipation of magnetic energy is substantial, providing most of the heat that maintains the X-ray emitting gas trapped in the bipolar fields (Parker, 1983a). The individual reconnection fluctuations — nanoflares — are mostly below

the limit of detection, so that the observer sees only the general glow that represents the sea of nanoflares. Similarly the magnetic bipoles that provide flares contain internal discontinuities that dissipate rapidly when bipoles collide to produce a flare. The observations of Machado, et al. (1988) indicate that well over half the energy of a flare comes from the nanoflares within the bipoles, as distinct from the initial explosive reconnection at the interface between two colliding bipoles.

With this general principle of spontaneous discontinuities in hand, it would appear that substantial internal generation of heat by magnetic dissipation in most active astronomical magnetic fields is inevitable. Planetary magnetospheres are subject to varying deformation in the fluctuating solar wind, while the footpoints of the planetary magnetic field move about in the ionosphere. The magnetic fields of galaxies are subject to the motion of the interstellar gas, and particularly to rapid inflation by the cosmic ray gas generated by supernovae, etc. We conjecture that the X-ray emission from the halos of many spiral galaxies is a consequence of the magnetic dissipation in the discontinuities associated with magnetostatic equilibrium of the deformed fields (Parker, 1990a, 1992). Needless to say, the X-ray emission from the Sun is the only case available for detailed telescopic observation, so it is the primary focus for the theoretical development.

1.4 Rapid Dissipation

It was realized from the outset that the explosive dissipation of magnetic energy to produce the flare must center around singular places in the magnetic field where dissipation can occur in spite of the small resistivity. Thus Giovanelli (1947) and Cowling (1953) considered the possibility of electrical discharges at neutral points in the field as the cause of a solar flare. Dungey (1953, 1958a,b) and Chapman and Kendall (1963) pursued the idea further, treating the stability of the (X-type) neutral point. Sweet (1958) considered the results of the neutral point created when, for instance, two bipolar regions on the Sun collide head to tail. Parker (1957b, 1963a, 1973) treated Sweet's scenario in the context of magnetohydrodynamics and showed that when two oppositely directed (antiparallel) magnetic fields $\pm B$ are pressed together over a width ℓ in the presence of an incompressible fluid of density ρ and resistive diffusion coefficient η, the fluid is squeezed out from between the two opposing fields, causing the field gradient to steepen until resistive dissipation creates a steady state. The configuration is sketched in Fig. 1.1. The steady state occurs when the characteristic thickness δ of the transition from $-B$ to $+B$ is reduced to the order of $\ell/N_L^{\frac{1}{2}}$, where again N_L is the Lundquist number $\ell C/\eta$, in terms of the characteristic Alfven speed $C = B/(4\pi\rho)^{\frac{1}{2}}$. With this steep field gradient (when $N_L \gg 1$) the two oppositely directed fields move into the transition layer from

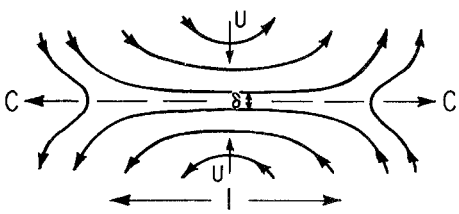

Fig. 1.1. A schematic drawing of the field lines undergoing rapid reconnection across the dashed center line.

either side with a speed u of the order of $\eta/\delta = C/N_L^{\frac{1}{2}}$. The fluid swept into the transition layer with the fields is expelled (by the magnetic tension and pressure) out the ends of the layer at a speed of the order of C. This expulsion of fluid is responsible for maintaining the small thickness δ so that the dissipation continues at a rapid pace $C/N_L^{\frac{1}{2}}$. In the absence of the expulsion of fluid, the characteristic scale is ℓ, rather than δ, and the rate of dissipation is C/N_L, smaller by the large factor $N_L^{\frac{1}{2}}$.

The essential point is that the magnetic stresses throughout the region of field are of such form as to push the thickness of the transition layer (from $-B$ to $+B$) continually toward zero, in an attempt to produce a tangential discontinuity. If the resistivity η were zero, then $N_L = \infty$ and δ would fall to zero, achieving a true discontinuity. Insofar as η is small but not zero, the enhanced field gradient, $O(B/\delta)$, remains large but finite, with the fields flowing into the region of dissipation from either side at a speed $u \cong C/N_L^{\frac{1}{2}}$. So, the discontinuities are really thin transition layers of small but nonvanishing thickness $O(\delta)$. The magnetic stresses throughout the continuous fields on either side of the transition layer drive the thickness toward zero while the resistivity tends to thicken the layer. A dynamical balance arises when δ is of the order of $\ell/N_L^{\frac{1}{2}}$. Thus the original theory of reconnection represents the essential feature of the spontaneous formation of the tangential discontinuity. What was lacking until recently was an understanding of the general occurrence of the scenario in all magnetic fields subject to continuous deformation.

Now the magnetic energy is dissipated by the reconnection speed u at a rate $uB^2/8\pi$ ergs/cm^2 sec. Some of the energy is converted directly into heat, with the rest into the kinetic energy of the ejected fluid depending upon detailed conditions. It is expected that the jet of ejected fluid is turbulent, so that it is quickly thermalized and converted to heat. The phenomenon is sketched in Fig. 1.1, and is referred to as rapid, or neutral point, reconnection, because the lines of force of the two initially separate and oppositely oriented fields $\pm B$ (parallel to the transition layer) are reconnected by the resistivity so as to lie across the transition layer of thickness δ. Magnetic energy is dissipated into heat across δ faster by a factor of the order of $N_L^{\frac{1}{2}}$ than by resistive diffusion across the scale ℓ.

The increased diffusion rate, by a factor $N_L^{\frac{1}{2}}$, is large and interesting, but entirely inadequate to account for the vigorous dissipation represented by a flare. The next step came from Petschek (1964) and Petschek and Thorne (1967) who suggested that two opposite fields $\pm B$ of scale ℓ may, as a consequence of the dynamics of the inflow and outflow, come into contact across only a narrow width $h(\ll \ell)$, rather than across the full width ℓ of the field as assumed by Parker and Sweet. Petschek argued from dynamical considerations that u might then be as large as $C/\ln N_L$, with h as small as $\ell(\ln N_L)^2/N_L$ and δ as small as $\ell \ \ln N_L/N_L$, in order of magnitude, in the limit of large N_L. The dynamics involve resistive diffusion in a small neighborhood of the neutral point, with oblique standing Alfven waves extending out from each of the four corners of the small central diffusion region (with dimensions $h \times \delta$) sketched in Fig. 1.2. It was clear that any intermediate merging rate $C/N_L^{\frac{1}{2}} < u < C/\ln N_L$ is possible, depending upon the boundary conditions (see discussion in Vasyliunas, 1975). This has been placed in a formal context recently by Priest and Forbes (1986) and particularly elegant analytical solutions have been provided by Hassam (1991) and Craig and Clymont (1991).

Sonnerup (1970, 1971) showed that with a somewhat different field profile, determined by the forces pushing the two opposite fields together (Fig. 1.3), the rate

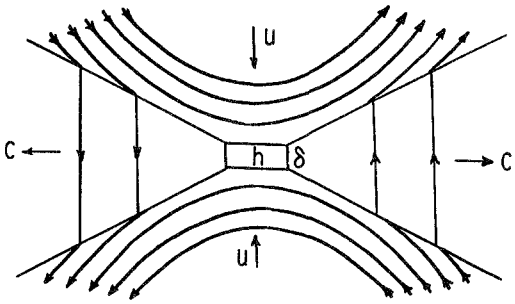

Fig. 1.2. A sketch of the field configuration for Petschek's model of rapid reconnection, with the configuration of Fig. 1.1 in the small central rectangle, where ℓ is replaced by h.

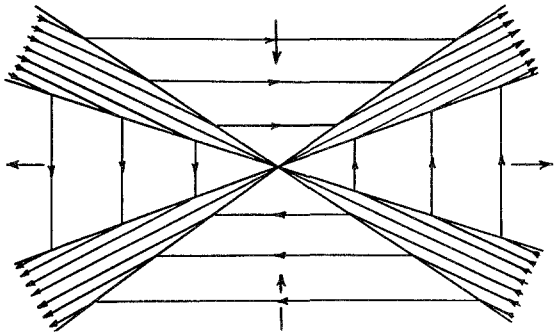

Fig. 1.3. A sketch of the field configuration in Sonnerup's model of rapid reconnection, with the central diffusion region shrunk to zero.

of reconnection u may be as large or as small as desired, depending upon how firmly the opposite fields are pushed together. In this case there are two slow waves in each quadrant extending obliquely into the origin from some (unexplained) point of creation at the periphery of the flow. The formal mathematical solution involves an infinitely sharp corner in the fluid velocity and in the magnetic field at the origin so that the resistivity of the fluid does not enter into the considerations.

Biskamp and Welter (1980) and Biskamp (1984, 1986) have constructed 2D-numerical simulations of reconnection and, so far, have found only slower reconnection rates, along the lines of Syrovatskii's (1971) current sheet model. Whether the failure to find the more rapid reconnection rates obtained by the analytical solutions can be attributed to the boundary conditions or to the modest Lundquist numbers to which numerical simulations are restricted, remains to be seen.

The theoretical developments provide reconnection velocities u that readily account for the initial intense phase of a solar flare as the rapid reconnection between two large-scale lobes of magnetic field pushed together by the motion of their footpoints in the photospheric convection. Priest (1981, pp. 139–216) Low (1987, 1989, 1991), Low and Wolfson (1987), and Jensen (1989), among others, provide explicit examples of the current sheets formed in this way.

To write down some specific numbers, note that with $\ell = 10^9$ cm, $\eta = 10^3$ cm^2/sec and $C = 10^8$ cm/sec the Lundquist number $N_L = \ell C/\eta$ may be as large as 10^{14} within a bipolar magnetic region on the Sun, so that the minimum reconnection rate $C/N_L^{\frac{1}{2}}$ is small, of the order of 10 cm/sec. But $\ln N_L$ is only about 30, so that the reconnection rate may be 10^6 times larger, or 10^2 km/sec, determined by the Alfven speed C. Thus the explosive burst of energy release at the onset of a flare (in a period of the order of 10^2 sec) is comprehensible in spite of the small resistivity.

The geomagnetic substorm, associated with geomagnetic reconnection with a southward component of the magnetic field in the solar wind and with the magnetic reconnection across the neutral sheet between the north and south lobes of the geomagnetic tail, is also understandable, if not precisely defined, by the theoretical rapid reconnection (cf. Hones, 1984).

In summary, the large-scale Fourier components of the magnetic fields in astronomical objects are preserved by the small resistivity. On the other hand, the magnetostatic theorem asserts that in the absence of resistivity the magnetic stresses cannot avoid creating small-scale Fourier components, in the form of tangential discontinuities, in almost all field topologies. The discontinuities arise spontaneously and asymptotically in time throughout the interior of each lobe of field simply as a consequence of an overall continuous deformation of the field. So small reconnection events are expected throughout any magnetic field subject to continuing deformation. The small and generally unresolved bursts of reconnection at the ubiquitous internal surfaces of discontinuity appear to be the principal heat source for the ongoing thermal X-ray emission from the solar corona (Parker, 1975, 1979, p. 359, 1981, 1983a; Glencross 1975, 1980) and appear (Parker, 1987; Machado, et al. 1988) to be responsible for the continuing X-ray emission of a solar flare, following the brief intense initial phase. One infers that the general X-ray emission from other stars and the intense flare phenomena of the *dM* dwarf stars are to be understood on a similar basis.

Now, it is apparent from this discussion that in the presence of a small resistivity the magnetostatic theorem leads to conditions that are anything but static at the surfaces of discontinuity. This does not invalidate the application of the theorem, however, because it is the quasi-static balance of magnetic pressure and tension throughout the continuous field filling the volume between surfaces of tangential discontinuity that drives the formation of the tangential discontinuities. The field throughout the volume moves only with the speed u that is small compared to the Alfven speed, and the rapid reconnection at the discontinuity does not significantly disturb the form of the quasi-static equilibrium throughout the volume. On the other hand the dynamical conditions within the actual transition layer δ, representing the potential surface of discontinuity, are not at all like the static form, and there is nothing that the magnetostatic theorem has to say about these dynamical conditions, except that they are created by the static conditions throughout the volume of the field. The situation is closely analogous to the strong shock in an otherwise continuous hydrodynamic flow. The hydrodynamic equations apply to the volume of continuous flow outside the thin shock transition and determine where the shock appears, without describing the structure of the shock transition.

In practice it appears that the reconnection proceeds somewhere in the neighborhood of the minimum rate $C/N_L^{\frac{1}{2}}$, except for sudden bursts of more rapid reconnection (Finn and Kaw, 1977; Van Hoven, 1981; Montgomery, 1982; Lichtenberg, 1984; Dahlburg, et al. 1986) perhaps triggered by passing magnetohydrodynamic waves (Sakai, 1983a,b; Matthaeus and Lamkin, 1985, 1986; Tajima and Sakai, 1986). We have referred to the individual (usually unresolved) small reconnection events as *nanoflares* (Parker, 1988) because it is estimated indirectly that in the solar X-ray corona they are 10^{-8}–10^{-9} of a large solar flare (at 10^{32} ergs/cm^2). We wish to distinguish them from microflares (typically 10^{-5}–10^{-6} of a large flare), which are small but individually observed where small magnetic bipoles collide in the conver-

ging flows at the boundaries of supergranule cells on the solar surface. The true nature of the nanoflare remains to be determined by direct means.

For the nanoflares, then, it appears that the internal winding and interweaving of the field lines slowly accumulates without much reconnection (perhaps at the slow rate $C/N_L^{\frac{1}{2}}$) until an individual discontinuity exceeds some critical strength, whereupon a local burst of reconnection (perhaps at a rate as large as $C/\ln N_L$) reduces the strength to where the rapid reconnection falls back toward $C/N_L^{\frac{1}{2}}$. Then the slow accumulation begins again, to be followed after a time by another burst of reconnection, etc.

1.5 The Magnetostatic Theorem

Consider the basic magnetostatic theorem on which the foregoing inferences are based. The theorem can be understood in a variety of ways. To begin, note the magnetostatic equation

$$4\pi \nabla p = (\nabla \times \mathbf{B}) \times \mathbf{B} \tag{1.1}$$

describing the balance between the gas pressure $p(\mathbf{r})$ and the Maxwell stresses in the magnetic field $\mathbf{B}(\mathbf{r})$. Arnold (1965, 1966, 1974) pointed out from formal mathematical considerations that almost all solutions to equations of the form of equation (1.1) contain discontinuities. He dealt explicitly with the Euler equation

$$-\nabla(p/\rho + \frac{1}{2} v^2) = (\nabla \times \mathbf{v}) \times \mathbf{v} \tag{1.2}$$

for the stationary flow of an ideal inviscid incompressible fluid, but, as emphasized by Moffatt (1985, 1990), the solutions of equations (1.1) and (1.2) have identical form. So there is a formal mathematical basis for the ubiquitous tangential discontinuity.

Moffatt (1986) showed that the solutions to equation (1.2) are dynamically unstable, in contrast to the general stability of many solutions of equation (1.1). The difference arises because the Maxwell stress provides a tension $B^2/4\pi$ along the field lines whereas the Reynolds stress provides a compressive force ρv^2 along the streamlines.

So far as we are aware, Syrovatskii (1971, 1978, 1981) was the first to argue that current sheets and rapid reconnection are a general and unavoidable consequence of the quasi-static deformation of a magnetic field. To understand his approach consider the projection of the field lines of $\mathbf{B}(\mathbf{r})$ onto the plane perpendicular to $\mathbf{B}(\mathbf{r})$ at some specified point $\mathbf{r}_0$ in the field. Projection of $\mathbf{B}(\mathbf{r})$ onto the plane perpendicular to $\mathbf{B}(\mathbf{r}_0)$ provides either an O-type or an X-type neutral point in the 2D field in that plane (see discussion in Parker, 1979, pp. 383–391). Consider a point $\mathbf{r}_0$ such that the neutral point is an X-type (Fig. 1.4a). Then apply external forces to the field so as to squash the whole region, and hence squash the X-type vertex, in one direction or another, splitting the X-type vertex into two Y-type vertices, as sketched in Fig. 1.4(b). There is then a tangential discontinuity extending from one Y-vertex to the other. There is also a weaker discontinuity extending away along the two arms of each Y. Syrovatskii pointed out that this effect occurs at one or more points in any quasi-static field subjected to anisotropic or inhomogeneous deformation.

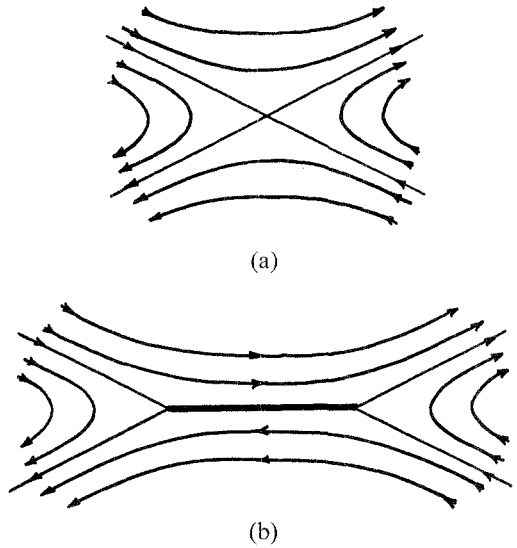

Fig. 1.4. (a) An X-type neutral point in the 2D field (on the plane perpendicular to $\mathbf{B}(\mathbf{r}_0)$), which is squeezed from above and below into two Y-type neutral points in (b). The heavy line indicates the location of the tangential discontinuity.

Subsequent development of the theory of tangential discontinuities shows that Syrovatskii's example illustrates the basic effect. The presentation of specific examples of the formation of discontinuities in Chapter 6 encounters the deformation of one X-type neutral point into two Y-type points in every case. The optical analogy, presented in Chapter 7, shows that the tangential discontinuity arises from gaps created by local field maxima in the flux surfaces, and Syrovatskii's example proves to be the elevation of the magnetic field whose plan view is the gap in the flux surface.

We begin, then, with a sketch of the standard setting for the construction of the magnetic field and the spontaneous appearance of the tangential discontinuities. The model is presented in detail in Chapter 3 and the sketch outlined here is intended only to define the physical context of the discussion. The essential physics is most simply described by starting with a uniform magnetic field B_0 extending in the z–direction through an infinitely conducting ($\eta = 0$) fluid between the boundary planes $z = 0$ and $z = L$ (Parker, 1972). Then at time $t = 0$ the fluid is set into the prescribed two dimensional incompressible transverse motion

$$v_x = +kz\partial\psi/\partial y,\ v_y = -kz\partial\psi/\partial x,\ v_z = 0, \tag{1.3}$$

determined by the arbitrary stream function $\psi(x, y, kzt)$. The function ψ is chosen to be a well-behaved, bounded, smooth, continuous, n–times differentiable function of its arguments. The deformation of the fluid and field is strong, and it is convenient to think of $\psi(x, y, kLt)$ as representing the introduction of a succession of unrelated mixing patterns of the footpoints of the field at the boundary $z = L$ while the footpoints are held fixed at $z = 0$.

After a time t the magnetic field, which is carried with the fluid, has the form

$$B_x = +B_0\, kt\partial\psi/\partial y,\ B_y = -B_0\, kt\partial\psi/\partial x,\ B_z = B_0 \tag{1.4}$$

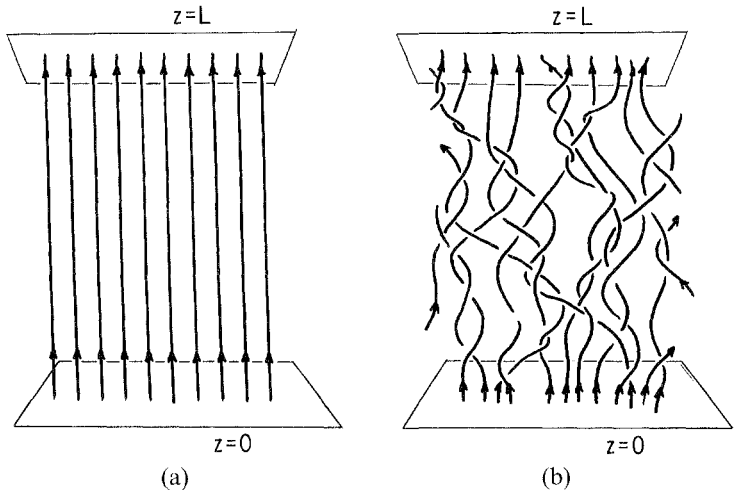

Fig. 1.5. A sketch of the arbitrary winding of the field lines of the continuous field described by equation (1.4), beginning with (a) the uniform field B_0 and become mixed and interlaced after a time t in (b).

The field lines wind and interweave among each other, following the same mixing in passing from $z = 0$ (where the footpoints remain fixed) to $z = L$ as the mapping $\psi(x, y, kLt)$ of the footpoints on $z = L$. Thus the field lines are strongly wrapped and randomly intermixed in passing from $z = 0$ to $z = L$. But note that the field is continuous, because of the prescribed smooth behavior of ψ. Fig. 1.5 is a sketch of the initial field and the interwoven state of that field after some time t.

The essential point is that any arbitrary, well-behaved function ψ is a physical possibility. To fix ideas suppose that $\psi(x, y, kLt)$ passes through n successive arbitrary, random swirling patterns at each value of (x, y) as t increases from 0 to τ. If the transverse swirling and mixing represented by ψ has a characteristic scale ℓ, then a footpoint of the field at $z = L$ moves a distance of the order of ℓ or more in each swirl, thereby accumulating a total path length $O(n\ell)$ or more on $z = L$. For purely random swirling the final distance of almost all footpoints from their initial positions is $O(n^{\frac{1}{2}}\ell)$. Note that L may be so large that $L \gg n\ell$ in the presence of strong winding, so that the inclination of the field to the z–direction may be, but need not be, small.

Stop the fluid motion at $t = \tau$, then, and hold the footpoints of the field lines fixed at $z = 0$ and at $z = L$. Release the fluid so that the continuous field is free to relax to the lowest available energy state, ultimately achieving static equilibrium after some sufficiently long period of time. The magnetostatic theorem states that almost all strongly deformed fields develop internal tangential discontinuities in the process of relaxing to static equilibrium (Parker, 1986a,b).

Consider some of the implications of this assertion. The first point is that the choice of the stream function ψ determines the topology — the winding and interweaving — of the field lines, and the topology is preserved by the infinite electrical conductivity of the fluid, i.e., by the vanishing of the electric field in the frame of reference moving with the fluid. The function ψ does not provide the final equilibrium field distribution $\mathbf{B}(\mathbf{r})$, which arises only in the relaxation of the field from the continuous form in equation (1.4) to static equilibrium.

1.6 Continuous Magnetostatic Fields

It comes to mind that there are extensive families of continuous solutions to the magnetostatic equilibrium equation (1.1). Infinitely many different continuous equilibrium fields! However, the known continuous equilibrium solutions involve either only weak deformation of the field from a uniform state (cf. the interesting numerical solutions by Van Ballegooijen, 1985, 1986; Mikic, Schnack, and Van Hoven, 1989 and the analytic solutions by Zweibel and Li, 1987) or a symmetry, degeneracy, or invariance (ignorable coordinate) of some form (see discussion in Grad, 1967, 1984; Parker, 1979, pp. 359–378; Tsinganos, 1981, 1982a,b). The requirement of invariance is not dissimilar to the well-known Taylor–Proudman theorem of hydrodynamics (Proudman, 1916; Taylor, 1917; see also discussion in Chandrasekhar, 1961) arising from the identical form of equations (1.1) and (1.2), to the effect that a stationary velocity field in a system rotating with angular velocity Ω must be invariant in the direction of Ω.

A noteworthy example of an equilibrium based on degeneracy was pointed out by Rosner and Knobloch (1982), involving force-free fields described by the solutions of the field equation (Lundquist, 1950)

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} \tag{1.5}$$

for a single value of α, say $\alpha = q$. The equation is then linear and solutions may be superposed without limit, providing no end of complexity in the total field $\mathbf{B}(\mathbf{r})$ throughout the region.

Another class of continuous solution was found by Van Ballegooijen (1985) who showed that small perturbations $\delta\mathbf{B}$ about a uniform field $\mathbf{B}_0$ (which permits strong mixing over the length of the field in the limit of large L) provide a field described by an equation that is an exact analogy to the 2D time-dependent vorticity equation. The equation is nonlinear and few analytic solutions are known, but one expects that the equation has a variety of continuous solutions. This interesting class of solutions is possible because the zero-order uniform field has no scale of its own, on which more will be said in the sequel.

It is clear that strong winding of the field, wherein a substantial fraction of the field lines extend through at least two different wrapping patterns between $z = 0$ and $z = L$, permits neither an ignorable coordinate, nor a uniform α, nor an undetermined scale in any direction. It is readily seen that such a field, if it were to remain continuous in static equilibrium, would pose a contradiction. The nature of the contradiction is most readily exhibited in the case that the pressure externally applied to the fluid at the boundaries $z = 0, L$ is uniform ($p = p_0$). The scalar product of $\mathbf{B}$ with equation (1.1) yields the condition $\mathbf{B} \cdot \nabla p = 0$, that the fluid pressure extends uniformly along each field line. Then since all the field lines connect to the uniform pressure at $z = 0, L$, it follows that p is uniform throughout the entire region ($0 \leqslant z \leqslant L$). Equation (1.1) reduces to the force-free form (1.5). Equation (1.5) has some interesting mathematical properties. The coefficient $\alpha = \alpha(\mathbf{r})$ represents the torsion in the field, with

$$\alpha = \mathbf{B} \cdot \nabla \times \mathbf{B}/B^2 \tag{1.6}$$

The analogous quantity $\mathbf{v} \cdot \nabla \times \mathbf{v}$ in hydrodynamics is called the *helicity*. Since $\nabla \cdot \mathbf{B} = 0$, the divergence of equation (1.5) gives the well-known result

$$\mathbf{B} \cdot \nabla \alpha = 0, \tag{1.7}$$

stating that the torsion α is uniform along each field line. It is obvious that field lines extending through two or more unrelated winding patterns may be subjected to quite different torsions in the different patterns, because as each flux bundle wraps in arbitrary ways around the neighboring flux bundles, its own internal torsion must match the wrapping if it is to fit continuously against its neighbors. But the internal torsion is fixed at some uniform value which cannot accommodate different topological wrappings. The only reconciliation with equation (1.7) is the development of tangential discontinuities, across which there is a finite difference in field direction. The net torsion across the tangential discontinuity is not restricted by equation (1.7) because the surface of discontinuity is only the surface of contact between two adjacent regions of field. The surface contains no magnetic flux — no field lines — and the restriction posed by equation (1.7) is evaded.

The arbitrarily complicated continuous fields constructed by Rosner and Knobloch (1982) are not restricted by equation (1.7) because they are based on $\nabla\alpha = 0$, for which equation (1.7) is trivially satisfied. On the other hand, if any variation of α is present, the degeneracy is removed, superposition of solutions is not possible, and equation (1.7) becomes a nontrivial restriction, requiring tangential discontinuities if the field topology involves a change in the torsion along any flux bundle anywhere in $0 < z < L$.

Another way to state the problem is to note the two requirements (a) that the torsion α is uniform along each field line while (b) an arbitrary topology is imposed on the field through our choice of ψ in equation (1.4). These two conditions are not irreconcilable unless we also arbitrarily impose the condition that the field is continuous everywhere. To insist on continuity is to restrict the function $\psi(x, y, kzt)$ to special invariant or degenerate forms that are incompatible with the physical possibility of arbitrary weaving and wrapping of the field lines.

We are accustomed to continuous fields in classical field theory, enforced by the fully elliptical character of the basic field equations, and leading to a unique determination of the field throughout a volume by specification of the field on the boundary of the volume. The field equation and the boundary conditions determine the topology of the field. But with a magnetic field in an infinitely conducting fluid the topology is determined ahead of time by the choice of ψ, and the normal component of the field on the boundary can be manipulated at will, by local compression or expansion of the distribution of footpoints of the field, without affecting the topology. So there can be no unique determination of the field topology throughout the volume by the field at the boundaries. As already noted, the field equations reflect this different situation by possessing an extra set of characteristics, in addition to the two imaginery characteristics of the familiar elliptic field equations in a vacuum. The subject is taken up at some length in Chapter 3, but the essential point is simply that the field lines represent a family of real characteristics, across which the field may be discontinuous, thereby accommodating the physics.

1.7 Perturbations of a Continuous Field

An obvious mathematical approach to the problem is to consider magnetostatic fields in some functional neighborhood of a known invariant equilibrium solution. This subject is taken up in Chapter 4 in detail, but it is appropriate to make some

general comments here to establish perspective. A convenient example (Parker, 1979, pp. 370–378) is the field with linear invariance ($\partial/\partial z = 0$), satisfying the familiar Grad–Shafranov equations

$$B_x = +\partial A/\partial y, B_y = -\partial A/\partial x, B_z = B_z(A), p = p(A), \tag{1.8}$$

and

$$\nabla^2 A + 4\pi p'(A) + B_z'(A)\, B_z(A) = 0 \tag{1.9}$$

(see Tsinganos, 1981, 1982a,b for the field equations and solutions in other invariant geometries). Treat the total field $\mathbf{B}(x, y) + \delta\mathbf{B}(x, y, z)$ where the perturbation does not share the invariance of $\mathbf{B}(x, y)$. The first order perturbation equations are

$$4\pi\delta p = (\nabla \times \mathbf{B}) \times \delta\mathbf{B} + (\nabla \times \delta\mathbf{B}) \times \mathbf{B}.$$

This equation can then be integrated along the real characteristics, which are the projection of the field lines of $\mathbf{B}(x, y)$ onto any plane $z = constant$, given by $A(x, y) = constant$. In the case of interest, most field lines of $\mathbf{B}$ do not extend to $x, y = \pm\infty$, and, hence, close locally on themselves. The equilibrium equation for $\delta\mathbf{B}$ can be integrated along any field line, and it is found that generally the $\delta\mathbf{B}$ computed after carrying the integration once around a closed field line is different from the initial $\delta\mathbf{B}$ from which the integration started. The special conditions necessary to produce the same final and initial $\delta\mathbf{B}$ can generally be met only on special field lines and not over a finite region of field. Hence there must be a discontinuity in $\delta\mathbf{B}$ somewhere along the contour of integration.

The only way to satisfy the equations and avoid a discontinuity is to require that $\partial\delta\mathbf{B}/\partial z = 0$, which provides a $\mathbf{B} + \delta\mathbf{B}$ that is a member of the same general class of invariant solutions, described by equations (1.8) and (1.9). In other words, there are discontinuous solutions $\delta\mathbf{B}(x, y, z)$ in the neighborhood of the continuous solutions $\mathbf{B}(x, y)$, but the only continuous solutions $\mathbf{B} + \delta\mathbf{B}$ necessarily possess the same invariance ($\partial/\partial z = 0$) as $\mathbf{B}$. So if $\delta\mathbf{B}$ were produced by winding the field lines (by moving the footpoints of the field at the end plates $z = 0, L$) in a pattern that varied along the field, the resulting equilibrium of $\mathbf{B} + \delta\mathbf{B}$ would possess internal discontinuities. The same general result has been demonstrated by Vainshtein and Parker (1986) for rotational invariance ($\partial/\partial\varphi = 0$) and more broadly by Tsinganos (1982c) in the presence of a stationary velocity field.

Tsinganos, Distler, and Rosner (1984) have taken a somewhat different approach, using a Hamiltonian formulation of the toroidal field, so that the field lines are precisely analogous to the dynamical trajectories of a mechanical system in phase space. They use the Kolmogoroff–Arnold–Moser theorem to show that if a symmetric continuous magnetostatic equilibrium is subjected to a perturbation of arbitrary symmetry, then for a finite range of pressure values, the 2D isobaric surfaces do not necessarily coincide with magnetic flux surfaces. But, as already noted, the scalar product of $\mathbf{B}$ with the magnetostatic equation (1.1) leads to $\mathbf{B} \cdot \nabla p = 0$, stating that the isobaric surfaces lie along the field. Hence magnetostatic equilibrium is no longer satisfied everywhere. They show that spatially symmetric magnetostatic equilibria are topologically unstable to finite amplitude perturbations which do not have the original symmetry properties. In other words, symmetric continuous magnetostatic equilibria are special (and hence unlikely) states in which to find a magnetic field.

This development makes Grad's (1967, 1984) earlier point, that magnetostatic equilibrium is unlikely in toroidal fields. More important, it extends the foregoing perturbation calculation to finite amplitudes, and it provides a quantitative demonstration of the very special form required for a field to avoid discontinuities while in a magnetostatic state. Invariance (an ignorable coordinate) is the simplest condition providing continuity in static equilibrium, although, as already noted, there are other special degenerate conditions permitting both static equilibrium and an absence of discontinuities throughout the field.

Coming back to the perturbation $\delta\mathbf{B}$ about a solution $\mathbf{B}$ to the Grad–Shafranov equations, the simplest example of $\mathbf{B}$ is a uniform field B_0 in the z–direction. As already noted this case is degenerate in the sense that there is no characteristic scale. In any field but a uniform field the winding of the field lines of $\mathbf{B}$ about each other provides a scale length in the z–direction that is related to the transverse scale, in the x- and y–directions. But the scale disappears for the uniform field and the scales of variation of $\delta\mathbf{B}$ along and across $\mathbf{B}$ are independent of each other. Thus, if the two scales are treated as being of the same order, the calculation is carried forward in the manner just described, integrating along the field lines of the unperturbed field. The necessary condition for equilibrium of a continuous field is again $\partial\delta\mathbf{B}/\partial z = 0$ (Parker, 1972; 1979 pp. 363–368).

More recently Van Ballegooijen (1985) pointed out that the scale of variation of $\delta\mathbf{B}$ along the uniform field $\mathbf{B}_0$ may be one order larger than across $\mathbf{B}_0$ providing a different ordering of the terms in the field equation (1.5) and an interesting variation in the requirement for solution (noted in §1.6) already noted in a different context in §1.6. In particular, the perturbation $\delta\mathbf{B}$ with a transverse scale ℓ and a magnitude εB_0 ($\varepsilon \ll 1$) has a winding pattern that extends a distance $O(\ell/\varepsilon)$ along $\mathbf{B}$ if the mutual wrapping of the field lines proceeds through one or more radians before the winding pattern changes. Then $\partial/\partial z$ is small $O(\varepsilon)$ compared to $\partial/\partial x$ and $\partial/\partial y$ so that $\partial\delta\mathbf{B}/\partial z$ is small $O(\varepsilon^2)$ and should be dropped from the first-order equations. Then if $\delta\alpha$ represents the torsion coefficient (the zero-order torsion coefficient being identically zero for the uniform field) equation (1.7) reduces to

$$\delta B_x \frac{\partial\delta\alpha}{\partial x} + \delta B_y \frac{\partial\delta\alpha}{\partial y} + B_0 \frac{\partial\delta\alpha}{\partial z} = 0, \tag{1.10}$$

each term being small to second order. Thus by making $\delta\mathbf{B}$ almost invariant $O(\varepsilon)$ with respect to z, the mathematics recovers a condition that is less stringent than the basic $\partial/\partial z = 0$. Equation (1.10) asserts that $\delta\alpha$ is invariant along the field lines of the perturbed field, rather than requiring that $\delta\alpha$ be invariant along the lines of the zero-order field. Van Ballegooijen goes on to point out the exact analogy between equation (1.10) and the equation for the time-dependent vorticity in the 2D motion of an ideal inviscid incompressible fluid. The variable t in the vorticity equation plays the role of z in equation (1.7), in this case, because the torsion $\delta\alpha$ is invariant along each field line and the vorticity is invariant along the world line of each element of fluid. This example is interesting because it establishes the existence of continuous fields which are not degenerate or symmetric or invariant in any simple way. Instead the solutions are characterized by a transverse scale small compared to the longitudinal scale.

There are no known continuous-time-dependent analytical solutions to the 2D vorticity equation, apart from the motion associated with one or two vortex lines

with potential flow everywhere between, or the motion associated with many moving vortex lines arranged in special symmetric patterns that do not change with time. The equation describes the time behavior of a 2D fluid, which can be set in arbitrary continuous motion at time $t = 0$, and which continues to move for an indefinite period afterward. There is obviously a diversity of initially continuous-time-dependent flows. One may wonder whether an initially continuous flow of an ideal fluid may in a finite time form tangential discontinuities. The strict mathematical analogy with the magnetostatic equation suggests that there are no such solutions. As we shall see in Chapter 8, the structure of tangential discontinuities in magnetic fields in static equilibrium indicates that they do not begin or end somewhere in the volume between the rigid end plates, $z = 0, L$. The formal analogy between z in the magnetostatic field and t in the evolving hydrodynamic field implies that discontinuities do not begin or end in finite time t. Note, however, that this analogy does not exclude the asymptotic formation of vortex sheets in the limit of large t. On the other hand, in the presence of a small but nonvanishing viscosity, the final asymptotic approach of a vortex layer to zero thickness is irrelevant, and it is well known that a slightly viscous fluid in inhomogeneous motion becomes turbulent in a finite time, with the vortex sheet thickness limited by the viscosity. In a similar way, a magnetostatic field may have an arbitrary winding pattern at $z = 0$, but if the field is continuous, the pattern at $z = 0$ determines the continuous pattern for all finite $z > 0$, producing whatever topology is compatible with the invariance of α along each field line, just as the vorticity ω is transported bodily along the world line of each element of fluid so that $d\omega/dt = 0$.

1.8 Tangential Discontinuities

In nature the winding of the field lines (carried out by random convective transport of the footpoints of the field) is arbitrary so that the field is not expected to have the symmetry, invariance, or degeneracy necessary for continuous solutions. To repeat the point made earlier (following equation (1.7) in §1.6) for arbitary ψ a flux bundle threads around first one way and then the other through the spaghetti of neighboring interwoven flux bundles. The torsion within each elemental flux bundle must vary along the bundle in precise coordination with the winding around the neighboring flux bundles if the bundle is to fit continuously against each flux bundle that it encounters. But $\mathbf{B} \cdot \nabla\alpha = 0$ requires that there is no variation in the torsion α along the flux bundle. So the condition for a continuous field cannot generally be met. There are unavoidable tangential discontinuities where the flux bundle winds around neighboring flux bundles. There is no way that an arbitary winding and interweaving of the field lines can relax to equilibrium and preserve continuity (Parker, 1986a,b,c, 1989a).

The braiding of three flux bundles provides an elementary illustration of the principles. A single wavelength of braiding is sketched in Fig. 1.6, in which each flux bundle wraps first one way and then the other around each of the other two bundles. Hence the net torsion along each flux bundle is zero, so α, being constant along each line of force, can be set equal to zero. Then since $\nabla \times \mathbf{B} = \alpha\mathbf{B}$, it follows that $\nabla \times \mathbf{B} = 0$ so that $\mathbf{B} = -\nabla\phi$ and $\nabla^2\phi = 0$. Within each flux bundle the field is a potential field without torsion. There are no continuous solutions to Laplace's

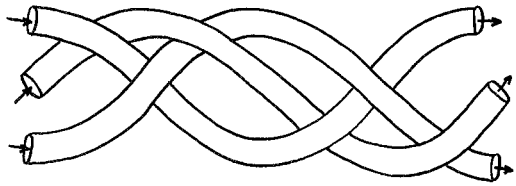

Fig. 1.6. A sketch of one wavelength of the braiding of three flux bundles.

equation representing a braided field. It follows that the fields are not aligned where each flux bundle passes obliquely across a neighbor, so that there is necessarily a tangential discontinuity. Note again, then, that the net torsion (i.e., the discontinuity in field direction) at each mathematical surface of tangential discontinuity does not violate $\mathbf{B} \cdot \nabla\alpha = 0$ because there is no magnetic flux in the mathematical surface. The surface of discontinuity is merely the surface of contact between two regions of nonparallel field.

The tangential discontinuity makes its appearance in the theory of magnetostatics in other ways than through considerations on $\mathbf{B} \cdot \nabla\alpha = 0$. For instance, one may construct any continuous magnetostatic field (e.g., a solution of equation (1.9)) containing more than one winding pattern, i.e., with two or more distinct topological regions, and show from elementary considerations that almost any overall deformation of that field creates tangential discontinuities, by upsetting the X-type vertices of the topological separatrices (Parker, 1982, 1983b, 1990b). The process can be seen in a variety of formal mathematical examples of tangential discontinuities from simple compression or expansion of a continuous field (Priest, 1981, pp. 144–171; Kulsrud and Hahm, 1982; Hahm and Kulsrud, 1985; Low, 1987, 1989; Low and Wolfson, 1987; Jensen, 1989). The discontinuities form along the topological separatrices of the original field. Chapter 6 treats several examples in detail.

A variation of this is to consider the construction of a continuous field by the actual physical displacement of the fluid and field in a hypothetical laboratory apparatus. It is then easy to show that the slightest random error in the mechanical manipulation misses the desired mathematical continuity, producing tangential discontinuities with two or more Y-type neutral points where one X-type is necessary for continuity (Parker, 1982, 1990b).

As already noted, the magnetostatic equation (1.1) has two imaginery characteristics (Parker, 1979, pp. 361–363). This fact in itself would make equation (1.1) a fully elliptic equation, so that the internal field would be uniquely determined by the arbitrary distribution of field at the boundaries. This is physically absurd, of course, because the internal winding and intertwining of the field, i.e., the internal topology, is arbitary, determined separately by the sequence of random swirling of the footpoints of the field (at $z = 0, L$). Following the random intertwining of the field lines the footpoints can be pushed together so that the normal component of the field on $z = 0, L$ has an arbitary distribution, without affecting the internal topology. We remark again that the escape from this contradiction lies in the additional set of characteristics, represented by the field lines. Discontinuities along these real characteristics are permitted. The discontinuities avoid the uniqueness implied by the imaginery characteristics alone, because the uniqueness depends on the assumption that the field is continuous throughout. Thus, in the foregoing example of the braided field, in which the field reduces to a solution of Laplace's equation,

$\nabla^2\phi = 0$, the well known uniqueness of $\nabla^2\phi = 0$ is avoided by the surfaces of tangential discontinuity (the topological separatrices) between the three individual flux bundles.

A direct approach to the formal problem is through integration of the nonlinear magnetostatic equation (1.1). As already noted, there are many families of known continuous solutions. Standard mathematical techniques automatically generate continuous solutions, so they provide the special topologies required to achieve continuity everywhere, but at the price of an ignorable coordinate or degeneracy. Fortunately, there are some simple cases where the integration of equation(1.1) can be carried through in the absence of the symmetry required for continuity (Parker, 1990c). One finds that, if the solution has the specified topology and fits the boundary conditions, then it contains surfaces of tangential discontinuity. If, on the other hand, the solution is required to be continuous, then there appears a unique solution which satisfies the essential boundary condition on the normal component of the field, but which necessarily has an internal topology different from the specified topology. Yet the specified topology is physically well posed, because it can be produced by simple precise hypothetical physical manipulation of a magnetic field in a ponderable fluid under ideal circumstances of zero resistivity. So complete continuity of a field simply does not conform to equation (1.1). An extended example is explored in Chapter 5.

A straightforward integration of $\mathbf{B} \cdot \nabla\alpha = 0$ provides a general expression for α in terms of $\mathbf{B}$, in the form of a contour integral along the field lines and along the orthogonal family of lines. It is shown (Parker, 1986a,b) that α is generally discontinuous, which comes about through the discontinuity of $\mathbf{B}$.

1.9 The Optical Analogy

Finally, we construct the optical analogy (Parker, 1981, 1989b,c, 1991b) to show how the tangential discontinuity arises from the local properties of the magnetic field in the neighborhood of a maximum in the field magnitude $\mathbf{B}(\mathbf{r})$. The optical analogy applies to the projection $\mathbf{F}_s$ of any vector field $\mathbf{F}(\mathbf{r})$ onto the local flux surfaces of $\nabla \times \mathbf{F}$, because $\mathbf{F}_s$ can be described as the gradient of a scalar potential ϕ in that surface. For the force-free field $\nabla \times \mathbf{B} = \alpha\mathbf{B}$, the local flux surfaces of $\nabla \times \mathbf{B}$ coincide with the local flux surfaces of $\mathbf{B}$. So in any flux surface of $\mathbf{B}(\mathbf{r})$ the field can be written as $\mathbf{B} = -\nabla\phi$, (noting that $\nabla \cdot \mathbf{B} = -\nabla^2\phi \neq 0$ in the 2D flux surface). The equations for the field lines in the flux surface are the same as the equations for the optical ray paths in a medium with index of refraction $B(\mathbf{r}) = |\mathbf{B}(\mathbf{r})|$. This is the optical analogy in its form for magnetostatic fields. Its importance lies in the fact that it shows the direct connection between the field pattern and the local variation of $B(\mathbf{r})$. In particular, the field lines are refracted around a local maximum in $B(\mathbf{r})$.

If the maximum is sufficiently concentrated, the optical pathlength around the maximum is shorter than across the maximum. So Fermat's principle specifies a bifurcation in the field pattern where the field lines pass around the maximum rather than across the maximum. This bifurcation or gap in the field pattern is the singular property that leads to the formation of the tangential discontinuity. The gap permits the fields on either side of the flux surface to come in contact, as already remarked. These otherwise separate fields are generally not parallel, with the result that the

contact surface, or separatrix, between the regions becomes a surface of tangential discontinuity throughout the gap.

There is a theorem by Yu (1973) that demonstrates the association of a bifurcation in the field lines with a tangential discontinuity. Yu considered field lines in the isobaric surfaces ($p = constant$) in a magnetic field in static equilibrium, described by equation (1.1). Since $\mathbf{B} \cdot \nabla p = \nabla \times \mathbf{B} \cdot \nabla p = 0$, the isobaric surfaces are flux surfaces of both $\mathbf{B}$ and $\nabla \times \mathbf{B}$. Treating two near neighboring field lines he showed that the current density $j_\perp$ perpendicular to the field varies in direct proportion to the separation of the lines. A bifurcation of the field pattern in an isobaric surface represents an increase in the separation of adjacent field lines from infinitesimal to finite distances in a finite distance along the field. Hence the bifurcation is automatically associated with an infinite current density, i.e., a current sheet or tangential discontinuity. It follows from the optical analogy that any field with localized internal maxima produces internal tangential discontinuities, unless there are rigid boundaries so close at to prevent any bifurcation in the field pattern.

1.10 General Discussion

The magnetostatic theorem is a statement about the relation between the topology and the continuity of a magnetostatic field, and as such it touches upon several other properties of the magnetic fields to be found in nature. Consider first the general stochastic nature of the field lines of any magnetic field in the physical universe. The field lines are defined as the integrals of

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z}$$

at any instant in time. Pick any two neighboring points $\mathbf{r}_0$ and $\mathbf{r}_0 + \delta\mathbf{r}_0$ with $\delta\mathbf{r}_0$ perpendicular to $\mathbf{B}(\mathbf{r}_0)$. These two points define two field lines to the specified field $\mathbf{B}(\mathbf{r})$ with characteristic scale ℓ. Assuming that $|\delta\mathbf{r}| \ll \ell$, the separation $\delta\mathbf{r}(s)$ of the two field lines increases more or less exponentially with distance s measured along either line from $\mathbf{r}_0$, as a consequence of the fluctuating gradients in $\mathbf{B}(\mathbf{r})$. The exponential separation continues until $|\delta\mathbf{r}(s)|$ becomes comparable to ℓ, beyond which the two field lines random walk relative to each other and their separation increases more like $s^{\frac{1}{2}}$ (Jokipii and Parker, 1968; Parker, 1979, pp. 274–297, 1992).

A Hamiltonian formulation of the field, in which the individual field line corresponds to the trajectory of a particle in phase space, is particularly useful to treat the topology when the field closes on itself, i.e., toroidal geometry. In any such closed configuration the field lines are ergodic, as may be seen from the Hamiltonian formulation (Kerst, 1962; Parker, 1969; Boozer, 1983; Cary and Littlejohn, 1983). Since $\mathbf{B} \cdot \nabla p = 0$ it follows that $\nabla p = 0$ throughout an ergodic toroidal field in static equilibrium, from which it follows that there can be no true static equilibrium if the plasma is confined by the field or vice versa (Moser, 1962; Grad, 1967, 1984). The construction of global flux surfaces and isobaric surfaces, which project along the field, is problematical because any such surface fills the entire toroidal volume. Needless to say a *local* flux surface is generated by the projection of any transverse curve along the field lines.

The presence of a suprathermal plasma component (e.g., the multimillion degree

gas of the solar corona, the 10^8 K plasma generated in a solar flare, or the cosmic rays that fill the gaseous disk of the galaxy) provides another source of nonequilibrium in any field rooted in the dense thermal gas of a self-gravitating body such as a star or galaxy. For the free flow of the suprathermal gas (without significant gravitational confinement) along the field lines inflates without limit the outer lobes of the field, where the field falls asymptotically to zero. This nonequilibrium is the means by which the open magnetic field regions are formed on the Sun to provide the coronal holes and the solar wind (Parker, 1963b). It appears to be, at least in part, the basis for the extended galactic magnetic halos (Parker, 1965, 1968, 1969, 1979, pp. 274–297, 1992) and for the resulting tangential discontinuities, magnetic dissipation, and X-ray emission from those halos (Parker, 1990a, 1992).

In summary, the stochastic field lines extending between $z = 0$ and $z = L$ (equivalent to the mirror geometry employed in the plasma laboratory) become the ergodic field of toroidal geometry (in which the field circles incommensurably through itself).

The present writing is concerned primarily with the bounded "mirror geometry" of a magnetic field extending in the z–direction between "end plates" $z = 0$ and $z = L$ (Parker, 1972, 1979, p. 364), thereby avoiding questions of the existence of global flux surfaces, global isobaric surfaces, and static equilibrium and avoiding the absence of equilibrium when a suprathermal gas component is present. The flux surfaces are simply and uniquely defined (extending only between the end plates $z = 0, L$) and play a key role in the development of the theory.

1.11 Hydrodynamic Turbulence

Consider the fact discussed in §1.5 that the magnetostatic equation (1.1) and the stationary Euler equation (1.2) are analogous. Thus the surfaces of tangential discontinuity (current sheets) in magnetostatic fields have an exact counterpart in the surfaces of tangential discontinuity (vortex sheets) in stationary flows. Arnold pointed out that almost all solutions of the Euler equation, and hence of the magnetostatic equation, involve tangential discontinuities. It can now be stated that there are no continuous solutions for almost all field topologies.

Now, we are familiar with the idea that the dynamical instability of a stationary flow at high Reynolds number leads to turbulence and the formation of vortex sheets (Batchelor, 1947). The interesting point is that the formation of vortex sheets is already an intrinsic part of the initial stationary flow, arising from the nature of the static Reynolds stresses (Parker, 1989b, 1991b). Turbulence is, then, the unstable time-dependent form of the ubiquitous vortex sheet formation, driven by the Reynolds stress in both the time-dependent and time-independent flow. Specifically, the vorticity ω is transported bodily with the fluid velocity $\mathbf{v}$, with

$$\partial\omega/\partial t = \nabla \times (\mathbf{v} \times \omega),$$

and regions of different vorticity, i.e., separate eddies, coming into contact without their velocities being compelled to match smoothly, creating a vortex sheet at the surface of contact, i.e., at the topological separatrix.

The formation of vortex sheets in a turbulent flow is conventionally referred to as a "cascade" to large wave number $\mathbf{k}$. The large wave number arises mainly from

the abrupt change of $\mathbf{v}$ across the vortex sheet, remembering that the sheet itself is relatively broad, involving small wave numbers. The broad vortex sheet is continually complicated by the Kelvin–Helmholtz instability which produces corrugations at increasingly large wave number, of course. The cascade to large wave numbers in statistically stationary turbulence may be thought of as the time-dependent unstable form of the tangential discontinuities of the stationary flow with any but the simplest topology.

The intrinsic role of tangential discontinuities in stationary velocity fields and in static magnetic fields could also be referred to as a cascade to large wave numbers, but in practice the term *cascade* suggests the concept of dynamical effects in turbulence, which is absent in the quasi-static formation of tangential discontinuities. On the other hand, the optical analogy, which is not restricted to quasi-static fields provides an appropriate mathematical tool for treating the dynamical tendency toward tangential discontinuities in the time-dependent turbulent flow, showing that any sufficiently localized maximum in v_s presses toward a gap in the pattern of $\mathbf{v}_s$, where $v_s = |\mathbf{v}_s|$ and $\mathbf{v}_s$ represents the instantaneous projection of the fluid velocity $\mathbf{v}$ onto a flux surface of the vorticity $\nabla \times \mathbf{v}$. This is developed at length in Chapter 7.

2

The Field Equations

2.1 Appropriate Field Concepts

Maxwell's equations can be written in terms of various field vectors **E**, **D**, **H**, **B**, **A**, **j**, etc. The first step is to establish the most convenient vector quantities for formulating the basic equations of magnetohydrodynamics (MHD). We find it simplest to deal directly with the electric field **E** and the magnetic field **B**, in a system of units (e.g., cgs) that is compatible with the basic symmetry of Maxwell's equations. In these units Maxwell's equations are

$$\nabla \cdot \mathbf{E} = 4\pi s, \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.2}$$

$$4\pi \mathbf{j} + \partial \mathbf{E}/\partial t = +c\nabla \times \mathbf{B}, \tag{2.3}$$

$$\partial \mathbf{B}/\partial t = -c\nabla \times \mathbf{E}, \tag{2.4}$$

where s is the electric charge density and **j** is the electric current density. The electric field **E** is defined in terms of the force on a static charge, and **B** can be defined by the Lorentz force $\mathbf{j} \times \mathbf{B}/c$ or by the equivalent Lorentz transformation in the presence of an electric field, there being no experimental evidence of magnetic charges in the universe. The field pressures $E^2/8\pi$ and $B^2/8\pi$ and the field tensions $E^2/4\pi$ and $B^2/4\pi$ are measured in dynes/cm^2 and the electric and magnetic fields E and B are of equal magnitude (gm$^{\frac{1}{2}}$ cm$^{-\frac{1}{2}}$ sec^{-1}) when their stress densities are equal. The effect of matter on the electromagnetic fields **E** and **B** appears through s and **j**.

The molecular electric and magnetic polarization of ionized gases is negligible in most circumstances, in contrast to the dominant effect of the inertia of the bulk motion **v** of the plasma. Thus, while the dielectric coefficient ε and the magnetic permeability μ of gaseous materials are close to one, the inertia of the plasma coupled to the magnetic field causes the phase velocity of a plane transverse MHD (low frequency) wave to be of the order of the Alfven speed $C = B/(4\pi\rho)^{\frac{1}{2}}$, which is small compared to the speed of light in any case with which we shall be concerned. So the formulation of the MHD equations is carried out in the nonrelativistic regime, neglecting the atomic polarization. That is to say, the development neglects $\varepsilon - 1$, $\mu - 1$, and v^2/c^2 compared to one. The effect of the inertia of the plasma is to produce an effective plasma index of refraction kc/ω large compared to one, even if $\varepsilon, \mu \cong 1$.