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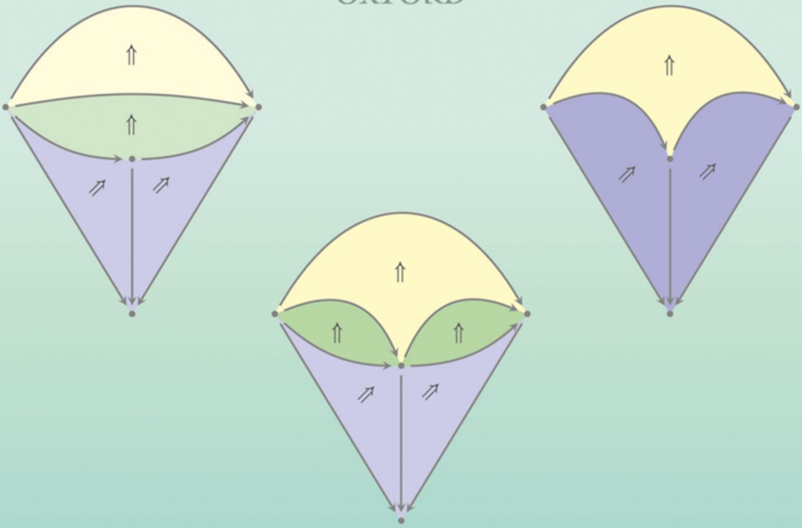

2-Dimensional CATEGORIES

NILES JOHNSON
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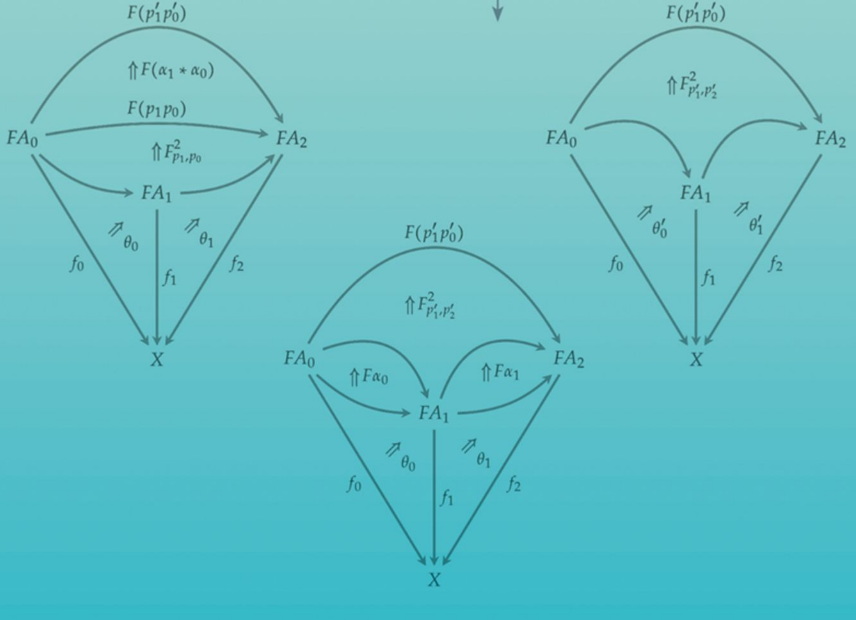

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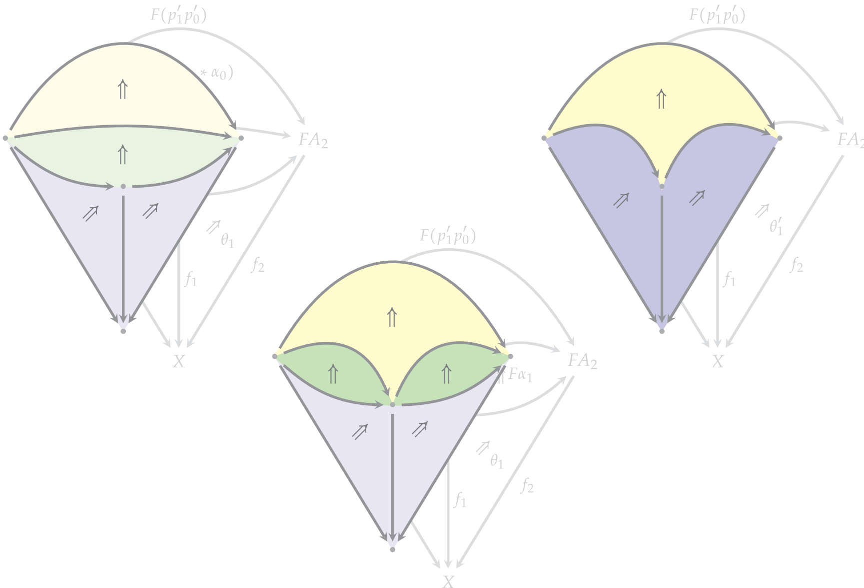

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The first author dedicates this book to his wife, Nemili.
The second author dedicates this book to Eun Soo and Jacqueline.

Preface

2-Dimensional Categories

The theory of 2-dimensional categories, which includes 2-categories and bicategories, is a fundamental part of modern category theory with a wide range of applications not only in mathematics but also in physics [**BN96**, **KV94a**, **KV94b**, **KTZ20**, **Par18**, **SP∞**], computer science [**PL07**], and linguistics [**Lam04**, **Lam11**]. The basic definitions and properties of 2-categories and bicategories were introduced by Bénabou in [**Bén65**] and [**Bén67**], respectively. The one-object case is illustrative: a monoid, which is a set with a unital and associative multiplication, is a one-object category. A monoidal category, which is a category with a product that is associative and unital up to coherent isomorphisms, is a one-object bicategory. The definition of a bicategory is obtained from that of a category by replacing the hom sets with hom categories, the composition and identities with functors, and the associativity and unity axioms with natural isomorphisms called the associator and the unitors. These data satisfy unity and pentagon axioms that are conceptually identical to those in a monoidal category. A 2-category is a bicategory in which the associator and the unitors are identities.

For example, small categories, functors, and natural transformations form a 2-category Cat. As we will see in Sections 2.4 and 2.5, there are similar 2-categories of multicategories and of polycategories. An important bicategory in algebra is Bimod, with rings as objects, bimodules as 1-cells, and bimodule homomorphisms as 2-cells. Another important bicategory is $\mathsf{Span}(\mathsf{C})$ for a category C with all pullbacks. This bicategory has the same objects as C and has spans in C as 1-cells. We will see in Example 6.4.9 that internal categories in C are monads in the bicategory $\mathsf{Span}(\mathsf{C})$.

Purpose and Audience

The literature on bicategories and 2-categories is scattered in a large number of research papers that span over half a century. Moreover, some fundamental results, well known to experts, are mentioned with little or no detail in the research literature. This presents a significant obstruction for beginners in the study of 2-dimensional categories. Varying terminology across the literature compounds the difficulty.

This book is a self-contained introduction to bicategories and 2-categories, assuming only the most elementary aspects of category theory, which is summarized in Chapter 1. The content is written for non-expert readers and provides complete details in both the basic definitions and fundamental results about bicategories and 2-categories. It aims to serve as both an entry point for students and a reference for researchers in related fields.

A review of basic category theory is followed by a systematic discussion of 2-/bicategories, pasting diagrams, morphisms (functors, transformations, and modifications), 2-/bilimits, the Duskin nerve, the 2-nerve, internal adjunctions, monads in bicategories, 2-monads, biequivalences, the Bicategorical Yoneda Lemma, and the Coherence Theorem for bicategories. The next two chapters discuss Grothendieck fibrations and the Grothendieck construction. The last two chapters provide introductions to more advanced topics, including tricategories, monoidal bicategories, the Gray tensor product, and double categories.

Features

Details: As mentioned above, one aspect that makes this subject challenging for beginners is the lack of detailed proofs, or sometimes even precise statements, of some fundamental results that are well known to experts. To make the subject of 2-dimensional categories as widely accessible as possible, this text presents precise statements and completely detailed proofs of the following fundamental but hard-to-find results:

- The Bicategorical Pasting Theorem 3.6.6, which shows that every pasting diagram has a well-defined and unique composite.
- The Whitehead Theorem 7.4.1, which gives a local characterization of a biequivalence, and a 2-categorical version in Theorem 7.5.8.
- The Bicategorical Yoneda Lemma 8.3.16 and the corresponding Coherence Theorem 8.4.1 for bicategories.
- The Grothendieck Fibration Theorem 9.5.6: cloven and split fibrations are, respectively, pseudo and strict $\mathcal{F}$-algebras for a 2-monad $\mathcal{F}$.

- The Grothendieck Construction Theorem 10.6.16: the Grothendieck construction is a 2-equivalence from the 2-category of pseudofunctors $\mathsf{C}^{\mathsf{op}} \longrightarrow \mathsf{Cat}$ to the 2-category of fibrations over C.
- The Grothendieck construction is a lax colimit (Theorem 10.2.3).
- The Gray tensor product is symmetric monoidal with an adjoint hom, providing a symmetric monoidal closed structure on the category of 2-categories and 2-functors (Theorem 12.2.31).

2-Categorical restrictions: The special case of 2-categories is both simpler and of independent importance. There is an extensive literature for 2-categories in their own right, some of which does not have a bicategorical analog. Whenever appropriate, the 2-categorical version of a bicategorical concept is presented. For example, Definition 2.3.2 of a 2-category is immediately unpacked into explicit data and axioms and then restated in terms of a Cat-enriched category. Another example is the Whitehead Theorem in Chapter 7, which is first discussed for bicategories and then restricted to 2-categories.

Motivation and explanation: Definitions of main concepts are preceded by motivational discussion that makes the upcoming definitions easier to understand. Whenever useful, main definitions are immediately followed by a detailed explanation that helps the reader interpret and unpack the various components. In the text, these are clearly marked as *Motivation* and *Explanation*, respectively.

Review: To make this book self-contained and accessible to beginners, definitions and facts in basic category theory are summarized in Chapter 1.

Exercises and notes: Exercises are collected in the final section of each chapter. Most of them involve proof techniques that are already discussed in detail in that chapter or earlier in this book. At the end of each chapter, we provide additional notes regarding references, terminology, related concepts, or other information that may be inessential but helpful to the reader.

Organization: Extensive and precise cross-references are given when earlier definitions and results are used. Near the end of this book, in addition to a detailed index, we also include a list of main facts and a list of notations, each organized by chapters.

Related Literature

The literature on bicategories and 2-categories is extensive, and a comprehensive review is beyond our scope. Here we mention only a selection of key references for background or further reading. The Notes section at the end of each chapter provides additional references for the content of that chapter.

1-**Categories:** [**Awo10**, **Gra18**, **Lei14**, **Rie16**, **Rom17**, **Sim11**]. These are introductory books on basic category theory at the advanced undergraduate and beginning graduate level. The standard reference for enriched category theory is [**Kel05**].

2-**Categories:** A standard reference is [**KS74**].

Bicategories: Besides the founding paper [**Bén67**], the papers [**Lac10a**, **Lei∞**, **Str80**, **Str87**, **Str96**] are often used as references.

Tricategories: The basic definitions and coherence of tricategories are discussed in [**GPS95**, **Gur13**].

$(\infty,1)$-**Categories:** Different models of $(\infty,1)$-categories are discussed in the books [**Ber18**, **Cis19**, **Lei04**, **Lur09**, **Pao19**, **Rie14**, **Sim12**].

Chapter Summaries

A brief description of each chapter follows.

Chapter 1: To make this book self-contained and accessible to beginners, in the first chapter we review basic concepts of category theory. Starting from the definitions of a category, a functor, and a natural transformation, we review limits, adjunctions, equivalences, the Yoneda Lemma, and monads. Then we review monoidal categories, which serve as both examples and motivation for bicategories, and Mac Lane's Coherence Theorem. Next we review enriched categories, which provide one characterization of 2-categories.

Chapter 2: The definitions of a bicategory and of a 2-category, along with basic examples, are given in this chapter. Section 2.2 contains several useful unity properties in bicategories, generalizing those in monoidal categories. These unity properties underlie many fundamental results in bicategory theory and are often used implicitly in the literature.

They will be used many times in later chapters. Examples include the uniqueness of lax and pseudo bilimits in Theorem 5.1.19, an explicit description of the Duskin nerve in Section 5.4, mates in Lemma 6.1.13, the Whitehead Theorem 7.4.1, the Bicategorical Yoneda Lemma 8.3.16, and the tricategory of bicategories in Chapter 11, to name a few.

Chapter 3: This chapter provides pasting theorems for 2-categories and bicategories. We discuss a 2-categorical pasting theorem first, although our bicategorical pasting theorem does not depend on the 2-categorical version. Each pasting theorem says that a pasting diagram, in a 2-category or a bicategory, has a unique composite. We refer the reader to Note 3.8.9 for a discussion of why it is important to *not* base a bicategorical pasting theorem on a 2-categorical version, the Whitehead Theorem (i.e., local characterization of a biequivalence) or the Bicategorical Coherence Theorem. String diagrams, which provide another way to visualize and manipulate pasting diagrams, are discussed in Section 3.7.

Chapter 4: This chapter presents bicategorical analogues of functors and natural transformations. We introduce lax functors between bicategories, lax transformations between lax functors, and modifications between lax transformations. We discuss important variations, including pseudofunctors, strong transformations, and icons. The representable pseudofunctors, representable transformations, and representable modifications in Section 4.5 will be important in Chapter 8 when we discuss the Bicategorical Yoneda Lemma 8.3.16.

Chapter 5: This chapter is about bicategorical analogs of limits and nerves. Using lax functors and pseudofunctors, we define lax cones and pseudocones with respect to a lax functor. These concepts are used to define lax and pseudo versions of bilimits and limits. We show in Theorem 5.1.19 that, like the 1-categorical fact that limits are unique up to an isomorphism, lax and pseudo (bi)limits are unique up to an equivalence and an invertible modification. We also discuss the dual concepts of lax and pseudo (bi)colimits, and 2-(co)limits. Next we describe the Duskin nerve and the 2-nerve, which associate to each small bicategory a simplicial set and a simplicial category, respectively. These are two different generalizations of the 1-categorical Grothendieck nerve, and for each we give an explicit description of their simplices.

Chapter 6: In this chapter we discuss bicategorical analogs of adjunctions, adjoint equivalences, and monads. After defining an internal adjunction in a bicategory and discussing some basic properties and examples, we discuss the theory of mates, which is a useful consequence of adjunctions. The basic concept of sameness between bicategories is that of a biequivalence, which is defined using adjoint equivalences in bicategories. Biequivalences between bicategories will play major roles in Chapters 7, 8, and 10. The second half of this chapter is about monads in a bicategory, 2-monads on a 2-category, and various concepts of algebras of a 2-monad. In Chapter 9 we will use pseudo and strict algebras of a 2-monad $\mathcal{F}$ to characterize cloven and split fibrations.

Chapter 7: In this chapter we provide a careful proof of a central result in basic bicategory theory, namely, the local characterization of a biequivalence between bicategories, which we call the Whitehead Theorem. This terminology comes from homotopy theory, with the Whitehead Theorem stating that a continuous map between CW complexes is a homotopy equivalence if and only if it induces an isomorphism on all homotopy groups. In 1-category theory, a functor is an equivalence if and only if it is essentially surjective on objects and fully faithful on morphisms. Analogously, the Bicategorical Whitehead Theorem 7.4.1 says that a pseudofunctor between bicategories is a biequivalence if and only if it is essentially surjective on objects (i.e., surjective up to adjoint equivalences), essentially full on 1-cells (i.e., surjective up to isomorphisms), and fully faithful on 2-cells (i.e., a bijection). Although the statement of this result is similar to the 1-categorical version, the actual details in the proof are much more involved. We give an outline in the introduction of the chapter. The Bicategorical Whitehead Theorem 7.4.1 will be used in Chapter 8 to prove the Coherence Theorem 8.4.1 for bicategories. Furthermore, the 2-Categorical Whitehead Theorem 7.5.8 will be used in Chapter 10 to establish a 2-equivalence between a 2-category of Grothendieck fibrations and a 2-category of pseudofunctors.

Chapter 8: The Yoneda Lemma is a central result in 1-category theory, and it entails several related statements about represented functors and natural transformations. In this chapter we discuss their bicategorical analogues. In Section 8.1 we discuss several versions of the 1-categorical Yoneda Lemma, both as a refresher and as motivation for the bicategorical versions. In Section 8.2 we construct a bicategorical version of the Yoneda embedding for a bicategory, which we call the Yoneda pseudofunctor. In Section 8.3 we first establish the Bicategorical Yoneda Embedding in Lemma 8.3.12, which states that the Yoneda pseudofunctor is a local equivalence. Then we prove the Bicategorical Yoneda Lemma 8.3.16, which describes a pseudofunctor $F : \mathsf{B}^{\mathsf{op}} \longrightarrow \mathsf{Cat}$ in terms of strong transformations from the Yoneda pseudofunctor to F. A consequence of the Bicategorical Whitehead Theorem 7.4.1 and the Bicategorical Yoneda Embedding is the Bicategorical Coherence Theorem 8.4.1, which states that every bicategory is biequivalent to a 2-category.

Chapter 9: This chapter is about Grothendieck fibrations. A functor is called a fibration if, in our terminology, every pre-lift has a Cartesian lift. A fibration with a chosen Cartesian lift for each pre-lift is called a cloven fibration, which is, furthermore, a split fibration if it satisfies a unity property and a multiplicativity property. After discussing some basic properties and examples of fibrations, we observe that there is a 2-category $\mathsf{Fib}(\mathsf{C})$ with fibrations over a given small category C as objects. In Theorem 9.1.20 we observe that fibrations are closed under pullbacks and that equivalences of 1-categories are closed under pullbacks along fibrations. The rest of this chapter contains the construction of a 2-monad $\mathcal{F}$ on the overcategory Cat/C and a detailed proof of the Grothendieck Fibration Theorem 9.5.6. The latter provides an explicit bijection between cloven fibrations and pseudo $\mathcal{F}$-algebras, and also between split fibrations and strict $\mathcal{F}$-algebras.

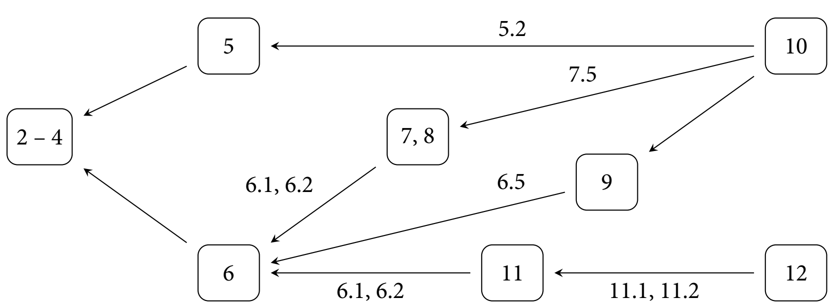
Chapter 10: This chapter presents the fundamental concept of the Grothendieck construction $\int F$ of a lax functor $F : \mathsf{C}^{\mathsf{op}} \longrightarrow \mathsf{Cat}$. For a pseudofunctor F, the category $\int F$ is equipped with a fibration $U_F : \int F \longrightarrow \mathsf{C}$ over C, which is split precisely when F is a strict functor. Using the concepts from Chapter 5, next we show that the Grothendieck construction is a lax colimit of F. Most of the rest of this chapter contains a detailed proof of the Grothendieck Construction Theorem 10.6.16: the Grothendieck construction is part of a 2-equivalence from the 2-category of pseudofunctors $\mathsf{C}^{\mathsf{op}} \longrightarrow \mathsf{Cat}$, strong transformations, and modifications, to the 2-category of fibrations over C, Cartesian functors, and vertical natural transformations. Section 10.7 briefly discusses a generalization of the Grothendieck construction that applies to an indexed bicategory.

Chapter 11: This chapter is about a 3-dimensional generalization of a bicategory called a tricategory. After a preliminary discussion of whiskerings of a lax transformation with a lax functor, we define a tricategory. The Bicategorical Pasting Theorem 3.6.6 plays a crucial role in interpreting the axioms of a tricategory, which are all stated in terms of pasting diagrams. The rest of this chapter contains the detailed definitions and a proof of the existence of a tricategory $\mathfrak{B}$ with small bicategories as objects, pseudofunctors as 1-cells, strong transformations as 2-cells, and modifications as 3-cells.

Chapter 12: Other 2-dimensional categorical structures are discussed in this chapter. Motivated by the fact that monoidal categories are one-object bicategories, a monoidal bicategory is defined as a one-object tricategory. Then we discuss the braided, sylleptic, and symmetric versions of monoidal bicategories. Just as it is for tricategories, the Bicategorical Pasting Theorem 3.6.6 is crucial in interpreting their axioms. Next we discuss the Gray tensor product on 2-categories, which provides a symmetric monoidal structure that is different from the Cartesian one, and the corresponding Gray monoids. The last part of this chapter discusses double categories and monoidal double categories.

Chapter Interdependency

The core concepts in Chapters 2 through 4 are used in all the subsequent chapters. Chapters 6 through 9 are independent of Chapter 5. Chapters 7 and 8 require internal adjunctions, mates, and internal equivalences from Sections 6.1 and 6.2. Chapter 9 uses 2-monads from Section 6.5. Chapter 10 depends on all of Chapter 9, and Section 10.2 uses lax colimits from Section 5.2. The rest of Chapter 10 uses the 2-Categorical Whitehead Theorem 7.5.8. Chapters 11 and 12 use internal adjunctions, mates, and internal equivalences from Sections 6.1 and 6.2 but none of the other material after Chapter 4. Chapter 12 depends on the whiskerings of Section 11.1 and the definition of a tricategory from Section 11.2. The following graph summarizes these dependencies.



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Contents

1
Categories

In this chapter we recall some basic concepts of category theory, including monads, monoidal categories, and enriched categories. For a more detailed discussion of basic category theory, the reader is referred to the references mentioned in Section 1.4.

1.1 Basic Category Theory

In this section we recall the concepts of categories, functors, natural transformations, adjunctions, equivalences, the Yoneda Lemma, (co)limits, and monads. We begin by fixing a set-theoretic convention.

Definition 1.1.1. A *Grothendieck universe*, or just a *universe*, is a set $\mathcal{U}$ with the following properties:

(1) If $x \in \mathcal{U}$ and $y \in x$, then $y \in \mathcal{U}$.
(2) If $x \in \mathcal{U}$, then $\mathcal{P}(x) \in \mathcal{U}$, where $\mathcal{P}(x)$ is the power set of x.
(3) If $I \in \mathcal{U}$ and $x_i \in \mathcal{U}$ for each $i \in I$, then the union $\bigcup_{i \in I} x_i \in \mathcal{U}$.
(4) The set of finite ordinals $\mathbb{N} \in \mathcal{U}$. ◇

Convention 1.1.2. We assume the following axiom:

Axiom of Universes: every set belongs to some universe.

We fix a universe $\mathcal{U}$. From now on, an element in $\mathcal{U}$ is called a *set*, and a subset of $\mathcal{U}$ is called a *class*. These conventions allow us to make the usual set-theoretic constructions, including Cartesian products, disjoint unions, and function sets. ◇

Proposition 1.1.3. *A universe $\mathcal{U}$ has the following properties:*

(a) If $x, y \in \mathcal{U}$, then $\{x, y\} \in \mathcal{U}$.
(b) If $x \in \mathcal{U}$ and $y \subset x$, then $y \in \mathcal{U}$.

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(c) *If $I \in \mathcal{U}$ and $x_i \in \mathcal{U}$ for each $i \in I$, then the Cartesian product $\prod_{i \in I} x_i \in \mathcal{U}$.*
(d) *If $I \in \mathcal{U}$ and $x_i \in \mathcal{U}$ for each $i \in I$, then the disjoint union $\coprod_{i \in I} x_i \in \mathcal{U}$.*
(e) *If $x, y \in \mathcal{U}$, then $y^x \in \mathcal{U}$, where y^x denotes the collection of functions $x \longrightarrow y$.*

Proof. Combining Axioms (4), (2), and (1) of Definition 1.1.1, we see that $\mathcal{U}$ contains an n-element set for each $n \in \mathbb{N}$. For Property (a), we therefore have $x \cup y \in \mathcal{U}$ by Axiom (3). Since $\{x, y\} \in \mathcal{P}(x \cup y)$, we have $\{x, y\} \in \mathcal{U}$ by Axiom (1). For Property (b), $y \subset x$ means that $y \in \mathcal{P}(x)$ and thus the assertion follows from Axioms (1) and (2). For Properties (c) and (d), we first note that, for any $x, y \in \mathcal{U}$, we have $x \times y \subset \mathcal{P}(\mathcal{P}(x \cup y))$ and therefore $x \times y \in \mathcal{U}$ by Property (b). Hence, using Axiom (3), the product $I \times \bigcup_{i \in I} x_i$ is an element of $\mathcal{U}$. The assertions of Properties (c) and (d), respectively, now follow because

$$\prod_{i \in I} x_i \subset \mathcal{P}(I \times \bigcup_{i \in I} x_i) \quad \text{and} \quad \coprod_{i \in I} x_i \subset I \times \bigcup_{i \in I} x_i.$$

Property (e) follows because $y^x \subset \mathcal{P}(x \times y)$. □

Definition 1.1.4. A *category* C consists of

- a class $\mathsf{Ob}(\mathsf{C})$ of *objects* in C;
- a set $\mathsf{C}(X, Y)$, also denoted by $\mathsf{C}(X; Y)$, of *morphisms* with *domain* $X = \mathsf{dom}(f)$ and *codomain* $Y = \mathsf{cod}(f)$ for any objects $X, Y \in \mathsf{Ob}(\mathsf{C})$;
- an assignment called *composition*

$$\mathsf{C}(Y, Z) \times \mathsf{C}(X, Y) \xrightarrow{\ \circ\ } \mathsf{C}(X, Z), \qquad \circ(g, f) = g \circ f$$

  for objects X, Y, Z in C; and
- an *identity morphism* $1_X \in \mathsf{C}(X, X)$ for each object X in C.

These data are required to satisfy the following two conditions:

Associativity: For morphisms f, g, and h, the equality

$$h \circ (g \circ f) = (h \circ g) \circ f$$

holds, provided the compositions are defined.

Unity: For each morphism $f \in \mathsf{C}(X, Y)$, the equalities

$$1_Y \circ f = f = f \circ 1_X$$

hold.

In subsequent chapters, a category is sometimes called a 1-*category*. ◇

In a category C, the class of objects $\mathsf{Ob}(\mathsf{C})$ is also denoted by C_0, and the collection of morphisms is denoted by either $\mathsf{Mor}(\mathsf{C})$ or C_1. For an object $X \in \mathsf{Ob}(\mathsf{C})$ and a morphism $f \in \mathsf{Mor}(\mathsf{C})$, we often write $X \in \mathsf{C}$ and $f \in \mathsf{C}$. We also denote a morphism $f \in \mathsf{C}(X, Y)$ as

$$f : X \longrightarrow Y, \qquad X \xrightarrow{\ f\ } Y, \quad \text{and} \quad X -f\to Y.$$

Morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are called *composable*, and $g \circ f \in \mathsf{C}(X, Z)$ is often abbreviated to gf, called their *composite*.

The identity morphism 1_X of an object X is also denoted by 1 or even just X. A morphism $f : X \longrightarrow Y$ in a category C is called an *isomorphism* if there exists a morphism $g : Y \longrightarrow X$ such that $gf = 1_X$ and $fg = 1_Y$. An isomorphism is sometimes denoted by $X \xrightarrow{\cong} Y$. A category is *discrete* if it contains no nonidentity morphisms. A *groupoid* is a category in which every morphism is an isomorphism. The *opposite category* of a category C is denoted by C^{op}. It has the same objects as C and morphism sets $\mathsf{C}^{\mathsf{op}}(X, Y) = \mathsf{C}(Y, X)$, with identity morphisms and composition inherited from C. A morphism $f : Y \longrightarrow X$ in C is denoted by $f^{\mathsf{op}} : X \longrightarrow Y$ in $\mathsf{C}^{\mathsf{op}}(X, Y)$. A *small category* is a category whose class of objects forms a set. A category is *essentially small* if its isomorphism classes of objects form a set.

Definition 1.1.5. For categories C and D, a *functor* $F : \mathsf{C} \longrightarrow \mathsf{D}$ consists of

- an assignment on objects

$$\mathsf{Ob}(\mathsf{C}) \longrightarrow \mathsf{Ob}(\mathsf{D}), \qquad X \longmapsto F(X); \text{ and}$$

- an assignment on morphisms

$$\mathsf{C}(X, Y) \longrightarrow \mathsf{D}\big(F(X), F(Y)\big), \qquad f \longmapsto F(f).$$

These data are required to satisfy the following two conditions:

Composition: The equality

$$F(gf) = F(g)F(f)$$

of morphisms holds, provided the compositions are defined.

Identities: For each object $X \in \mathsf{C}$, the equality

$$F(1_X) = 1_{F(X)}$$

in $\mathsf{D}\big(F(X), F(X)\big)$ holds. ◇

We often abbreviate $F(X)$ and $F(f)$ to FX and Ff, respectively. Functors are composed by composing the assignments on objects and on morphisms. The *identity functor* of a category C is determined by the identity assignments on objects and morphisms and is written as either Id_{C} or 1_{C}. We write Cat for the category with small categories as objects, functors as morphisms, identity functors as identity morphisms, and composition of functors as composition. For categories C and D, the collection of functors $\mathsf{C} \longrightarrow \mathsf{D}$ is denoted by $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$. For a functor $F : \mathsf{C} \longrightarrow \mathsf{D}$, the functor

$$\mathsf{C}^{\mathsf{op}} \xrightarrow{F^{\mathsf{op}}} \mathsf{D}^{\mathsf{op}}, \quad \begin{cases} \mathsf{Ob}(\mathsf{C}^{\mathsf{op}}) \ni X & \longmapsto FX \in \mathsf{Ob}(\mathsf{D}^{\mathsf{op}}), \\ \mathsf{C}^{\mathsf{op}}(X, Y) \ni f & \longmapsto (Ff)^{\mathsf{op}} \in \mathsf{D}^{\mathsf{op}}(FX, FY) \end{cases} \tag{1.1.6}$$

is called the *opposite functor*.

Definition 1.1.7. Suppose $F, G : \mathsf{C} \longrightarrow \mathsf{D}$ are functors. A *natural transformation* $\theta : F \longrightarrow G$ consists of a morphism $\theta_X : FX \longrightarrow GX$ in D for each object $X \in \mathsf{C}$ such that the diagram

$$\begin{tikzcd} FX \arrow[r, "\theta_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\theta_Y"] & GY \end{tikzcd}$$

in D is commutative for each morphism $f : X \longrightarrow Y$ in C. ◇

In other words, the equality

$$Gf \circ \theta_X = \theta_Y \circ Ff$$

holds in $\mathsf{D}(FX, GY)$. The collection of natural transformations $F \longrightarrow G$ is denoted by $\mathsf{Nat}(F,G)$. Each morphism θ_X is called a *component* of θ. The *identity natural transformation* $1_F : F \longrightarrow F$ of a functor F has each component an identity morphism. A *natural isomorphism* is a natural transformation in which every component is an isomorphism.

Definition 1.1.8. Suppose $\theta : F \longrightarrow G$ is a natural transformation for functors $F, G : \mathsf{C} \longrightarrow \mathsf{D}$.

(1) Suppose $\phi : G \longrightarrow H$ is a natural transformation for another functor $H : \mathsf{C} \longrightarrow \mathsf{D}$. The *vertical composition*

$$\phi\theta : F \longrightarrow H$$

is the natural transformation with components

$$(\phi\theta)_X = \phi_X \circ \theta_X : FX \longrightarrow HX \quad \text{for} \quad X \in \mathsf{C}. \tag{1.1.9}$$

(2) Suppose $\theta' : F' \longrightarrow G'$ is a natural transformation for functors $F', G' : \mathsf{D} \longrightarrow \mathsf{E}$. The *horizontal composition*

$$\theta' * \theta : F'F \longrightarrow G'G$$

is the natural transformation whose component $(\theta' * \theta)_X$ for an object $X \in \mathsf{C}$ is defined as either composite in the commutative diagram

$$\begin{tikzcd} F'FX \arrow[r, "\theta'_{FX}"] \arrow[d, "F'\theta_X"'] & G'FX \arrow[d, "G'\theta_X"] \\ F'GX \arrow[r, "\theta'_{GX}"] & G'GX \end{tikzcd} \tag{1.1.10}$$

in D. ◇

For a category C and a small category D, a D-*diagram in* C is a functor $\mathsf{D} \longrightarrow \mathsf{C}$. The *diagram category* C^{D} has D-diagrams in C as objects, natural transformations between such functors as morphisms, and vertical composition of natural transformations as composition.

Definition 1.1.11. For categories C and D, an *adjunction* from C to D is a triple (L, R, ϕ) consisting of

- a pair of functors in opposite directions

$$\mathsf{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{D}, \text{ and}$$

- a family of bijections

$$\mathsf{D}(LX, Y) \xrightarrow[\cong]{\phi_{X,Y}} \mathsf{C}(X, RY)$$

  that is natural in the objects $X \in \mathsf{C}$ and $Y \in \mathsf{D}$.

Such an adjunction is also called an *adjoint pair*, with L the *left adjoint*, and R the *right adjoint*. ◇

We also denote such an adjunction by $L \dashv R$. We always display the left adjoint on top, pointing to the right. If an adjunction is displayed vertically, then the left adjoint is written on the left-hand side.

In an adjunction $L \dashv R$ as in Definition 1.1.11, setting $Y = LX$ or $X = RY$, the natural bijection ϕ yields natural transformations

$$1_{\mathsf{C}} \xrightarrow{\eta} RL \quad \text{and} \quad LR \xrightarrow{\varepsilon} 1_{\mathsf{D}}, \tag{1.1.12}$$

called the *unit* and the *counit*, respectively. The vertically composed natural transformations

$$R \xrightarrow{\eta R} RLR \xrightarrow{R\varepsilon} R \quad \text{and} \quad L \xrightarrow{L\eta} LRL \xrightarrow{\varepsilon L} L \tag{1.1.13}$$

are equal to 1_R and 1_L, respectively. Here

$$\eta R = \eta * 1_R, \qquad R\varepsilon = 1_R * \varepsilon,$$

and similarly for $L\eta$ and εL. The identities in 1.1.13 are known as the *triangle identities*. Characterizations of adjunctions are given in [**Bor94a**, Chapter 3] and [**ML98**, IV.1], one of which is the following: an adjunction (L, R, ϕ) is completely determined by

- the functors $L : \mathsf{C} \longrightarrow \mathsf{D}$ and $R : \mathsf{D} \longrightarrow \mathsf{C}$, and
- the natural transformation $\eta : 1_{\mathsf{C}} \longrightarrow RL$

such that for each morphism $f : X \longrightarrow RY$ in C with $X \in \mathsf{C}$ and $Y \in \mathsf{D}$, there exists a unique morphism $f' : LX \longrightarrow Y$ in D such that the diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, equal] & RLX \arrow[d, "Rf'"] \\ X \arrow[r, "f"] & RY \end{tikzcd}$$

in C is commutative.

Definition 1.1.14. A functor $F : \mathsf{C} \longrightarrow \mathsf{D}$ is called an *equivalence* if there exist

- a functor $G : \mathsf{D} \longrightarrow \mathsf{C}$, and
- natural isomorphisms $\eta : 1_{\mathsf{C}} \xrightarrow{\cong} GF$ and $\varepsilon : FG \xrightarrow{\cong} 1_{\mathsf{D}}$.

If, in addition, F is left adjoint to G with unit η and counit ε, then $(F, G, \eta, \varepsilon)$ is called an *adjoint equivalence*. ◇

Equivalences can be characterized locally as follows. A functor F is an equivalence if and only if it is both

- *fully faithful*, which means that each function $\mathsf{C}(X, Z) \longrightarrow \mathsf{D}(FX, FZ)$ on morphism sets is a bijection; and
- *essentially surjective*, which means that, for each object $Y \in \mathsf{D}$, there exists an isomorphism $FX \xrightarrow{\cong} Y$ for some object $X \in \mathsf{C}$.

Definition 1.1.15. Suppose C is a category, and A is an object in C. The functor

$$\mathcal{Y}_A = \mathsf{C}(-, A) : \mathsf{C}^{\mathsf{op}} \longrightarrow \mathsf{Set}$$

defined by

$$\begin{aligned} \mathsf{Ob}(\mathsf{C}) \ni X &\longmapsto \mathcal{Y}_A(X) = \mathsf{C}(X, A), \\ \mathsf{C}(X, Y) \ni f &\longmapsto \mathcal{Y}_A(f) = (-) \circ f : \mathsf{C}(Y, A) \longrightarrow \mathsf{C}(X, A) \end{aligned}$$

is called the *representable functor induced by* A. ◇

The Yoneda Lemma states that there is a bijection

$$\mathsf{Nat}(\mathcal{Y}_A, F) \cong F(A), \tag{1.1.16}$$

defined by

$$\big(\theta : \mathcal{Y}_A \longrightarrow F\big) \longmapsto \theta_A(1_A) \in F(A),$$

that is natural in the object $A \in \mathsf{C}$ and the functor $F : \mathsf{C}^{\mathsf{op}} \longrightarrow \mathsf{Set}$.

A special case of the Yoneda Lemma is the natural bijection

$$\mathsf{Nat}(\mathcal{Y}_A, \mathcal{Y}_B) \cong \mathcal{Y}_B(A) = \mathsf{C}(A, B)$$

for objects $A, B \in \mathsf{C}$. The *Yoneda embedding* is the functor

$$\mathcal{Y}_- : \mathsf{C} \longrightarrow \mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Set}). \tag{1.1.17}$$

This functor is fully faithful by the previous bijection.

Definition 1.1.18. Suppose $F : \mathsf{D} \longrightarrow \mathsf{C}$ is a functor. A *colimit of* F, if it exists, is a pair $(\mathsf{colim}\, F, \delta)$ consisting of

- an object $\mathsf{colim}\, F \in \mathsf{C}$, and
- a morphism $\delta_d : Fd \longrightarrow \mathsf{colim}\, F$ in C for each object $d \in \mathsf{D}$

that satisfies the following two conditions:

(1) For each morphism $f : d \longrightarrow d'$ in D, the diagram

$$\begin{tikzcd} Fd \arrow[r, "\delta_d"] \arrow[d, "Ff"'] & \mathsf{colim}\, F \arrow[d, equal] \\ Fd' \arrow[r, "\delta_{d'}"] & \mathsf{colim}\, F \end{tikzcd}$$

in C is commutative. A pair $(\mathsf{colim}\, F, \delta)$ with this property is called a *cocone of* F.

(2) The pair $(\mathsf{colim}\, F, \delta)$ is *universal* among cocones of F. This means that if (X, δ') is another such pair that satisfies Property (1), then there exists a unique morphism $h : \mathsf{colim}\, F \longrightarrow X$ in C such that the diagram

$$\begin{tikzcd} Fd \arrow[r, "\delta_d"] \arrow[d, equal] & \mathsf{colim}\, F \arrow[d, "h"] \\ Fd \arrow[r, "\delta'_d"] & X \end{tikzcd}$$

is commutative for each object $d \in \mathsf{D}$. ◇

A *limit of* F $(\mathsf{lim}\, F, \delta)$, if it exists, is defined dually by turning the morphisms δ_d for $d \in \mathsf{D}$ and h backward. A *small (co)limit* is a (co)limit of a functor whose domain category is a small category. A category C is *(co)complete* if it has all small (co)limits. For a functor $F : \mathsf{D} \longrightarrow \mathsf{C}$, its colimit, if it exists, is also denoted by $\operatorname{colim}_{x\in\mathsf{D}} Fx$ and $\operatorname{colim}_{\mathsf{D}} F$, and similarly for limits.

A left adjoint $F : \mathsf{C} \longrightarrow \mathsf{D}$ preserves all the colimits that exist in C. In other words, if $H : \mathsf{E} \longrightarrow \mathsf{C}$ has a colimit, then $FH : \mathsf{E} \longrightarrow \mathsf{D}$ also has a colimit, and the natural morphism

$$\operatorname*{colim}_{e\in\mathsf{E}} FHe \longrightarrow F\Big(\operatorname*{colim}_{e\in E} He\Big) \tag{1.1.19}$$

is an isomorphism. Similarly, a right adjoint $G : \mathsf{D} \longrightarrow \mathsf{C}$ preserves all the limits that exist in D.

Example 1.1.20. Here are some special types of colimits in a category C:

(1) An *initial object* $\varnothing^{\mathsf{C}}$ in C is a colimit of the functor $\varnothing \longrightarrow \mathsf{C}$, where $\varnothing$ is the empty category with no objects and no morphisms. It is characterized by the universal property that, for each object X in C, there is a unique morphism $\varnothing^{\mathsf{C}} \longrightarrow X$ in C.
(2) A *coproduct* is a colimit of a functor whose domain category is a discrete category. We use the symbols $\coprod$ and $\amalg$ to denote coproducts.
(3) A *pushout* is a colimit of a functor whose domain category has the form

$$\bullet \longleftarrow \bullet \longrightarrow \bullet$$

with three objects and two nonidentity morphisms.
(4) A *coequalizer* is a colimit of a functor whose domain category has the form

$$\bullet \rightrightarrows \bullet$$

with two objects and two nonidentity morphisms.

Terminal objects, products, pullbacks, and equalizers are the corresponding limit concepts. ◇

Notation 1.1.21. We let **1** denote the terminal category; it has a unique object $*$ and a unique 1-cell 1_*. ◇

Definition 1.1.22. A *monad* on a category C is a triple (T, μ, η) in which

- $T : \mathsf{C} \longrightarrow \mathsf{C}$ is a functor, and
- $\mu : T^2 \longrightarrow T$, called the *multiplication*, and $\eta : 1_{\mathsf{C}} \longrightarrow T$, called the *unit*, are natural transformations

such that the associativity and unity diagrams

$$
\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
1_{\mathsf{C}} \circ T \arrow[r, "\eta T"] \arrow[d, equal] & T^2 \arrow[d, "\mu"'] & T \circ 1_{\mathsf{C}} \arrow[l, "T\eta"'] \arrow[d, equal] \\
T \arrow[r, equal] & T \arrow[r, equal] & T
\end{tikzcd}
$$

are commutative. We often refer to such a monad as simply T. ◇

Definition 1.1.23. Suppose (T, μ, η) is a monad on a category C.

(1) A *T-algebra* is a pair (X, θ) consisting of

- an object X in C and
- a morphism $\theta : TX \longrightarrow X$, called the *structure morphism*,

such that the associativity and unity diagrams

$$
\begin{tikzcd}
T^2X \arrow[r, "T\theta"] \arrow[d, "\mu_X"'] & TX \arrow[d, "\theta"] \\
TX \arrow[r, "\theta"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, equal] & TX \arrow[d, "\theta"] \\
& X
\end{tikzcd}
\tag{1.1.24}
$$

are commutative.

(2) A *morphism of T-algebras*

$$f : (X, \theta^X) \longrightarrow (Y, \theta^Y)$$

is a morphism $f : X \longrightarrow Y$ in C such that the diagram

$$
\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "\theta^X"'] & TY \arrow[d, "\theta^Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
$$

is commutative.

(3) The category of T-algebras is denoted by $\mathsf{Alg}(T)$. ◇

Definition 1.1.25. For a monad (T, μ, η) on a category C, the *Eilenberg-Moore adjunction* is the adjunction

$$
\begin{tikzcd}
\mathsf{C} \arrow[r, shift left, "T"] & \mathsf{Alg}(T) \arrow[l, shift left, "U"]
\end{tikzcd}
$$

in which:

- The right adjoint U is the forgetful functor $U(X,\theta) = X$.
- The left adjoint sends an object $X \in \mathsf{C}$ to the free T-algebra

$$\big(TX, \mu_X : T^2X \longrightarrow TX\big). \qquad \diamond$$

1.2 Monoidal Categories

In this section we recall the definitions of a monoidal category, a monoidal functor, a monoidal natural transformation, and their symmetric and braided versions. We also recall Mac Lane's Coherence Theorem for monoidal categories and discuss some examples. One may think of a monoidal category as a categorical generalization of a monoid, in which there is a way to multiply together objects and morphisms.

Definition 1.2.1. A *monoidal category* is a tuple

$$(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$$

consisting of

- a category C;
- a functor $\otimes : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$, called the *monoidal product*;
- an object $\mathbb{1} \in \mathsf{C}$, called the *monoidal unit*;
- a natural isomorphism

$$(X \otimes Y) \otimes Z \xrightarrow[\cong]{\alpha_{X,Y,Z}} X \otimes (Y \otimes Z) \tag{1.2.2}$$

  for all objects $X, Y, Z \in \mathsf{C}$, called the *associativity isomorphism*; and
- natural isomorphisms

$$\mathbb{1} \otimes X \xrightarrow[\cong]{\lambda_X} X \quad \text{and} \quad X \otimes \mathbb{1} \xrightarrow[\cong]{\rho_X} X \tag{1.2.3}$$

  for all objects $X \in \mathsf{C}$, called the *left unit isomorphism* and the *right unit isomorphism*, respectively.

These data are required to satisfy the following axioms:

The Unity Axioms: The *middle unity diagram*

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[r, "\alpha_{X,\mathbb{1},Y}"] \arrow[d, "\rho_X \otimes Y"'] & X \otimes (\mathbb{1} \otimes Y) \arrow[d, "X \otimes \lambda_Y"] \\
X \otimes Y \arrow[r, equal] & X \otimes Y
\end{tikzcd} \tag{1.2.4}$$

is commutative for all objects $X, Y \in \mathsf{C}$. Moreover, the equality

$$\lambda_{\mathbb{1}} = \rho_{\mathbb{1}} : \mathbb{1} \otimes \mathbb{1} \xrightarrow{\cong} \mathbb{1}$$

holds.

The Pentagon Axiom: The pentagon

$$\begin{tikzcd}
& (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "\alpha_{W,X,Y \otimes Z}"] & \\
\big((W \otimes X) \otimes Y\big) \otimes Z \arrow[ur, "\alpha_{W \otimes X,Y,Z}"] \arrow[d, "\alpha_{W,X,Y} \otimes Z"'] & & W \otimes \big(X \otimes (Y \otimes Z)\big) \\
\big(W \otimes (X \otimes Y)\big) \otimes Z \arrow[rr, "\alpha_{W,X \otimes Y,Z}"] & & W \otimes \big((X \otimes Y) \otimes Z\big) \arrow[u, "W \otimes \alpha_{X,Y,Z}"']
\end{tikzcd} \tag{1.2.5}$$

is commutative for all objects $W, X, Y, Z \in \mathsf{C}$.

A *strict monoidal category* is a monoidal category in which the components of α, λ, and ρ are all identity morphisms. ◇

Convention 1.2.6. In a monoidal category, an *empty tensor product*, written as $X^{\otimes 0}$ or $X^{\otimes \varnothing}$, means the monoidal unit $\mathbb{1}$. We sometimes use concatenation as an abbreviation for the monoidal product, so for example

$$XY = X \otimes Y, \qquad (XY)Z = (X \otimes Y) \otimes Z,$$

and similarly for morphisms. We usually suppress α, λ, and ρ from the notation and refer to a monoidal category as simply $(\mathsf{C}, \otimes, \mathbb{1})$ or C. To emphasize the ambient monoidal category C, we decorate the monoidal structure accordingly as $\otimes^{\mathsf{C}}$, $\mathbb{1}^{\mathsf{C}}$, α^{C}, λ^{C}, and ρ^{C}. ◇

Remark 1.2.7. In a monoidal category:

(1) The axiom $\lambda_{\mathbb{1}} = \rho_{\mathbb{1}}$ is actually a consequence of the middle unity diagram 1.2.4 and the pentagon axiom 1.2.5.
(2) The diagrams

$$\begin{tikzcd} (\mathbb{1} \otimes X) \otimes Y \arrow[r, "\alpha_{\mathbb{1},X,Y}"] \arrow[d, "\lambda_X \otimes Y"'] & \mathbb{1} \otimes (X \otimes Y) \arrow[d, "\lambda_{X \otimes Y}"] \\ X \otimes Y \arrow[r, equal] & X \otimes Y \end{tikzcd} \qquad \begin{tikzcd} (X \otimes Y) \otimes \mathbb{1} \arrow[r, "\alpha_{X,Y,\mathbb{1}}"] \arrow[d, "\rho_{X \otimes Y}"'] & X \otimes (Y \otimes \mathbb{1}) \arrow[d, "X \otimes \rho_Y"] \\ X \otimes Y \arrow[r, equal] & X \otimes Y \end{tikzcd} \tag{1.2.8}$$

are commutative. They are called the *left unity diagram* and the *right unity diagram*, respectively. ◇

Example 1.2.9 (Reversed Monoidal Category)**.** Every monoidal category

$$(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$$

induces another monoidal category with the order of the monoidal product reversed. More precisely, we define the following structures:

- First we define the composite functor

$$\begin{tikzcd} \mathsf{C} \times \mathsf{C} \arrow[r, "\tau"] \arrow[rr, bend left, "\otimes'"] & \mathsf{C} \times \mathsf{C} \arrow[r, "\otimes"] & \mathsf{C}, \end{tikzcd}$$

  which is called the *reversed monoidal product*, in which τ switches the two arguments.
- Next we define the natural isomorphism

$$(X \otimes' Y) \otimes' Z \xrightarrow[\cong]{\alpha'_{X,Y,Z}} X \otimes' (Y \otimes' Z)$$

  as

$$\alpha'_{X,Y,Z} = \alpha^{-1}_{Z,Y,X}.$$

- Then we define the natural isomorphisms

$$\mathbb{1} \otimes' X \xrightarrow[\cong]{\lambda'_X} X \quad \text{and} \quad X \otimes' \mathbb{1} \xrightarrow[\cong]{\rho'_X} X$$

as

$$\lambda'_X = \rho_X \quad \text{and} \quad \rho'_X = \lambda_X,$$

respectively.

Then

$$\mathsf{C}^{\mathsf{rev}} = (\mathsf{C}, \otimes', \mathbb{1}, \alpha', \lambda', \rho')$$

is a monoidal category, called the *reversed monoidal category* of C. For example, the middle unity diagram 1.2.4 in $\mathsf{C}^{\mathsf{rev}}$ is the diagram

$$\begin{tikzcd} Y \otimes (\mathbb{1} \otimes X) \arrow[r, "\alpha^{-1}_{Y,\mathbb{1},X}"] \arrow[d, "Y \otimes \lambda_X"'] & (Y \otimes \mathbb{1}) \otimes X \arrow[d, "\rho_Y \otimes X"] \\ Y \otimes X \arrow[r, equal] & Y \otimes X \end{tikzcd}$$

in C, which is commutative by the middle unity diagram in C. A similar argument proves the pentagon axiom in $\mathsf{C}^{\mathsf{rev}}$. We will come back to this example in Chapter 2 when we discuss dualities of bicategories. ◇

Example 1.2.10 (Opposite Monoidal Category)**.** For each monoidal category C, its opposite category C^{op} has a monoidal structure

$$(\mathsf{C}^{\mathsf{op}}, \otimes^{\mathsf{op}}, \mathbb{1}, \alpha^{-1}, \lambda^{-1}, \rho^{-1})$$

with monoidal product

$$\mathsf{C}^{\mathsf{op}} \times \mathsf{C}^{\mathsf{op}} \cong (\mathsf{C} \times \mathsf{C})^{\mathsf{op}} \xrightarrow{\otimes^{\mathsf{op}}} \mathsf{C}^{\mathsf{op}}$$

the opposite functor of $\otimes$, and with the same monoidal unit. Its associativity isomorphism, left unit isomorphism, and right unit isomorphism are the inverses of their counterparts in C. ◇

Definition 1.2.11. A *monoid* in a monoidal category C is a triple $(X, \mu, 1)$ with

- X an object in C;
- $\mu : X \otimes X \longrightarrow X$ a morphism, called the *multiplication*; and
- $1 : \mathbb{1} \longrightarrow X$ a morphism, called the *unit*.

These data are required to make the following associativity and unity diagrams commutative.

$$\begin{tikzcd}
(X \otimes X) \otimes X \arrow[r, "\alpha"] \arrow[dd, "\mu \otimes X"'] & X \otimes (X \otimes X) \arrow[d, "X \otimes \mu"] \\
& X \otimes X \arrow[d, "\mu"] \\
X \otimes X \arrow[r, "\mu"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \otimes X \arrow[r, "1 \otimes X"] \arrow[d, "\lambda"', "\cong"] & X \otimes X \arrow[d, "\mu"] & X \otimes \mathbb{1} \arrow[l, "X \otimes 1"'] \arrow[d, "\rho", "\cong"'] \\
X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}$$

A morphism of monoids

$$f : (X, \mu^X, 1^X) \longrightarrow (Y, \mu^Y, 1^Y)$$

is a morphism $f : X \longrightarrow Y$ in C that preserves the multiplications and the units in the sense that the diagrams

$$\begin{tikzcd}
X \otimes X \arrow[r, "f \otimes f"] \arrow[d, "\mu^X"'] & Y \otimes Y \arrow[d, "\mu^Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \arrow[r, "1^X"] \arrow[d, equal] & X \arrow[d, "f"] \\
\mathbb{1} \arrow[r, "1^Y"] & Y
\end{tikzcd}$$

are commutative. The category of monoids in a monoidal category C is denoted by $\mathsf{Mon}(\mathsf{C})$. ◇

Example 1.2.12. Suppose $(X, \mu, 1)$ is a monoid in the category Set, with sets as objects, functions as morphisms, and monoidal product given by the Cartesian product. There are two ways to regard $(X, \mu, 1)$ as a category:

(1) There is a category ΣX with one object $*$, morphism set $\Sigma X(*, *) = X$, composition $\mu : X \times X \longrightarrow X$, and identity morphism $1_* = 1$. The associativity and unity of the monoid $(X, \mu, 1)$ become those of the category ΣX.
(2) We may also regard the set X as a discrete category X^{dis}, so there are no nonidentity morphisms. This discrete category is a strict monoidal category with monoidal product μ on objects, and monoidal unit 1.

We will come back to the category ΣX in Chapter 2. ◇

Example 1.2.13. In the context of Example 1.2.12, consider the *opposite monoid*

$$X^{\mathsf{op}} = (X, \mu^{\mathsf{op}}, 1)$$

in which

$$\mu^{\mathsf{op}}(a,b) = \mu(b,a) \quad \text{for} \quad a,b \in X.$$

(1) There is an equality

$$\Sigma(X^{\mathsf{op}}) = (\Sigma X)^{\mathsf{op}}$$

of categories. This means that the one-object category of the opposite monoid is the opposite category of the one-object category of $(X, \mu, 1)$.

(2) Recall the reversed monoidal category in Example 1.2.9. Then there is an equality

$$(X^{\mathsf{op}})^{\mathsf{dis}} = (X^{\mathsf{dis}})^{\mathsf{rev}}$$

of strict monoidal categories. In other words, the discrete strict monoidal category of the opposite monoid is the reversed monoidal category of the discrete strict monoidal category of $(X, \mu, 1)$. ◇

Definition 1.2.14. For monoidal categories C and D, a *monoidal functor*

$$(F, F_2, F_0) : \mathsf{C} \longrightarrow \mathsf{D}$$

consists of

- a functor $F : \mathsf{C} \longrightarrow \mathsf{D}$;
- a natural transformation

$$FX \otimes FY \xrightarrow{F_2} F(X \otimes Y) \in \mathsf{D}, \tag{1.2.15}$$

  where X and Y are objects in C; and
- a morphism

$$\mathbb{1}^{\mathsf{D}} \xrightarrow{F_0} F\mathbb{1}^{\mathsf{C}} \in \mathsf{D}. \tag{1.2.16}$$

These data are required to satisfy the following associativity and unity axioms:

Associativity: The diagram

$$\begin{tikzcd}
\big(FX \otimes FY\big) \otimes FZ \arrow[r, "\alpha^{\mathsf{D}}"] \arrow[d, "F_2 \otimes FZ"'] & FX \otimes \big(FY \otimes FZ\big) \arrow[d, "FX \otimes F_2"] \\
F(X \otimes Y) \otimes FZ \arrow[d, "F_2"'] & FX \otimes F(Y \otimes Z) \arrow[d, "F_2"] \\
F\big((X \otimes Y) \otimes Z\big) \arrow[r, "F\alpha^{\mathsf{C}}"] & F\big(X \otimes (Y \otimes Z)\big)
\end{tikzcd} \tag{1.2.17}$$

is commutative for all objects $X, Y, Z \in \mathsf{C}$.

Left Unity: The diagram

$$\begin{tikzcd}
\mathbb{1}^{\mathsf{D}} \otimes FX \arrow[r, "\lambda^{\mathsf{D}}_{FX}"] \arrow[d, "F_0 \otimes FX"'] & FX \\
F\mathbb{1}^{\mathsf{C}} \otimes FX \arrow[r, "F_2"] & F(\mathbb{1}^{\mathsf{C}} \otimes X) \arrow[u, "F\lambda^{\mathsf{C}}_X"']
\end{tikzcd} \tag{1.2.18}$$

is commutative for all objects $X \in \mathsf{C}$.

Right Unity: The diagram

$$\begin{tikzcd}
FX \otimes \mathbb{1}^{\mathsf{D}} \arrow[r, "\rho^{\mathsf{D}}_{FX}"] \arrow[d, "FX \otimes F_0"'] & FX \\
FX \otimes F\mathbb{1}^{\mathsf{C}} \arrow[r, "F_2"] & F(X \otimes \mathbb{1}^{\mathsf{C}}) \arrow[u, "F\rho^{\mathsf{C}}_X"']
\end{tikzcd} \tag{1.2.19}$$

is commutative for all objects $X \in \mathsf{C}$.

A monoidal functor (F, F_2, F_0) is often referred to as simply F.

A *strong monoidal functor* is a monoidal functor in which the morphisms F_0 and F_2 are all isomorphisms. A *strict monoidal functor* is a monoidal functor in which the morphisms F_0 and F_2 are all identity morphisms. ◇

Definition 1.2.20. For monoidal functors $F, G : \mathsf{C} \longrightarrow \mathsf{D}$, a *monoidal natural transformation* $\theta : F \longrightarrow G$ is a natural transformation between the underlying functors such that the diagrams

$$\begin{tikzcd} FX \otimes FY \arrow[r, "\theta_X \otimes \theta_Y"] \arrow[d, "F_2"'] & GX \otimes GY \arrow[d, "G_2"] \\ F(X \otimes Y) \arrow[r, "\theta_{X \otimes Y}"] & G(X \otimes Y) \end{tikzcd} \tag{1.2.21}$$

and

$$\begin{tikzcd} \mathbb{1}^{\mathsf{D}} \arrow[r, "F_0"] \arrow[d, equal] & F\mathbb{1}^{\mathsf{C}} \arrow[d, "\theta_{\mathbb{1}^{\mathsf{C}}}"] \\ \mathbb{1}^{\mathsf{D}} \arrow[r, "G_0"] & G\mathbb{1}^{\mathsf{C}} \end{tikzcd} \tag{1.2.22}$$

are commutative for all objects $X, Y \in \mathsf{C}$. $\diamond$

The following strictification result for monoidal categories is due to Mac Lane; see [**ML63**], [**ML98**, XI.3 Theorem 1], and [**JS93**]:

Theorem 1.2.23 (Mac Lane's Coherence)**.** *For each monoidal category* C*, there exist a strict monoidal category* C_{st} *and an adjoint equivalence*

$$\begin{tikzcd} \mathsf{C} \arrow[r, shift left, "L"] & \mathsf{C}_{\mathsf{st}} \arrow[l, shift left, "R"] \end{tikzcd}$$

with (i) both L *and* R *strong monoidal functors and (ii)* $RL = 1_{\mathsf{C}}$.

In other words, every monoidal category can be strictified via an adjoint equivalence consisting of strong monoidal functors. Note:

- Another version of the Coherence Theorem [**ML98**, VII.2 Theorem 1] describes explicitly the free monoidal category generated by one object.
- A third version of the Coherence Theorem [**ML98**, VII.2 Corollary] states that every *formal diagram* in a monoidal category is commutative. A formal diagram is a diagram that involves only the associativity isomorphism, the unit isomorphisms, their inverses, identity morphisms, the monoidal product, and composites.
- A fourth version of the Coherence Theorem states that, for each category C, the unique strict monoidal functor from the free monoidal category generated by C to the free strict monoidal category generated by C is an equivalence of categories [**JS93**, Theorem 1.2].

Next we consider symmetric monoidal categories.

Definition 1.2.24. A *symmetric monoidal category* is a pair (C, ξ) in which:

- $\mathsf{C} = (\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ is a monoidal category as in Definition 1.2.1.
- ξ is a natural isomorphism

$$X \otimes Y \xrightarrow[\cong]{\xi_{X,Y}} Y \otimes X \tag{1.2.25}$$

  for objects $X, Y \in \mathsf{C}$, called the *symmetry isomorphism*.

These data are required to satisfy the following three axioms:

The Symmetry Axiom: The diagram

$$\begin{tikzcd} X \otimes Y \arrow[r, "\xi_{X,Y}"] \arrow[dr, equal] & Y \otimes X \arrow[d, "\xi_{Y,X}"] \\ & X \otimes Y \end{tikzcd} \tag{1.2.26}$$

is commutative for all objects $X, Y \in \mathsf{C}$.

The Unit Axiom: The diagram

$$\begin{tikzcd} X \otimes \mathbb{1} \arrow[r, "\xi_{X,\mathbb{1}}"] \arrow[d, "\rho_X"'] & \mathbb{1} \otimes X \arrow[d, "\lambda_X"] \\ X \arrow[r, equal] & X \end{tikzcd} \tag{1.2.27}$$

is commutative for all objects $X \in \mathsf{C}$.

The Hexagon Axiom: The diagram

$$\begin{tikzcd} & X \otimes (Z \otimes Y) \arrow[r, "X \otimes \xi_{Z,Y}"] & X \otimes (Y \otimes Z) \arrow[dr, "\alpha^{-1}"] & \\ (X \otimes Z) \otimes Y \arrow[ur, "\alpha"] \arrow[dr, "\xi_{X \otimes Z, Y}"'] & & & (X \otimes Y) \otimes Z \\ & Y \otimes (X \otimes Z) \arrow[r, "\alpha^{-1}"] & (Y \otimes X) \otimes Z \arrow[ur, "\xi_{Y,X} \otimes Z"'] & \end{tikzcd} \tag{1.2.28}$$

is commutative for all objects $X, Y, Z \in \mathsf{C}$.

A symmetric monoidal category is said to be *strict* if the underlying monoidal category is strict. ◇

Definition 1.2.29. A *commutative monoid* in a symmetric monoidal category (C, ξ) is a monoid $(X, \mu, 1)$ in C such that the multiplication μ is commutative in the sense that the diagram

$$\begin{tikzcd} X \otimes X \arrow[r, "\xi_{X,X}"] \arrow[d, "\mu"'] & X \otimes X \arrow[d, "\mu"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

is commutative. A morphism of commutative monoids is a morphism of the underlying monoids. The category of commutative monoids in C is denoted by $\mathsf{CMon}(\mathsf{C})$. ◇

Definition 1.2.30. For symmetric monoidal categories C and D, a *symmetric monoidal functor* $(F, F_2, F_0) : \mathsf{C} \longrightarrow \mathsf{D}$ is a monoidal functor between the underlying monoidal categories that is compatible with the symmetry isomorphisms, in the sense that the diagram

$$\begin{tikzcd} FX \otimes FY \arrow[r, "\xi_{FX,FY}", "\cong"'] \arrow[d, "F_2"'] & FY \otimes FX \arrow[d, "F_2"] \\ F(X \otimes Y) \arrow[r, "F\xi_{X,Y}", "\cong"'] & F(Y \otimes X) \end{tikzcd} \tag{1.2.31}$$

is commutative for all objects $X, Y \in \mathsf{C}$. A symmetric monoidal functor is said to be *strong* (resp., *strict*) if the underlying monoidal functor is so. ◇

The symmetric version of the Coherence Theorem 1.2.23 states that every symmetric monoidal category can be strictified to a strict symmetric monoidal category via an adjoint equivalence consisting of strong symmetric monoidal functors. The following variations from [**JS93**] are also true:

- Every *formal diagram* in a symmetric monoidal category is commutative. Here a formal diagram is defined as in the nonsymmetric case by allowing the symmetry isomorphism as well.
- The unique strict symmetric monoidal functor from the free symmetric monoidal category generated by a category C to the free strict symmetric monoidal category generated by C is an equivalence of categories.

Example 1.2.32. Here are some examples of symmetric monoidal categories:

- $(\mathsf{Set}, \times, *)$: The category of sets and functions. A monoid in Set is a monoid in the usual sense.
- $(\mathsf{Cat}, \times, \mathbf{1})$: The category of small categories and functors. Here $\mathbf{1}$ is a category with one object and only the identity morphism
- $(\mathsf{Hilb}, \widehat{\otimes}, \mathbb{C})$: The category of complex Hilbert spaces and bounded linear maps, with $\widehat{\otimes}$ the completed tensor product of Hilbert spaces [**Wei80**]. ◇

Definition 1.2.33. A symmetric monoidal category C is *closed* if for each object X, the functor

$$- \otimes X : \mathsf{C} \longrightarrow \mathsf{C}$$

admits a right adjoint, which is denoted by by $[X, -]$ and called the *internal hom*. ◇

Next we turn to braided monoidal categories:

Definition 1.2.34. A *braided monoidal category* is a pair (C, ξ) in which:

- $(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ is a monoidal category as in Definition 1.2.1.
- ξ is a natural isomorphism

$$X \otimes Y \xrightarrow[\cong]{\xi_{X,Y}} Y \otimes X \tag{1.2.35}$$

  for objects $X, Y \in \mathsf{C}$, called the *braiding*.

These data are required to satisfy the following axioms:

The Unit Axiom: The diagram

$$\begin{tikzcd} X \otimes \mathbb{1} \arrow[r, "\xi_{X,\mathbb{1}}"] \arrow[d, "\rho"'] & \mathbb{1} \otimes X \arrow[d, "\lambda"] \\ X \arrow[r, equal] & X \end{tikzcd} \tag{1.2.36}$$

is commutative for all objects $X \in \mathsf{C}$.

The Hexagon Axioms: The following two hexagon diagrams are required to be commutative for objects $X, Y, Z \in \mathsf{C}$.

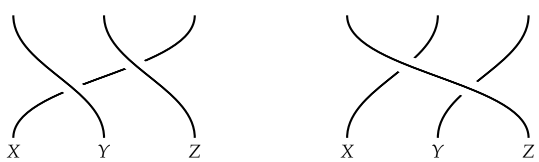
$$\begin{tikzcd}
& (Y \otimes X) \otimes Z \arrow[r, "\alpha"] & Y \otimes (X \otimes Z) \arrow[dr, "Y \otimes \xi_{X,Z}"] & \\
(X \otimes Y) \otimes Z \arrow[ur, "\xi_{X,Y} \otimes Z"] \arrow[dr, "\alpha"'] & & & Y \otimes (Z \otimes X) \\
& X \otimes (Y \otimes Z) \arrow[r, "\xi_{X,Y \otimes Z}"] & (Y \otimes Z) \otimes X \arrow[ur, "\alpha"'] &
\end{tikzcd} \tag{1.2.37}$$

$$\begin{tikzcd}
& X \otimes (Z \otimes Y) \arrow[r, "\alpha^{-1}"] & (X \otimes Z) \otimes Y \arrow[dr, "\xi_{X,Z} \otimes Y"] & \\
X \otimes (Y \otimes Z) \arrow[ur, "X \otimes \xi_{Y,Z}"] \arrow[dr, "\alpha^{-1}"'] & & & (Z \otimes X) \otimes Y \\
& (X \otimes Y) \otimes Z \arrow[r, "\xi_{X \otimes Y,Z}"] & Z \otimes (X \otimes Y) \arrow[ur, "\alpha^{-1}"'] &
\end{tikzcd} \tag{1.2.38}$$

A braided monoidal category is said to be *strict* if the underlying monoidal category is strict.

A *braided monoidal functor* is defined in the same way that a symmetric monoidal functor is, and the same is true for the *strong* and *strict* versions. ◇

Explanation 1.2.39. The two hexagon diagrams (1.2.37) and (1.2.38) may be visualized as the braids, read bottom to top,



in the braid group B_3 [**Art47**], with the braiding ξ interpreted as the generator ⤫ in the braid group B_2. On the left, the two strings labeled by Y and Z cross over the string labeled by X. The two composites along the boundary of the hexagon diagram (1.2.37) correspond to passing Y and Z over X, either one at a time or both at once. On the right, the string labeled by Z crosses over the two strings labeled by Y and X. The two composites along the boundary of (1.2.38) likewise correspond to the two ways of passing Z over X and Y. ◇

The braided version of the Coherence Theorem 1.2.23 states that every braided monoidal category can be strictified to a strict braided monoidal category via an adjoint equivalence consisting of strong braided monoidal functors. The following variations from [**JS93**] are also true:

- A *formal diagram*, defined as in the symmetric case with the braiding in place of the symmetry isomorphism, in a braided monoidal category is commutative if and only if composites with the same (co)domain have the same underlying braid.
- For each category C, the unique strict braided monoidal functor from the free braided monoidal category generated by C to the free strict braided monoidal category generated by C is an equivalence of categories.

1.3 Enriched Categories

In this section we recall some basic definitions regarding enriched categories, which will be useful when we discuss 2-categories. While a category has morphism sets, an enriched category has morphism objects in another category V. The composition, identity morphisms, associativity, and unity are all phrased in the category V. Fix a monoidal category $(\mathsf{V}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ as in Definition 1.2.1; then we can obtain the following definition.

Definition 1.3.1. A V-*category* C, also called a *category enriched in* V, consists of

- a class $\mathsf{Ob}(\mathsf{C})$ of objects in C;
- for each pair of objects X, Y in C, an object $\mathsf{C}(X, Y)$ in V, called the *hom object*, with domain X and codomain Y;
- for each triple of objects X, Y, Z in C, a morphism

$$\mathsf{C}(Y, Z) \otimes \mathsf{C}(X, Y) \xrightarrow{m_{XYZ}} \mathsf{C}(X, Z)$$

  in V, called the *composition*; and
- for each object X in C, a morphism

$$\mathbb{1} \xrightarrow{i_X} \mathsf{C}(X, X)$$

  in V, called the *identity* of X.

These data are required to make the *associativity diagram*

$$\begin{tikzcd}
\big(\mathsf{C}(Y,Z)\otimes\mathsf{C}(X,Y)\big)\otimes\mathsf{C}(W,X) \arrow[r, "\alpha"] \arrow[dd, "m\otimes 1"'] & \mathsf{C}(Y,Z)\otimes\big(\mathsf{C}(X,Y)\otimes\mathsf{C}(W,X)\big) \arrow[d, "1\otimes m"] \\
& \mathsf{C}(Y,Z)\otimes\mathsf{C}(W,Y) \arrow[d, "m"] \\
\mathsf{C}(X,Z)\otimes\mathsf{C}(W,X) \arrow[r, "m"] & \mathsf{C}(W,Z)
\end{tikzcd} \tag{1.3.2}$$

and the *unity diagram*

$$\begin{tikzcd}
\mathbb{1}\otimes\mathsf{C}(X,Y) \arrow[r, "\lambda"] \arrow[d, "i_Y\otimes 1"'] & \mathsf{C}(X,Y) \arrow[d, equal] & \mathsf{C}(X,Y)\otimes\mathbb{1} \arrow[l, "\rho"'] \arrow[d, "1\otimes i_X"] \\
\mathsf{C}(Y,Y)\otimes\mathsf{C}(X,Y) \arrow[r, "m"] & \mathsf{C}(X,Y) & \mathsf{C}(X,Y)\otimes\mathsf{C}(X,X) \arrow[l, "m"']
\end{tikzcd} \tag{1.3.3}$$

commute for objects W, X, Y, Z in C. This finishes the definition of a V-category. A V-category C is *small* if $\mathsf{Ob}(\mathsf{C})$ is a set. $\diamond$

Example 1.3.4. Here are some examples of enriched categories:

(1) A Set-category, for the symmetric monoidal category $(\mathsf{Set}, \times, *)$ of sets, is precisely a category in the usual sense.

(2) A Top-category, for the symmetric monoidal category $(\mathsf{Top}, \times, *)$ of topological spaces, is usually called a *topological category*. If we restrict to compactly generated Hausdorff spaces, then an example of a Top-category is Top itself. For two spaces X and Y, the set $\mathsf{Top}(X, Y)$ of continuous maps from X to Y is given the compact-open topology.

(3) An Ab-category, for the symmetric monoidal category $(\mathsf{Ab}, \otimes, \mathbb{Z})$ of abelian groups, is sometimes called a *pre-additive category* in the literature. Explicitly, an Ab-category C is a category in which each morphism set $\mathsf{C}(X, Y)$ is equipped with the structure of an abelian group such that composition distributes over addition, in the sense that

$$h(g_1 + g_2)f = hg_1f + hg_2f$$

when the compositions are defined.

(4) For a commutative ring R, suppose $(\mathsf{Ch}, \otimes_R, R)$ is the symmetric monoidal category of chain complexes of R-modules. A Ch-category is usually called a *differential graded category*.
(5) A symmetric monoidal closed category V becomes a V-category with hom objects the internal hom $[X, Y]$ for objects X, Y in V. The composition m is induced by the adjunction between $- \otimes X$ and $[X, -]$. The identity i_X is adjoint to the left unit isomorphism $\lambda_X : \mathbb{1} \otimes X \cong X$.
(6) We will see in Chapter 2 that Cat-categories are locally small 2-categories.

Although the definition of a V-category does not require V to be symmetric, in practice V is often a symmetric monoidal category. ◇

Next we recall functors, natural transformations, adjunctions, and monads in the enriched setting. In the next few definitions, the reader will notice that we recover the usual notions in Section 1.1 when $\mathsf{V} = \mathsf{Set}$.

Definition 1.3.5. Suppose C and D are V-categories. A V*-functor* $F : \mathsf{C} \longrightarrow \mathsf{D}$ consists of

- an assignment on objects

$$\mathsf{Ob}(\mathsf{C}) \longrightarrow \mathsf{Ob}(\mathsf{D}), \qquad X \longmapsto FX; \quad \text{and}$$

- for each pair of objects X, Y in C, a morphism

$$\mathsf{C}(X, Y) \xrightarrow{F_{XY}} \mathsf{D}\big(FX, FY\big)$$

  in V.

These data are required to satisfy the following two conditions:

Composition: For each triple of objects X, Y, Z in C, the diagram

$$\begin{tikzcd}
\mathsf{C}(Y,Z) \otimes \mathsf{C}(X,Y) \arrow[r, "m"] \arrow[d, "F\otimes F"'] & \mathsf{C}(X,Z) \arrow[d, "F"] \\
\mathsf{D}(FY,FZ) \otimes \mathsf{D}(FX,FY) \arrow[r, "m"] & \mathsf{D}(FX,FZ)
\end{tikzcd}$$

in V is commutative.

Identities: For each object $X \in \mathsf{C}$, the diagram

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "i_X"] \arrow[d, equal] & \mathsf{C}(X,X) \arrow[d, "F"] \\
\mathbb{1} \arrow[r, "i_{FX}"] & \mathsf{D}(FX,FX)
\end{tikzcd}$$

in V is commutative.

Moreover:

- For V-functors $F : \mathsf{C} \longrightarrow \mathsf{D}$ and $G : \mathsf{D} \longrightarrow \mathsf{E}$, their composition

$$GF : \mathsf{C} \longrightarrow \mathsf{E}$$

is the V-functor defined by composing the assignments on objects and forming the composite

$$(GF)_{XY} = G_{FX,FY}F_{XY} : \mathsf{C}(X,Y) \longrightarrow \mathsf{E}(GFX,GFY)$$

in V on hom objects.

- The *identity* V*-functor* of C, denoted $1_{\mathsf{C}} : \mathsf{C} \longrightarrow \mathsf{C}$, is given by the identity map on $\mathsf{Ob}(\mathsf{C})$ and the identity morphism $1_{\mathsf{C}(X,Y)}$ for objects X,Y in C. $\diamond$

Definition 1.3.6. Suppose $F,G : \mathsf{C} \longrightarrow \mathsf{D}$ are V-functors between V-categories C and D.

(1) A V-*natural transformation* $\theta : F \longrightarrow G$ consists of a morphism

$$\theta_X : \mathbb{1} \longrightarrow \mathsf{D}(FX,GX)$$

in V, called a *component* of θ, for each object X in C, such that the diagram

$$\begin{tikzcd}
\mathsf{C}(X,Y) \arrow[r, "\lambda^{-1}", "\cong"'] \arrow[d, "\rho^{-1}"', "\cong"] & \mathbb{1} \otimes \mathsf{C}(X,Y) \arrow[d, "\theta_Y \otimes F"] \\
\mathsf{C}(X,Y) \otimes \mathbb{1} \arrow[d, "G \otimes \theta_X"'] & \mathsf{D}(FY,GY) \otimes \mathsf{D}(FX,FY) \arrow[d, "m"] \\
\mathsf{D}(GX,GY) \otimes \mathsf{D}(FX,GX) \arrow[r, "m"] & \mathsf{D}(FX,GY)
\end{tikzcd} \tag{1.3.7}$$

is commutative for objects X,Y in C.

(2) The *identity* V*-natural transformation of* F, denoted by $1_F : F \longrightarrow F$, is defined by the component

$$(1_F)_X = i_{FX} : \mathbb{1} \longrightarrow \mathsf{D}(FX, FX)$$

for each object X in C. ◇

As for natural transformations, there are two types of compositions for V-natural transformations.

Definition 1.3.8. Suppose $\theta : F \longrightarrow G$ is a V-natural transformation for V-functors $F, G : \mathsf{C} \longrightarrow \mathsf{D}$.

(1) Suppose $\phi : G \longrightarrow H$ is another V-natural transformation for a V-functor $H : \mathsf{C} \longrightarrow \mathsf{D}$. The *vertical composition*

$$\phi\theta : F \longrightarrow H$$

is the V-natural transformation whose component $(\phi\theta)_X$ is the composite

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "(\phi\theta)_X"] \arrow[d, "\lambda^{-1}"', "\cong"] & \mathsf{D}(FX, HX) \\
\mathbb{1} \otimes \mathbb{1} \arrow[r, "\phi_X \otimes \theta_X"] & \mathsf{D}(GX, HX) \otimes \mathsf{D}(FX, GX) \arrow[u, "m"']
\end{tikzcd}$$

in V for each object X in C.

(2) Suppose $\theta' : F' \longrightarrow G'$ is a V-natural transformation for V-functors $F', G' : \mathsf{D} \longrightarrow \mathsf{E}$ with E a V-category. The *horizontal composition*

$$\theta' * \theta : F'F \longrightarrow G'G$$

is the V-natural transformation whose component $(\theta' * \theta)_X$, for an object X in C, is defined as the composite

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "(\theta' * \theta)_X"] \arrow[dd, "\lambda^{-1}"', "\cong"] & \mathsf{E}(F'FX, G'GX) \\
& \mathsf{E}(F'GX, G'GX) \otimes \mathsf{E}(F'FX, F'GX) \arrow[u, "m"'] \\
\mathbb{1} \otimes \mathbb{1} \arrow[r, "\theta'_{GX} \otimes \theta_X"] & \mathsf{E}(F'GX, G'GX) \otimes \mathsf{D}(FX, GX) \arrow[u, "1 \otimes F'"']
\end{tikzcd} \tag{1.3.9}$$

in V. ◇

For ordinary categories, adjunctions can be characterized in terms of the unit, the counit, and the triangle identities (1.1.13). In the enriched setting, we use the triangle identities as the definition for an adjunction.

Definition 1.3.10. Suppose C and D are V-categories, and suppose $L : \mathsf{C} \longrightarrow \mathsf{D}$ and $R : \mathsf{D} \longrightarrow \mathsf{C}$ are V-functors. A V-*adjunction* $L \dashv R$ consists of

- a V-natural transformation $\eta : 1_{\mathsf{C}} \longrightarrow RL$, which is called the *unit*, and
- a V-natural transformation $\varepsilon : LR \longrightarrow 1_{\mathsf{D}}$, which is called the *counit*,

such that the diagrams

$$\begin{tikzcd} & RLR \arrow[dr, "1_R * \varepsilon"] & \\ R \arrow[ur, "\eta * 1_R"] \arrow[rr, "1_R"] & & R \end{tikzcd} \qquad \begin{tikzcd} & LRL \arrow[dr, "\varepsilon * 1_L"] & \\ L \arrow[ur, "1_L * \eta"] \arrow[rr, "1_L"] & & L \end{tikzcd}$$

commute. In this case, L is called the *left adjoint*, and R is called the *right adjoint*. ◇

Definition 1.3.11. Suppose $F, G : \mathsf{C} \longrightarrow \mathsf{D}$ are V-functors.

(1) A V-natural transformation $\theta : F \longrightarrow G$ is called a V-*natural isomorphism* if there exists a V-natural transformation $\theta^{-1} : G \longrightarrow F$ such that the equalities

$$\theta^{-1}\theta = 1_F \quad \text{and} \quad \theta\theta^{-1} = 1_G$$

hold.

(2) F is called a V-*equivalence* if there exist
- a V-functor $F' : \mathsf{D} \longrightarrow \mathsf{C}$ and
- V-natural isomorphisms $\eta : 1_{\mathsf{C}} \xrightarrow{\cong} F'F$ and $\varepsilon : FF' \xrightarrow{\cong} 1_{\mathsf{D}}$.

◇

Definition 1.3.12. A V-*monad* in a V-category C is a triple (T, μ, η) consisting of

- a V-functor $T : \mathsf{C} \longrightarrow \mathsf{C}$,
- a V-natural transformation $\mu : T^2 \longrightarrow T$, which is called the *multiplication*, and
- a V-natural transformation $\eta : 1_{\mathsf{C}} \longrightarrow T$, which is called the *unit*,

such that the associativity and unity diagrams

$$\begin{tikzcd} T^3 \arrow[r, "1_T*\mu"] \arrow[d, "\mu*1_T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} 1_{\mathsf{C}}T \arrow[r, "\eta*1_T"] \arrow[d, equal] & T^2 \arrow[d, "\mu"'] & T1_{\mathsf{C}} \arrow[l, "1_T*\eta"'] \arrow[d, equal] \\ T \arrow[r, equal] & T \arrow[r, equal] & T \end{tikzcd}$$

are commutative. We often refer to such a monad as simply T. ◇

Definition 1.3.13. Suppose (T, μ, η) is a V-monad in a V-category C.

(1) A T-*algebra* is a pair (X, θ) consisting of
- an object X in C and
- a morphism $\theta : \mathbb{1} \longrightarrow \mathsf{C}(TX, X)$ in V, which is called the *structure morphism*,

such that the associativity diagram

$$\begin{tikzcd} \mathbb{1} \arrow[rr, "\lambda^{-1}", "\cong"'] \arrow[d, "\lambda^{-1}"', "\cong"] & & \mathbb{1} \otimes \mathbb{1} \arrow[d, "\theta\otimes\mu_X"] \\ \mathbb{1} \otimes \mathbb{1} \arrow[d, "\theta\otimes\theta"'] & & \mathsf{C}(TX, X) \otimes \mathsf{C}(T^2X, TX) \arrow[dd, "m"] \\ \mathsf{C}(TX, X) \otimes \mathsf{C}(TX, X) \arrow[d, "1\otimes T"'] & & \\ \mathsf{C}(TX, X) \otimes \mathsf{C}(T^2X, TX) \arrow[rr, "m"] & & \mathsf{C}(T^2X, X) \end{tikzcd}$$

and the unity diagram

$$\begin{tikzcd} \mathbb{1} \arrow[r, "i_X"] \arrow[d, "\lambda^{-1}"', "\cong"] & \mathsf{C}(X, X) \\ \mathbb{1} \otimes \mathbb{1} \arrow[r, "\theta\otimes\eta_X"] & \mathsf{C}(TX, X) \otimes \mathsf{C}(X, TX) \arrow[u, "m"'] \end{tikzcd}$$

are commutative.

(2) For T-algebras (X,θ^X) and (Y,θ^Y), a *morphism of T-algebras*

$$f : (X,\theta^X) \longrightarrow (Y,\theta^Y)$$

is a morphism $f : \mathbb{1} \longrightarrow \mathsf{C}(X,Y)$ in V such that the diagram

$$\begin{tikzcd}
\mathbb{1} \arrow[rr, "\lambda^{-1}", "\cong"'] \arrow[d, "\lambda^{-1}"', "\cong"] & & \mathbb{1}\otimes\mathbb{1} \arrow[r, "\theta^Y\otimes f"] & \mathsf{C}(TY,Y)\otimes\mathsf{C}(X,Y) \arrow[d, "1\otimes T"] \\
\mathbb{1}\otimes\mathbb{1} \arrow[d, "f\otimes\theta^X"'] & & & \mathsf{C}(TY,Y)\otimes\mathsf{C}(TX,TY) \arrow[d, "m"] \\
\mathsf{C}(X,Y)\otimes\mathsf{C}(TX,X) \arrow[rrr, "m"] & & & \mathsf{C}(TX,Y)
\end{tikzcd}$$

is commutative. ◇

1.4 Exercises and Notes

Exercise 1.4.1. Check that the vertical composition of two natural transformations, when it is defined, is actually a natural transformation and that vertical composition is associative and unital. Do the same for horizontal composition.

Exercise 1.4.2. Repeat the previous exercise for V-natural transformations for a monoidal category V.

Exercise 1.4.3. Suppose $\theta : F \longrightarrow G$ is a natural transformation. Prove that θ is a natural isomorphism if and only if there exists a unique natural transformation $\phi : G \longrightarrow F$ such that $\phi\theta = 1_F$ and $\theta\phi = 1_G$.

Exercise 1.4.4. For an adjunction $L \dashv R$, prove the triangle identities (1.1.13).

Exercise 1.4.5. Prove the alternative characterization of an adjunction stated at the end of the paragraph containing (1.1.12).

Exercise 1.4.6. Prove that, for a functor F, the following statements are equivalent:

(i) F is part of an adjoint equivalence.
(ii) F is an equivalence.
(iii) F is both fully faithful and essentially surjective.

Exercise 1.4.7. Prove that adjunctions can be composed.

Exercise 1.4.8. Prove the Yoneda Lemma 1.1.16.

Exercise 1.4.9. Prove that the limit of a functor, if it exists, is unique up to a unique isomorphism. Do the same for the colimit.

Exercise 1.4.10. Prove that a left adjoint preserves colimits and that a right adjoint preserves limits.

Exercise 1.4.11. Suppose C is a monoidal category, except that the axiom $\lambda_{\mathbb{1}} = \rho_{\mathbb{1}}$ is not assumed. Prove that this axiom follows from the unity axiom (1.2.4) and the pentagon axiom (1.2.5).

Exercise 1.4.12. Prove that the unity diagrams (1.2.8) are commutative in a monoidal category.

Exercise 1.4.13. In Example 1.2.9, check that $\mathsf{C}^{\mathsf{rev}}$ satisfies the pentagon axiom.

Exercise 1.4.14. Prove that each monoidal functor $(F, F_2, F_0) : \mathsf{C} \longrightarrow \mathsf{D}$ induces a functor

$$\mathsf{Mon}(\mathsf{C}) \xrightarrow{F} \mathsf{Mon}(\mathsf{D})$$

that sends a monoid $(X, \mu, 1)$ in C to the monoid $\left(FX, \mu^{FX}, 1^{FX}\right)$ in D, with unit the composite

$$\mathbb{1}^{\mathsf{D}} \xrightarrow{F_0} F\mathbb{1}^{\mathsf{C}} \xrightarrow{F1} FX$$

and multiplication the composite

$$FX \otimes FX \xrightarrow{F_2} F(X \otimes X) \xrightarrow{F\mu} FX.$$

In other words, monoidal functors preserve monoids.

Exercise 1.4.15. Repeat the previous exercise for a symmetric monoidal functor. In other words, prove that each symmetric monoidal functor $(F, F_2, F_0) : \mathsf{C} \longrightarrow \mathsf{D}$ induces a functor defined as in the previous exercise,

$$\mathsf{CMon}(\mathsf{C}) \xrightarrow{F} \mathsf{CMon}(\mathsf{D}),$$

between the categories of commutative monoids.

Exercise 1.4.16. Suppose that G is a group, and M is a G-module. A *normalized* 3-*cocycle* for G with coefficients in M is a function $h : G^3 \longrightarrow M$ such that the following two equalities hold in M for all x, y, z, w in G:

$$h(x,1,y) = 0$$
$$w \cdot h(x,y,z) + h(w,xy,z) + h(w,x,y) = h(w,x,yz) + h(wx,y,z).$$

Given such an h, define a category $T = T(G,M,h)$ as follows. The objects of T are given by the elements of G. For each x, the set of endomorphisms $T(x,x) = M$, and for $x \neq y$ the morphism set $T(x,y)$ is empty. Identities and composition (of endomorphisms) are given by the identity and addition in M.

Show that the following structure makes T a monoidal category:

- The product of objects is given by multiplication in G.
- The product of morphisms $p : x \longrightarrow x$ and $q : y \longrightarrow y$ is given by

$$p + x \cdot q : xy \longrightarrow xy.$$

- The unit element of G is the monoidal unit, and the unit isomorphisms are trivial.
- The associativity isomorphism components $(xy)z \longrightarrow x(yz)$ are defined to be $h(x,y,z)$.

Exercise 1.4.17. Suppose C is a small category. Prove that monads on C are precisely the monoids in a certain strict monoidal category.

Exercise 1.4.18. Repeat the previous exercise for V-monads on a small V-category C.

Exercise 1.4.19. Show that the composite (1.3.9) that defines the component $(\theta' * \theta)_X$ of the horizontal composition is equal to the following composite.

$$\begin{tikzcd}
\mathbb{1} \arrow[dd, "\lambda^{-1}"', "\cong"] & \mathsf{E}(F'FX, G'GX) \\
 & \mathsf{E}(G'FX, G'GX) \otimes \mathsf{E}(F'FX, G'FX) \arrow[u, "m"'] \\
\mathbb{1} \otimes \mathbb{1} \arrow[r, "\theta_X \otimes \theta'_{FX}"] & \mathsf{D}(FX, GX) \otimes \mathsf{E}(F'FX, G'FX) \arrow[u, "G' \otimes 1"']
\end{tikzcd}$$

Exercise 1.4.20. For a V-monad T in a V-category C, show that T-algebras and their morphisms form a category.

Notes

1.4.21 (General References). For more detailed discussion of basic category theory, we refer the reader to the introductory books [**Awo10**, **Gra18**, **Lei14**, **Rie16**, **Rom17**, **Sim11**]. $\diamond$

1.4.22 (Set-Theoretic Foundations). Our set-theoretic convention using Grothendieck universes is from [**AGV72**]. For more discussion of set-theoretic foundation in the context of category theory, the reader is referred to [**ML69**, **Shu∞b**, **Low∞**]. $\diamond$

1.4.23 (The Yoneda Embedding). In the literature, the Yoneda embedding of an object A is sometimes denoted by h_A. We chose the symbol $\mathcal{Y}_A$ to make it easier for the reader to remember that $\mathcal{Y}$ stands for Yoneda. $\diamond$

1.4.24 (Monads). For further discussion of monads, the reader may consult [**BW05**, **Bor94b**, **God58**, **ML98**, **Rie16**]. Monads are also called *triples* and *standard constructions* in the literature. $\diamond$

1.4.25 (Monoidal Categories and Functors). What we call a strict symmetric monoidal category is sometimes called a *permutative category* in the literature. What we call a (symmetric/braided) monoidal category is what Joyal and Street [**JS93**] called a *(symmetric/braided) tensor category*. A monoidal functor is sometimes called a *lax monoidal functor* in the literature, to emphasize the condition that the morphisms F_2 and F_0 are not necessarily invertible. A strong monoidal functor is also known as a *tensor functor*. Discussion of monoidal categories and their coherence can be found in [**JS93**, **Kel64**, **ML63**, **ML98**, **Yau∞**]. Exercise 1.4.16 appears in one of the works by Joyal and Street [**JS86**, Section 6]. $\diamond$

1.4.26 (Enriched Categories). The standard comprehensive reference for enriched category theory is Kelly's book, [**Kel05**]. Some discussion can also be found in [**Bor94b**, Chapter 6]. For the theory of enriched monads, the reader is referred to [**BKP89**, **LS02**, **Str72a**]. $\diamond$

2
Bicategories and 2-Categories

In this chapter we define bicategories and 2-categories. The definition of a bicategory and a series of examples are given in Section 2.1. Several useful unity properties in bicategories are presented in Section 2.2. The definition of a 2-category and a series of examples are given in Section 2.3. In Sections 2.4 and 2.5 we discuss the 2-categories of multicategories and polycategories, generalizing the 2-category of small categories, functors, and natural transformations. Dualities of bicategories are discussed in Section 2.6.

2.1 Bicategories

In this section we give a detailed definition of a bicategory, and some examples.

Convention 2.1.1. Recall from Notation 1.1.21 that $\mathbf{1}$ denotes the category with one object $*$ and only its identity morphism. For a category C, we usually identify the categories $\mathsf{C} \times \mathbf{1}$ and $\mathbf{1} \times \mathsf{C}$ with C and regard the canonical isomorphisms between them as 1_{C}. For an object X in C, its identity morphism 1_X is also denoted by X. ◇

Motivation 2.1.2. As we pointed out in Example 1.2.12, a monoid $(X, \mu, 1)$ in Set may be regarded as a category ΣX with one object $*$, morphism set $\Sigma X(*, *) = X$, identity morphism $1_* = 1$, and composition μ. The associativity and unity axioms of the monoid X become the associativity and unity axioms of the category ΣX. So a category is a multi-object version of a monoid. In a similar way, a bicategory, to be defined shortly, is a multi-object version of a monoidal category as in Definition 1.2.1. ◇

Definition 2.1.3. A *bicategory* is a tuple

$$\big(\mathsf{B}, 1, c, a, \ell, r\big)$$

2-Dimensional Categories. Niles Johnson and Donald Yau, Oxford University Press (2021).
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consisting of the following data:

Objects: B is equipped with a class $\mathsf{Ob}(\mathsf{B}) = \mathsf{B}_0$, whose elements are called *objects* or 0-*cells* in B. If $X \in \mathsf{B}_0$, we also write $X \in \mathsf{B}$.

Hom Categories: For each pair of objects $X, Y \in \mathsf{B}$, B is equipped with a category $\mathsf{B}(X, Y)$, called a *hom category*:

- Its objects are called 1-*cells* in B. The collection of all the 1-cells in B is denoted by B_1.
- Its morphisms are called 2-*cells* in B. The collection of all the 2-cells in B is denoted by B_2.
- Composition and identity morphisms in the category $\mathsf{B}(X, Y)$ are called *vertical composition* and *identity* 2-*cells*, respectively.
- An isomorphism in $\mathsf{B}(X, Y)$ is called an *invertible* 2-*cell*, and its inverse is called a *vertical inverse*.
- For a 1-cell f, its identity 2-cell is denoted by 1_f.

Identity 1-Cells: For each object $X \in \mathsf{B}$,

$$1_X : \mathbf{1} \longrightarrow \mathsf{B}(X, X)$$

is a functor. We identify the functor 1_X with the 1-cell $1_X(*) \in \mathsf{B}(X, X)$, called the *identity* 1-*cell of* X.

Horizontal Composition: For each triple of objects $X, Y, Z \in \mathsf{B}$,

$$c_{XYZ} : \mathsf{B}(Y, Z) \times \mathsf{B}(X, Y) \longrightarrow \mathsf{B}(X, Z)$$

is a functor, called the *horizontal composition*. For 1-cells $f \in \mathsf{B}(X, Y)$ and $g \in \mathsf{B}(Y, Z)$, and 2-cells $\alpha \in \mathsf{B}(X, Y)$ and $\beta \in \mathsf{B}(Y, Z)$, we use the notations

$$\begin{aligned} c_{XYZ}(g, f) &= g \circ f \quad \text{or} \quad gf, \\ c_{XYZ}(\beta, \alpha) &= \beta * \alpha. \end{aligned}$$

Associator: For objects $W, X, Y, Z \in \mathsf{B}$,

$$a_{WXYZ} : c_{WXZ}\big(c_{XYZ} \times \mathrm{Id}_{\mathsf{B}(W,X)}\big) \longrightarrow c_{WYZ}\big(\mathrm{Id}_{\mathsf{B}(Y,Z)} \times c_{WXY}\big)$$

is a natural isomorphism, called the *associator*, between functors

$$\mathsf{B}(Y,Z) \times \mathsf{B}(X,Y) \times \mathsf{B}(W,X) \longrightarrow \mathsf{B}(W,Z).$$

Unitors: For each pair of objects $X, Y \in \mathsf{B}$,

$$c_{XYY}\big(1_Y \times \mathrm{Id}_{\mathsf{B}(X,Y)}\big) \xrightarrow{\ell_{XY}} \mathrm{Id}_{\mathsf{B}(X,Y)} \xleftarrow{r_{XY}} c_{XXY}\big(\mathrm{Id}_{\mathsf{B}(X,Y)} \times 1_X\big)$$

are natural isomorphisms, called the *left unitor* and the *right unitor*, respectively.

The subscripts in c will often be omitted. The subscripts in a, ℓ, and r will often be used to denote their components. The above data are required to satisfy the following two axioms for 1-cells $f \in \mathsf{B}(V,W)$, $g \in \mathsf{B}(W,X)$, $h \in \mathsf{B}(X,Y)$, and $k \in \mathsf{B}(Y,Z)$:

The Unity Axiom: The middle unity diagram

$$\begin{tikzcd}
(g1_W)f \arrow[rr, "a"] \arrow[dr, "r_g * 1_f"'] & & g(1_W f) \arrow[dl, "1_g * \ell_f"] \\
& gf &
\end{tikzcd} \tag{2.1.4}$$

in $\mathsf{B}(V,X)$ is commutative.

The Pentagon Axiom: The diagram

$$\begin{tikzcd}
& (kh)(gf) \arrow[dr, "a_{k,h,gf}"] & \\
((kh)g)f \arrow[ur, "a_{kh,g,f}"] \arrow[d, "a_{k,h,g} * 1_f"'] & & k(h(gf)) \\
(k(hg))f \arrow[r, "a_{k,hg,f}"] & k((hg)f) \arrow[ur, "1_k * a_{h,g,f}"'] &
\end{tikzcd} \tag{2.1.5}$$

in $\mathsf{B}(V,Z)$ is commutative.

This finishes the definition of a bicategory. ◇

Explanation 2.1.6. We usually abbreviate a bicategory as above to B.

(1) We assume the hom categories $\mathsf{B}(X, Y)$ for objects $X, Y \in \mathsf{B}$ are disjoint. If not, we tacitly replace them with their disjoint union.

(2) In each hom category $\mathsf{B}(X, Y)$, the vertical composition of 2-cells is associative and unital in the strict sense. In other words, for 1-cells f, f', f'', and f''' in $\mathsf{B}(X, Y)$, and 2-cells $\alpha : f \longrightarrow f'$, $\alpha' : f' \longrightarrow f''$, and $\alpha'' : f'' \longrightarrow f'''$, the equalities

$$\begin{aligned} (\alpha''\alpha')\alpha &= \alpha''(\alpha'\alpha), \\ \alpha = \alpha 1_f &= 1_{f'}\alpha \end{aligned} \tag{2.1.7}$$

hold.

(3) For 1-cells $f, f' \in \mathsf{B}(X, Y)$, we display each 2-cell $\alpha : f \longrightarrow f'$ in diagrams as

$$\begin{tikzcd} X \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "f'"'{name=D}] & Y \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

with a double arrow for the 2-cell. With this notation, the horizontal composition c_{XYZ} is the assignment

$$\begin{tikzcd} X \arrow[r, bend left, "f"{name=U1}] \arrow[r, bend right, "f'"'{name=D1}] & Y \arrow[r, bend left, "g"{name=U2}] \arrow[r, bend right, "g'"'{name=D2}] & Z \arrow[from=U1, to=D1, phantom, "\Downarrow\alpha"] \arrow[from=U2, to=D2, phantom, "\Downarrow\beta"] \end{tikzcd} \longmapsto \begin{tikzcd} X \arrow[r, bend left, "gf"{name=U}] \arrow[r, bend right, "g'f'"'{name=D}] & Z \arrow[from=U, to=D, phantom, "\Downarrow\beta * \alpha"] \end{tikzcd}$$

for 1-cells $f, f' \in \mathsf{B}(X, Y)$, $g, g' \in \mathsf{B}(Y, Z)$, and 2-cells $\alpha : f \longrightarrow f'$, $\beta : g \longrightarrow g'$.

(4) The fact that the horizontal composition c_{XYZ} is a functor means:

  (a) It preserves identity 2-cells, that is,

$$1_g * 1_f = 1_{gf} \tag{2.1.8}$$

  in $\mathsf{B}(X, Z)(gf, gf)$.

  (b) It preserves vertical composition, that is,

$$(\beta'\beta) * (\alpha'\alpha) = (\beta' * \alpha')(\beta * \alpha) \tag{2.1.9}$$

  in $\mathsf{B}(X, Z)(gf, g''f'')$ for 1-cells $f'' \in \mathsf{B}(X, Y)$, $g'' \in \mathsf{B}(Y, Z)$, and 2-cells $\alpha' : f' \longrightarrow f''$, $\beta' : g' \longrightarrow g''$.

The equality (2.1.9) is called the *middle four exchange*. It may be visualized as the equality of the two ways to compose the diagram

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"] \arrow[r, "f'" description] \arrow[r, bend right=50, "f''"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\alpha'"] & Y \arrow[r, bend left=50, "g"] \arrow[r, "g'" description] \arrow[r, bend right=50, "g''"'] \arrow[r, phantom, shift left=3, "\Downarrow\beta"] \arrow[r, phantom, shift right=3, "\Downarrow\beta'"] & Z \end{tikzcd}$$

down to a single 2-cell.

(5) Horizontal composition is associative up to the specified natural isomorphism a. So for 1-cells $f \in \mathsf{B}(W,X)$, $g \in \mathsf{B}(X,Y)$, and $h \in \mathsf{B}(Y,Z)$, the component of a is an invertible 2-cell

$$a_{h,g,f} : (hg)f \xrightarrow{\cong} h(gf) \tag{2.1.10}$$

in $\mathsf{B}(W,Z)$. The naturality of a means that, for 2-cells $\alpha : f \longrightarrow f'$, $\beta : g \longrightarrow g'$, and $\gamma : h \longrightarrow h'$, the diagram

$$\begin{tikzcd} (hg)f \arrow[r, "a_{h,g,f}"] \arrow[d, "(\gamma*\beta)*\alpha"'] & h(gf) \arrow[d, "\gamma*(\beta*\alpha)"] \\ (h'g')f' \arrow[r, "a_{h',g',f'}"] & h'(g'f') \end{tikzcd} \tag{2.1.11}$$

in $\mathsf{B}(W,Z)$ is commutative.

(6) Similarly, horizontal composition is unital with respect to the identity 1-cells up to the specified natural isomorphisms ℓ and r. So for each 1-cell $f \in \mathsf{B}(X,Y)$, their components are invertible 2-cells

$$\ell_f : 1_Y f \xrightarrow{\cong} f \quad \text{and} \quad r_f : f1_X \xrightarrow{\cong} f \tag{2.1.12}$$

in $\mathsf{B}(X,Y)$. The naturality of ℓ and r means the diagram

$$\begin{tikzcd} 1_Y f \arrow[r, "\ell_f"] \arrow[d, "1_{1_Y}*\alpha"'] & f \arrow[d, "\alpha"] & f1_X \arrow[l, "r_f"'] \arrow[d, "\alpha*1_{1_X}"] \\ 1_Y f' \arrow[r, "\ell_{f'}"] & f' & f'1_X \arrow[l, "r_{f'}"'] \end{tikzcd} \tag{2.1.13}$$

is commutative for each 2-cell $\alpha : f \longrightarrow f'$.

(7) The unity axiom (2.1.4) asserts the equality of 2-cells

$$r * 1_f = (1_g * \ell)a \in \mathsf{B}(V,X)\big((g1_W)f, gf\big).$$

The right-hand side is the vertical composition of a component of the associator a with the horizontal composition $1_g * \ell$.

(8) Similarly, the pentagon axiom (2.1.5) is an equality of 2-cells in the set

$$\mathsf{B}(V,Z)\big(((kh)g)f, k(h(gf))\big).$$

One of the 2-cells is the vertical composition of two instances of the associator a. The other 2-cell is the vertical composition of the 2-cells

$$a * 1_f, \qquad a, \quad \text{and} \quad 1_k * a,$$

the first and the last of which are horizontal compositions. ◇

Definition 2.1.14. Suppose P is a property of categories. A bicategory B is *locally* P if each hom category in B has property P. In particular, B is

- *locally small* if each hom category is a small category,
- *locally discrete* if each hom category is discrete, and
- *locally partially ordered* if each hom category is a partially ordered set regarded as a small category.

Finally, B is *small* if it is locally small and if B_0 is a set. ◇

Definition 2.1.15. Suppose B and B' are bicategories. Then B' is called a *sub-bicategory* of B if the following statements hold:

- B'_0 is a subclass of B_0.
- For objects $X, Y \in \mathsf{B}'$, $\mathsf{B}'(X,Y)$ is a subcategory of $\mathsf{B}(X,Y)$.
- The identity 1-cell of X in B' is equal to the identity 1-cell of X in B.
- For objects X, Y, Z in B', the horizontal composition c'_{XYZ} in B' makes the diagram

$$\begin{tikzcd}
\mathsf{B}'(Y,Z) \times \mathsf{B}'(X,Y) \arrow[r, "c'_{XYZ}"] \arrow[d] & \mathsf{B}'(X,Z) \arrow[d] \\
\mathsf{B}(Y,Z) \times \mathsf{B}(X,Y) \arrow[r, "c_{XYZ}"] & \mathsf{B}(X,Z)
\end{tikzcd}$$

commutative, in which the unnamed arrows are subcategory inclusions.

- Every component of the associator in B' is equal to the corresponding component of the associator in B, and similarly for the left unitors and the right unitors.

This finishes the definition of a sub-bicategory. ◇

The following special cases of the horizontal composition, in which one of the 2-cells is an identity 2-cell of some 1-cell, will come up often.

Definition 2.1.16. In a bicategory B, suppose given 1-cells $h \in \mathsf{B}(W,X)$, $f, f' \in \mathsf{B}(X,Y)$, $g \in \mathsf{B}(Y,Z)$, and a 2-cell $\alpha : f \longrightarrow f'$, as in the diagram

$$\begin{tikzcd} W \arrow[r, "h"] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Then the horizontal compositions $\alpha * 1_h$ and $1_g * \alpha$ are called the *whiskering of h and α*, and the *whiskering of α and g*, respectively. ◇

Explanation 2.1.17. The whiskering $\alpha * 1_h$ is a 2-cell $fh \longrightarrow f'h$ in $\mathsf{B}(W,Y)$. The whiskering $1_g * \alpha$ is a 2-cell $gf \longrightarrow gf'$ in $\mathsf{B}(X,Z)$. ◇

The rest of this section contains examples of bicategories.

Example 2.1.18 (Categories)**.** Categories are identified with locally discrete bicategories. Indeed, in each category C, each morphism set $\mathsf{C}(X,Y)$ may be regarded as a discrete category, that is, there are only identity 2-cells.

- The identity 1-cells are the identity morphisms in C.
- The horizontal composition of 1-cells is the composition in C.
- The horizontal composition and the vertical composition of identity 2-cells yield identity 2-cells.
- The natural isomorphisms a, ℓ, and r are defined as the identity natural transformations.

We write C_{bi} for this locally discrete bicategory.

Conversely, for a locally discrete bicategory B, the natural isomorphisms a, ℓ, and r are identities by (2.1.10) and (2.1.12). So the identification above yields a category $(\mathsf{B}_0, \mathsf{B}_1, 1, c)$. ◇

Example 2.1.19 (Monoidal Categories)**.** Monoidal categories are canonically identified with one-object bicategories. Indeed, suppose $(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ is a monoidal category as in Definition 1.2.1. Then it yields a bicategory $\Sigma\mathsf{C}$ with

- one object $*$,
- hom category $\Sigma\mathsf{C}(*, *) = \mathsf{C}$,
- identity 1-cell $1_* = \mathbb{1}$,
- horizontal composition $c = \otimes : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$,
- associator $a = \alpha$, and
- left unitor $\ell = \lambda$ and right unitor $r = \rho$.

The unity axiom (2.1.4) and the pentagon axiom (2.1.5) in $\Sigma\mathsf{C}$ are those of the monoidal category C in (1.2.4) and (1.2.5), respectively.

Conversely, for a bicategory B with one object $*$, the hom category $\mathsf{B}(*, *)$, along with the identification in the previous paragraph, is a monoidal category. ◇

Example 2.1.20 (Hom Monoidal Categories)**.** For each object X in a bicategory B, the hom category $\mathsf{C} = \mathsf{B}(X, X)$ is a monoidal category with

- monoidal unit $\mathbb{1} = 1_X$,
- monoidal product $\otimes = c_{XXX} : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$,
- associativity isomorphism $\alpha = a_{XXXX}$, and
- left and right unit isomorphisms $\lambda = \ell_{XX}$ and $\rho = r_{XX}$.

As in Example 2.1.19, the monoidal category axioms (1.2.4) and (1.2.5) in C follow from the bicategory axioms (2.1.4) and (2.1.5) in B. ◇

Example 2.1.21 (Products)**.** Suppose A and B are bicategories. The *product bicategory* $\mathsf{A} \times \mathsf{B}$ is the bicategory defined by the following data:

- $(\mathsf{A} \times \mathsf{B})_0 = \mathsf{A}_0 \times \mathsf{B}_0$.
- For objects $A, A' \in \mathsf{A}$ and $B, B' \in \mathsf{B}$, it has the Cartesian product hom category

$$(\mathsf{A} \times \mathsf{B})\big((A, B), (A', B')\big) = \mathsf{A}(A, A') \times \mathsf{B}(B, B').$$

- The identity 1-cell of an object (A, B) is $(1_A, 1_B)$.

- The horizontal composition is the composite functor

$$\begin{tikzcd}
\mathsf{A}(A',A'') \times \mathsf{B}(B',B'') \times \mathsf{A}(A,A') \times \mathsf{B}(B,B') \arrow[r, "c"] \arrow[d, "\cong"'] & \mathsf{A}(A,A'') \times \mathsf{B}(B,B'') \\
\mathsf{A}(A',A'') \times \mathsf{A}(A,A') \times \mathsf{B}(B',B'') \times \mathsf{B}(B,B') \arrow[ur, bend right, "c_{\mathsf{A}} \times c_{\mathsf{B}}"'] &
\end{tikzcd}$$

  with c_{A} and c_{B} the horizontal compositions in A and B, respectively, and the left vertical functor permuting the middle two categories.
- The associator, the left unitor, and the right unitor are all induced entrywise by those in A and B.

The unity axiom and the pentagon axiom follow from those in A and B. ◇

Example 2.1.22 (Spans)**.** Suppose C is a category in which all pullbacks exist. For each diagram in C of the form $X \xrightarrow{f} B \xleftarrow{g} Y$, we choose an arbitrary pullback diagram

$$\begin{tikzcd}
X \times_B Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & B
\end{tikzcd}$$

in C. A *span* in C from A to B is a diagram of the form

$$A \xleftarrow{f_1} X \xrightarrow{f_2} B. \tag{2.1.23}$$

There is a bicategory $\mathsf{Span}(\mathsf{C})$, or Span if C is clear from the context, consisting of the following data:

- Its objects are the objects in C.
- For objects $A, B \in \mathsf{C}$, the 1-cells in $\mathsf{Span}(A,B)$ are the spans in C from A to B. The identity 1-cell of A consists of two copies of the identity morphism 1_A.
- A 2-cell in $\mathsf{Span}(A,B)$ from the span (2.1.23) to the span $A \xleftarrow{f_1'} X' \xrightarrow{f_2'} B$ is a morphism $\phi : X \longrightarrow X'$ in C such that the diagram

$$\begin{tikzcd}
& X \arrow[dl, "f_1"'] \arrow[dr, "f_2"] \arrow[dd, "\phi"] & \\
A & & B \\
& X' \arrow[ul, "f_1'"] \arrow[ur, "f_2'"'] &
\end{tikzcd} \tag{2.1.24}$$