DYNAMIC ECONOMICS

OPTIMIZATION BY THE LAGRANGE METHOD

Gregory C. Chow

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GREGORY C. CHOW

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Preface

Classical economic ideas, beginning with those recorded in Adam Smith's *Wealth of Nations* (1776), have been refined, improved, and challenged. Two main ideas are that individuals pursue self-interests (or maximize utility and profits in a "rational" manner) and that markets can coordinate individual self-interests for the common good (or achieve a competitive equilibrium which is Pareto optimal) without government intervention. Observing the poverty of British workers in the 19th century, Karl Marx rejected these ideas. Witnessing the Great Depression in the 1930s Keynes wrote the *General Theory* (1936) and started a revolution against classical economics. Later the ideas of classical economics have been recovering from the Keynesian attack, to a large extent through the efforts of University of Chicago economists, beginning with Milton Freidman in the 1950s. Thus, the two main ideas of classical economics, the rationality of economic agents and the efficiency of markets, have gone through refinement and challenge. This book reports on the development of the economic ideas of self-interest and market equilibrium from the 1960s to the 1990s.

As the subtitle indicates, dynamic economic behavior is explained by assuming rationality of economic agents who optimize. Much of the book presents models in which markets are both clear and efficient. Both ideas are opposed to the main ideas of Keynes' *General Theory*. When I studied dynamic economics in the early 1970s, the idea of using optimization as the basis of dynamic economics was not as widely accepted as it is today. [See my *Analysis and Control of Dynamic Economic Systems* (1975, p. 22).] Today, some leading economists—including Robert Solow, who is the father of modern growth theory—do not believe that optimization together with market equilibrium is necessarily the way to model economic behavior. [See Solow's article "Perspective on Growth Theory," *Journal of Economic Perspectives* (1994, pp. 45–54) and his recent book *A Critical Essay on Modern Macroeconomic Theory* (MIT Press, 1995), coauthored with Frank Hahn.] Without taking a stand, I present in this book the optimization framework for dynamic economics so that readers can understand the approach and use it as they see fit.

Once optimization is selected as the main theme for dynamic economics, there is a choice of the method of dynamic optimization. In this book, the choice is clear. It is the method of Lagrange multipliers rather than dynamic programming. From the methodological point of view, much of this book contains applications of the method of Lagrange multipliers to modeling dynamic economic behavior. As a by-product, it will convince most readers that this method is easier and more efficient than dynamic programming. Being able to economize on the tool, the reader can understand the substance of dynamic economics better.

When I first worked on dynamic optimization as it is applied to macroeconomic policy (*Journal of Money, Credit and Banking*, 1970, pp. 291–302), I used the method of Lagrange multipliers. The Lagrange method and dynamic programming were presented in my 1975 book in chapters 7 and 8, respectively. I applied the method in special cases, mainly in the case of a quadratic objective function and a linear dynamic model. Not until 1991, did I discover that, for the general case, the Lagrange method can and should (for most applications) replace dynamic programming, as I wrote in *Economic Modelling* (January, 1992) and *Journal of Economic Dynamics and Control* (July, 1993). In the last three years, in research and teaching, I found the Lagrange method so much more convenient to use that I decided to record my experience in this book and share it with interested readers.

Dynamic economics has experienced rapid growth in the last three decades. This book provides a unified and simple treatment of the subject using dynamic optimization as the main theme and using the method of Lagrange multipliers to solve dynamic optimization problems. That economic agents try to maximize an objective function subject to resource constraints is a fundamental tenet of economic theory. This tenet is adopted to explain economic behavior at one instance of time and the evolution of economic behavior through time. The use of the Lagrange method allows the author to treat different topics in economics, including economic growth, macroeconomics, microeconomics, finance, and dynamic games. It also allows the treatment to be presented in an elementary level that makes the text accessible to first-year graduate students.

This book is written as a reference or a supplementary text for first-year graduate courses in macroeconomics (by using chapters 2, 3, 4, and 5), microeconomics (by using chapters 2 and 3, section 5.6, and chapters 6 and 8), mathematics for economists, and finance (by using chapters 2, 7, 8, and 9). It could be used as a text for a graduate course or an advanced undergraduate course on dynamic economics. It can also be used by researchers who would like to learn dynamic economics without the requirement of advanced mathematics. Because dynamic optimization or optimal control is a set of tools useful in operations research, engineering, and other applied fields, students and researchers in these other areas may also benefit from reading this book.

As human knowledge accumulates, the education process becomes more efficient. Holders of bachelor's and doctoral degrees today know much more than their counterparts three decades ago. For more knowledge to be acquired in the same interval of time, university education has to be and has become more efficient. Efficiency is achieved by presenting once difficult and advanced concepts in easier and more elementary forms. The writing of this book illustrates the simplification of knowledge in the study of economics.

Like many other fields of knowledge, dynamic economics is a field that has gone through a rapid development in the last three decades. Many ideas and concepts were discovered using advanced and difficult mathematical tools. As these ideas become better understood, they can be presented in a more elementary and simple fashion. A solid background in intermediate-level economics, in multivariate calculus, and in mathematical statistics is sufficient for reading most of this book. Of course, presenting the subject in an elementary fashion may sacrifice mathematical rigor and leaves some fundamental theorems unproved, as in the case of teaching calculus to beginning students. The gain is that the tools so presented can be more easily understood and hence more readily applied to solve economic problems. Most of this book should be accessible to first-year graduate students and a small fraction of advanced undergraduate students. It is hoped that a future edition of this book that incorporates further improvements in exposition should be accessible to a larger fraction of undergraduates.

Besides being elementary, this book teaches by using examples. Rather than introducing concepts in general forms, it begins by using the concepts and tools to solve simple problems. After learning the methods through examples, the reader is encouraged to apply them to solve other problems provided at the end of each chapter. Human beings learn by examples encountered in their daily experience. After they learn from particular examples, they generalize and summarize their experience in the form of general propositions. This process of learning and generalizing occurs not only in scientific inference through induction, but also in the discovery of mathematical theorems through deduction. Mathematicians have confirmed that, while they state their results in general theorems, they have often discovered them by encountering and thinking through simple examples in special cases. By introducing examples before stating general results, this book allows the reader to go through a common learning process. Some might argue that stating a general theorem first, before illustrating it by examples, is a more efficient way of teaching, even if it is not the way that the theorem was first discovered. It certainly is a more efficient way of presenting results in mathematics, but it may or may not be a more efficient way of teaching a subject. The author is not opposed to the other style of presentation, but merely makes a case for his own style and recognizes that both have merits. A list of mathematical statements is provided at the end of the book for lease of reference.

Teaching by using examples applies also to the discussion of substantive economics, in addition to the discussion of analytical techniques. The subject matter of each chapter is presented by discussing theories and models proposed by different authors, in addition to stating general approaches to the subject. Readers may learn to formulate their own theories and solve new problems in dynamic economics by studying how others have done so. Readers are encouraged to learn by using the techniques and formulations presented and illustrated here.

I acknowledge the congenial intellectual atmosphere provided at the Economics Department of Princeton University under the able chairmanship of Stephen Goldfeld and Harvey Rosen, which made the writing this book possible. Many students have worked through drafts of various chapters and helped me understand the subject and improve the writing. Colleagues and friends who have commented on and contributed ideas to this book include, among others, Andrew Abel, Dilip Abreu, Giuseppe Bertola, Alberto Bisin, Claudia Choi, Avinash Dixit, Y. K. Kwan, Robin Lumsdaine, Richard Quandt, Mark Watson, Harald Uhlig, and Dahai Yu. To all of them, I express my sincere thanks. Thanks also to Cynthia Cohen, who has typed drafts of the manuscript with efficiency and cheer.

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Introduction

1.1 Dynamic Economics and Optimization

Dynamic economics is concerned with explaining economic behavior through time. Because economic behavior of individuals and enterprises always occurs through time, should all of economics be dynamic economics? No, because of the power of abstraction in scientific theorizing. Of course, all behavior occurs through time. Yet, without explicitly taking into account the temporal aspects of behavior, economists are able to explain a great deal of economic behavior. For example, static demand theory implies that if the price goes up, the quantity demanded goes down. The price and quantity referred to are supposed to be observed in the same time period. The price is the price on a certain Wednesday, and the quantity is the amount purchased on that same Wednesday. The term *dynamic economics* is used because there was a long period in its development when much of economic theorizing ignored the dynamic aspects of economic behavior and failed to explain the time paths of economic variables. Built on the method of comparative statics, economic theory explains only the directions of change of economic variables resulting from a change in the parameters, but it does not date the changes at each instance of time. Dynamic economics can explain the time paths of economic variables.

How do economists explain the time paths of economic variables? At an early stage of the development of dynamic economics in the 1950s and 1960s, a significant portion of dynamic theories is ad hoc. Starting with a static theory explaining a variable y by a variable x, delayed effects of x can be introduced by using distributed lags to obtain a dynamic theory. For example if y_t at time t is assumed to equal $\beta x + \gamma y_{t-1}$, the time path y_t for $t = 1, 2, \ldots$ is determined once the initial value y_0 and the time path x_t for $t = 1, 2, \ldots$ are given. Baumol (1951) is a good exposition of dynamic economics as it was known at the time, as is Allen (1956), which appeared a few years later. Many of the models introduced are based on ad hoc assumptions. The main tool of analysis was difference equations. The building of dynamic econometric models from the 1950s to the 1970s was also characterized by the introduction of ad hoc assumptions concerning the lagged effects of economic variables. Distributed lags were estimated from the data empirically without much guidance from economic theory. Not until the late 1970s did economists begin to adopt, as a general principle, the method of economic theorizing based on maximizing behavior in the study of dynamic economic phenomena in the same way that the method had been applied in the formulation of static theories.

My opinion concerning the role of optimization in dynamic economic theory in the early 1970s was as follows (Chow, 1975, p. 20):

> From the practical point of view we have dynamic macroeconomic models that are built from a combination of theory (of whatever kind) and statistical measurement, and these econometric models deserve to be studied by the methods of dynamics described. Furthermore, we introduce methods of optimization over time for the purpose of improving the dynamic performance of an economy—methods that will be useful for those theorists who wish to derive dynamic economic theories from optimization over time and presumably under uncertainty. There is no need to entertain the question whether the basis of static economics under certainty, namely maximizing behavior, should always be extended to the study of dynamic behavior over time under uncertainty. Possibly, or possibly not, maximizing behavior is a better approximation to human situations in which time and uncertainty are less relevant.

An important concept in economics, perhaps the most important, is optimization. Because economics deals with the optimum allocation of resources to achieve a given objective, economic agents are assumed to maximize an objective function subject to resource constraints. In dynamic economics, the choice is among the use of resources in different time periods. Maximizing a multiperiod objective function of variables in many periods, subject to budget and resource constraints on these variables through time, constitutes a basic method for theorizing in dynamic economics.

In this book, I use dynamic optimization as a unifying theme to study some important problems in dynamic economics, just as constrained maximization has served as a unifying theme for much of static economics since the publication of Hicks (1939) and Samuelson (1948). The topics to be studied include economic growth, theories of market equilibrium, business cycles, dynamic games, models in finance, models of investment, and numerical methods of dynamic optimization. Because these topics are studied from one unifying point of view, the coverage of each topic is selective and not complete. Nevertheless, the use of dynamic optimization as a unifying theme enables an understanding of the important elements of each topic and the relationship among different topics. Another objective of this book is to provide an elementary exposition of dynamic economics and to make the topics covered more easily accessible. Drawing from a variety of sources, some of which are difficult to read, the exposition provided facilitates understanding these topics. A final objective is to introduce the Lagrange method for dynamic optimization under uncertainty as an alternative and a supplement to dynamic programming. In the middle 1990s, researchers and teachers have been trained to use dynamic programming and will continue to use it for some time because they are familiar with it. However, in this book, I will demonstrate that for most applications the Lagrange method is analytically simpler and numerically more accurate. The Lagrange method suggested here is essentially a generalization of Pontryagin's maximum principle to the case of uncertainty in which the dynamic model in the optimization problem is stochastic. To solve dynamic optimization problems under certainty, students and researchers have found Pontryagin's maximum principle useful and often more convenient than dynamic programming. For similar reasons, which are partly stated in section 1.2, students and researchers should also find the Lagrange method useful and often more convenient than dynamic programming in solving many dynamic optimization problems under uncertainty.

Dynamic economics experienced rapid growth in the 1960s and 1970s. The growth was partly stimulated by the analysis of dynamic econometric models and the use of these models for policy analysis. Chow (1975) reports on some of the work done since the 1960s, and on the analysis and formulation of optimum policies based on dynamic econometric models. The method of dynamic programming advanced by Bellman (1957) has been a main tool for optimization over time under uncertainty. In addition, since the early 1970s, the use of optimal control techniques for the formulation and analysis of macroeconomic policy and for the characterization of macroeconometric models has been an important activity in applied economics.

In the middle 1970s, in academic circles, doubts were expressed on such uses of optimal control methods by some critics, notably Lucas (1976). Following the early ideas of Marschak (1953), Lucas points out that if government policy changes, rational economic agents will take that change into account and make corresponding changes in their own decision rules that are behavioral equations of econometric models. Hence, an econometrician cannot assume that the equations of the econometric models are fixed when assisting the government in designing an optimal decision rule. Applied economists continued to use econometric models for policy analysis, believing that the changes in the parameters of the behavioral equations of econometric models alluded to by Lucas are too small or too slow in practice to seriously affect the usefulness of optimal control calculations using econometric models. Lucas is correct in principle, but how relevant or how frequently is his point encountered in practice?

Sims (1980, pp. 12–13) answers this question by stating that Lucas' point is rarely relevant because, in practice, econometricians seldom assist the government in changing a policy rule or regime. A policy rule can be expressed in the form of a reaction function. For example, a reaction function may specify a monetary policy variable as a function of observed states of the economy including, for example, real output, inflation rate, and unemployment rate. Sims observes that such policy rules or reaction functions are seldom changed. Otherwise, econometric models estimated by using data that span several decades without structural breaks cannot be useful. In performing optimal control exercises using an existing econometric model, an econometrician merely assists the government in implementing an existing policy rule and not a change in the policy rule. In other words, a reaction function is a means by which econometricians describe government behavior, just as a demand function is a means for describing consumer behavior. To derive a reaction function or a demand function, the economist assumes that the government or the consumer is implicitly performing some maximization exercises; the former maximizes a social objective function, and the latter maximizes a utility function. The econometricians are assisting the government in computing its existing reaction function and not, in most econometric policy exercises, in designing a new reaction function or policy regime. The application of optimal control techniques for government policy analysis is like the application of mathematical programming and other techniques taught in business schools to assist firms to maximize profits. While the debate on the usefulness and limitations of macroeconometric models in policy analysis goes on in academic circles, practitioners continue to use such models and optimal control methods, partly to understand the dynamic characteristics of the econometric models used and partly to assist the government in implementing a given policy regime.

Econometric models are defined as "structural" if the parameters do not change when a policy regime changes. Such a definition of structural parameters underlied the research of the members of the Cowles Commission at the University of Chicago when they proposed simultaneous structural equations in econometrics, as aptly pointed out by Marschak (1953). Some economists later believed that simultaneous equations models built by profitmaking model builders are not structural in the sense of being invariant after policy regime shifts. If a regime shift does occur (though rarely), to build models with invariant structural parameters, one should start from a model of optimizing behavior on the part of economic agents who are assumed to solve a dynamic optimization problem under uncertainty. The parameters of the optimization problem are the structural parameters. Once these structural parameters are known, the behavioral equations of economic agents, or the reduced-form equations of the model, can be derived by using dynamic optimization techniques. When government decision rules change, because the rules are part of the environment facing the optimizing agents, the behavioral equations of the agents will also change. The new behavioral equations can be rederived by solving the dynamic optimization problem by using the new government decision rules as part of the environment. Hence, in addition to macroeconomic policy analysis from the viewpoint of the government under a fixed regime, dynamic optimization techniques can be used to study the behavior of nongovernment economic agents, which are assumed to optimize, over time, under uncertainty and to evaluate the economic effects of government policy changes viewed as changes in regimes. Dynamic programming and optimal control methods remain an indispensable tool of dynamic economics in either case. The former case assumes that the government optimizes and the latter assumes that the nongovernment economic agents optimize.

A theme of this book is that as long as optimization remains a key concept in economics, the method of Lagrange multipliers is a convenient tool to use. Dynamic optimization can be viewed as a constrained maximization problem. The tool to solve this problem is the method of Lagrange multipliers; the same tool used in static economics as treated by Hicks (1939) and Samuelson (1948). In this book, I examine many dynamic economic problems by using the method of Lagrange multipliers. The reader will be able to learn the Lagrange method and to use it in future research. Each of the following sections introduces or summarizes the contents of one chapter of this book.

1.2 Methods of Dynamic Optimization

Consider a Robinson Crusoe economy in which the economic agent is assumed to live only two periods. Capital stock x_1 at the beginning of period 1 is given. The economic agent derives utility in period t equal to a differentiable function $r(x_t, u_t)$, where x_t is the quantity of capital stock and u_t is the quantity of consumption at period t. As a special case, x_t may not enter the utility function, but in general a capital good, such as an apple tree, may provide shade and a nice view besides yielding apples for consumption. Capital stock evolves through time according to

$$x_{t+1} = f(x_t, u_t) = \mathrm{A}x_t^{\alpha} - u_t$$

if one assumes that output (including depreciated capital stock from the previous period) is produced by a Cobb-Douglas production function $\mathrm{A}x_t^{\alpha}$, given one unit of labor, and can be used either for consumption u_t or as future capital stock x_{t+1}. The economic agent's problem is to maximize

$$r(x_1, u_1) + \beta r(x_2, u_2)$$

in which β is a subjective discount factor by choosing the optimum amounts of consumption u_1 and u_2 for the two periods. As u_t is subject to the control of the agent, it is called a *control variable*; x_t is called a *state variable*, the dynamic evolution of which has to be specified in an optimization problem.

Assume that both r and f are differentiable and concave (i.e., having negative second partial derivatives) and that the solution is an interior solution. By the method of *dynamic programming*, one begins by solving the problem for the last period and completes the following five steps.

Step 1. Maximize $\beta r(x_2, u_2)$ with respect to u_2 by differentiation and obtain an *optimum decision function* for u_2 as a function $g_2(x_2)$ of x_2.

$$\frac{\partial r(x_2, u_2)}{\partial u_2} = 0 \quad \Rightarrow \quad u_2 = g_2(x_2)$$

Step 2. Evaluate the *value function* $\mathrm{V}_2(x_2)$ for the second period by substituting the optimum decision function $g_2(x_2)$ for u_2 in the objective function in the problem of Step 1.

$$V_2(x_2) = r(x_2, g_2(x_2))$$

Step 3. Form an objective function for the 2-period problem of periods 1 and 2 by substituting $V_2(x_2)$ for $r(x_2, u_2)$.

$$r(x_1, u_1) + \beta r(x_2, u_2) = r(x_1, u_1) + \beta V_2(x_2)$$

Step 4. Using the dynamic model, substitute $f(x_1, u_1)$ for x_2 in the objective function in Step 3.

$$r(x_1, u_1) + \beta V_2(f(x_1, u_1))$$

Step 5. Maximize the objective function of Step 4 with respect to u_1 by differentiation and obtain an optimum decision function for u_1 as a function of x_1.

$$\frac{\partial r(x_1, u_1)}{\partial u_1} + \beta \frac{\partial f(x_1, u_1)}{\partial u_1} \cdot \frac{\partial V_2}{\partial x_2} = 0 \quad \Rightarrow \quad u_1 = g_1(x_1)$$

When $g_1(x_1)$ is substituted for u_1 in the above expression, the result is the value function $V_1(x_1)$ for period 1. The result can be summarized by the following *Bellman equation*

$$V_1(x_1) = \max_{u_1} \{ r(x_1, u_1) + \beta V_2(x_2) \}$$

For problems involving more than 2 periods, the subscript 1 above is replaced by t and the subscript 2 by $t + 1$. If the value function $V_t(x_t)$ converges to some function $V(x_t)$ after solving the problem backward in time for many periods starting from the last period T, the subscripts t and $t + 1$ of V_t and V_{t+1} can be dropped. When the value function reaches a *steady state* the *Bellman equation* becomes

$$V(x_t) = \max_{u_t} \{ r(x_t, u_t) + \beta V(x_{t+1}) \}$$

Solving this functional equation for $V(\cdot)$ yields the solution for the dynamic optimization problem.

By the method of Lagrange multipliers, one forms a Lagrangean expression by using the constraint of the dynamic model and by treating x_1 as given.

$$\mathcal{L} = r(x_1, u_1) + \beta r(x_2, u_2) - \beta \lambda_2 (x_2 - f(x_1, u_1))$$

Setting to zero the derivatives with respect to u_2, u_1, and x_2 one obtains three first-order conditions

$$\frac{\partial \mathcal{L}}{\partial u_2} = \beta \frac{\partial r(x_2, u_2)}{\partial u_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial u_1} = \frac{\partial r(x_1, u_1)}{\partial u_1} + \beta \frac{\partial f(x_1, u_1)}{\partial u_1} \lambda_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \beta \frac{\partial r(x_2, u_2)}{\partial x_2} - \beta \lambda_2 = 0$$

which can be solved for u_1, u_2, and λ_2.

To compare the two methods, note that the first condition of the Lagrange method is identical with the condition from maximizing $\beta r(x_2, u_2)$ in Step 1 using dynamic programming. The second condition is also identical with the condition from maximizing $r(x_1, u_1) + \beta V_2(x_2)$ with respect to u_1 in Step 5 of dynamic programming, provided that $\lambda_2 = \partial V_2(x_2)/\partial x_2$. The difference is that the Lagrange method uses the third first-order condition to obtain λ_2 and avoids Steps 2, 3, and 4 of dynamic programming. If we consider

$$r(x_1, u_1) + \beta V_2(x_2)$$

as a function of two variables u_1 and x_2, subject to the constraint $x_2 = f(x_1, u_1)$, the Lagrange method introduces the constraint by using the multiplier λ_2, whereas dynamic programming substitutes $f(x_1, u_1)$ for x_2 in the value function $V_2(x_2)$. The Bellman equation fails to exploit an important first-order condition: $\partial \mathcal{L}/\partial x_2 = 0$. It is important to note that by either method, there is no need to know the value function $V_2(x_2)$. By dynamic programming, Step 5 requires only $\partial V_2/\partial x_2$ and not V_2 itself. Thus, trying to obtain the value function either analytically or numerically in Step 2 and to set up the objective function in Steps 3 and 4, as is done in practice when dynamic programming is used, amounts to finding a function and introducing steps that are unnecessary for obtaining the optimum. It is sometimes claimed that *backward induction* (solving a dynamic optimization problem backward in time) is a virtue of dynamic programming. The preceding example illustrates that backward induction is also applied in the Lagrange method, because the optimum control function for u_2 must be obtained first. Backward induction has nothing to do with keeping the value function. This is true for decision problems of one person or several persons, the latter case being demonstrated in section 6.4, equations (6.33) and (6.34).

Knowledge of the value function may have other uses than for deriving an optimum decision function. Some of its uses are discussed in this book. However, for solving the dynamic optimization problem knowledge of the Lagrange multiplier, or the Lagrange function, is necessary. One must find the Lagrange function to exploit all the first-order conditions for optimality by either method of dynamic programming or Lagrange multipliers. Hence, one's effort should be concentrated on obtaining the Lagrange function rather than the value function. There are circumstances in which the value function is easy to obtain by solving the Bellman equation. Under these circumstances one should, by all means, obtain the value function. To find the necessary Lagrange function, one simply differentiates the value function. In almost all economic applications that I have encountered, including those discussed in this book, the value function is more difficult, or at least not easier, to find than the Lagrange function. Hence, it is more efficient to seek the Lagrange function directly by the Lagrange method. I have applied the Lagrange method to solve a number of problems that were originally solved by dynamic programming. Notable examples showing the efficiency of the Lagrange method are contained in sections 3.1, 3.2, 4.1, 4.4, 6.2, and 7.5. Possible computational advantages of the Lagrange method are discussed in sections 2.4, 7.4, and chapter 9.

The Lagrange method is easier, because the value function demands and contains more information than is required. Given the value function, one can always find its derivatives, but more work is often required to obtain the value function itself from its derivatives, and such work is unnecessary. In the univariate case when the state variable x_t is a scalar, often, the derivative of the value function is simpler than the value function itself. For example, if the value function is quadratic, the Lagrange function is linear. Even if the derivative is not simpler, one still needs the Lagrange function to solve the optimization problem using first-order conditions. In the multivariate case where x_t is a vector, often the derivatives of the value function with respect to only a subset of but not all components of x_t are required. This is the case when the dynamic evolution of some components of the vector of state variables x_t does not depend on the vector of control variables u_t. The Lagrange multipliers are introduced only for those state variables that are constrained by the control variables. Notable examples are given in sections 3.1 and 3.2. In these cases, because the value function contains much more knowledge than required, it is likely to be more difficult to obtain than a subset of its partial derivatives. After the optimization problem is solved by the Lagrange method, the value function can be obtained by integrating the Lagrange functions or by substituting the optimum control function into the dynamic model to evaluate the objective function. These advantages of the Lagrange method remain valid for an optimization problem involving a stochastic model for the evolution of x_t, in both discrete and continuous time. In the stochastic case, the objective function is a mathematical expectation and, using the Lagrange method, one simply differentiates the Lagrange expression on the right side of the expectation operator. Optimization in discrete time and in continuous time is discussed in chapters 2 and 7, respectively.

In chapter 2, I explain both methods of dynamic optimization in discrete time more fully, provide a numerical method for solving the first-order conditions derived from the Lagrange method, discuss sufficient conditions for optimum, and relate the Lagrange method to well-known results on optimization in economics.

1.3 Economic Growth

A key problem of allocation of resources over time is the choice between current consumption and future consumption. By reducing current consumption, more resource is made available to augment the capital stock that can be used to produce consumption goods in the future. This statement is illustrated by the relationship discussed in section 1.2, namely

$$k_{t+1} = k_t^{\alpha} z_t - c_t \tag{1.1}$$