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Multivariable Calculus

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Preface

This book is a basic introduction to multivariable calculus, intended for a one-semester course. For the student, my hope is that it sparks an appreciation and (dare I say?) enthusiasm for both this subject and further investigations in mathematics. For the instructor, my hope is that this book serves as a resource to augment the irreplaceable contribution your own energy and expertise makes to your students' education. For all who use it, I hope you engage with and benefit from this text.

The transition to three dimensions is a significant hurdle for most students of multivariable calculus. Students spend years learning about two dimensions, from plotting points and connecting the dots in middle school to using derivatives to discuss concavity in calculus. Upon entering a course in multivariable calculus students have had substantial exposure to, and experience with, all things planar. As a result, students have developed a fair intuition for thinking in two dimensions, an intuition and comfort level that is usually underdeveloped in dimension three. Several features of this text are intended to facilitate the transition to three dimensions.

One of these features is that the first chapter is an attempt to familiarize students with three dimensions. Curves and surfaces are studied from a variety of perspectives, and students should come away with the ability to relate analytic formulas to the geometric objects they represent. A wide range of descriptions for curves and surfaces are presented, not all of which need emphasis, and a judicious choice of techniques to focus on will streamline this chapter. Since the graph of a function is a special case of a parametric surface, formulas for graphs (e.g. normal vectors, flux integrals, etc.) are presented as special cases of those for parametric surfaces later in the text.

A second feature that helps develop three-dimensional acumen is the use of Math Apps regularly throughout the text. Math Apps are interactive graphics, developed using Maple software, that highlight geometric aspects of the topic under discussion. Readers must have Maple Player (a free download), or a full version of Maple, to take advantage of the Math Apps. Math Apps can be downloaded from the companion website, please see below for further instructions.

In addition to the geometric appeal of multivariable calculus, the subject has many applications to other fields of study. This text uses applications to motivate and illustrate mathematical techniques. Streamlines from an ideal fluid flow, for example, are used to illustrate level curves, while the integral of a vector field on a surface is motivated by the flux of a flow. Bézier curves, used in computer imaging, are introduced as an application of vector algebra, and the physical notion of work as an application of the dot product. Applications, while not the focus of this text, are sprinkled throughout to demonstrate the utility of the subject. Students understanding the material here should be well-poised to pursue interests in related fields.

Students: It has been my goal to make this a text you can *read* and learn from, and I hope that I have accomplished that to a certain extent. My recommendation is to grab a coffee and allow yourself the luxury of ruminating over the topics in this (and any other) text. There are a lot of formulas in this text, which I've tried to motivate to varying degrees. Do not content yourselves with memorization, rather ponder the concept behind the formalism. Some of the ideas might require two cups of coffee, and that's Ok.

Faculty: I've tried to produce a valuable resource to supplement your course. The topics are standard, and hopefully the text is flexible enough for you to put your own spin on it, and to take advantage of the plethora of other resources now available to educators.

Maple Player: A word about viewing Math Apps is in order before we get started. Maple Player is a free application that allows you to use the Math Apps you encounter in this text. There are downloadable versions of Maple Player for Windows, Macintosh, and Linux which can be found on the Maplesoft website (a quick search for Maple Player will find it for you). After downloading the application, go to where it is downloaded and double-click to start the installation wizard. Following the prompts should get you up and running.

Math Apps: The Math Apps accompanying this text must be downloaded from the companion website:

https://global.oup.com/booksites/content/9780198835172/

Once downloaded, eBook users should make sure the text and the Math App files are in the same folder. Simply clicking the figure hyperlink in the text will open the corresponding Math App in Maple Player. As they encounter a Math App in the text, hardcopy users will have to open the corresponding Math App file manually on their computer.

I apologize in advance for the mistakes that undoubtedly lurk in the following pages (hopefully they don't "glare"), and thank you in advance for forwarding corrections to me, and for your patience.

Acknowledgments

I am very grateful to many people who have helped me while writing this book. Professionally I'd like to thank Jeremy Aiken, Corey Dunn, Giovanna Llosent, Jeff Meyer, Lynn Scow, and Wenxiang Wang. I've benefitted from discussions with these folks, and from their willingness to pilot various versions of these notes as they developed. I also thank the many students who were willing (forced?) to follow some circuitous routes as I formalized these ideas. Their patience, questions, and input are greatly appreciated. I am thankful to Katherine Ward and Dan Taber at Oxford University Press for their patience and guidance, and for their efforts on my behalf.

On a personal level, thanks goes to Rich and Adele Kehoe, Harold and Karen Sprague, Kevin and Patti Hogan, Claudia Bouslough, and Chuck and Cindy Peterson—a small group of friends from church. Their prayers and encouragement were invaluable during the final year of preparing this manuscript—as was the fact that they never asked "Aren't you done yet?"!

Thanks goes to my kids, Ben, Jake and Ellen. They *did* ask "Aren't you done yet?", but I could take it from them. Thanks to Ellen for encouragement and prayer, to Jake for offering to market my book in his sphere of influence, and to Ben, who gave me the idea of using Math Apps during a conversation we had driving back from school.

My wife, Becky, deserves my greatest thanks. Bec enriches every aspect of my life, and is my best friend. She supported me when I started this project, encouraged me when it didn't progress as I thought it should, and sacrificed a great deal to allow me to finish. I'm looking forward to getting reacquainted! Thanks for your help, Bec. This book is dedicated to you.

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1 Introduction to Three Dimensions

In this chapter we begin our foray into multivariable calculus by getting comfortable with three dimensions. Section one introduces coordinate systems for describing points in three space. We see that Cartesian and polar coordinates in the plane extend naturally to Cartesian and cylindrical coordinates for space. A third coordinate system, spherical coordinates, is also introduced, rounding out those systems most used in this text. Sections 1.2, 1.3 and 1.5 study three different methods of describing surfaces in $\mathbb{R}^3$. Section 1.2 focuses on surfaces arising from graphs of functions of two variables, using level curves to aid understanding. Level curves turn out to be the equivalent of a topographic map for the graph of $z = f(x,y)$. Solution sets of equations in three variables also give rise to surfaces in $\mathbb{R}^3$, and these are considered in Section 1.3. Parametric curves are the topic of Section 1.4, and parametric surfaces are that of Section 1.5. Each method of defining surfaces yields a different insight into the geometry of $\mathbb{R}^3$, and will be used regularly when discussing differentiation and integration. The chapter ends with a section on describing regions in space using systems of inequalities. This skill will be useful when determining limits of integration for multiple integrals.

1.1 Describing Points in 3-Space

Before describing coordinate systems in three dimensions we recall what we know about points in $\mathbb{R}^2$, the Cartesian plane. The common coordinate systems in $\mathbb{R}^2$ will extend naturally to coordinates for $\mathbb{R}^3$.

Coordinates for $\mathbb{R}^2$ In two dimensions there are two familiar methods for describing points. The most common coordinate system is Cartesian coordinates in which a point is described by how far horizontally and vertically it is from the origin. To walk to the point (x,y) from the origin $(0,0)$ simply walk x units horizontally, then y units vertically.

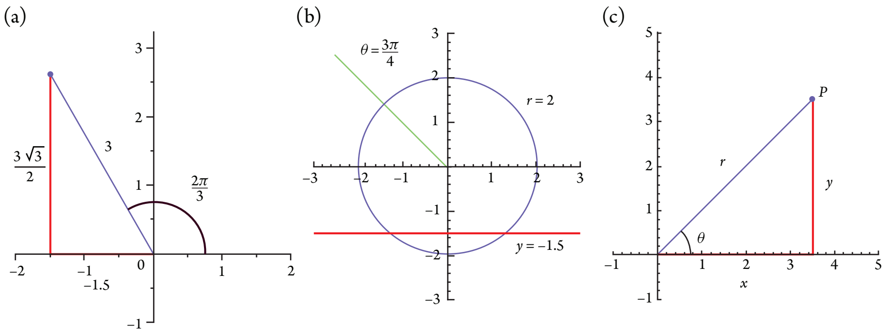


Figure 1.1.1 Review of planar coordinates

Polar coordinates also describe points in the plane. Rather than using horizontal and vertical distances, polar coordinates tell you how far to walk and in what direction. To walk from $(0,0)$ to the point with polar coordinates (r,θ), simply walk r units at an angle θ with the positive x-axis. Thus in polar coordinates, r is the distance to the origin and θ is the angle with the positive x-axis.

There is some ambiguity when using polar coordinates to describe points in the plane. Typically one chooses $r \geq 0$, but we also make the convention that if $r < 0$, go $|r|$ units in the opposite direction from θ. Thus $\left(3, \frac{2\pi}{3}\right)$ and $\left(-3, \frac{5\pi}{3}\right)$ represent the same point in polar coordinates, namely, the point with Cartesian coordinates $\left(-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$. You'll also remember that polar coordinates with the same r and angles that differ by a multiple of 2π represent the same point. In general this ambiguity will not cause confusion.

Constant Coordinate Curves: Equations for curves in the plane can be given in either coordinate system, and we recall some particularly simple ones here. In particular, consider the curves obtained by fixing just one of the coordinates. Solution sets to the Cartesian equations $x = c$ and $y = d$ are vertical and horizontal lines, respectively (here c and d are constants). Moreover, since r denotes the distance to the origin, the polar equation $r = c$ denotes a circle centered at $(0,0)$ of radius c. Finally, the polar equation $\theta = d$ defines a ray emanating from the origin and making an angle of d with the positive x-axis (we assume $r \geq 0$). Since we get these curves by fixing one coordinate and letting the other vary, we call them constant coordinate curves.

Finally, the triangle pictured in Figure 1.1.1(c) verifies the relationships between Cartesian and polar coordinates. Trigonometry leads to the familiar change of coordinate formulas:

Change of Coordinates

Polar to Cartesian	Cartesian to Polar
$x = r\cos\theta$	$r = \sqrt{x^2+y^2}$
$y = r\sin\theta$	$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

The conversion $\theta = \tan^{-1}(y/x)$ is valid as long as $x > 0$, since the range of the arctangent is $-\pi/2 < \theta < \pi/2$. If $x < 0$, you must use $\theta = \tan^{-1}(y/x) + \pi$.

Coordinates for $\mathbb{R}^3$ To describe points in three dimensions, it stands to reason that a third coordinate is needed. The most direct way is to add a third coordinate to the two-dimensional coordinate systems just discussed. The result will be Cartesian and cylindrical coordinates for $\mathbb{R}^3$. We will describe a third coordinate system for $\mathbb{R}^3$, called spherical coordinates, which is useful as well.

Cartesian Coordinates

To obtain Cartesian coordinates for $\mathbb{R}^3$, start with an xy-plane and add a third axis through the origin which is perpendicular to both the x- and y-axes. Call the third axis the z-axis. Typically we think of the xy-plane as lying horizontally in space, and the z-axis as being the vertical direction. Cartesian coordinates (x, y, z) of the point P in $\mathbb{R}^3$ mean the same as they did in two dimensions, with the z-coordinate giving the height of P above or below the xy-plane.

Constant Coordinate Surfaces: In three dimensions, the Cartesian equation $z = c$ represents all points a fixed height c from the xy-plane. Thus $z = c$ is an equation for a plane parallel to the xy-plane but c units from it. This is analogous to the two-dimensional situation, where the equation $y = c$ describes a line parallel to, and c units from, the x-axis.

There are three coordinate planes in $\mathbb{R}^3$, the xy-, xz-, and yz-planes, which slice space into octants, pictured in Figure 1.1.2(a). Can you think of equations for them (Hint: the equation for the y-axis in $\mathbb{R}^2$ is $x = 0$)? It should be clear that the three-dimensional Cartesian equation $x = c$ describes a plane parallel to, and c units from, the yz-plane. See Figure 1.1.2(b) for the planes obtained by fixing a single Cartesian coordinate. Notice that

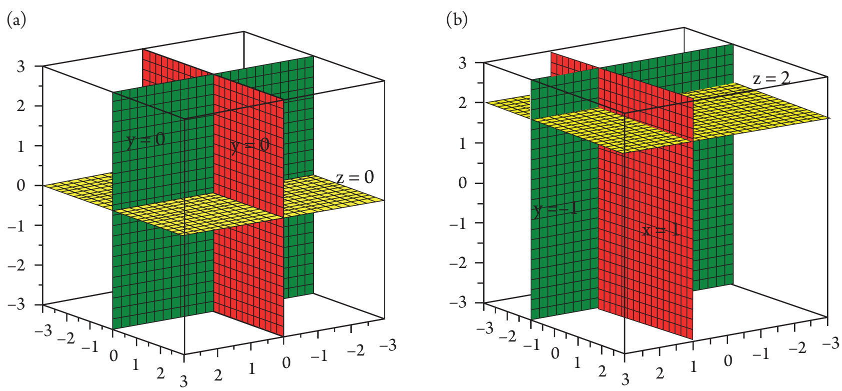


Figure 1.1.2 Cartesian Coordinates

the solution set to the equation $x = c$ depends on what dimension you're in. In $\mathbb{R}^2$ it is a line while in $\mathbb{R}^3$ it's a plane. The context of the situation will dictate which interpretation to use.

Example 1.1.1. *Describing a plane in* $\mathbb{R}^3$

The phrase "the horizontal plane two units above the xy-plane" describes the horizontal plane in Figure 1.1.2 in English. Geometrically this surface is a plane, and analytically it can be described as (the solution set of) the equation $z = 2$. We now know that equations like $z = c$ describe planes! ▲

Example 1.1.2. *Describing a line in* $\mathbb{R}^3$

The simplest lines to describe in $\mathbb{R}^3$ are the coordinate axes. The x-axis can be described as the set of all points whose y- and z-coordinates are both zero, so the set of all points of the form $(x, 0, 0)$. Similarly the y- and z-axes are all points of the form $(0, y, 0)$ and $(0, 0, z)$, respectively. We now consider lines parallel to the coordinate axes.

In $\mathbb{R}^3$, fixing one coordinate gives a plane. Fixing two coordinates, however, will give a line. For example, the solution set of the Cartesian system of equations $x = 1, y = -1$ is the set of all points $(1, -1, z)$ where z is a variable. Thus it is a line parallel to the z-axis, and is the intersection of the planes $x = 1$ and $y = -1$ pictured in Figure 1.1.2(b). It is interesting that a single Cartesian equation yields a surface in $\mathbb{R}^3$ (e.g. $y = -1$ is a plane), while a system of two Cartesian equations yields a curve (e.g. $x = 1, y = -1$ describes a line). We call $\mathbf{C}(t) = (1, -1, t)$, $-\infty < t < \infty$ parametric equations for the line. ▲

Example 1.1.3. *Describing solids in space—Cartesian coordinates*

We describe the portion of $\mathbb{R}^3$ defined by the system of inequalities

$$\begin{aligned} 0 &\le x \le 2 \\ 0 &\le y \le 3 \\ 0 &\le z \le 5 \end{aligned}$$

The restrictions on x indicate the solid is between the planes $x = 0$ and $x = 2$. Similarly it is between the xz-plane and the plane $y = 3$, as well as between $z = 0$ and $z = 5$. Thus it is a rectangular box (see Figure 1.1.3). ▲

We will usually think of the yz-plane as the plane of the paper, with the x-axis pointing out of the paper toward you. This is helpful to keep in mind when viewing static pictures, but software allows more flexibility.

We also mention that the distance formula for $\mathbb{R}^3$ is a natural generalization of the two-dimensional one. Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be points in space, then the distance d between them is given by

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

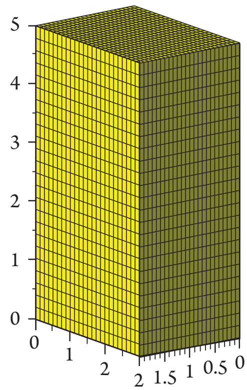


Figure 1.1.3 A rectangular box

Example 1.1.4. *Equations for spheres*

This distance formula gives rise to Cartesian equations for spheres in $\mathbb{R}^3$. Indeed, a sphere is all points a fixed distance from a given point. The equation for a sphere radius r and centered at (a,b,c) is

$$(x-a)^2 + \left(y-b\right)^2 + (z-c)^2 = r^2,$$

where we squared the distance formula to simplify the equation. Note the similarity between equations for spheres in three dimensions and those of circles in two. ▲

Cylindrical Coordinates

Cylindrical coordinates for $\mathbb{R}^3$ are obtained by adding the z-coordinate to polar coordinates for the xy-plane in space. To get from the origin to the point (r,θ,z), first walk in the xy-plane r units at an angle of θ with the x-axis. Then jump z units vertically. Note that, while r represented distance to the origin in polar coordinates, it represents distance to the z-axis in cylindrical coordinates. The right triangle pictured in Figure 1.1.4(a) indicates that the distance from (r,θ,z) to the origin in $\mathbb{R}^3$ is given by $\rho = \sqrt{r^2+z^2}$.

Constant Coordinate Surfaces: As in Cartesian coordinates, let's analyze what we get by fixing one cylindrical coordinate. Since Cartesian and cylindrical have the same z-coordinate, fixing z yields a horizontal plane. Letting r be constant describes the set of all points a fixed distance from the z-axis. To get a feel for what this is, recall that in polar coordinates fixing r gave a circle. In three dimensions, that circle can be translated up and down the z-axis without changing the distance to the z-axis (i.e. without changing r). Thus fixing r yields a cylinder in $\mathbb{R}^3$ whose axis of symmetry is the z-axis. Hence the name "Cylindrical" coordinates. Finally, if you fix θ in polar coordinates, you get a ray emanating from the origin. As in the case of the cylinder, translate this up and

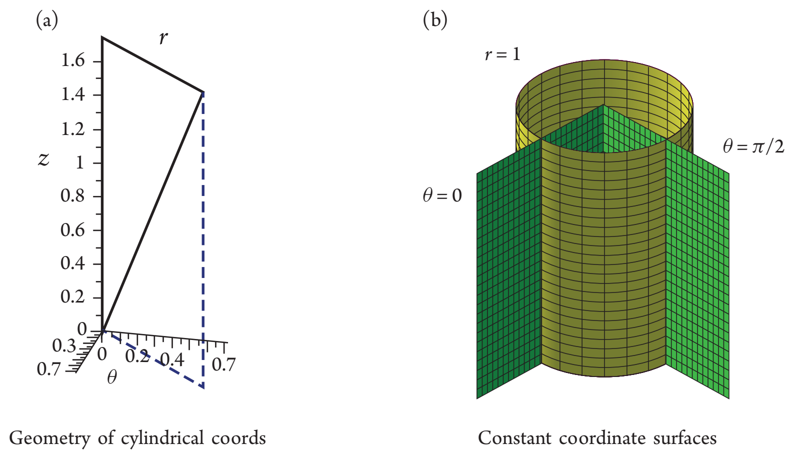
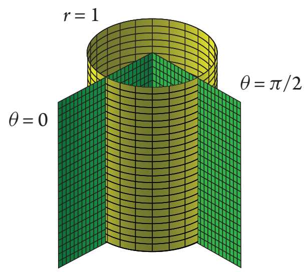


Figure 1.1.4 Cylindrical coordinates

down the z-axis to find what you get in space. The result is a half-plane that makes an angle of θ with the positive x-axis, and whose boundary is the z-axis.

Math App 1.1.1. *Cylindrical constant coordinate surfaces*

Throughout this text Math Apps will be used to enhance geometric understanding. Please refer to Page vi of the preface for instructions on downloading Math Apps. Print users then open the app manually while eBook users click the figure hyperlink below.



Since both Cartesian and Cylindrical coordinates for $\mathbb{R}^3$ extend coordinate systems for $\mathbb{R}^2$, converting between them is the same as between Cartesian and polar. There is the obvious addition that the z-coordinates are the same. Thus the Cartesian coordinates for the cylindrical coordinates (r,θ,z) are

$$(x,y,z) = (r\cos\theta, r\sin\theta, z). \tag{1.1.1}$$

Example 1.1.5. *Constant Coordinate Surfaces*

We again emphasize the English, geometric, and analytic descriptions of a surface. The infinite cylinder with radius one and z-axis as core is pictured in Figure 1.1.4(b). The given cylinder is the solution set of the cylindrical equation $r = 1$.

Cylinders with z-axis as core are constant coordinate surfaces when using cylindrical coordinates, as are half-planes with z-axis as boundary. The half-plane that contains the positive y-axis is given by the equation $\theta = \pi/2$. ▲

Example 1.1.6. *Curves of intersection in cylindrical coordinates*

In Cartesian coordinates we found the solution set to the system $x = 1$, $y = -1$ was the line of intersection of the corresponding planes. Interesting curves also arise from intersecting constant coordinate surfaces in cylindrical coordinates.

Figure 1.1.4(b) indicates that the system of cylindrical equations $r = 1$, $\theta = \pi/2$ describes the intersection of the cylinder $r = 1$ and the half-plane $\theta = \pi/2$. The result is a vertical line. ▲

Remark: While discussing cylindrical coordinates, we should mention that there is nothing special about using the z-axis as the third coordinate. We could have just as easily used polar coordinates in the yz-plane, and the x-axis as the third coordinate. Then r would be the distance to the x-axis, θ the angle with the positive y-axis and x would just be x. Unless otherwise stated, however, cylindrical coordinates will mean (r,θ,z).

We finish our initiation into cylindrical coordinates by looking at the solution set of a system of inequalities. The idea of describing regions in space using a system of inequalities will be useful when setting up limits of integration in triple integrals.

Example 1.1.7. *Describing solids in space—cylindrical coordinates*

We describe the portion of $\mathbb{R}^3$ defined by the system of inequalities

$$\begin{gathered} 0 \le r \le 2 \\ 0 \le \theta \le \frac{\pi}{2} \\ 0 \le z \le 5 \end{gathered}$$

The restrictions on r, which is the distance to the z-axis, describe an infinite solid cylinder with core along the z-axis and radius 2. The restrictions on z cut it down to a solid cylinder radius 2 and height 5, with base in the xy-plane. Finally, the restriction on θ reduces it to that portion which is in the first octant (See Figure 1.1.5). ▲

Spherical Coordinates

In cylindrical coordinates, two coordinates describe distances and one describes a direction. We now introduce spherical coordinates, in which two describe directions and only

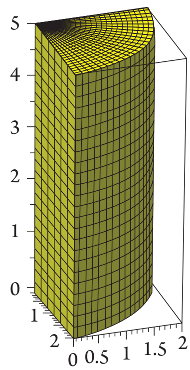


Figure 1.1.5 The solid wedge

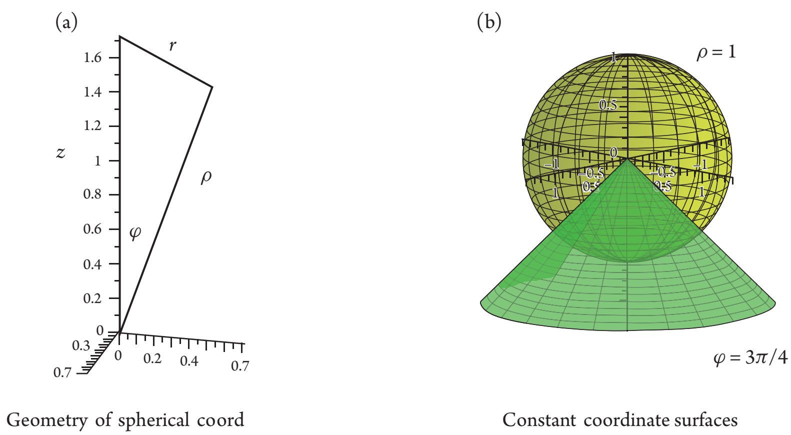


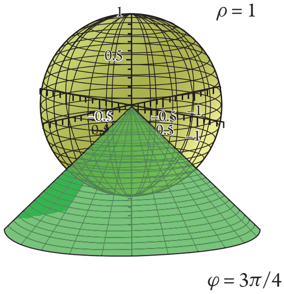
Figure 1.1.6 Spherical coordinates

one is a distance. Spherical coordinates are denoted (ρ,θ,ϕ), where ρ is the distance to the origin, θ is our old friend from polar and cylindrical coordinates, and ϕ is the angle with the positive z-axis. See Figure 1.1.6.

Constant Coordinate Surfaces: Fixing θ just gives us a half-plane with boundary on the z-axis and making an angle of θ with the positive x-axis, as before. Fixing ρ focuses on all points a fixed distance from the origin; take a guess at what shape that might be. The set of all points in $\mathbb{R}^3$ satisfying $\phi = c$ is the set of all points making a fixed angle with the positive z-axis. This is actually a cone with vertex at the origin and making an angle of c with the positive z-axis.

Math App 1.1.2. *Spherical constant coordinate surfaces*

Click the following hyperlink, or print users open manually, to view and manipulate a Math App illustrating constant coordinate surfaces $\phi = c$.



Now consider curves of intersection of constant coordinate surfaces. Fixing both θ and ϕ results in a ray through the origin. Indeed, fixing θ results in a half-plane, while fixing ϕ yields a cone. Fixing both is equivalent to taking the intersection of the half-plane and cone (convince yourself I'm not lying), which is a ray in the half-plane that makes the given angle with the z-axis.

Example 1.1.8. *Spherical equation from geometric description*

Find a spherical equation for the cone with vertex at the origin and that makes an angle of $\pi/3$ with the positive z-axis.

Since the angle with the positive z-axis is the coordinate ϕ in spherical coordinates, the spherical equation is $\phi = \pi/3$. ▲

Example 1.1.9. *Curve of intersection between constant coordinate surfaces*

Describe, as carefully as possible, the curve of intersection of the surfaces $\phi = 3\pi/4$ and $\rho = 1$.

The surfaces are pictured in Figure 1.1.6(b), and the intersection of the cone and sphere will be a circle. Since the cone is pointing straight down, it will be a horizontal circle. Further, using the trigonometry of the triangle corresponding to that of Figure 1.1.6(a) we can determine the radius and height below the xy-plane. In the triangle we have hypotenuse 1, since $\rho = 1$, and an angle of $3\pi/4$ with the positive z-axis. Trigonometry implies that $r = \sqrt{2}/2$ and $z = -\sqrt{2}/2$.

In summary, the intersection of $\phi = 3\pi/4$ and $\rho = 1$ is a horizontal circle at height $z = -\sqrt{2}/2$ with radius $r = \sqrt{2}/2$. ▲

Example 1.1.10. *Describing solids in space—spherical coordinates*

We describe the portion of $\mathbb{R}^3$ defined by the system of inequalities

$$\begin{aligned} 2 &\le \rho \le 3 \\ 0 &\le \phi \le \pi/2 \\ 0 &\le \theta \le 3\pi/2 \end{aligned}$$

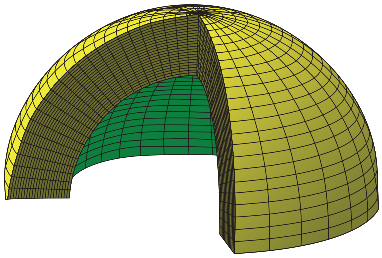


Figure 1.1.7 A solid shell

The restrictions on ρ indicate that the solid is between spheres of radius 2 and 3 centered at the origin. The region of $\mathbb{R}^3$ described by $0 \le \phi \le \pi/2$ is the top half of space, while the restriction $0 \le \theta \le 3\pi/2$ lets you got three-quarters of the way around the z-axis. Thus it is the solid pictured in Figure 1.1.7. ▲

Coordinate Conversion: Converting between spherical and other coordinates is easily achieved using trigonometry and the right triangle illustrated in Figure 1.1.6(a). Notice that $r = \rho \sin\phi$ and $z = \rho\cos\phi$, giving the conversion from spherical to cylindrical. A simple substitution then gives $(x,y,z) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$. We summarize converting between the different coordinate systems in the following table.

Converting between coordinate systems

Cartesian		Cylindrical		Spherical
x	$=$	$r\cos\theta$	$=$	$\rho\sin\phi\cos\theta$
y	$=$	$r\sin\theta$	$=$	$\rho\sin\phi\sin\theta$
z	$=$	z	$=$	$\rho\cos\phi$

Other useful conversions

$$r^2 = x^2 + y^2, \quad \rho^2 = x^2 + y^2 + z^2, \quad r = \rho\sin\phi$$

It will frequently be helpful to translate between coordinate systems, and these conversions facilitate that translation.

Example 1.1.11. *Verifying a conversion analytically*

By direct substitution, and simplification, we verify $x^2 + y^2 + z^2 = \rho^2$.

$$\begin{aligned} x^2 + y^2 + z^2 &= (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 + (\rho\cos\phi)^2 \\ &= \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi \end{aligned}$$

$$= \rho^2 \sin^2\phi\left(\cos^2\theta + \sin^2\theta\right) + \rho^2\cos^2\phi = \rho^2\sin^2\phi + \rho^2\cos^2\phi$$
$$= \rho^2. \ \blacktriangle$$

Example 1.1.12. *Equations in different coordinate systems*

The conversions in the above table allow one to translate an equation from one coordinate system to another. For example, the Cartesian equation for a sphere centered at the origin radius 2 is $x^2 + y^2 + z^2 = 4$. Using the appropriate above conversion, we replace $x^2 + y^2 + z^2$ with ρ^2 to get the spherical equation $\rho^2 = 4$, which simplifies to $\rho = 2$.

Similarly, we can derive cylindrical and spherical equations for the Cartesian equation $x = 5$. Replacing x with $r\cos\theta$ we obtain the cylindrical equation $r\cos\theta = 5$ for the plane $x = 5$. Using the appropriate spherical substitution we find the spherical equation is $\rho\sin\phi\cos\phi = 5$. ▲

Example 1.1.13. *Equations for surface described geometrically*

Find an equation in each coordinate system for the plane parallel to the xz-plane, and 7 units to the right of it.

Since such planes have Cartesian equations $y = c$ for some constant c, we have $y = 7$ as the Cartesian equation of the plane. Substitutions yield:

Cartesian	$y = 7$
Cylindrical	$r\sin\theta = 7$
Spherical	$\rho\sin\phi\sin\theta = 7$ ▲

In this section we've introduced coordinate systems in three dimensions. We now summarize the important points.

Things to know/Skills to have

- The interpretation of each coordinate in Cartesian, cylindrical and spherical coordinates (e.g. z is the height above the xy-plane, while r is the distance to the z-axis).
- Converting between different coordinate systems, and why they work using trig.
- Be able to sketch and describe in English constant coordinate surfaces in each coordinate system.
- Be able to describe in English and sketch the intersection of two constant coordinate surfaces.
- Be able to give equations in each coordinate system for constant coordinate surfaces.
- Be able to use a system of inequalities to describe solids in $\mathbb{R}^3$.

Exercises

1. Sketch the following constant coordinate surfaces; include the coordinate planes in your sketch.
   (a) $z = -2$.
   (b) $x = 4$.
   (c) $y = -5$.
2. Sketch the following constant coordinate surfaces; include the coordinate axes in your sketch.
   (a) $z = -2$.
   (b) $r = 4$.
   (c) $\theta = -3\pi/4$.
3. Sketch the following constant coordinate surfaces; include the coordinate axes in your sketch.
   (a) $\rho = 2$.
   (b) $\phi = \pi/3$.
   (c) $\theta = -3\pi/4$.
4. Sketch the constant coordinate surfaces $x = -1$, $x = 0$, and $x = 3$ on the same set of axes.
5. Sketch the constant coordinate surfaces $r = 1$, $r = 3$, and $r = 5$ on the same set of axes.
6. Sketch the constant coordinate surfaces $\theta = 0$, $\theta = \pi/4$, and $\theta = 3\pi/4$ on the same set of axes.
7. Sketch the constant coordinate surfaces $\phi = \pi/6$, $\phi = \pi/2$, and $\phi = 3\pi/4$ on the same set of axes.
8. Sketch the constant coordinate surfaces $\rho = 1$, $\rho = 2$, and $\rho = 5$ on the same set of axes.
9. Give a one-sentence English description of each of the following constant coordinate surfaces:
   (a) $z = 5$.
   (b) $r = 4$.
   (c) $\rho = 2$.
   (d) $\phi = \pi/3$.
   (e) $\theta = -3\pi/4$.
10. Find Cartesian equations for the following surfaces.
   (a) $\rho = 2$.
   (b) $\phi = \pi/2$.
   (c) $r = 3$.

11. Find cylindrical equations for the following surfaces.
    (a) $x^2 + y^2 = 9$.
    (b) $z = 3$.
    (c) $\rho \sin\phi = 6$.

12. Find spherical equations for the following surfaces.
    (a) $r^2 + z^2 = 4$.
    (b) $z = 3$.
    (c) $x^2 + y^2 + z^2 = 9$.

13. Find a Cartesian equation for the set S of all points 6 units above the xy-plane. Now find cylindrical and spherical equations for S.

14. Find a Cartesian equation for the set S of all points 3 units to the left of the xz-plane. Now find cylindrical and spherical equations for S.

15. Find a cylindrical equation for the set of all points in $\mathbb{R}^3$ that are 4 units from the z-axis.

16. Find a spherical equation for the set S of all points 5 units from the origin. Now find Cartesian and cylindrical equations for S.

17. Find a Cartesian equation for the sphere centered at $(0,0,1)$ with radius 1. Now find a cylindrical equation for it.

18. Describe, as carefully as you can, the intersection of the constant coordinate surfaces given below. Include what geometric shape it is (e.g. a line, ray, circle, etc.), and how it sits in $\mathbb{R}^3$ (e.g. horizontally, parallel to the y-axis, etc.).
    (a) $y = 3, z = -2$.
    (b) $z = 5, r = 2$.
    (c) $z = -2, \theta = \frac{3\pi}{4}$.
    (d) $r = 5, \theta = -\frac{\pi}{3}$.
    (e) $\rho = 3, \phi = \frac{\pi}{4}$.
    (f) $\rho = 3, \phi = \frac{\pi}{2}$.
    (g) $\rho = 5, \theta = \frac{4\pi}{3}$.
    (h) $\phi = \frac{\pi}{6}, \theta = \frac{\pi}{6}$.

19. Use the Pythagorean theorem to prove the distance formula in $\mathbb{R}^3$.

20. Justify in English the conversion $z = \rho\cos\phi$. (Hint: use Figure 1.1.6)

21. Justify in English the conversion $r = \rho\sin\phi$. (Hint: use Figure 1.1.6)

22. Justify, analytically and in English, the conversion $r^2 = x^2 + y^2$.

23. Justify, in English, the conversion $\rho^2 = x^2 + y^2 + z^2$.

24. Sketch the solid determined by the system of inequalities:

$$0 \le x \le 3; \quad 0 \le y \le 5; \quad 0 \le z \le 1.$$

25. Sketch the solid determined by the system of inequalities:

$$0 \le x \le 4; \quad -2 \le y \le 0; \quad 0 \le z \le 3.$$

26. Sketch the solid determined by the system of inequalities:

$$0 \le r \le 4; \quad 0 \le \theta \le \pi; \quad 0 \le z \le 3.$$

27. Sketch the solid determined by the system of inequalities:

$$2 \le r \le 3; \quad 0 \le \theta \le \pi/2; \quad 0 \le z \le 4.$$

28. Sketch the solid determined by the system of inequalities:

$$0 \le \rho \le 2; \quad 0 \le \theta \le \pi/2; \quad 0 \le \phi \le \pi.$$

29. Sketch the solid determined by the system of inequalities:

$$1 \le \rho \le 2; \quad 0 \le \theta \le \pi; \quad 0 \le \phi \le \pi/2.$$

30. A cylindrical can has radius 5 and height 2. A coordinate system is introduced so that the center of mass of the can is at the origin, and its axis is the z-axis. What system of inequalities on cylindrical coordinates describes the region of space occupied by the can?

31. A spherical shell is centered at the origin. Its inner radius is 2 and it is half a unit thick. What system of inequalities in spherical coordinates describes the region of space occupied by the shell?

32. Define a different set of cylindrical coordinates, where r is the distance to the x-axis and θ is the angle made with the positive y-axis. What are the change-of-coordinate functions from this system to Cartesian coordinates?

33. Let T be rotation of space counterclockwise around the z-axis through an angle of $\frac{\pi}{2}$, and let $(\rho, \theta, \phi) = \left(2, -\frac{\pi}{3}, \frac{\pi}{4}\right)$ be the spherical coordinates of the point P. Find the spherical coordinates of the rotated point $T(P)$.

1.2 Surfaces from Graphs

In single-variable calculus, considerable effort is spent on studying curves defined as graphs of functions $y = f(x)$. The derivative $f'(x)$ is the instantaneous rate of change of f, and can be used to determine when the graph of f is increasing or decreasing, the concavity of f, and extreme values of f. The integral of f can represent area or distance traveled, and can be used to find physical quantities like arclength and centers of mass. Indeed, much of single-variable calculus is concerned with analyzing properties of functions $f(x)$ and their graphs. In multivariable calculus we will be concerned with functions of several variables, and we begin our study in this section with analyzing graphs of functions of two variables.

Recall that the graph of a function, say

$$f(x) = \sinh x = \frac{e^x - e^{-x}}{2},$$

is the set of all points of the form $\big(x, f(x)\big)$ as pictured in Figure 1.2.1.

In this context the domain of the function is the horizontal axis—a subset of the plane. The range consists of y-values, which are on the vertical axis. Combining the domain and range in the same space is so familiar, we rarely think about it. We take our cue from the two-dimensional case, and make the following definition:

Definition 1.2.1. *The graph of a function $f(x,y)$ is the set of all points in $\mathbb{R}^3$ of the form $\big(x, y, f(x,y)\big)$.*

To sketch the graph of $z = f(x,y)$, consider the domain to be the xy-plane in $\mathbb{R}^3$ and the function value $f(x,y)$ determines the height above it. For example, if $f(x,y) = x^2 - y^2$, the point $(2, 1, f(2,1)) = (2,1,3)$ is on the graph of f. We illustrate the graphs of two functions below, then describe the level curve technique for understanding the graph of a function of two variables.

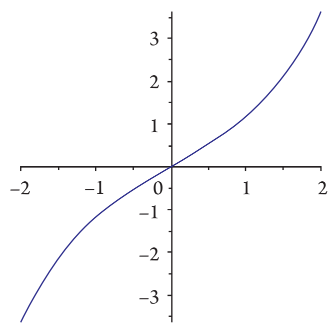


Figure 1.2.1 The hyperbolic sine function

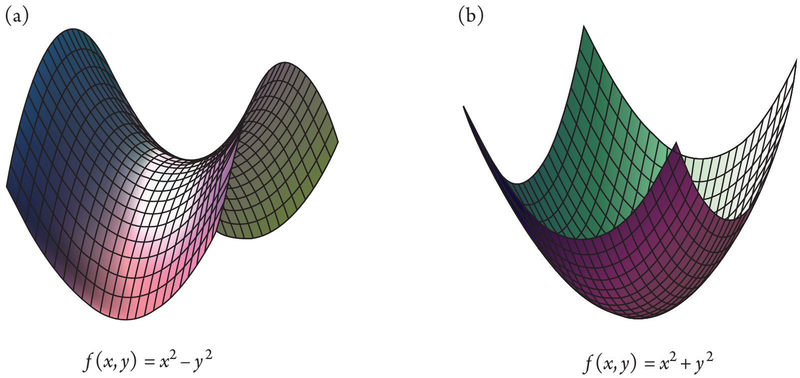


Figure 1.2.2 The graphs of two functions

Example 1.2.1. *The graphs of two functions*

We give, without justification, the graphs of two functions. These surfaces represent all points in $\mathbb{R}^3$ of the form $\left(x, y, f(x,y)\right)$ for the respective functions. ▲

Our current task is to develop a method for determining the pictures of Figure 1.2.2 from the formulas. This method uses level curves, which we will want to understand analytically, geometrically, and conversationally. We motivate level curves by considering topographic maps.

A topographic map is one that also illustrates the topography of a region using what are called contour lines. Contour lines are constant altitude curves, or curves that connect points of the same altitude. Figure 1.2.3 is a topographic map of the area around Longs Peak in Rocky Mountain National Park, Colorado. The contour lines, together with shading and colors, help give us an idea of what the terrain is like. For example, consecutive darker curves represent an altitude gain of 250 feet. The closer they are together, the steeper the terrain in that area. The label on a curve tells you its altitude, and we can use them to determine which peak is higher, Longs Peak in the lower center of the region, or Mount Meeker, south east of Longs. One could also use contour lines to determine the direction a hiker at the middle of the map should walk to get downhill the fastest. A lot of information is encoded in a topographic map.

We wish to use similar techniques to understand graphs of functions $z = f(x,y)$. The "mathematical mountain" in Figure 1.2.4(a), for example, is represented by the topographic map in Figure 1.2.4(b). Notice that the contour lines are labeled with their corresponding altitudes. If we had a big can of paint and a lot of time we could paint the curves on the mountain that correspond to the contour lines on the map, as in Figure 1.2.4(c). The contour lines of the map are not actually on the mountain, they live in two dimensions. However, looking only at the topographic map, the contour lines do give us an idea of what the mountain looks like.

To construct a topographic map from a formula, note that the constant altitude curves *on the mountain* in Figure 1.2.4(c) can be thought of as the curve of intersection of the mountain and a horizontal plane. For example, the highest curve is 630 feet above sea

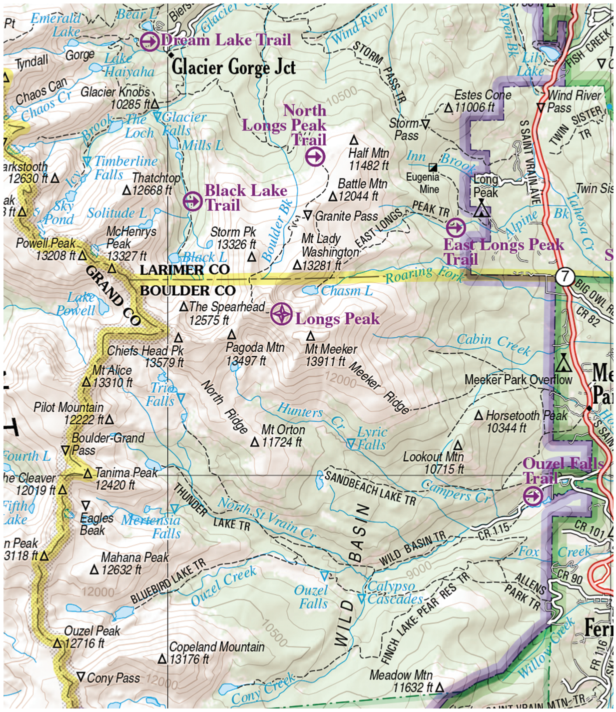
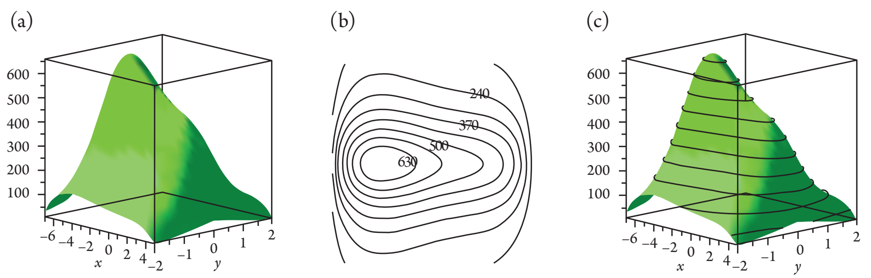


Figure 1.2.3 Longs Peak area



Mathematical mountain | Topographic map | Curves on mountain

Figure 1.2.4 Topographic maps motivate level curves

level, and can be thought of as the intersection of the mountain with the plane $z = 630$. Thus constant altitude curves on the mountain $z = f(x,y)$ at height c can be thought of as the solution set in $\mathbb{R}^3$ of the system of equations

$$\begin{cases} z = f(x,y) \\ z = c. \end{cases} \tag{1.2.1}$$

The contour lines of Figure 1.2.4(b) are obtained by dropping the curves on the mountain down into the xy-plane.

This "dropping" is more precisely called *projecting* into the xy-plane, and is accomplished analytically by eliminating the z-coordinate. To eliminate the z-coordinates from the system of equations 1.2.1, merely turn them into a single equation by substituting c for z, obtaining $c = f(x,y)$. Thus we have accomplished our goal. We have determined how to find equations for contour lines from a formula for the function f! Let's look at a concrete example before going further.

Example 1.2.2. *A contour line, or level curve*

Let $f(x,y) = x^2 + y^2$, and find a contour line at height 4 for the graph of $z = x^2 + y^2$.

As described above, the curve on the "mountain" $z = x^2 + y^2$ at height 4 is the solution set to the system of equations

$$\begin{cases} z = x^2 + y^2 \\ z = 4 \end{cases}$$

To find the contour line, merely substitute 4 for z in the first equation yielding

$$x^2 + y^2 = 4.$$

The curve in the xy-plane defined by this equation is the desired contour line, and we will call it the *level curve* of the function $z = x^2 + y^2$ at level 4. It is obtained by projecting into the xy-plane the curve of intersection of the surface $z = x^2 + y^2$ and the plane $z = 4$. Figure 1.2.5 illustrates what's going on geometrically. ▲

After this concrete example, we are ready to make a general definition. We remark that, although the term "contour line" is used in the context of topographic maps, we will transition to using the term *level curve*.

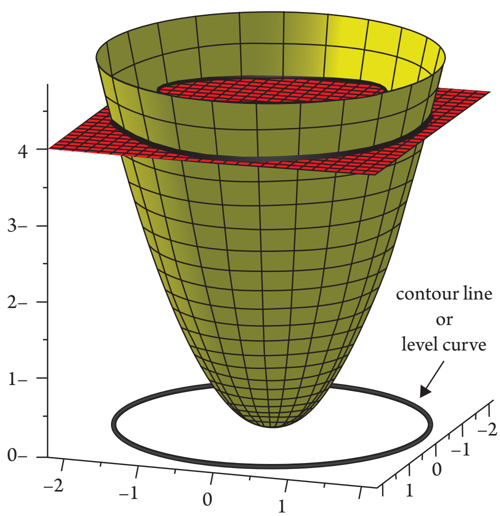


Figure 1.2.5 Geometric understanding of the $\mathbb{R}^3$ problem

Definition 1.2.2. *The* level curve *of $f(x,y)$ at level c is the curve in the xy-plane given by the equation $c = f(x,y)$.*

We found that the level curve of $f(x,y) = x^2 + y^2$ at level $c = 4$ is the circle $x^2 + y^2 = 4$ (see Figure 1.2.5). The equation for the level curve is found by setting $f(x,y)$ equal to the given level. Equivalently, we substitute the level c for z in the equation $z = f(x,y)$. Let's investigate this further with some examples.

Example 1.2.3. *Level curves of $f(x,y) = x^2 + y^2$*

Using the above strategy we find that the level curve at level $c = 0$ is the solution set of

$$x^2 + y^2 = 0,$$

which is a single point—the origin. Thus a level curve need not be a curve at all, but can be a single point. In fact, the situation can be more extreme. Level curves can be intersecting lines, multiple curves, or even fail to exist. For example, the level curve of $f(x,y) = x^2 + y^2$ at level $c = -1$ does not exist since it is the solution set of the equation

$$x^2 + y^2 = -1.$$

If one is considering several level curves simultaneously, it is sometimes convenient to summarize the equations in tabular form. For example, we see the following equations corresponding to different levels:

Level	**Level Curve**
$c = -1$	$x^2 + y^2 = -1$
$c = 0$	$x^2 + y^2 = 0$
$c = 0.5$	$x^2 + y^2 = 0.5$
$c = 1$	$x^2 + y^2 = 1$
$c = 2$	$x^2 + y^2 = 2$
$c = 4$	$x^2 + y^2 = 4$

Analytically, then, it is easy to find equations for level curves—just let the function equal the level. To develop a greater geometric understanding of the graph of $z = f(x,y)$, we plot the level curves in the xy-plane. When plotting level curves in the plane, it is customary to label each curve with its corresponding level, as in a topographic map (see Figure 1.2.6). It is also common to consider several level curves at the same time. Doing this allows us to analyze the surface $z = f(x,y)$. For example, the level curves in Figure 1.2.6 are concentric circles centered at the point at level zero. Take a moment to compare the surface in Figure 1.2.5 with its corresponding level curves in Figure 1.2.6.▲

Math App 1.2.1. *Visualizing level curves*

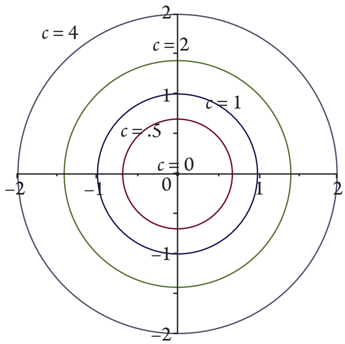
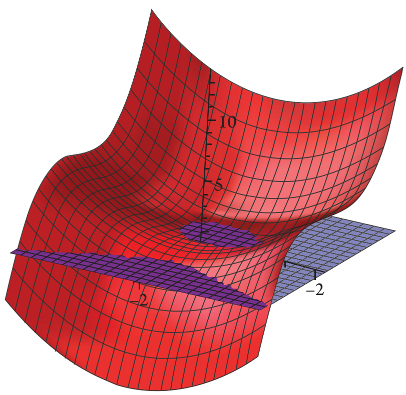


Figure 1.2.6 Level curves of $f(x,y) = x^2 + y^2$

Click the hyperlink below, or print users open manually, to visualize the process of slicing the surface $z = f(x,y)$ and graphing the level curve side by side. Use the slider to change the level, and see how the level curves change.

Surface with Plane



Example 1.2.4. *Level curves of $f(x,y) = x^2 - y^2$*

Let's now analyze the surface in Figure 1.2.2(a). The level curve of $f(x,y) = x^2 - y^2$ at level $c = 1$ is the hyperbola $x^2 - y^2 = 1$. It has asymptotes $y = \pm x$, vertices $(\pm 1, 0)$, and opens sideways. The level curves of $f(x,y) = x^2 - y^2$ at levels $c = -1, 0, 1, 2$ are the curves in the xy-plane given by the equation $f(x,y) = c$. They are

Level	**Level Curve**
$c = -1$	$x^2 - y^2 = -1$
$c = 0$	$x^2 - y^2 = 0$
$c = 1$	$x^2 - y^2 = 1$
$c = 2$	$x^2 - y^2 = 2$

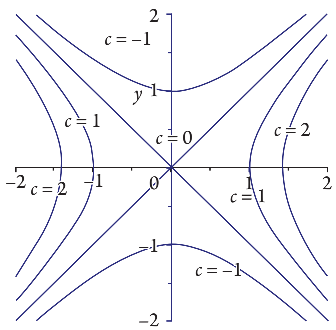
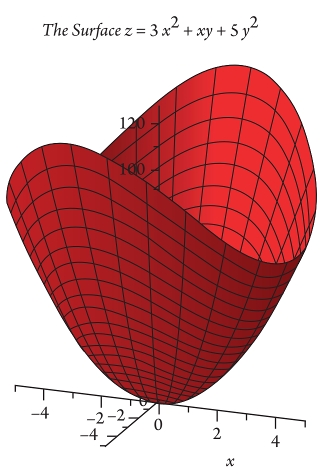


Figure 1.2.7 Level curves of $f(x,y) = x^2 - y^2$

Most of these curves are hyperbolas in the plane, with one exception. At level $c = 0$, the level curve is given by $x^2 - y^2 = 0$. The left-hand side factors to $(x+y)(x-y) = 0$, which holds when either $y = x$ or $y = -x$. Thus the level curve at level zero is really a pair of intersecting lines. Plotting the level curves in the xy-plane yields Figure 1.2.7. ▲

Math App 1.2.2. *Level curves of quadratic functions*

The previous two examples have both involved quadratic functions. Click the hyperlink below, or print users open manually, to explore level curves of the function $f(x,y) = Ax^2 + Bxy + Cy^2$ for different values of A, B, C. It turns out that the discriminant $4AC - B^2$ is a magic number! See if you can tell what the level curves look like when $4AC - B^2$ is positive, negative, and zero.



You now have two examples of level curves under your belt, and it's time to see how to use them to visualize the surface $z = f(x,y)$. Recall that Figures 1.2.6 and 1.2.7 are topographic maps of their corresponding surfaces. To use these maps to visualize your surface, you just need to add the third dimension. We now illustrate how level curves can tell us when a surface is above or below the xy-plane.

Example 1.2.5. *Interpreting level curves of* $f(x,y) = x^2 - y^2$.

In Figure 1.2.7 the level curves at level 0 cut the plane into four pieces: the left, right, top, and bottom. On each piece the function $f(x,y) = x^2 - y^2$ is either always positive or always negative, telling us if the graph of f is above or below that portion of its domain. The levels are positive on the left and right pieces, indicating the graph is above the xy-plane over those regions. Similarly, the graph of f is below the xy-plane in the top and bottom regions. ▲

The surface of Figure 1.2.2(a) is called a saddle (it kinda looks like one, doesn't it: the sides go down but the front and back go up), and the origin is a saddle point of the surface. Now if a circus wanted to design a saddle for a monkey who rode horses in the show, they'd have to consider its tail. A *monkey saddle* would have to go down in the back so its tail could relax and the monkey would be comfortable. The surface given by $f(x,y) = x(y-x)(x+y)$ is a monkey saddle. What are the level curves at level $c = 0$ for the monkey saddle? These curves divide the xy-plane into regions on which the graph of f is always above or always below the xy-plane. Over which regions is it above? below?

We now include an example where a little algebra is helpful before sketching level curves.

Example 1.2.6. *Level curves of* $f(x,y) = x^2 - 2x + y^2 + 4y + 1$

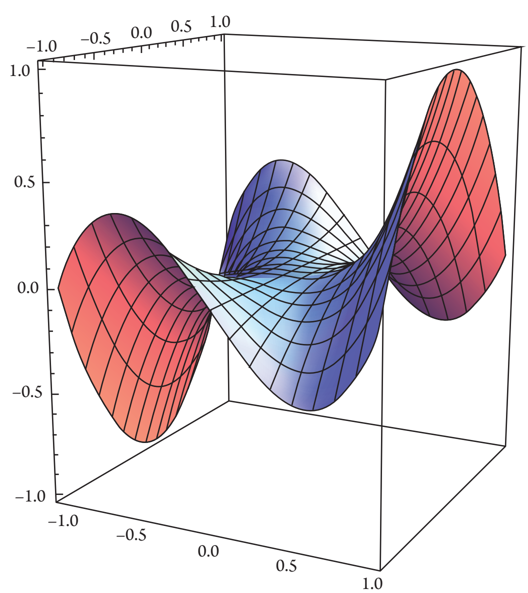


Figure 1.2.8 A monkey saddle

Sketch the level curves of $f(x,y) = x^2 - 2x + y^2 + 4y + 1$ at levels $c = -1, 0, 1$. First, we complete the square on x and y to see that $f(x,y) = (x-1)^2 + (y+2)^2 - 4$. Setting the function equal to each level gives the equation for the level curve in the xy-plane. We have

Level	**Level Curve**	**Simplified Equation**
$c = -1$	$-1 = (x-1)^2 + (y+2)^2 - 4$	$3 = (x-1)^2 + (y+2)^2$
$c = 0$	$0 = (x-1)^2 + (y+2)^2 - 4$	$4 = (x-1)^2 + (y+2)^2$
$c = 1$	$1 = (x-1)^2 + (y+2)^2 - 4$	$5 = (x-1)^2 + (y+2)^2$

Thus the level curves are all circles centered at the point $(1,-2)$ with radii $\sqrt{3}, 2, \sqrt{5}$, respectively. See Figure 1.2.9(a) for a sketch of the level curves. It turns out that this surface is again a paraboloid, seen in Figure 1.2.9(b), but be forewarned that not every surface with circular level curves is a paraboloid! ▲

Example 1.2.7. *Level curves and optimization*

In the previous example we sketched some level curves of $f(x,y) = x^2 - 2x + y^2 + 4y + 1$. In this example we show how level curves can be used to solve constrained optimization problems. In a constrained optimization problem there is an *objective function*, which needs to be maximized or minimized. The "constrained" part of the problem means we want to optimize our objective function, but subject to some *constraint equation*. A general method for doing so, called the method of Lagrange multipliers, will be outlined in Section 3.8. For now we content ourselves with an example.

We address the following question:

An ant crawls on the surface $z = f(x,y) = x^2 - 2x + y^2 + 4y + 1$ directly "above" the unit circle in the xy-plane. What is the maximum altitude that the ant achieves? This is a constrained optimization problem, where the objective function is $f(x,y)$, and we want to maximize it on the unit circle. The constraint equation is therefore $x^2 + y^2 = 1$ (see Figure 1.2.9(a) for the level curves together with the constraint equation).

This question can be completely rephrased in terms of level curves. The ant is one unit up when it is above the intersection of the unit circle and the level curve $c = 1$. More generally, the level of a level curve through a point on the unit circle tells you the height of the ant there. Woah! Stop and ruminate on that sentence—it's worth understanding.

To find its maximum height, then, one has to find the highest level of any level curve intersecting the unit circle. Figure 1.2.9(b) shows the actual path of the ant on the surface "above" the constraint curve.

In this case, level curves are concentric circles centered at $(1,-2)$ and the levels are increasing as the radius increases. If a level curve C intersects the unit circle, it either does so in two points or one. If C intersects the unit circle in two points, then there are level curves with slightly larger radius that still intersect the unit circle. Therefore the largest level curve intersecting the unit circle will be the large one tangent to it. This occurs when the radius of the large circle centered at $(1,-2)$ goes through the center of the unit

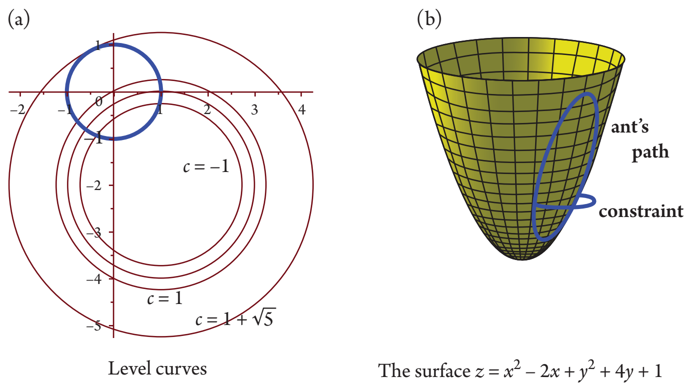


Figure 1.2.9 A shifted paraboloid

circle. Since the distance from $(1,-2)$ to the origin is $\sqrt{5}$ units and the distance from the origin to the unit circle is 1, we have that the radius of the level curve giving the maximum altitude of the ant is $1+\sqrt{5}$.

We now have to find the level associated with this radius. The equation of the level curve at level c is $c+4=(x-1)^2+(y+2)^2$. In general, then, the relationship between the radius r and level c is given by $r^2=c+4$. Substituting $1+\sqrt{5}$ for r gives $c=2+2\sqrt{5}$. Thus the maximum height achieved by the ant is $2+2\sqrt{5}$.

We pause to point out the critical observation in solving this problem. We wanted to maximize the function $f(x,y)=x^2-2x+y^2+4y+1$, subject to the constraint that $x^2+y^2=1$. The key observation was that $f(x,y)$ would be maximized at a point where the constraint curve (the unit circle) was tangent to the level curve. More generally, the maximum (or minimum) of a function subject to a constraint will occur at a point where the constraint curve and level curves are tangent to each other. This will be expanded on in Section 3.8. ▲

Level curves and critical points: One can look at a topographic map and tell where the mountain peaks are. Similarly, level curves can tell us where relative maxima and minima of our function are, as well as saddle points. More precisely, level curves near extreme values look like concentric ellipses (for nice functions $f(x,y)$), while near a saddle they look more like hyperbolas. We illustrate with two examples.

Example 1.2.8. *Extreme values from level curves*

Some level curves of the function $f(x,y)=e^{-x^2}(y^3-y)$ are given in Figure 1.2.10, and are labeled with their corresponding heights. The points P and Q in this example are degenerate level curves. Since the level curves are elliptical in shape around the points P and Q, those points are relative extreme points for the surface. The levels are decreasing toward P, so P is a relative minimum of the surface. Similarly, the levels are increasing toward Q so it is a relative maximum. ▲

In the previous example we used the fact that if a level curve is a point (like P or Q above), and the curves around them are elliptical in shape, the points correspond to

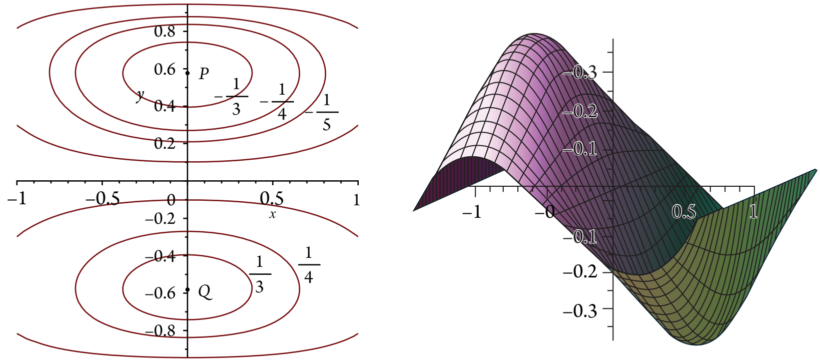


Figure 1.2.10 Some level curves of $f(x,y) = e^{-x^2}(y^3 - y)$

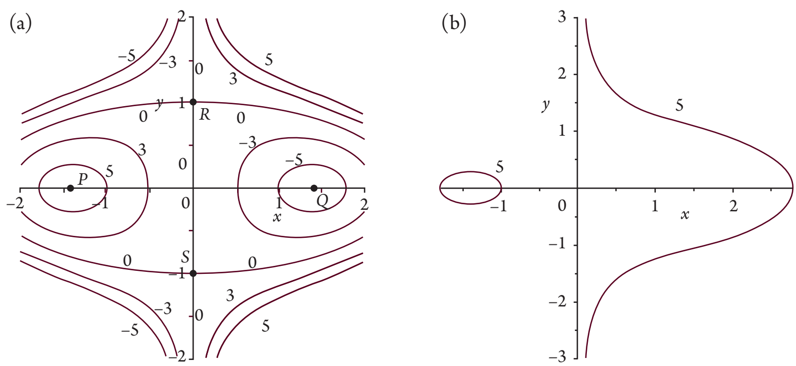


Figure 1.2.11 Some level curves of $f(x,y) = x^3 + 6xy^2 - 6x$

relative extrema of $f(x,y)$ (some additional assumptions on the function f are necessary, but we ignore the details for now). We've also seen that near a saddle point the level curves look like hyperbolas (see Figure 1.2.7). We use these facts to interpret the following level curve diagram.

Example 1.2.9. *Interpreting level curves, again*

The level curves in Figure 1.2.11 are for the function $f(x,y) = x^3 + 6xy^2 - 6x$. Notice that they are not connected. For example, in Figure 1.2.11(a) the level curves at level 5 appear in the upper and lower right, then close to the point P on the left. There are actually only two components of the level curve $5 = x^3 + 6xy^2 - 6x$, as in Figure 1.2.11(b), the display just wasn't wide enough to show it. In any case, this illustrates that level curves can have several pieces.

Reasoning as in the previous example and discussion, we wish to classify the points P, Q, R, S as relative maxima, minima, or saddles. Since levels are increasing toward P, $f(x,y)$ attains a relative maximum there. Similarly, $f(x,y)$ has a relative minimum at Q. The level curve at level 0 intersects itself at R and S, and look like hyperbolas nearby, so R and S are saddles of f. ▲

Fluid Flow: We introduce two-dimensional fluid flow here, since streamlines (the paths of particles in the flow) are level curves of the *stream function* corresponding to the flow. After defining some terminology from the field of fluid mechanics, we discuss a model for ideal fluid flow around a corner (for a more thorough discussion see [9]).

Viscosity is a property of the fluid itself, rather than the flow, which basically describes how thick it is. Pouring caramel on a bowl of ice cream or honey in a cup of tea illustrates that these are high-viscosity fluids. Water and air are low viscosity fluids, and pour quite readily. A second property of fluids is that of *incompressibility*. Air and helium are compressible in that they can be forced into smaller spaces at the expense of increasing pressure. Conversely, compressed gasses can expand to fill larger spaces, reducing pressure. You witness this every time you inflate a balloon from a tank of helium. Water, on the other hand, is incompressible. Viscosity and compressibility are properties that help describe and distinguish fluids.

In order to understand a fluid flow one must also consider properties of the flow itself, not just of the fluid. A flow can cause particles to both move along with it, as well as spin while moving. An example illustrates what I mean. The surface of a pond can be thought of as a two-dimensional fluid flow. In the fall, if you drop a leaf into the pond, the leaf will begin to drift along with the flow. If the leaf starts spinning as well, the flow is *rotational*. An *irrotational* flow is one in which all leaves drift along, but none spin.

It turns out that models that are both incompressible and irrotational closely imitate the flow of low viscosity fluids. We will consider such models, which are called *ideal fluid flows*, as they are excellent applications of much of the mathematics discussed in this book. (By the way, the notions of incompressible and irrotational will be made quite precise when we discuss divergence and curl in Section 5.5.)

Example 1.2.10. *Ideal flow around a right-angle*

With a bit of intuition in hand, we begin a more formal discussion of fluid flow. An ideal fluid flow is described using two functions: the velocity potential and stream functions. We focus on the two-dimensional ideal flow around a corner. The velocity potential and stream functions for this situation are

$$\begin{aligned}\varphi(x,y) &= x^2 - y^2, \text{ and}\\ \psi(x,y) &= 2xy,\end{aligned}$$

respectively. The level curves of the stream function ψ are called *streamlines*, and are the paths particles travel along in the flow. Thus the flow is along hyperbolas with equations $2xy = c$, which we sketch in the first quadrant in Figure 1.2.12(a), thinking of the axes as the corner the flow goes around. ▲

Before leaving this topic, we take a moment to generalize Example 1.2.10 to flows around a corner with angle π/n as it is a particularly nice application of polar coordinates.

Example 1.2.11. *Flows around arbitrary corners*

To motivate the generalization, we first find polar equations for a right-angled flow. Using double-angle formulas from trig, we see

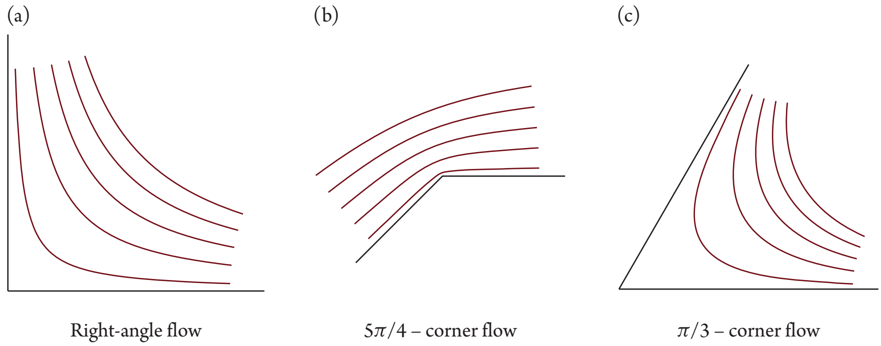


Figure 1.2.12 Streamlines of ideal fluid flow

$$\begin{aligned}\varphi(x,y) &= x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta,\\ \psi(x,y) &= 2xy = r^2 2\cos\theta\sin\theta = r^2\sin 2\theta.\end{aligned} \tag{1.2.2}$$

To generalize from a corner angle of $\pi/2$ to one of π/n, it would be great to change the 2's to n's in Equation 1.2.2. Somewhat miraculously, this is indeed the proper generalization, so that an ideal flow around a corner π/n has velocity potential and stream function given by:

$$\begin{aligned}\varphi(r,\theta) &= r^n\cos n\theta,\\ \psi(r,\theta) &= r^n\sin n\theta.\end{aligned} \tag{1.2.3}$$

To be more specific, let's sketch some streamlines for an ideal fluid flow around a 60° corner (see Figure 1.2.12(c)). In this case, the angle is $\pi/3$ and the stream function is $\psi = r^3\sin 3\theta$. Note that $\psi = 0$ when θ is 0 or $\pi/3$ (among other values), so those rays are level curves of ψ at level 0. These curves represent the boundary of the flow, and for this reason we only sketch the curves for $0 < \theta < \pi/3$.

For $c > 0$, the streamline equation $\psi = c$ can be solved for r to be

$$r = \sqrt[3]{\frac{c}{\sin 3\theta}}.$$

For θ near 0 and $\pi/3$ the sine approaches 0 and $r \to \infty$, so the streamlines are asymptotic to the boundary of the flow. Additionally, the distance r is minimized when $\sin 3\theta$ is maximized at $3\theta = \pi/2$, or when $\theta = \pi/6$, which is half-way through the domain. These features are illustrated in Figure 1.2.12(c).

Finally, since one might be more comfortable with Cartesian coordinates, we use angle sum identities to translate φ and ψ into Cartesian coordinates. While these may be more familiar, they are definitely less illuminating, thereby illustrating the usefulness of polar coordinates!

$$\begin{aligned}
\varphi &= r^3 \cos 3\theta = r^3(\cos 2\theta \cos\theta - \sin 2\theta \sin\theta) \\
&= r^3\left((\cos^2\theta - \sin^2\theta)\cos\theta - 2\sin^2\theta\cos\theta\right) \\
&= r^3\cos^3\theta - 3r^3\sin^2\theta\cos\theta \\
&= x^3 - 3xy^2, \\
\psi &= r^3\sin 3\theta = r^3(\sin 2\theta\cos\theta + \cos 2\theta \sin\theta) \\
&= r^3\left(2\sin\theta\cos^2\theta + (\cos^2\theta - \sin^2\theta)\sin\theta\right) \\
&= 3r^3\cos^2\theta\sin\theta - r^3\sin^3\theta \\
&= 3x^2y - y^3. \blacktriangle
\end{aligned}$$

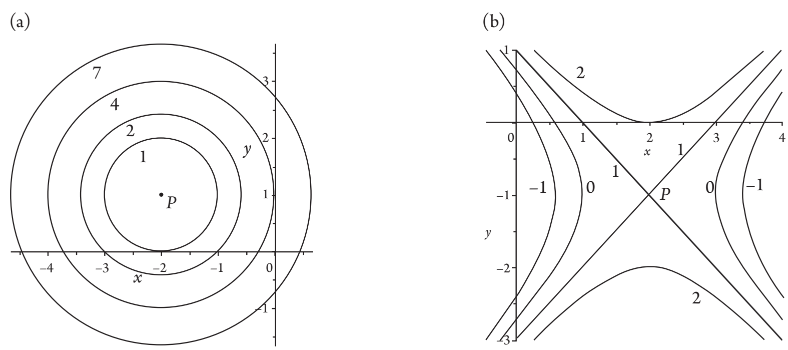
In the above generalization to flows around π/n-corners, n need not be an integer. As an example, the streamlines for a $5\pi/4$-corner and are shown in Figure 1.2.12(b).

Things to know/Skills to have

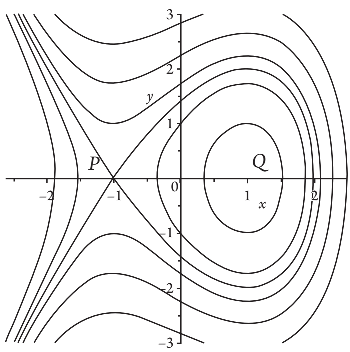
- Know the definition of the graph of a function of two variables.
- Be able to sketch level curves for a given function and level.
- Solve a constrained optimization problem using level curves.
- Be able to determine maxima, minima, and saddle points for a function from its level curves. More generally, relate the level curves to the geometry of a surface.
- Sketch streamlines given the stream function for an ideal fluid flow.

Exercises

1. Decide whether each point P is a relative maximum, relative minimum, or a saddle point. Give a one-sentence justification for your answer.



2. Sketch the level curves of $f(x,y) = xy$ at levels $c = -1,0,1,2$. What kind of surface is this?
3. Sketch the level curves of $f(x,y) = x^2 + 4y^2$ at levels $c = 0,4,16$. For what levels c does the level curve $f(x,y) = c$ not exist? What does this imply about the intersection of the surface $z = f(x,y)$ and the plane $z = c$ for these levels?
4. Some level curves of the function $f(x,y) = x^3 - 3x + y^2$ are pictured below. Guess whether the points P and Q represent maxima, minima, or saddles. What keeps you from being definite in your answer?



5. Sketch the level curves of $f(x,y) = 3x + 2y + 7$ for levels $c = -2,1,8$. Can you guess at the shape of the surface?
6. Sketch the level curves of $f(x,y) = x^2 - y^2 - 4x - 2y$ at levels $c = -4,-3,-2,0$. What kind of surface do you get?
7. Sketch the level curves of $f(x,y) = \sqrt{16 - x^2 - y^2}$ for levels $c = 0,1,2,3,4$. Can you describe the surface?
8. Sketch the level curves of $f(x,y) = x^2 + y^2 - 2xy$ for levels $c = 0,1,4$. Can you guess at the shape of the surface? (Hint: factor $f(x,y)$ first)
9. Maximize $f(x,y) = x - \sqrt{3}y$ subject to the constraint $x^2 + y^2 = 1$.
10. Sketch the level curves for $f(x,y) = 3x - 3y + 4$ at levels $c = 0,4,16$. An ant is crawling on the surface $z = f(x,y)$ above the unit circle in the xy-plane. What are the highest and lowest elevations the ant attains? Sketch the surface in $\mathbb{R}^3$.
11. Sketch the level curves for $f(x,y) = x^2 + y^2 - 6x + 16y$ at levels $c = 0,4,16$. An ant is crawling on the surface $z = f(x,y)$ above the unit circle in the xy-plane. What are the highest and lowest elevations the ant attains? Sketch the surface in $\mathbb{R}^3$.
12. Different surfaces can have very similar level curves!
    (a) Sketch the level curves of $f(x,y) = x^2 + y^2$ at levels $c = 0,1,2$.
    (b) Sketch the level curves of $f(x,y) = \sqrt{x^2 + y^2}$ at levels $c = 0,1,2$.
    (c) The level curves for both surfaces are concentric circles centered at the origin. How can you tell the surfaces apart from their level curves?

13. The saddle $f(x,y) = (x-y)(x+y)$ has two portions sloping down and two sloping up. The monkey saddle $f(x,y) = x(x-y)(x+y)$ has three portions of the surface sloping down (two for the legs and one for the tail) and three up. Can you guess a formula for a surface with four upward and four downward sloping portions? Can you generalize?
14. Sketch some level curves of $h(x,y) = \ln(x^2+y^2+1)$. Describe the surface in a sentence or two.
15. Sketch some level curves of $h(x,y) = \cos(x^2+y^2)$. Describe the surface in a sentence or two.
16. Sketch some level curves of $h(x,y) = 1/(1+x^2+y^2)$, and describe what the surface does near infinity.
17. An ideal fluid flow is modeled with the velocity potential $\varphi = 4x - 3y$ and stream function $\psi = 3x + 4y$. Sketch some streamlines for this flow. Can you describe this flow in a sentence or two?
18. The velocity potential $\varphi = x + 2y$ and stream function $\psi = -2x + y$ model a two-dimensional flow in the plane. Sketch a few streamlines for the flow.
19. Find the velocity potential and stream function for an ideal fluid flow around a $\pi/4$-corner. Sketch some streamlines for the flow.
20. Guess what the streamlines around a π-corner would look like (write your guess down first—AND why you think so!). Now find the velocity potential and stream function for this case in polar and in Cartesian coordinates, and sketch some streamlines. Write a sentence or two comparing your guess to the analysis that followed.
21. A point source in two-dimensional ideal fluid flow emits fluid at a constant rate μ uniformly in all directions (think of a fountain, for example). A point source at the origin can be modeled in polar coordinates with velocity potential and stream function

$$\varphi = \frac{\mu}{2\pi}\ln r,\ \psi = \frac{\mu}{2\pi}\theta.$$

Letting $\mu = 1$, sketch some streamlines for the flow.
22. A vortex in a two-dimensional flow has velocity and stream functions

$$\varphi = K\theta,\ \psi = -K\ln r.$$

Where K is the strength of the vortex. Letting $K = 1$, sketch some streamlines for the flow. Compare the streamlines for a source to those of a vortex.

1.3 Surfaces from Equations

In this section we present a potpourri of surfaces arising as the solution sets of equations, and techniques to study them. We study slicing surfaces with planes, quadric surfaces, generalized cylinders, planes, level surfaces, and equations in other coordinate systems. The basic technique for studying quadric surfaces and level surfaces will be to slice the surfaces with planes parallel to coordinate planes. Generalized cylinders turn out to be easy to recognize from their equations, and easy to sketch. Geometric understanding will help analyze surfaces given by equations in other coordinate systems. Enjoy the variety!

Planes: Lines are some of the first curves you learn to graph in the plane. Analogously, we begin this section on surfaces with planes. The general form for an equation of a line is $Ax + By = C$, so it is no surprise that any plane in $\mathbb{R}^3$ has an equation of the form

$$Ax + By + Cz = D,$$

where A through D are constants and at least one of A, B, or C is not zero. The constants in the equation $Ax + By + Cz = D$ turn out to have significant geometric meaning, which we will see after discussing vectors in Section 2.3... something to look forward to!

Example 1.3.1. *The plane* $5x + 2y + z = 10$

An easy way to visualize the plane $5x + 2y + z = 10$ is to plot the intersections with the coordinate axes, connect the dots to form a triangle. When sketching planes, the triangle might be enough to visualize it as long as you remember that the plane extends without bound in all directions.

The y- and z-coordinates of points on the x-axis in $\mathbb{R}^3$ are both zero. This implies that letting $y = z = 0$ in the equation for the plane gives the x-coordinate, and we get the intercept $(2, 0, 0)$. Similarly, we get y-intercept $(0, 5, 0)$ and z-intercept $(0, 0, 10)$. Plotting these points and connecting the dots gives a triangle that lives in the plane (see Figure 1.3.1(a)). Extending the triangle in all directions gives the plane itself, as in Figure 1.3.1(b). ▲

Example 1.3.2. *Relating a planar equation to the graph*

Let $\mathcal{P}$ be the plane given by $Ax + By + Cz = D$.

1. Suppose $D = 0$, what can you say about the plane?
   If $D = 0$, then the origin $(0, 0, 0)$ satisfies the equation, so $\mathcal{P}$ goes through the origin. In this case the above technique for visualizing planes will not work, but Section 2.3 will use vectors perpendicular to the plane.
2. Suppose $A \cdot B < 0$ and $D \neq 0$, what can you say about the plane $\mathcal{P}$?
   Since $A \cdot B < 0$, we know A and B have opposite signs. If the x-intercept is positive, then the y-intercept will be negative, and vice versa. Thus we can say the x- and y-intercepts have opposite signs. ▲

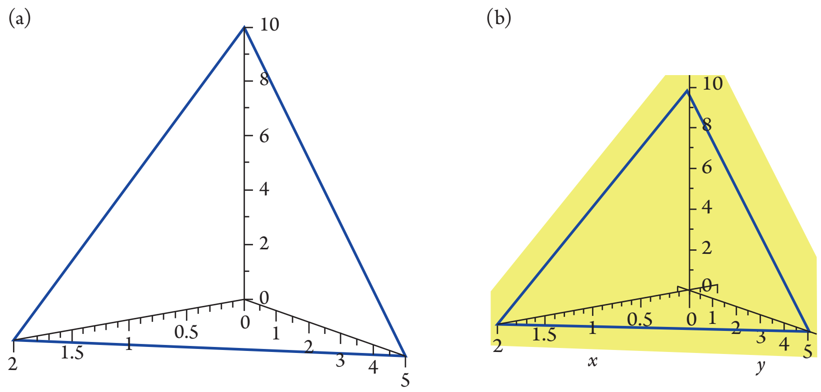


Figure 1.3.1 Visualizing the plane $Ax+By+Cz=D$

Generalized Cylinders: We have already seen that the cylindrical coordinate equation $r=2$ describes a cylinder in space. Translating to Cartesian coordinates for $\mathbb{R}^3$ this corresponds to the equation $\sqrt{x^2+y^2}=2$, or $x^2+y^2=4$, which is missing the z-coordinate entirely. One way to think of $r=2$, then, is that it is the curve $x^2+y^2=4$ in the xy-plane translated up and down along the z-axis. More generally, we'll say that:

Definition 1.3.1. *Any surface whose Cartesian equation is missing one variable is a* generalized cylinder.

To visualize generalized cylinders:

1. Sketch the curve in the appropriate plane.
2. Translate the curve along the axis of the missing variable.

Example 1.3.3. *A hyperbolic cylinder*

Sketch the surface $x^2-4z^2=1$. We know this is a generalized cylinder since the y-coordinate is missing from the equation. The first step is to sketch the curve $x^2-4z^2=1$ in the xz-plane. This is a hyperbola with vertices $(\pm 1,0,0)$, and asymptotes $z=\pm x/2$ (see the dark curve of intersection in Figure 1.3.2). To get the surface, simply translate the hyperbola back and forth along the y-axis, as in Figure 1.3.2. ▲

Example 1.3.4. *A washboard*

You can sketch the surface $y=\sin z$ in $\mathbb{R}^3$ in two steps. First sketch the curve $y=\sin z$ in the yz-plane, then translate the curve back and forth along the x-axis. This process is illustrated in Figure 1.3.3. ▲

Plane Sections: In Section 1.2 we used level curves to make topographic maps for surfaces arising from the graphs of functions $z=f(x,y)$. Geometrically, this corresponded to considering the intersection of the surface $z=f(x,y)$ with a horizontal plane $z=c$.

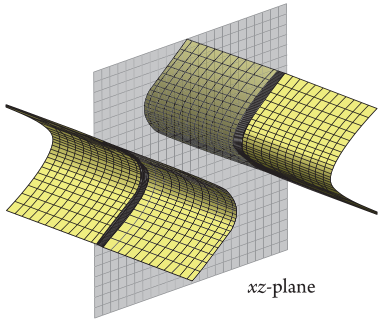


Figure 1.3.2 The generalized cylinder $x^2 - 4z^2 = 1$

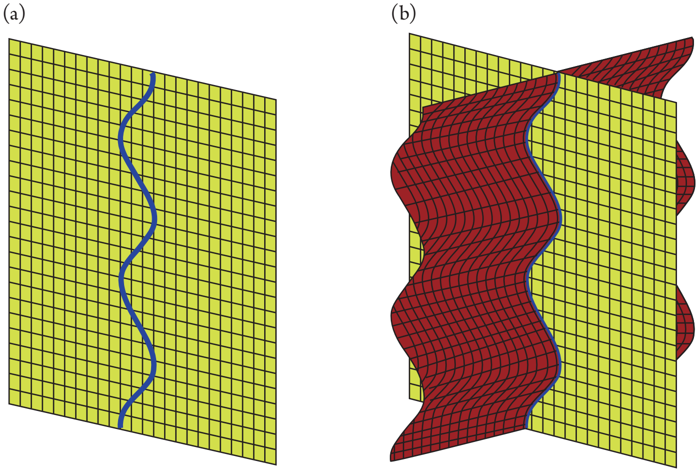


Figure 1.3.3 A "cylinder"

We generalize Section 1.2 in two ways. One generalization is to consider surfaces that need not be graphs of functions. This is analogous to considering curves in the plane defined as solution sets to equations, such as the ellipse $\frac{x^2}{9} + 4y^2 = 1$, which is not the graph of a function. The second generalization is that the planes we slice them with need not be horizontal. Allowing for this flexibility can give greater insight into the structure of some surfaces. Which planes give you greatest insight into a surface is usually dictated by the equation that defines it.

We begin with some examples of slicing graphs of functions with other planes, which could be done using the level curve technique but are easier to understand by slicing with other planes. We then proceed to surfaces whose defining equations are not functions of one of the variables.

Example 1.3.5. *Slicing surfaces for geometric understanding*

Level curves of the function $f(x,y) = x^3 - x + y^2$ are curves in the xy-plane defined by

$$x^3 - x + y^2 = c,$$

for some constant c. Since these curves are hard to visualize, it may be easier to find the intersections with other planes instead. Let's investigate the graph of $f(x,y) = x^3 - x + y^2$ by slicing it with planes parallel to the xz-plane.

First, the curve of intersection of the surface $z = x^3 - x + y^2$ with the plane $y = 0$ is the solution to the system of equations

$$\begin{cases} z = x^3 - x + y^2 \\ y = 0. \end{cases}$$

An equation for the curve is obtained by substituting 0 for y in the first equation, yielding $z = x^3 - x$. One can plot this curve in the xz-plane, as in Figure 1.3.4(b). The intersection with the plane $y = -3$ has equation $z = x^3 - x + (-3)^2 = x^3 - x + 9$, and is pictured in 1.3.4(a). Similarly the $y = 2$ cross-section is in 1.3.4(c), and they are put together in Figure 1.3.5.

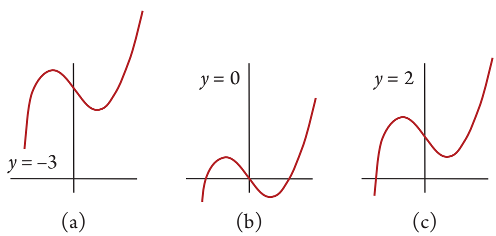


Figure 1.3.4 Plane sections of $f(x,y) = x^3 - x + y^2$

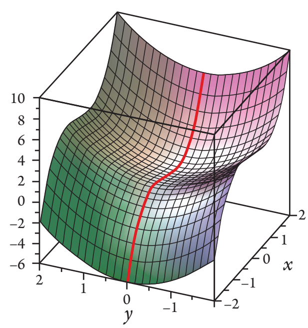
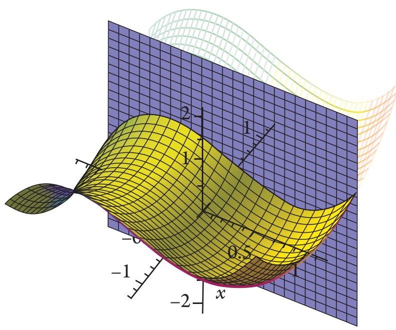


Figure 1.3.5 The surface $z = x^3 - x + y^2$

More generally, to describe the intersection of the surface with the plane $y = c$ analytically we substitute c for y, obtaining $z = x^3 - x + c^2$. This is a vertical translation of the curve $z = x^3 - x$ by c^2 units. So as you move back and forth along the y-axis, the curve $z = x^3 - x$ gets translated up by the appropriate amount, sweeping out the surface as in Figure 1.3.5. The grid lines on the surface that are "parallel" to the highlighted one in the middle are all intersections with planes $y = c$. This technique is convenient because the x's and y's in $f(x, y) = x^3 - x + y^2$ are added together, making the vertical translation easy to see. ▲

Math App 1.3.1. *Surface sections*

This Math App gives more hands-on experience with plane sections. Click on the hyperlink, or print users open manually, to improve your geometric intuition.



Example 1.3.6. *Fun with trigonometric functions*

Equations for the level curves of $f(x, y) = \cos x + \sin y$ are of the form

$$\cos(x) + \sin(y) = c,$$

which are definitely not fun to work with. To make this example more enjoyable, we'll choose other planes to get some two-dimensional slices with. If we choose to slice the surface $z = \cos x + \sin y$ with planes parallel to the yz-plane, that amounts to replacing x with a constant and analyzing the curves. The curve of intersection with the yz-plane (or $x = 0$) is given by $z = \cos 0 + \sin x = 1 + \sin y$, which is a translation of the sine curve one unit up. In fact, slicing with the plane $x = c$ gives the curve

$$z = \cos c + \sin y,$$

which is a vertical translation of $z = \sin y$ by the value $\cos c$. The surface $z = \cos x + \sin y$ can be thought of, then, as taking the sine curve in the y-direction, and having it ride along a roller coaster in the x-direction. The roller coaster is the cosine curve (see Figure 1.3.6). ▲

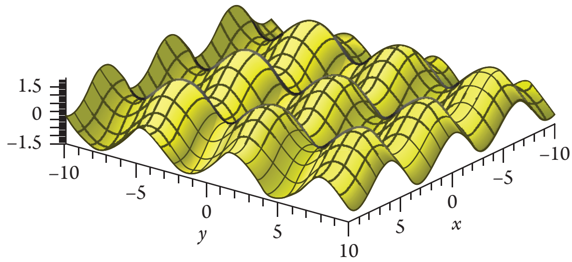


Figure 1.3.6 The surface $z = \cos x + \sin y$

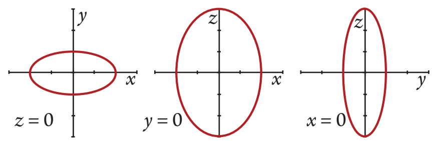


Figure 1.3.7 Planar cross-sections of $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 1$

The preceeding two examples involved surfaces that were graphs of functions, but didn't have particularly nice level curves. The strategy was to get two-dimensional slices by cutting with planes parallel to one of the coordinate planes, other than the xy-plane. We now consider some surfaces arising from equations that can't be solved for one of the variables. We will find that slicing with a variety of planes gives geometric insight to the surfaces.

A quadric surface is the solution to a quadratic equation in three variables. The quadratic equations we will consider have the form

$$Ax^2 + By^2 + Cz^2 + Dx + Ey + Fz = G$$

(we avoid equations with mixed terms xy, xz, or yz). Our basic strategy for understanding such surfaces will be to sketch a "skeleton" by intersecting the surface with coordinate planes, then "connect the curves". We illustrate this with two examples.

Example 1.3.7. *An ellipsoid*

The solution set to $\frac{x^2}{4} + y^2 = 1$ is an ellipse. Analogously, the solution set to the three-variable equation $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 1$ is an ellipsoid. To get a feel for what it looks like, consider the intersections with coordinate planes. When first sketching these surfaces, it might help to sketch the intersections in separate planes, as in Figure 1.3.7, then piece them together. To find equations for these curves of intersection, set one of the variables to zero and sketch the resulting curve in the appropriate coordinate plane. The intersection of the surface $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 1$ with the xy-plane is gotten by setting $z = 0$ (technically, we're solving the *system* of equations consisting of both, and getting the intersection of the surfaces). The intersection with the xz-and yz-planes are the ellipses $\frac{x^2}{4} + \frac{z^2}{9} = 1$ and $y^2 + \frac{z^2}{9} = 1$, respectively. The planes pieced together form a "skeleton" for the whole ellipsoid pictured in Figure 1.3.8. ▲

Example 1.3.8. *The one-sheeted hyperboloid*

The solution set to the equation $x^2 + y^2 - z^2 = 1$ is called a hyperboloid of one sheet. Intersecting it with the coordinate planes yields the circle $x^2 + y^2 = 1$ in the xy-plane, and the hyperbolas $x^2 - z^2 = 1$ and $y^2 - z^2 = 1$ in the xz- and yz-planes, respectively (see Figure 1.3.9).

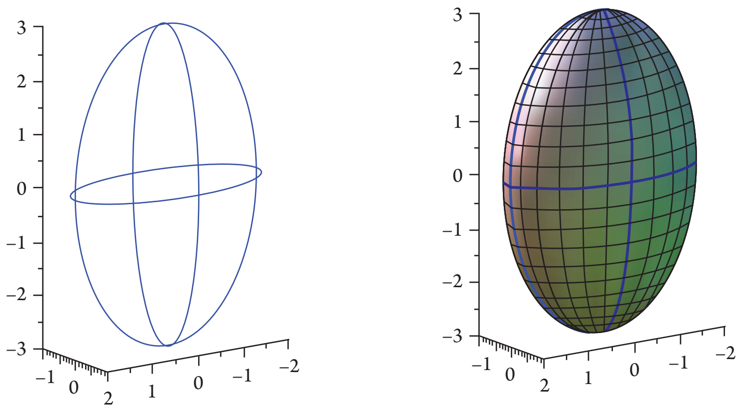
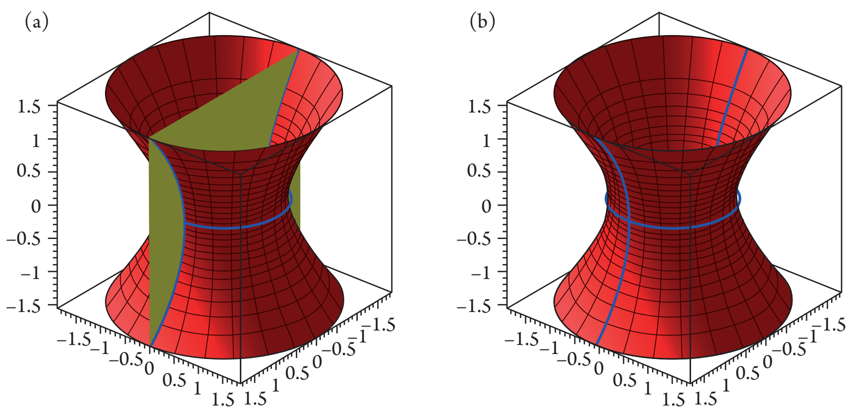


Figure 1.3.8 An ellipsoid and its skeleton



Hyperboloid intersectingy = 0 | Hyperboloid and skeleton

Figure 1.3.9 A one-sheeted hyperboloid