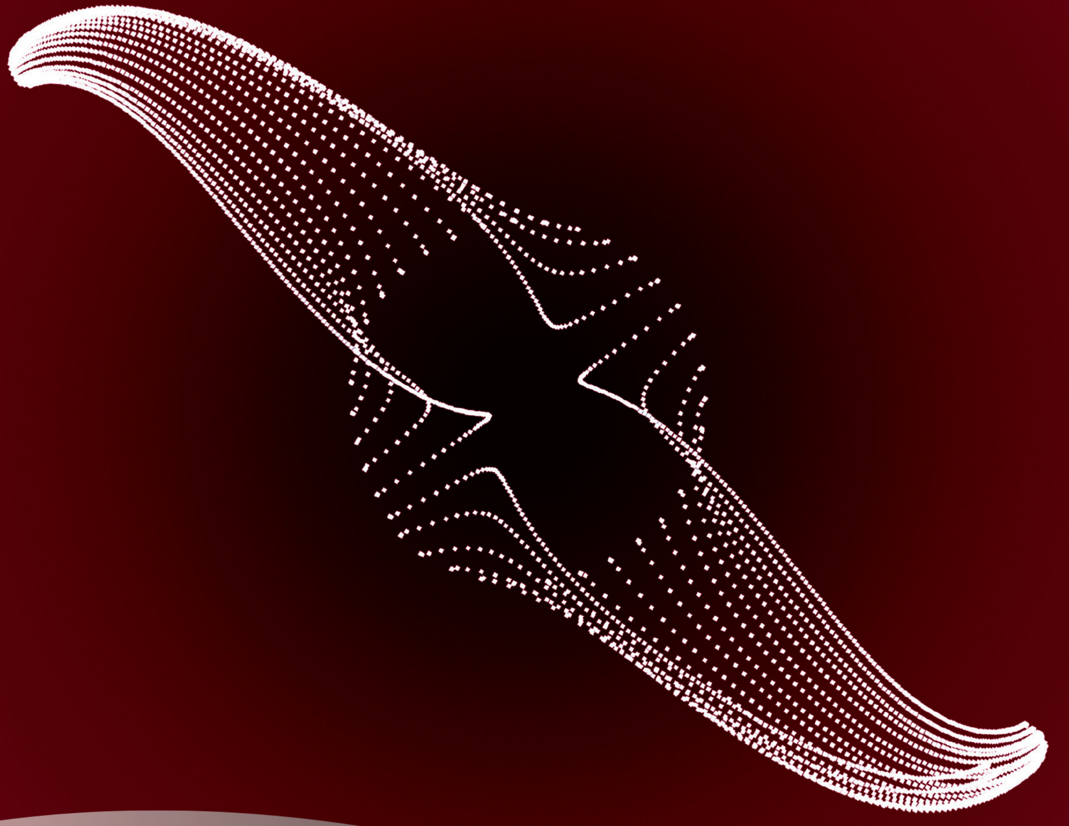

Classical and Quantum Parametric Phenomena

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and Oded Zilberberg

OXFORD GRADUATE TEXTS

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To our wives and daughters

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Preface

This book provides an overview of the phenomena arising when parametric pumping is applied to oscillators. These phenomena include parametric amplification, noise squeezing, spontaneous symmetry breaking, activated switching, cat states, and synthetic Ising spin lattices. To understand these effects, we introduce topics such as nonlinear and stochastic dynamics, mode coupling, and quantum mechanics. Throughout the book, we keep these introductions as succinct as possible and focus our attention on understanding parametric oscillators. As a result, we familiarize ourselves with many aspects of parametric systems and understand the common theoretical origin of nanomechanical sensors, optical amplifiers, and superconducting qubits.

Parametric phenomena have enabled important scientific breakthroughs over recent decades and are still the focus of intense research efforts. Our intention is to provide a resource for experimental and theoretical physicists entering the field or wishing to gain a deeper understanding of the underlying connections. As such, we combine formal and intuitive explanations, accompanied by exercises based on numerical python codes. This combination allows the reader to experience parametric phenomena from various directions and to apply their understanding directly to their own research projects. For lecturers, the book supplies all the material necessary for an advanced class on the topic.

Contents

List of Important Symbols

(in approximate order of appearance)

H	Hamiltonian
E_{pot}	potential energy
E_{kin}	kinetic energy
E_{tot}	$= E_{\mathrm{pot}} + E_{\mathrm{kin}}$; total energy
x	displacement
p	momentum
t	time
m	mass
ω_0	angular resonance frequency
ν_0	$= \omega_0/(2\pi)$; temporal resonance frequency
k	$= m\omega_0^2$; spring constant
T_0	$= 2\pi/\omega_0 = 1/\nu_0$; unforced oscillator period
Q	quality factor
Γ	$= \omega_0/Q$; damping rate
τ_0	$= 2/\Gamma$; amplitude decay time constant
ω_Γ	$= \left(\omega_0^2 - \Gamma^2/4\right)^{1/2} \approx \omega_0$; dissipation-shifted angular resonance frequency
μ	characteristic exponent
F_0	amplitude of external force
F	in Chapters 1 to 7: all force terms acting on the bare resonator
	in Chapters 9 and 10: $= \frac{F_0}{2}\sqrt{\hbar/2m\omega_0}$; rotating-frame quantum force term
ω	angular frequency of external force
θ	phase offset of external force
χ	susceptibility function of driven resonator
X	oscillation amplitude
$\mathbf{x}$	vector of a system's degrees of freedom
G	in Chapters 1 to 7: matrix containing the coefficients of the differential equation
	in Chapter 10: parametric drive in the rotating-frame quantum Hamiltonian
W	in Chapters 1 to 7: Wronskian matrix
	in Chapters 8 to 10: Wigner quasiprobability density
Φ	state transfer matrix
T_p	period of parametric pump
ω_p	$= 2\pi/T_p$; angular frequency of parametric pump
λ	parametric modulation depth
λ_{th}	$= 2/Q$; parametric pumping threshold at $\omega_p = 2\omega_0$
β_3	coefficient of cubic (Duffing) nonlinearity
β_2	coefficient of quadratic nonlinearity
β	$= \beta_3 - \frac{10}{9}\frac{\beta_2^2}{\omega^2}$; coefficient of effective Duffing nonlinearity
u	in-phase oscillation quadrature
v	out-of-phase oscillation quadrature

ψ	phase offset of parametric pump
η	coefficient of nonlinear damping
k_B	$\approx 1.38 \times 10^{-23}\,\mathrm{J\,T^{-1}}$; Boltzmann constant
T	temperature
E_{eq}	equilibrium energy
ξ	force noise term
ς_D	standard deviation of force noise
σ_x	standard deviation of x (for any variable x)
S_F	power spectral density of force noise
Ξ_u	in-phase quadrature of force noise
Ξ_v	out-of-phase quadrature of force noise
ρ	probability density
	in Chapters 8 to 10: density operator
J	coefficient of coupling between resonators
Δk	detuning spring force
U	in Chapter 6: normal-mode transformation matrix
	in Chapter 10: Kerr nonlinearity
ω_Δ	$= \frac{J}{\omega_0 m}$; angular exchange rate and spectral splitting
t_Δ	$= \frac{2\pi}{\omega_\Delta}$; energy exchange time
g	parametric coupling modulation depth
ω_R	angular Rabi frequency
$\hbar$	$\approx 1.05 \times 10^{-34}\,\mathrm{J\,s^{-1}}$; reduced Planck constant
σ_τ	state lifetime
σ_E	energy uncertainty
Ψ	wave function
n	in Chapters 8 to 10: Fock state number
a	$= \hat{a}$; annihilation operator
$a^\dagger$	$= \hat{a}^\dagger$; creation operator
x_{dl}	$= \frac{1}{2}(a^\dagger + a)$; dimensionless x operator
p_{dl}	$= \frac{i}{2}(a^\dagger - a)$; dimensionless x operator
α	amplitude of coherent state
P_j	probability of measuring the system in the state j
κ	$= \Gamma$; system-environment coupling rate
n_{th}	mean thermal excitation
U_{rot}	rotating-frame transformation matrix
Δ	$\omega - \omega_0$; angular frequency detuning
$\tilde{a}$	annihilation operator in the rotating frame
$\tilde{a}^\dagger$	creation operator in the rotating frame
α_R	real part of coherent state amplitude
α_I	imaginary part of coherent state amplitude
Δ_U	$= \Delta + U$; detuning shifted by the Kerr nonlinearity

Introduction

"It's still magic even if you know how it's done."
(Terry Pratchett, *A Hat Full of Sky*)

About This Work

This book emerged from a master-level course on "Parametric Phenomena" that the authors held together at ETH Zurich between 2018 and 2021, and individually at their respective universities since then. The course was organized as a reverse-classroom event: students would prepare by reading material at home, and then use the time in class to solve exercises and discuss with the teaching team. With this approach, we hoped to present the topic in much the same way as we experience it during our own research, and to encourage the students to formulate (and solve) their own questions. In line with this philosophy, the graded deliverable that every student handed in for passing the course was a poster that approximated one particular system as a parametric oscillator, including physical units and estimated numerical values. We saw many creative results, ranging from an airplane wobbling in the wind and a ship rolling in the sea to a nanoparticle trapped in an optical potential, a Josephson superconducting resonator, an optical ring resonator, a yo-yo, and even the predator–prey dynamics between a pack of wolves and a flock of sheep. This book is meant to provide all that is necessary to hold such a course, including the reading material, exercises, codes to solve the exercises, and a tutorial of how to map realistic physical systems onto the desired equations.

In this book, we perform a diagonal cut through many different topics. We follow a path from the deterministic mechanics of a harmonic oscillator all the way to the non-deterministic physics of coupled nonlinear quantum oscillators. Along this trajectory, we encounter many ideas and concepts that can fill entire books of their own accord. Our discussions of these concepts are guided by the wish to build an understanding without dealing with all possible details. This book is clearly *not* an exhaustive resource on topics such as nonlinear mechanics, stochastic physics, or the quantum oscillator. These topics have been treated in much more detail in other articles and books which we cite where appropriate. Rather, we want to focus on the combination of all these fundamental theories to gain a balanced and comprehensive view of the parametric oscillator.

In Chapter 1, we start with the deterministic behavior of the classical harmonic oscillator subject to damping and driving, and later to parametric pumping. Building on this foundation, we add nonlinearities in Chapter 2, and combine them with a parametric pump in Chapter 3. In Chapter 4, we introduce fluctuating forces for the example of the harmonic oscillator, which we generalize to the nonlinear parametric oscillator in Chapter 5. Coupling between oscillators is discussed in Chapter 6 and applied to stochastic, nonlinear parametric oscillators in Chapter 7. The quantum harmonic oscillator follows in Chapter 8, which leads to the driven and damped quantum harmonic oscillator in Chapter 9, and the quantum parametric oscillators in Chapter 10. Finally, in Chapter 11 we explain with several examples how mechanical, electrical, and optical systems can all serve as parametric oscillators.

0.1 Historical Review

In this Introduction, we review historical examples of parametric phenomena and understand why this topic is still the focus of so many research fields today. Before we can embark on our tour through the centuries, we must clarify what we mean by the term *parametric*. In our usage of the word, it refers to a periodic modulation of a resonator's potential — physically, the modulation could originate from a change in the tension of a mechanical string, a child alternatively standing and squatting on a swing, the effect of waves hitting a ship to change its buoyancy center, a variation in the effective capacitance of an electrical resonator, or an increase of the polarization of an optical medium in response to electromagnetic waves. All of these seemingly disparate examples obey very similar equations, and many of them can be used for similar technological applications (although so far no applications have been developed for children on swings . . .). The phenomena that arise as a consequence of parametric modulation are as varied as the physical systems in which they appear. At first glance, the menagerie of parametric phenomena may appear endless, but we will see that they all follow a few intuitive rules and can be classified accordingly.

The earliest examples of parametric oscillation are found, not surprisingly, in the mechanical domain. To our knowledge, the first experimental description of parametric resonance is ascribed to the works of Michael Faraday in 1831 [1] and Franz Melde in 1860 [2]. However, applications of the effect are much older: the big censer "O Botafumeiro" used for certain rituals in the Cathedral of Santiago de Compostela in Spain is set into pendulum motion by periodically modulating (i.e. parametrically pumping) the length of its rope [3]. As the censer weights about 60 kg and moves 20 m up and down during its largest oscillations, a team of operators is needed for this pumping, and their actions have to be coordinated in time to achieve the desired effect. Reports of parametric pumping of O Botafumeiro reach back to the 13th century. A mathematical treatment of parametric oscillation was not attempted until 1883, when Lord Rayleigh published his paper "On maintained vibrations" [4]. He analyzed the different types of driving that a system can experience and showed that parametric modulations can explain Faraday's experimental observations [1].

Technological applications of parametric pumping in electronics began to appear in the 20th century with the development of the Mag Amp [5] and the Klystron [6] amplifiers, both of which were based on time-dependent modulation of a control parameter. The Mag Amp found application in early radio telephones around 1915, and the Klystron allows high-power microwave generation and is still in use today for niche applications such as spacecraft communication and synchrotrons. In the second half of the century, inventions like the *Parametric Amplifier* by Arthur Ashkin and colleagues in 1959 [7] and the *Broadband cavity parametric amplifier with tuning* by Closson in 1962 [8] opened up new perspectives for electrical signal amplification. It was understood that a modulation of the reactance (i.e., the capacitance or inductance) of a resonant electrical circuit can lead to strong signal amplification without adding Nyquist noise which is unavoidable in resistor-based operational amplifiers [9, 10].

With the advent of superconducting circuits and the possibility of a strong nonlinear inductance imposed by Josephson junctions, the parametric amplifier was brought to its logical culmination, offering signal amplification with no more noise than what is absolutely required by the laws of quantum mechanics [11–13]. However, it was only after the turn of the millennium that these *Josephson parametric amplifiers* moved fully into the focus of the experimental quantum physics community [14–20], enabling experiments that previously were unfeasible [21]. Around the same time, parametric amplification [22–27] and coupling [28–32] were also explored in the growing nanomechanics community. A particularly important application arose in *cavity optomechanics*, where the parametric coupling between a mechanical and an optical degree of freedom can be used for precise control of the resonator and for cooling it down to its quantum ground state [33]. Parametric squeezing can be used to reduce fluctuations [22, 34–36] and has been employed as a means to generate nonclassical optical [37, 38] or mechanical [39–41] states. Importantly, parametric squeezing has been proposed as a way to boost the sensitivity of optical interferometers for gravitational wave detection [42–44].

Most of the above applications are achieved for relatively weak parametric modulation. By contrast, when the pumping exceeds a certain threshold, entirely new phenomena appear. Under strong parametric pumping at a frequency close to twice its resonance frequency, a resonator experiences a negative effective damping, such that it will ring up to large amplitudes and be stabilized only by nonlinear potential terms [45]. Such parametric instability appears in many contexts; for instance, it is held responsible for the dreaded *parametric rolling* of ships that has caused catastrophic accidents [46]. It is also considered as a possible mechanism for particle creation in models of the early universe [47, 48].

Beyond the instability threshold, the parametrically driven resonator can select one of two oscillation phases that are separated by a phase of π. This causes a spontaneous breaking of the time-translation symmetry of the system — oscillations with either phase are equivalent solutions in response to the drive, but only one of them can be realized at the same time (in a classical system) [45, 49]. Around 1960, Eiichi Goto [50] and John von Neumann [51] independently realized that these phase states offer a way to encode digital information. The *parametron* was indeed used as a memory unit for electrical computers in Japan until the invention of the transistor provided a more efficient solution.

0.2 Present and Future

Over the last few years, the development of novel resonators in the electrical, mechanical and optical domain has led to a revival of interest in the parametric oscillator[1] and the idea of *parametron phase logic*, in both the classical and quantum domains [52–65]. Of particular interest is the idea of coupling many parametrons into a configurable

[1] Other terms for the parametric oscillator are *Kerr parametric oscillator* or *two-photon driven Kerr resonator*.

Hopfield-type network [66, 67]. Here, the phase states of a single parametron represent the two polarization states of a spin, and the entire network can be used to simulate the behavior of the corresponding many-body Ising model [68]. Many optimization problems, such as the MAX-CUT problem [69, 70] or the number partitioning problem [71], are isomorphic to finding the ground state of an Ising network, and at the same time are nearly intractable with classical (sequential) computers [72]. Recent years have therefore seen a surge of ideas related to parametron logic control [55, 57, 73–77] and parametric network operation [62, 69–71, 78–86].

Whether the complexity of a multimode nonlinear oscillator network can be tamed to enable parallel computing and quantum simulations is an open question and will be the subject of intensive research over the coming years. What is safe to predict is that every new physical implementation of the harmonic oscillator sooner or later rediscovers parametric phenomena and applies it to a new purpose. A concept that is so versatile and useful will remain important in science and technology, no matter what the future brings.

1 The Harmonic Resonator

Harmonic oscillators are ubiquitous in nature and have been treated in many textbooks in depth [87, 88]. We briefly repeat in this first chapter those features that are important for the rest of the book. To facilitate an intuitive approach, we adopt the language of a mechanical oscillator, but the discussion may easily be translated to any oscillating system, cf. Chapter 11. Examples will be calculated without units, to preserve the spirit of a general treatment.

1.1 Newton's Equation of Motion

Consider a mass on a spring, see Fig. 1.1. The system has kinetic and potential energy, where the latter is stored in the spring proportional to the square of the displacement x, such that

$$H = E_{\text{tot}} = E_{\text{kin}} + E_{\text{pot}} = \frac{p^2}{2m} + \frac{1}{2}kx^2 \,. \tag{1.1}$$

Here, H is the Hamiltonian of the system, p the momentum and canonical conjugate of the displacement x, k the spring constant, and m the mass. The Hamiltonian is a function that describes the total kinetic and potential energy of a closed system. From Hamiltonian mechanics, we can calculate the force that acts on the mass at any given time t as

$$F \equiv \dot{p} \equiv \frac{dp}{dt} = -\frac{\partial H}{\partial x} = -kx \,, \tag{1.2}$$

where dots denote differentiation with respect to time t. The quadratic potential, thus, corresponds to a linear spring force. Combining eqn (1.1) with the second one of Hamilton's equations of motion (EOM),

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \,, \tag{1.3}$$

we obtain a second-order differential equation that is known as Newton's EOM,

$$\ddot{x} + \frac{k}{m}x = 0 \,. \tag{1.4}$$

Equation (1.4) is solved using the ansatz $x(t) = x_{\text{ini}}e^{i\omega_0 t}$, where x_{ini} is determined by the initial boundary conditions, $\omega_0 = (k/m)^{1/2} = 2\pi\nu_0 = 2\pi/T_0$ is the angular resonance frequency, T_0 is the unforced periodicity of the oscillator, and we refer to

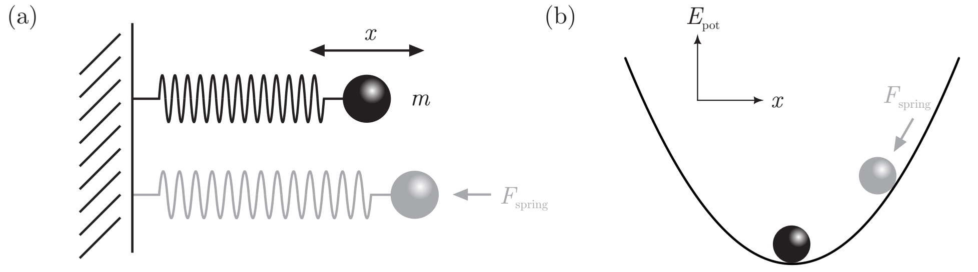

Fig. 1.1 (a) As an example of a harmonic oscillator, we use a mass on a spring. Displacing the mass from its rest position by x results in a restoring spring force $F_{\mathrm{spring}} = -kx$. A displaced mass is shown in gray. (b) The potential energy of a harmonic oscillator is quadratic in displacement, $E_{\mathrm{pot}} = \frac{1}{2}kx^2$, cf. eqn (1.2).

ν_0 as *natural frequency*. Note that eqn (1.4) describes an oscillator that is isolated from its environment, that is, Hamiltonian evolution is energy-conserving and does not feature damping terms.

Finding the microscopic origin of damping terms is an important topic on its own [89]. For now, we assume a phenomenological source of dissipation that enters Newton's EOM and can stabilize the oscillator's motion,

$$\ddot{x} + \Gamma\dot{x} + \frac{k}{m}x = 0\,, \tag{1.5}$$

where Γ is the coefficient corresponding to the dissipative (linear) damping enacted by the environment. Note that from a mathematical point of view, we can account for the added damping term through the transformation [88]

$$x(t) = e^{-\Gamma t/2}y(t) = e^{-t/\tau_0}y(t)\,, \tag{1.6}$$

where we define a decay time $\tau_0 = 2/\Gamma$. The equation of motion for $y(t)$ then takes the form of a closed harmonic oscillator,

$$\ddot{y} + \omega_\Gamma^2 y = 0\,, \tag{1.7}$$

in an exponentially expanding or shrinking coordinate system and with a slightly shifted resonance frequency

$$\omega_\Gamma^2 = \omega_0^2 - \frac{\Gamma^2}{4}\,. \tag{1.8}$$

From the transformation in eqn (1.6), we observe that for $2\omega_0 > \Gamma > 0$ the oscillator coordinate $x(t)$ decays exponentially in time in addition to an harmonic oscillation. However, we can already guess that something different must happen once $2\omega_0 \leq \Gamma$.

A direct treatment of the homogeneous dissipative case in eqn (1.5) is possible starting from the same ansatz that any particular solution has the form

$$x(t) = x_{\mathrm{ini}}e^{\mu t} \tag{1.9}$$

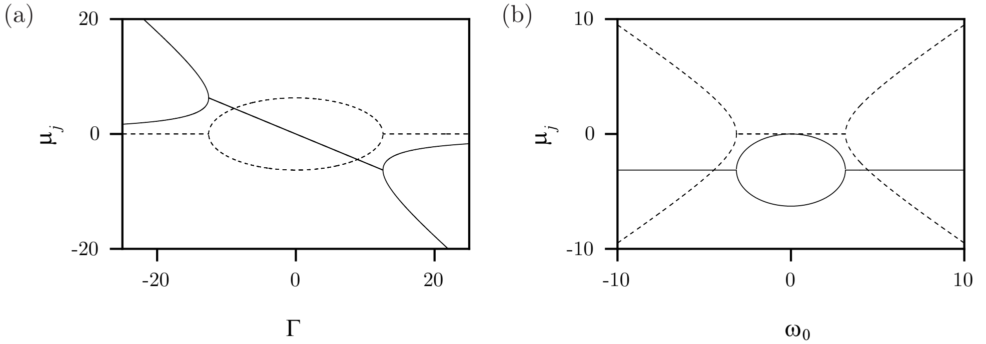

Fig. 1.2 The real (solid) and imaginary (dashed) parts of the characteristic exponents, cf. eqn (1.11), as a function of (a) damping coefficient Γ for a bare angular resonance frequency $\omega_0 = 2\pi$, and of (b) ω_0 for $\Gamma = 2\pi$.

with a complex **characteristic exponent** $\mu \in$. Inserting eqn (1.9) into eqn (1.5) leads to

$$x\left(\mu^2 + \mu\Gamma + \omega_0^2\right) = 0\,, \tag{1.10}$$

which, for $x \neq 0$, results in a quadratic **characteristic equation** with the two roots

$$\mu_{a,b} = -\frac{\Gamma}{2} \pm \sqrt{\Gamma^2/4 - \omega_0^2} = -\frac{\Gamma}{2} \pm i\omega_\Gamma\,. \tag{1.11}$$

This is identical to what we obtained with the coordinate transformation method in eqn (1.6), see eqn (1.8) and the discussion thereafter.

We can identify several distinct regimes of motion: for damped oscillators ($\Gamma > 0$) we distinguish between overdamped ($\omega_\Gamma^2 < 0$), critically damped ($\omega_\Gamma^2 = 0$), and underdamped motion ($\omega_\Gamma^2 > 0$), where oscillation appears only for the latter.[1] For $\Gamma < 0$, the oscillator is unstable and the motion becomes unbounded. This is visualized by plotting the real and imaginary part of the characteristic exponents $\mu_{a,b}$, see Fig. 1.2. Note that, in many cases, the small correction to the bare frequency due to the damping term is neglected, such that $\omega_\Gamma^2 \approx \omega_0^2$.

1.2 Response of the Driven Resonator

In large parts of our treatment, we will use Newton's EOM to analyze the behavior of driven oscillating systems. For our mass on a spring, we can write

$$\ddot{x} + \frac{k}{m}x + \Gamma\dot{x} = \frac{F_0}{m}\cos(\omega t)\,, \tag{1.12}$$

where $F = F_0\cos(\omega t)$ is an external driving force that turns eqn (1.12) into an inhomogeneous differential equation.

[1] The critically damped point is an example of an *exceptional point* where the roots are degenerate and eqn (1.9) is insufficient [90].