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## Les Houches 2016 <br> 

## Integrability: From Statistical Systems to Gauge Theory

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Editors

INTEGRABILITY: FROM STATISTICAL SYSTEMS TO GAUGE THEORY

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 Gauge Theory}

Edited by<br>Patrick Dorey, Gregory Korchemsky, Nikita Nekrasov, Volker Schomerus, Didina Serban, Leticia Cugliandolo

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# Integrability in statistical physics and quantum spin chains 

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## Preface

The goal of these lectures is to illustrate some basic concepts of quantum integrable systems on two important models of statistical physics: the $Q$-state Potts model and the $\mathrm{O}(n)$ model. Both models have a geometric formulation as lattice models of fluctuating loops, hence making contact with the Temperley-Lieb algebra and its dilute generalization, respectively. They possess a conformally invariant continuum limit, that can be easily visualized in the Coulomb gas formalism by viewing the loops as level lines of a deformed Gaussian field theory. For particular values of $Q$ and $n$, the continuum limit is more subtle and gives rise to logarithmic conformal field theories [1]. Important challenges remain to be solved in this case; this is however beyond the scope of these introductory lectures.

We set out by transforming the Potts model into a loop model, and further into a six-vertex model. Both formulations unveil the underlying Temperley-Lieb algebra, and hence permit us to identify the quantum integrable $R$-matrix. This leads to the solution of the model by the algebraic Bethe ansatz technique. Extracting information about the continuum limit from the resulting Bethe ansatz equations calls instead for techniques of analysis. We focus on the critical case $-1<\Delta<1$, and derive the ground state free energy in detail. Then we discuss elementary excitations and establish the relationship with the Coulomb gas.

A final section provides a survey of some more recent material. We discuss the spinone Izergin-Korepin model and establish its relation with an $\mathrm{O}(n)$ loop model. We focus on the so-called regime III, which gives rise to an unusual, non-compact continuum limit, in which the spectrum of critical exponents contains both discrete and continuous components. The $n \rightarrow 0$ limit in this regime describes a collapse transition of polymers due to the critical self-attraction between monomers, an example of a so-called $\Theta$-point.

These lectures are based on sundry material, including standard textbooks and reviews on integrability [2-5] and conformal field theory [6], my lecture notes for the AIMES course [7] given at the ICFP graduate school at the Ecole Normal Supérieure in Paris since 2011, my review of CFT applied to loop models [8], a few theses [9, 10], and a lot of original research articles. The main text cites some of the latter, with no attempt at exhaustiveness, the principal aim being to provide the reader with a few entry points to a large and ever growing body of literature.

I take the opportunity here of expressing my deep gratitude towards my students, colleagues, and collaborators over the years for all they have taught me. I also thank the students of the Summer School for their suggestions to improve these lecture notes.

### 1.1 Potts model

In this section we define the Potts model and transform it into a loop model. By orientating the loops we then exhibit the equivalence with a six-vertex model. Both formulations provide a relation to the Temperley-Lieb algebra, which is the basis for solving the model by techniques of quantum integrability.

Even though we are mainly interested in the model defined on a square lattice, the main steps are valid on more general graphs [11]. Since it is hardly more complicatedand a lot more instructive-to work in the 'correct' generality, we shall choose to do so and specialize only when needed.

### 1.1.1 Spin representation

Let $G=(V, E)$ be an arbitrary connected graph with vertex set $V$ and edge set $E$. The $Q$-state Potts model [12] is initially defined by assigning a spin variable $\sigma_{i}$ to each vertex $i \in V$. Each spin can take $Q$ different values, by convention chosen as $\sigma_{i}=1,2, \ldots, Q$. We denote by $\sigma$ the collection of all spin variables on the graph. Two spins $i$ and $j$ are called nearest neighbours if they are incident on a common edge $e=(i j) \in E$. In any given configuration $\sigma$, a pair of nearest neighbour spins is assigned an energy $-\mathcal{F}$ if they take identical values, $\sigma_{i}=\sigma_{j}$. The Hamiltonian (dimensionless energy functional) of the Potts model is thus

$$
\begin{equation*}
\mathcal{H}=-K \sum_{(i, j) \in E} \delta\left(\sigma_{i}, \sigma_{j}\right) \tag{1.1}
\end{equation*}
$$

where the Kronecker delta function is defined as

$$
\delta\left(\sigma_{i}, \sigma_{j}\right)= \begin{cases}1 & \text { if } \sigma_{i}=\sigma_{j}  \tag{1.2}\\ 0 & \text { otherwise }\end{cases}
$$

and $K=\mathscr{F} / k_{\mathrm{B}} T$ is a dimensionless coupling constant (interaction energy).

The case $Q=2$ corresponds to the Ising model. Indeed, if $S_{i}= \pm 1$ we have

$$
\begin{equation*}
2 \delta\left(S_{i}, S_{j}\right)=S_{i} S_{j}+1 \tag{1.3}
\end{equation*}
$$

The second term amounts to an unimportant shift of the interaction energy, and so the models are equivalent if we set $K_{\text {Potts }}=2 K_{\text {Ising }}$.

The thermodynamic information about the Potts model is encoded in the partition function

$$
\begin{equation*}
Z=\sum_{\sigma} \mathrm{e}^{-\mathcal{H}}=\sum_{\sigma} \prod_{(i j) \in E} \mathrm{e}^{K \delta\left(\sigma_{i}, \sigma_{j}\right)} \tag{1.4}
\end{equation*}
$$

and in various correlation functions. By a correlation function we understand the probability that a given set of vertices are assigned fixed values of the spins.

In the ferromagnetic case $K>0$ the spins tend to align at low temperatures ( $K \gg 1$ ), defining a phase of ferromagnetic order. Conversely, at high temperatures $(K \ll 1)$ the spins are almost independent, leading to a paramagnetic phase where entropic effects prevail. On physical grounds, one expects the two phases to be separated by a critical point $K_{\mathrm{c}}$ where the effective interactions between spins becomes long ranged.

For certain regular planar lattices $K_{\mathrm{c}}$ can be determined exactly by duality considerations [12, 14]. Moreover, $K_{c}$ will turn out to be the locus of a second order phase transition if $0 \leq Q \leq 4$ [13]. In that case the Potts model enjoys conformal invariance in the limit of an infinite lattice, allowing its critical properties to be determined exactly by a variety of techniques. These properties turn out to be universal, i.e. independent of the lattice used for defining the model microscopically.

### 1.1.2 Fortuin-Kasteleyn cluster representation

The initial definition (1.1) of the Potts model requires the number of spins $Q$ to be a positive integer. It is possible to rewrite the partition function and correlation functions so that $Q$ appears only as a parameter [15]. This makes its possible to assign to $Q$ arbitrary real (or even complex) values.

Notice first that by (1.2) we have the identity

$$
\begin{equation*}
\mathrm{e}^{K \delta\left(\sigma_{i}, \sigma_{j}\right)}=1+v \delta\left(\sigma_{i}, \sigma_{j}\right) \tag{1.5}
\end{equation*}
$$

where we have defined $v=\mathrm{e}^{K}-1$. Now, it is obvious that for any edge-dependent factors $h_{e}$ one has

$$
\begin{equation*}
\prod_{e \in E}\left(1+h_{e}\right)=\sum_{E^{\prime} \subseteq E} \prod_{e \in E^{\prime}} h_{e}, \tag{1.6}
\end{equation*}
$$

where the subset $E^{\prime}$ is defined as the set of edges for which we have taken the term $h_{e}$ in the development of the product $\prod_{e \in E}$. In particular, taking $h_{e}=v \delta\left(\sigma_{i}, \sigma_{j}\right)$ we obtain for the partition function (1.4)

$$
\begin{equation*}
Z=\sum_{E^{\prime} \subseteq E} v^{\left|E^{\prime}\right|} \sum_{\sigma} \prod_{(i j) \in E^{\prime}} \delta\left(\sigma_{i}, \sigma_{j}\right)=\sum_{E^{\prime} \subseteq E} v^{\left|E^{\prime}\right|} Q^{k\left(E^{\prime}\right)} \tag{1.7}
\end{equation*}
$$

where $k\left(E^{\prime}\right)$ is the number of connected components in the graph $G^{\prime}=\left(V, E^{\prime}\right)$, i.e. the graph obtained from $G$ by removing the edges in $E \backslash E^{\prime}$. Those connected components are called clusters, and (1.7) is the Fortuin-Kasteleyn cluster representation of the Potts model partition function. The sum over spins $\sigma$ in (1.4) has now been replaced by a sum over edge subsets, and $Q$ appears as a parameter in (1.7) and no longer as a summation limit.

In (1.7) -and in all that follows-it presents no added complication to consider arbitrary edge-dependent couplings $v_{e}$. In that sense, $v^{\left|E^{\prime}\right|}$ is just a short-hand notation for $\prod_{e \in E^{\prime}} v_{e}$. We shall need this inhomogeneous generalization in section 1.1.7.

### 1.1.3 Duality of the partition function

Consider now the case where $G=(V, E)$ is a connected planar graph. Any planar graph possesses a dual graph $G^{*}=\left(V^{*}, E^{*}\right)$ which is constructed by placing a dual vertex $i^{*} \in V^{*}$ in each face of $G$, and connecting a pair of dual vertices by a dual edge $e^{*} \in E^{*}$ if and only if the corresponding faces are adjacent in $G$. In other words, there is a bijection between edges and dual edges, since each edge $e \in E$ intersects precisely one dual edge $e^{*} \in E^{*}$. Note that by the Euler relation,

$$
\begin{equation*}
|V|+\left|V^{*}\right|=|E|+2 . \tag{1.8}
\end{equation*}
$$

By construction, the dual graph is also connected and planar. Note also that duality is an involution, i.e. $\left(G^{*}\right)^{*}=G$.

The Euler relation can easily be proved by induction. If $E=\emptyset$, since $G$ was supposed connected we must have $|V|=\left|V^{*}\right|=1$, so (1.8) indeed holds. Each time a further edge is added to $E$, there are two possibilities. Either it connects an existing vertex to a new vertex, in which case $|V|$ increases by one and $\left|V^{*}\right|$ is unchanged. Or it connects two existing vertices, meaning that a cycle is closed in $G$. In this case $|V|$ is unchanged and $V^{*}$ increases by one. In both cases (1.8) remains valid.

Recalling the cluster representation (1.7)

$$
\begin{align*}
Z_{G}(Q, v) & =\sum_{E_{1} \subseteq E} v^{\left|E_{1}\right|} Q^{k\left(E_{1}\right)} \\
Z_{G^{*}}\left(Q, v^{*}\right) & =\sum_{E_{2} \subseteq E^{*}}\left(v^{*}\right)^{\left|E_{2}\right|} Q^{k\left(E_{2}\right)} \tag{1.9}
\end{align*}
$$

we now claim that it is possible to choose $v^{*}$ so that

$$
\begin{equation*}
Z_{G}(Q, v)=k Z_{G^{*}}\left(Q, v^{*}\right) \tag{1.10}
\end{equation*}
$$

where $k$ is an unimportant multiplicative constant.
To prove this claim, we show that the proportionality (1.10) holds term by term in the summations (1.9). To this end, we first define a bijection between the terms by $E_{2}=\left(E \backslash E_{1}\right)^{*}$, i.e. an edge is present in $E_{1}$ if its dual edge is absent from $E_{2}$, and vice versa. This implies

$$
\begin{equation*}
\left|E_{1}\right|+\left|E_{2}\right|=|E| . \tag{1.11}
\end{equation*}
$$

We have moreover the topological identity for the induced (not necessarily connected) graphs $G_{1}=\left(V, E_{1}\right)$ and $G_{2}=\left(V^{*}, E_{2}\right)$,

$$
\begin{equation*}
k\left(E_{1}\right)=|V|-\left|E_{1}\right|+c\left(E_{1}\right)=|V|-\left|E_{1}\right|+k\left(E_{2}\right)-1, \tag{1.12}
\end{equation*}
$$

where $k\left(E_{1}\right)$ and $c\left(E_{1}\right)$ are respectively the number of connected components and the number of independent cycles ${ }^{1}$ in the graph $G_{1}$.

The proof of (1.12) is again by induction. If $E_{1}=\emptyset$, we have $k\left(E_{1}\right)=|V|, c\left(E_{1}\right)=0$, and $k\left(E_{2}\right)=1$. Each time an edge is added to $E_{1}$ there are two possibilities. Either $c\left(E_{1}\right)$ stays constant, in which case $k\left(E_{1}\right)$ is reduced by one and $k\left(E_{2}\right)$ is unchanged. $\operatorname{Or} c\left(E_{1}\right)$ increases by one, in which case $k\left(E_{1}\right)$ is unchanged and $k\left(E_{2}\right)$ increases by one. In both cases (1.12) remains valid.

Combining (1.11)-(1.12) gives

$$
\begin{equation*}
v^{\left|E_{1}\right|} Q^{k\left(E_{1}\right)}=k\left(v^{*}\right)^{\left|E_{2}\right|} Q^{k\left(E_{2}\right)} \tag{1.13}
\end{equation*}
$$

where we have defined

$$
\begin{equation*}
k=Q^{1-\left|V^{*}\right|} v^{|E|}=Q^{|V|-|E|-1} v^{|E|} \tag{1.14}
\end{equation*}
$$

and $v^{*}=Q / v$. Comparing (1.13) with (1.9) completes the demonstration of (1.10) and furnishes the desired duality relation

$$
\begin{equation*}
v v^{*}=Q . \tag{1.15}
\end{equation*}
$$

The duality relation (1.15) is particularly useful when the graph is selfdual, $G^{*}=G$. This is the case of the regular square lattice. Assuming the uniqueness of the phase transition, the critical point is given by the selfdual coupling

$$
\begin{equation*}
v_{\mathrm{c}}= \pm \sqrt{Q} \quad \text { (square lattice) } \tag{1.16}
\end{equation*}
$$

[^0]
### 1.1.4 Special cases

One of the strengths of the $Q$-state Potts model is that it contains a large number of interesting special cases. Many of those make manifest the geometrical content of the partition function (1.7). The equivalence between $Q=2$ and the Ising model has already been discussed. We shall concentrate here on a couple of other subtle equivalences, that explicitly exploit the fact that $Q$ can now be used as a continuous variable.

### 1.1.4.1 Bond percolation

For $Q=1$ the Potts model is seemingly trivial, with partition function $Z=(1+v)^{|E|}$. Instead of setting $Q=1$ brutally, one can however consider taking the limit $Q \rightarrow 1$. This leads to the important special case of bond percolation.

Let $p \in[0,1]$ and set $v=p /(1-p)$. We then consider the rescaled partition function

$$
\begin{equation*}
\tilde{Z}(Q) \equiv(1-p)^{|E|} Z=\sum_{E^{\prime} \subseteq E} p^{\left|E^{\prime}\right|}(1-p)^{|E|-\left|E^{\prime}\right|} Q^{k\left(E^{\prime}\right)} \tag{1.17}
\end{equation*}
$$

We have of course $\tilde{Z}(1)=1$. But formally, what is written here is that each edge is present in $E^{\prime}$ (i.e. percolating) with probability $p$ and absent (i.e. non-percolating) with probability $1-p$. Appropriate correlation functions and derivatives of $\tilde{Z}(Q)$ in the limit $Q \rightarrow 1$ furnish valuable information about the geometry of the percolation clusters. For instance

$$
\begin{equation*}
\lim _{Q \rightarrow 1} Q \frac{\mathrm{~d} \tilde{Z}(Q)}{\mathrm{d} Q}=\left\langle k\left(E^{\prime}\right)\right\rangle \tag{1.18}
\end{equation*}
$$

gives the average number of clusters.

### 1.1.4.2 Trees and forests

Using (1.12), and defining $w=\frac{Q}{v}$, one can rewrite (1.7) as

$$
\begin{align*}
Z & =\sum_{E_{1} \subseteq E}\left(\frac{Q}{w}\right)^{|V|+c\left(E_{1}\right)-k\left(E_{1}\right)} Q^{|V|-\left|E_{1}\right|+c\left(E_{1}\right)} \\
& =v^{|V|} \sum_{E_{1} \subseteq E} w^{k\left(E_{1}\right)-c\left(E_{1}\right)} Q^{c\left(E_{1}\right)} \tag{1.19}
\end{align*}
$$

Take now the limit $Q \rightarrow 0$ and $v \rightarrow 0$ in such a way that the ratio $w=Q / v$ is fixed and finite, and consider the rescaled partition function $\tilde{Z}=Z v^{-|V|}$. The limit $Q \rightarrow 0$ will suppress any term with $c\left(E_{1}\right)>0$, and we are left with

$$
\begin{equation*}
\tilde{Z}=\sum_{E_{1} \subseteq E}^{\prime} w^{k\left(E_{1}\right)} \tag{1.20}
\end{equation*}
$$

where the prime indicates that the summation is over edge sets such that the graphs $G_{1}=$ ( $V, E_{1}$ ) have no cycles, $c\left(E_{1}\right)=0$. Such graphs are known as forests, or more precisely (since the vertex set $V$ is that of $G$ ), spanning forests of $G$. Each connected component carries a weight $w$.

For $w \rightarrow 0$, the surviving terms are spanning trees, i.e. forests with a single connected component. Note that the critical curve on the square lattice (1.16) goes through the point $(Q, v)=(0,0)$ with a vertical tangent (i.e. $w \rightarrow 0)$ and thus describes spanning trees. General values of $w$ have also been studied in detail [16-18].

### 1.1.5 Loop representation

We now transform the Potts model defined on a planar graph $G$ into a model of selfavoiding loops [19] on a related graph $\mathcal{M}(G)$, known in graph theory as the medial graph. Each term $E^{\prime}$ in the cluster representation (1.7) is in bijection with a term in the loop representation. The correspondence is, roughly speaking, that the loops turn around the connected components in $G^{\prime}=\left(V, E^{\prime}\right)$ as well as their elementary internal cycles. More precisely, the loops separate the clusters from their duals.

To make this transformation precise, first notice that we can draw a quadrangle around each pair of intersecting edges, $e \in E$ and $e^{*} \in E^{*}$. This set of quadrangles defines the quadrangulation $\hat{G}$ (see Figure 1.1b). The medial lattice is just the dual of this quadrangulation: $\mathcal{M}(G)=\hat{G}^{*}$ (see Figure 1.1c). Each vertex of $\mathcal{M}(G)$ thus stands at the intersection between $e$ and $e^{*}$.

In the case where $G$ is the square lattice, the medial $\mathcal{M}(G)$ is just another (tilted) square lattice.


Figure 1.1 (a) A planar graph $G$ (black circles and solid lines) and its dual graph $G^{*}$ (white circles and dashed lines). (b) The plane quadrangulation $\hat{G}=\mathcal{M}(G)^{*}$. (c) The medial graph $\mathcal{M}(G)=\mathcal{M}\left(G^{*}\right)$.

We recall that the partition function $Z_{G}(Q, v)$ and its dual $Z_{G^{*}}\left(Q, v^{*}\right)$ were given in (1.9) as sums over mutually dual subsets of edges $E_{1}$ and $E_{2}$. The last step of the transformation is to split the vertices of $\mathcal{M}(G)$ in the following way:

$$
\begin{equation*}
\text { If } e \in E_{1}, e \notin E_{2} \text { : If } e \notin E_{1}, e \in E_{2}: \tag{1.21}
\end{equation*}
$$

In concrete terms, this definition means that the loops bounce off all edges in $E_{1}$ and $E_{2}$, or, equivalently, they separate the FK clusters from their duals.

To complete the transformation, note that the number of loops $l\left(E_{1}\right)$ is the sum of the number of connected components $k\left(E_{1}\right)$ and the number of independent cycles $c\left(E_{1}\right)$,

$$
\begin{equation*}
l\left(E_{1}\right)=k\left(E_{1}\right)+c\left(E_{1}\right) \tag{1.22}
\end{equation*}
$$

Inserting this and the topological identity (1.12) into (1.7) we arrive at

$$
\begin{equation*}
Z=Q^{|V| / 2} \sum_{E_{1} \subseteq E} x^{\left|E_{1}\right|} Q^{l\left(E_{1}\right) / 2} \tag{1.23}
\end{equation*}
$$

where we have defined $x=v Q^{-1 / 2}$.
This is the loop representation of the Potts partition function. It importance stems from the fact that the loops, their connectivity properties, and the non-local quantity $l\left(E_{1}\right)$ all admit an algebraic interpretation within the Temperley-Lieb algebra [20].

In terms of the $x$ variables the duality relation (1.15) reads simply

$$
\begin{equation*}
x x^{*}=1 \tag{1.24}
\end{equation*}
$$

In the case of the square lattice, the self-dual points are $x_{\mathrm{c}}= \pm 1$, and the usual critical point is $x_{\mathrm{c}+}=1$. The loop model (1.23) then becomes extremely simple: there is just a weight $n=\sqrt{Q}$ for each loop.

### 1.1.6 Vertex model representation

In the definition of the $Q$-state Potts model, $Q$ was originally positive integers. However, in the corresponding loop model (1.23) it appears as formal parameters and may thus take arbitrary complex values. The price to pay for this generalization is the appearance of a non-locally defined quantity, the number of loops $l$. The locality of the model may be recovered at the expense of introducing complex Boltzmann weights, as we now show.

The following argument supposes that $G=(V, E)$ is a (connected) planar graph. Most applications however suppose a regular lattice, a situation to which we shall return shortly.

Consider any model of self-avoiding loops defined on $G$ (or some related graph, such as the medial graph $\mathcal{M}(G)$ for the Potts model). The Boltzmann weights are supposed to consist of a local piece-depending on if and how the loops pass through a given vertex-and a non-local piece of the form $n^{l}$, where $n$ is the loop weight and $l$ is the number of loops. In the case of the Potts model we have $n=\sqrt{Q}$.

In a first step, each loop is independently decorated by a global orientation, which by planarity and self-avoidance can be described as either anticlockwise or clockwise. Let us give a weight $\exp \left(i \gamma \frac{\alpha}{2 \pi}\right)$ whenever an orientated loop turns an angle $\alpha$ in the positive (trigonometric) direction. Summing over orientations this gives

$$
\begin{equation*}
n=\mathrm{e}^{i \gamma}+\mathrm{e}^{-i \gamma}=2 \cos \gamma \tag{1.25}
\end{equation*}
$$

Note that in the expected critical regime, $n \in[-2,2]$, we have $\gamma \in[0, \pi]$.
The loop model is now transformed into a local vertex model by assigning to each edge traversed by a loop the orientation of that loop. The total vertex weight equals the above local loop weights summed over the possible splittings of oriented loops which are compatible with the given edge orientations. In addition, one must multiply this by any loop-independent local weights, such as $x$ in (1.23).

### 1.1.7 Six-vertex model

To see how this is done, we finally specialize to the Potts model defined on the square lattice $G$. The loop model is defined on the corresponding medial lattice $\mathcal{M}(G)$ which is another (tilted) square lattice. Each edge of the lattice is visited by a loop, and two loop segments (possibly parts of the same loop) meet at each vertex. In the orientated loop representation, each vertex is therefore incident on two outgoing and two ingoing edges.

It is convenient for the subsequent discussion to make the couplings of the Potts model anisotropic. In its original spin formulation (1.4) we therefore let $K_{1}$ (resp. $K_{2}$ ) denote respectively the dimensionless coupling in the horizontal (resp. vertical) direction of the square lattice, and we let

$$
\begin{equation*}
x_{1}=\frac{\mathrm{e}^{K_{1}}-1}{\sqrt{Q}}, \quad x_{2}=\frac{\mathrm{e}^{K_{2}}-1}{\sqrt{Q}} \tag{1.26}
\end{equation*}
$$

be the corresponding parameters appearing in the loop representation (1.23).
The six possible configurations of arrows around a vertex of the medial lattice $\mathcal{M}(G)$ are shown in Figure 1.2. The corresponding vertex weights are denoted $\omega_{p}$ (resp. $\omega_{p}^{\prime}$ ) on the even (resp. odd) sublattice of $\mathcal{M}(G)$. By definition, a vertex of the even (resp. odd) sublattice of $\mathcal{M}(G)$ is the mid point of an edge with coupling $K_{1}$ (resp. $K_{2}$ ) of the original spin lattice $G$. With respect to Figure 1.2 we define the even sublattice to be such that

$\omega_{1}$


$\omega_{3}$

$\omega_{4}$


Figure 1.2 The allowed arrow arrangements (top) around a vertex that define the six-vertex model, with the corresponding particle trajectories (bottom).
an edge $e \in E$ runs horizontally, and the corresponding dual edge $e^{*} \in E^{*}$ is vertical. For the odd sublattice, exchange $e$ and $e^{*}$.

Using (1.23) we then have

$$
\begin{equation*}
Z=Q^{|V| / 2} \sum_{\text {arrows }} \prod_{p=1}^{6}\left(\omega_{p}\right)^{N_{p}}\left(\omega_{p}^{\prime}\right)^{N_{p}^{\prime}} \tag{1.27}
\end{equation*}
$$

where the sum is over arrow configurations satisfying the constraint 'two in, two out' at each vertex, and $N_{p}$ (resp. $N_{p}^{\prime}$ ) is the number of vertices on the even (resp. odd) sublattice with arrow configuration $p$. Thus, the square-lattice Potts model has been represented as a staggered six-vertex model. ${ }^{2}$ The weights read explicitly

$$
\begin{align*}
& \omega_{1}, \ldots, \omega_{6}=1,1, x_{1}, x_{1}, \mathrm{e}^{i \gamma / 2}+x_{1} \mathrm{e}^{-i \gamma / 2}, \mathrm{e}^{-i \gamma / 2}+x_{1} \mathrm{e}^{i \gamma / 2}  \tag{1.28}\\
& \omega_{1}^{\prime}, \ldots, \omega_{6}^{\prime}=x_{2}, x_{2}, 1,1, \mathrm{e}^{-i \gamma / 2}+x_{2} \mathrm{e}^{i \gamma / 2}, \mathrm{e}^{i \gamma / 2}+x_{2} \mathrm{e}^{-i \gamma / 2} \tag{1.29}
\end{align*}
$$

To see this, note that configurations $i=1,2,3,4$ are compatible with just one linking of the oriented loops:

whereas $i=5,6$ are compatible with two different linkings (and the weight is obtained by summing over these two):

Note that the even and odd sublattices are related by a $\pi / 2$ rotation of the vertices in Figure 1.2. This rotation interchanges configurations $\left(\omega_{1}, \omega_{2}\right) \leftrightarrow\left(\omega_{3}^{\prime}, \omega_{4}^{\prime}\right)$ and $\omega_{5} \leftrightarrow \omega_{6}^{\prime}$. On the level of the weights it corresponds to $x_{1} \leftrightarrow x_{2}$.

[^1]The staggered six-vertex model is not exactly solvable in general. However, if we impose that the couplings be mutually dual,

$$
\begin{equation*}
x_{2}=\left(x_{1}\right)^{-1} \tag{1.32}
\end{equation*}
$$

we have $\omega_{i}^{\prime}=\left(x_{1}\right)^{-1} \omega_{i}$ for any $i=1,2, \ldots, 6$. The factors $\left(x_{1}\right)^{-1}$ from each $\omega_{i}^{\prime}$ can be taken outside the summation in (1.27) and we have effectively $\omega_{i}^{\prime}=\omega_{i}$. The six-vertex model then becomes homogeneous.

The homogeneous six-vertex model turns out to be solvable when the weights $\omega_{i}$ are invariant under a global arrow reversal. The resulting weights are traditionally denoted

$$
\begin{equation*}
a=\omega_{1}=\omega_{2}, \quad b=\omega_{3}=\omega_{4}, \quad c=\omega_{5}=\omega_{6} \tag{1.33}
\end{equation*}
$$

The constraint $\omega_{5}=\omega_{6}$ is actually not necessary. Indeed, the corresponding vertices act as sources and sinks of arrows in either lattice direction, so with appropriate (periodic) boundary conditions there is an equal number of vertices of either type. Therefore $Z$ depends on $\omega_{5}, \omega_{6}$ only via their product, and we might as well set $c=\left(\omega_{5} \omega_{6}\right)^{1 / 2}$.

In the study of the six-vertex model, a special role is played by the so-called anisotropy parameter

$$
\begin{equation*}
\Delta=\frac{a^{2}+b^{2}-c^{2}}{2 a b} \tag{1.34}
\end{equation*}
$$

In our case we have simply

$$
\begin{equation*}
\Delta=-\cos \gamma \tag{1.35}
\end{equation*}
$$

We shall see that the Bethe ansatz equations depend only on $\Delta$.
The other independent ratio among $a, b, c$ is essentially $x_{1}$. We shall use it to parameterize the so-called spectral parameter $u$. It is seen to control the spatial anisotropy of the 2 D statistical model (the anisotropy corresponding to $\Delta$ refers instead to the equivalent 1 D quantum spin chain).

We stress once more that the square-lattice Potts model is solvable at its selfdual point, but not at arbitrary temperatures. ${ }^{3}$ This is in contrast with the Ising model, which is solvable at any temperature [24]. In that sense the Ising model is a rather untypical integrable model.

[^2]
### 1.2 Quantum integrability

Vertex models are statistical models in which integer-valued states are defined on each edge of some lattice, and the interaction takes place at the vertices. For a lattice of coordination number $z$ with $s$ possible states on each edge, each vertex can see $k_{0}=z^{s}$ possible arrangements of its incident edges. The Boltzmann weight at a vertex is taken to depend on this arrangement. Usually only $k \leq k_{0}$ arrangements correspond to a non-zero weight. We refer to the corresponding statistical model as a $k$-vertex model.

### 1.2.1 $R$-matrix

It is natural to think of the propagation through a vertex in a transfer matrix formalism. When $z$ is even, we can define a transfer direction so that $\frac{z}{2}$ consecutive edges define the in-state, and the remaining $\frac{z}{2}$ edges the out-state. The Boltzmann weights can then be regrouped in a $\left(k_{0}\right)^{1 / 2}$ dimensional matrix, called the $R$-matrix, with precisely $k$ non-zero entries.

Figure 1.2 defines the 6 -vertex model on a square lattice ( $z=4$ and $s=2$ ). We take the transfer direction to be upwards, so the two edges on the bottom (resp. top) define the in-state (resp. out-state). The state of an edge supporting a down-arrow (resp. an up-arrow) is denoted $|1\rangle$ (resp. $|0\rangle$ ). The state $|1\rangle$ can be interpreted as the presence of a particle. The particles are conserved by the time evolution, because the allowed vertices have two ingoing and two outgoing arrows.

The basis of in-states can now be written

$$
\begin{equation*}
\left(\mathbb{C}^{2}\right)^{2}=\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\} \tag{1.36}
\end{equation*}
$$

where the left in-state refers to the leftmost edge on the bottom. The out-states can be labelled similarly, but two different conventions are possible. In the first convention, the left out-state refers to the rightmost edge on the top, so that the labelling of spaces follows the lines (which intersect at the vertex). This defines the $R$-matrix:

$$
R=\left[\begin{array}{cccc}
\omega_{1} & 0 & 0 & 0  \tag{1.37}\\
0 & \omega_{3} & \omega_{6} & 0 \\
0 & \omega_{5} & \omega_{4} & 0 \\
0 & 0 & 0 & \omega_{2}
\end{array}\right]
$$

In the second convention, the left out-state refers to the leftmost edge on the top, so that in-states and out-states have the same left/right convention. This defines the $\check{R}$-matrix:

$$
\check{R}=\left[\begin{array}{cccc}
\omega_{1} & 0 & 0 & 0  \tag{1.38}\\
0 & \omega_{5} & \omega_{4} & 0 \\
0 & \omega_{3} & \omega_{6} & 0 \\
0 & 0 & 0 & \omega_{2}
\end{array}\right]
$$

Obviously $\check{R}=P R$, where $P$ is the operator that permutes the two spaces.

### 1.2.2 Spectral parameter

Let us parameterize the weights of the six-vertex model as follows:

$$
\begin{align*}
\omega_{1}=\omega_{2} & =\sin (\gamma-u), \\
\omega_{3}=\omega_{4} & =\sin u, \\
\omega_{5} & =\mathrm{e}^{-i(u-\eta)} \sin \gamma, \\
\omega_{6} & =\mathrm{e}^{i(u-\eta)} \sin \gamma . \tag{1.39}
\end{align*}
$$

We have then $\Delta=-\cos \gamma$. The gauge parameter $\eta$ can be chosen at will, since vertices of type 5 and 6 appear in pairs, and only the value of $\sqrt{\omega_{5} \omega_{6}}$ enters the computation of the partition function. The correspondence (1.28) with the selfdual Potts model then requires

$$
\begin{equation*}
x_{1}=\frac{1}{x_{2}}=\frac{\omega_{3}}{\omega_{1}}=\frac{\sin u}{\sin (\gamma-u)} . \tag{1.40}
\end{equation*}
$$

We note that $u$ parameterizes the spatial anisotropy of the coupling constants. The isotropic point at which $x_{1}=x_{2}=1$ corresponds to $u=\frac{\gamma}{2}$.

More generally, we wish to formalize some useful properties of the $R$-matrix for vertex models defined on a so-called Baxter lattice [25]. By this we mean any lattice that can be drawn in the plane as a collection of lines (one can think of them as straight lines, but this is not necessary) that undergo only pairwise intersections. We attribute a fixed orientation to each line in the lattice, and associate a so-called spectral parameter $u \in \mathbb{C}$ with each oriented line.

The Boltzmann weights for a vertex where two lines with spectral parameters $u$ and $v$ intersect are supposed to have the difference property: they depend only on the difference $u-v$. To be precise, when viewing the vertex along the forward direction of the two oriented lines, $u($ resp. $v)$ is the spectral parameter of the line to the left (resp. right) of the observer. In particular, inverting the direction of the line corresponds to $u \rightarrow-u$.

It is useful to turn the vertices through an angle $-\frac{\pi}{4}$, so that time flows in the north-easterly direction, rather than upwards. This is shown for the six-vertex model in Figure 1.3 where we also give the corresponding trajectories of particles (the state $|1\rangle$ ). The elements of the $R$-matrix can then be written,


Figure 1.3 The allowed arrow arrangements (top) around a vertex that define the six-vertex model, with the corresponding particle trajectories (bottom).

$$
\begin{equation*}
\mu_{i} \xrightarrow[v_{\uparrow}]{\substack{\beta_{i} \\ \alpha_{i}}} \mu_{i+1}=R_{\mu_{i} \alpha_{i}}^{\mu_{i+1} \beta_{i}}(u-v)={ }_{\mathrm{a}}\left\langle\mu_{i+1}\right| \otimes_{i}\left\langle\beta_{i}\right| R_{\mathrm{a} i}(u-v)\left|\mu_{i}\right\rangle_{\mathrm{a}} \otimes\left|\alpha_{i}\right\rangle_{i} . \tag{1.41}
\end{equation*}
$$

We have oriented the horizontal line towards the right and called the corresponding space a (for auxiliary space). The vertical line is oriented upwards, and its space is labelled $i$ (for $i$ 'th quantum space). The indices $\mu_{i}, \alpha_{i}$ and $\mu_{i+1}, \beta_{i}$ label states living on the lattice edges. The last notation $R_{\mathrm{a} i}(u-v)$ makes explicit both the labelling of spaces and the difference property of the spectral parameters.

More formally, the $R$-matrix is a linear operator (endomorphism)

$$
\begin{equation*}
R_{\mathrm{a} i}: V_{\mathrm{a}} \otimes V_{i} \mapsto V_{\mathrm{a}} \otimes V_{i}, \tag{1.42}
\end{equation*}
$$

where the vector spaces $V_{\mathrm{a}}$ (auxiliary) and $V_{i}$ (quantum) carry the edge degrees of freedom. For instance, in the six-vertex model they are both equal to the spin- $\frac{1}{2}$ representation space $\mathbb{C}^{2}$, since each arrow can be in two possible states; the $R$-matrix is then a $4 \times 4$ matrix.

### 1.2.3 Transfer matrix

We now wish to define the row-to-row transfer matrix for a system of width $L$ with periodic boundary directions in the horizontal direction.

The transfer matrix $t$ is an endomorphism on the tensor product of all quantum spaces,

$$
\begin{equation*}
t: V_{1} \otimes V_{2} \otimes \cdots \otimes V_{L} \mapsto V_{1} \otimes V_{2} \otimes \cdots \otimes V_{L} \tag{1.43}
\end{equation*}
$$

It can be written as

$$
\begin{equation*}
t=\operatorname{Tr}_{\mathrm{a}}\left(R_{\mathrm{a} L} R_{\mathrm{a} L-1} \cdots R_{\mathrm{a} 2} R_{\mathrm{a} 1}\right) \tag{1.44}
\end{equation*}
$$

where $\operatorname{Tr}_{\mathrm{a}}$ denotes the trace over the auxiliary space $V_{\mathrm{a}}$. For simplicity we have not written the dependence on the spectral parameters. Indeed, one has the possibility of taking different spectral parameters for each quantum space $V_{i}$, and also for $V_{\mathrm{a}}$, which will correspond to a completely inhomogeneous lattice model. The matrix elements of $t$ can be written very explicitly as

$$
\begin{equation*}
\langle\beta| t|\alpha\rangle=\sum_{\mu_{1}, \ldots, \mu_{L}} R_{\mu_{L} \alpha_{L}}^{\mu_{1} \beta_{L}} R_{\mu_{L-1} \alpha_{L-1}}^{\mu_{L} \beta_{L-1}} \cdots R_{\mu_{2} \alpha_{2}}^{\mu_{3} \beta_{2}} R_{\mu_{1} \alpha_{1}}^{\mu_{2} \beta_{1}} . \tag{1.45}
\end{equation*}
$$

We remark that $\mu_{1}$ appears both in the rightmost and the leftmost factor, so we indeed perform the operator $\operatorname{Tr}_{\mathrm{a}}$.

Notice that although the individual $R$-matrices evolve the system in the north-easterly direction, the result of the trace is that $t$ evolves the system upwards.

### 1.2.4 Commuting transfer matrices

The $R$-matrix formalism presented here makes sense on any lattice of orientated lines that make only pairwise intersections. Such a lattice is called a Baxter lattice. It is possible to impose periodic boundary conditions on some of the lines. Notice that the lines are not required to be straight, nor to be disposed in any regular fashion.

A statistical model defined on a Baxter lattice is said to be integrable provided its $R$-matrix satisfies the Yang-Baxter equation and the inversion relation. The Yang-Baxter relation reads pictorially


The algebraic transcription is

$$
\begin{equation*}
R_{12}(u) R_{13}(u+v) R_{23}(v)=R_{23}(v) R_{13}(u+v) R_{12}(u), \tag{1.47}
\end{equation*}
$$

where we have set $u=u_{1}-u_{2}$ and $v=u_{2}-u_{3}$, so that $u_{1}-u_{3}=u+v$. We stress again that the spectral parameters $u_{i}$ always follow the lines of the Baxter lattice. The same is true for the labels of the representation spaces, that appear as subscripts for the $R$-matrix. It is sometimes convenient for these labels to stay well ordered in space (i.e. with 1 on the left, 2 in the middle, and 3 on the right) at all times (in the diagram time flows upwards). In that case one uses instead the $\check{R}$-matrix, for which the Yang-Baxter equation reads

$$
\begin{equation*}
\check{R}_{23}(u) \check{R}_{12}(u+v) \check{R}_{23}(v)=\check{R}_{12}(v) \check{R}_{23}(u+v) \check{R}_{12}(u) . \tag{1.48}
\end{equation*}
$$

The inversion relation can be represented pictorially as

and reads algebraically

$$
\begin{equation*}
R_{21}(u) R_{12}(-u) \propto I \tag{1.50}
\end{equation*}
$$

The constant of proportionality could of course be set to unity by a suitable rescaling of $R$. Note also how the sign convention for spectral parameters comes into use when writing (1.50).

The relations (1.47) and (1.50) imply the commutation of two transfer matrices corresponding to different choices of spectral parameters on the auxiliary lines. This is best demonstrated graphically:



The first picture represents the product $t\left(u_{2}\right) t\left(u_{1}\right)$, since the two crossings to the left amount to the identity by (1.50). ${ }^{4}$ In the second picture we have used (1.47) to push the $v_{1}$ line to the left. This is repeated in the third picture for the next $v_{2}$ line. Repeating this operation $L$ times, we finally arrive at the last picture, which represents the product $t\left(u_{1}\right) t\left(u_{2}\right)$, apart from the crossings on the left and right. But the right crossing can be taken around the periodic boundary condition (more formally, we are using the cyclicity of the trace), and using once more (1.50) the two crossings annihilate. Summarizing, we have shown that

$$
\begin{equation*}
t\left(u_{2}\right) t\left(u_{1}\right)=t\left(u_{1}\right) t\left(u_{2}\right) \tag{1.52}
\end{equation*}
$$

The existence of an infinite family of commuting transfer matrices has important consequences. Indeed the Bethe ansatz technique permits us to diagonalize all these transfer matrices simultaneously.

[^3]Moreover, we can take derivatives of (1.52) with respect to $u_{2}$. All these derivatives commute with $t\left(u_{1}\right)$, hence are conserved by the time evolution process. In other words, an integrable system has an infinite number of conserved quantities. The first few derivatives can be identified with the Hamiltonian, the momentum operator, and so on. We shall present explicit examples later.

Note also that the various vector spaces in which the $R$-matrices act need not be isomorphic. In particular, one can have different representations on the quantum and auxiliary spaces. From a basic integrable model-such as the six-vertex model in the spin- $\frac{1}{2}$ representation-one can construct higher-spin solutions by appropriate fusions of representation spaces. One speaks in that case of descendent models. An example will be given in Chapter 6.

### 1.2.5 Six-vertex model

It is an instructive exercise to verify that the $R$-matrix of the six-vertex model indeed satisfies (1.47) and (1.50). Choosing the gauge $\eta=0$ in the parameterization (1.39), the $\check{R}$-matrix (1.38) reads

$$
\check{R}(u)=\left[\begin{array}{cccc}
\sin (\gamma-u) & 0 & 0 & 0  \tag{1.53}\\
0 & \mathrm{e}^{-i u} \sin \gamma & \sin u & 0 \\
0 & \sin u & \mathrm{e}^{i u} \sin \gamma & 0 \\
0 & 0 & 0 & \sin (\gamma-u)
\end{array}\right]
$$

We identify the uniformizing parameter $u$ with the difference of spectral parameters.
In tensor notation (1.48) reads in the space $\left(\mathbb{C}^{2}\right)^{3}$

$$
\begin{equation*}
(I \otimes \check{R}(u))(\check{R}(u+v) \otimes I)(I \otimes \check{R}(v))=(\check{R}(v) \otimes I)(I \otimes \check{R}(u+v))(\check{R}(u) \otimes I) \tag{1.54}
\end{equation*}
$$

This identity between $8 \times 8$ matrices is greatly simplified by the symmetries of the problem. Firstly, the number of particles is conserved. Secondly, the weights are invariant under a global negation $(0 \leftrightarrow 1)$ of the occupation numbers.

The only equations to be verified thus concern a $1 \times 1$ matrix (in the 0 -particle space $|000\rangle$ ) and a $3 \times 3$ matrix (in the 1 -particle space $|100\rangle,|010\rangle,|001\rangle$ ). Only the latter gives rise to non-trivial equations. They turn out to be verified, upon application of trigonometric identities.

The inversion relation (1.50) can be verified similarly. In particular, we obtain the proportionality factor,

$$
\begin{equation*}
\check{R}(u) \check{R}(-u)=\sin (\gamma-u) \sin (\gamma+u) I . \tag{1.55}
\end{equation*}
$$

There is an alternative route to deriving the integrable $R$-matrix of the six-vertex model, which has the advantage of revealing a rich underlying algebraic structure. To achieve this we first need a few definitions.

### 1.2.6 Temperley-Lieb algebra

The Temperley-Lieb algebra $T L_{N}(n)$ is a unital associative algebra over $\mathbb{C}$. Its $N-1$ generators are denoted $E_{m}$ for $m=1,2, \ldots, N-1$. They satisfy the relations [20]

$$
\begin{align*}
\left(E_{m}\right)^{2} & =n E_{m}, \\
E_{m} E_{m \pm 1} E_{m} & =E_{m}, \\
E_{m} E_{m^{\prime}} & =E_{m^{\prime}} E_{m} \text { for }\left|m-m^{\prime}\right|>1 . \tag{1.56}
\end{align*}
$$

As for any abstract algebra, $T L_{N}(n)$ can be represented in different ways. We shall be particularly interested in its loop-model representation, since this permits us to make contact with Section 1.1 .5 where the Potts model was formulated as a loop model. In this representation, $T L_{N}(n)$ is viewed as an algebra of diagrams acting on $N$ numbered vertical strands (for convenience depicted inside a dashed box) as


Multiplication in $T L_{N}(n)$ is defined by stacking diagrams vertically. More precisely, the product of two generators $g_{2} g_{1}$ is defined by placing the diagram for $g_{2}$ above the diagram for $g_{1}$, identifying the bottom points of $g_{2}$ with the top points of $g_{1}$. The resulting diagram is considered up to smooth isotopies that keep fixed the surrounding box, and any closed loop is replaced by the factor $n$.

In this way we have for instance (omitting strands on which the action is trivial)

$$
\left(E_{m}\right)^{2}=\bigcirc_{\Omega}=n \bigodot^{-}=n E_{m}
$$

and


It is thus readily seen that all the defining relations (1.56) are satisfied. Moreover, for generic values of $n$ no further relations hold; the loop-model representation is faithful.

The connection with the Potts model is provided by (1.31): the left-hand diagram is nothing but the generator $E_{m}$ of (1.57), while the right-hand diagram is the identity operator. We recall that according to (1.25) the loop weight is $n=2 \cos \gamma$.

### 1.2.6.1 Integrable R-matrix in the loop representation

Starting from first principles, we now construct an integrable model based on the TL algebra. Let us suppose that the $\check{R}$-matrix has the form

$$
\begin{equation*}
\check{R}_{m, m+1}(u)=f(u) I+g(u) E_{m}, \tag{1.58}
\end{equation*}
$$

where $f(u)$ and $g(u)$ are some functions of the spectral parameter $u$ to be determined. Inserting this into the Yang-Baxter equation (1.48) yields

$$
\begin{align*}
& \left(f(u) I+g(u) E_{2}\right)\left(f(u+v) I+g(u+v) E_{1}\right)\left(f(v) I+g(v) E_{2}\right)= \\
& \left(f(v) I+g(v) E_{1}\right)\left(f(u+v) I+g(u+v) E_{2}\right)\left(f(u) I+g(u) E_{1}\right) . \tag{1.59}
\end{align*}
$$

Using the algebraic relations (1.56) we can expand both sides of (1.59). The left-hand side produces

$$
\begin{aligned}
& f(u) f(u+v) f(v) I+f(u) g(u+v) f(v) E_{1}+ \\
& g(u) g(u+v) f(v) E_{2} E_{1}+f(u) g(u+v) g(v) E_{1} E_{2}+ \\
& {[g(u) g(v)(g(u+v)+n f(u+v))+f(u+v)(f(u) g(v)+f(v) g(u))] E_{2},}
\end{aligned}
$$

and the right-hand side becomes

$$
\begin{aligned}
& f(v) f(u+v) f(u) I+f(v) g(u+v) f(u) E_{2}+ \\
& f(v) g(u+v) g(u) E_{2} E_{1}+g(v) g(u+v) f(u) E_{1} E_{2}+ \\
& {[g(u) g(v)(g(u+v)+n f(u+v))+f(u+v)(f(u) g(v)+f(v) g(u))] E_{1} .}
\end{aligned}
$$

These expressions must be identical in $T L_{3}(n)$, and so we can identify the coefficients for each of the five possible words in the algebra. The relations resulting from the words $I, E_{1} E_{2}$, and $E_{2} E_{1}$ are trivial. The relations coming from $E_{1}$ and $E_{2}$ are identical—related via an exchange of the left- and right-hand sides-and read

$$
\begin{align*}
& g(u) g(v)(g(u+v)+n f(u+v))+f(u+v)(f(u) g(v)+f(v) g(u))= \\
& f(u) f(v) g(u+v) . \tag{1.60}
\end{align*}
$$

The functional relation (1.60) is a typical outcome of this way of solving the YangBaxter equations. It is in general not easy to solve this type of relation, and even if one finds solutions it is often difficult to make sure that one has found all the solutions. Worse, in more complicated cases than the one considered here the ansatz for the $\check{R}$-matrix will involve more terms and the functions $f(u), g(u), \ldots$ must satisfy several coupled functional equations.

It is useful to rewrite (1.60) in terms of the parameters $z=\mathrm{e}^{i u}, w=\mathrm{e}^{i v}$ and $q=\mathrm{e}^{i \gamma}$. That is, instead of the additive spectral parameters $u, v$ we have now multiplicative spectral parameters $z, w$. Thus,

$$
\begin{equation*}
g(z) g(w)\left(g(z w)+\left(q+q^{-1}\right) f(z w)\right)+f(z w)(f(z) g(w)+f(w) g(z))=f(z) f(w) g(z w) \tag{1.61}
\end{equation*}
$$

It is tempting to set $f(z)=1$, since the overall normalization of the $\check{R}$-matrix is unimportant, but in general this is not a good idea. A time proven strategy is to suppose that $f(z)$ and $g(z)$ are polynomials of some small degree in the variables $z, z^{-1}, q$ and $q^{-1}$. (In some cases one needs to try fractional powers of $q$ as well). In this case we are lucky, there is a solution of degree one,

$$
\begin{align*}
& f(z)=\frac{q}{z}-\frac{z}{q}  \tag{1.62}\\
& g(z)=z-z^{-1} \tag{1.63}
\end{align*}
$$

Going back to additive spectral parameters, we thus have a trigonometric solution of (1.59),

$$
\begin{equation*}
f(u)=\sin (\gamma-u), \quad g(u)=\sin (u) \tag{1.64}
\end{equation*}
$$

In general, solutions to the Yang-Baxter equation turn out to be polynomial, trigonometric, or elliptic (in order of increasing difficulty).

Summarizing, we have found the integrable $\check{R}$-matrix

$$
\begin{equation*}
\check{R}(u)=\sin (\gamma-u) I+\sin (u) E . \tag{1.65}
\end{equation*}
$$

### 1.2.6.2 Back to the vertex model

Combining (1.65) with (1.53) we obtain the TL generator in the vertex-model representation,

$$
E=\left[\begin{array}{cccc}
0 & 0 & 0 & 0  \tag{1.66}\\
0 & \mathrm{e}^{-i \gamma} & 1 & 0 \\
0 & 1 & \mathrm{e}^{i \gamma} & 0 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

By forming tensor products, one can of course verify that the defining relations (1.56) are indeed satisfied.

It is convenient to make manifest the spin- $\frac{1}{2}$ nature of the six-vertex model by reexpressing things in terms of the Pauli matrices

$$
\sigma^{x}=\left[\begin{array}{ll}
0 & 1  \tag{1.67}\\
1 & 0
\end{array}\right], \quad \sigma^{y}=\left[\begin{array}{rr}
0 & -i \\
i & 0
\end{array}\right], \quad \sigma^{z}=\left[\begin{array}{rr}
1 & 0 \\
0 & -1
\end{array}\right] .
$$

The arrow conservation then means that the transfer matrix $t(u)$ commutes with the total magnetization

$$
\begin{equation*}
S^{z}=\frac{1}{2} \sum_{m=1}^{L} \sigma_{m}^{z} . \tag{1.68}
\end{equation*}
$$

The Temperley-Lieb generator can be written as

$$
\begin{equation*}
E_{m}=\frac{1}{2}\left[\sigma_{m}^{x} \sigma_{m+1}^{x}+\sigma_{m}^{y} \sigma_{m+1}^{y}-\cos \gamma\left(\sigma_{m}^{z} \sigma_{m+1}^{z}-I\right)-i \sin \gamma\left(\sigma_{m}^{z}-\sigma_{m+1}^{z}\right)\right] . \tag{1.69}
\end{equation*}
$$

### 1.2.7 Spectral parameter and anisotropy

The physical meaning of the spectral parameter $u$ is that it controls the spatial anisotropy of the system. To see this qualitatively, note that in the $u \rightarrow 0$ limit, the $\check{R}$-matrix is proportional to the identity by (1.65). The transfer matrix $t(u)$ thus acts on a state just by shifting all spins one unit to the right (with periodic boundary conditions); note that this follows from the fact that time propagates in the north-easterly direction.

In a $1+1$ dimensional quantum mechanical analogy, the $u \rightarrow 0$ limit thus means that interactions between spins happen very slowly. Equivalently, the time direction has been stretched with respect to the spatial direction. A homogeneous system can be retrieved by rescaling time by a certain anisotropy factor $\zeta(u)$. Determining $\zeta(u)$ requires some more work: the result is [26]

$$
\begin{equation*}
\zeta(u)=\sin \left(\frac{\pi u}{\gamma}\right) . \tag{1.70}
\end{equation*}
$$

This predicts that the isotropic point $\zeta(u)=1$ occurs for $u=\frac{\gamma}{2}$, which is in accord with (1.65).

### 1.2.8 Spin chain Hamiltonian

Using (1.65) we thus see that in the completely anisotropic limit $u \rightarrow 0$ the transfer matrix becomes

$$
\begin{equation*}
t(0)=\sin ^{L}(\gamma) \mathrm{e}^{-i P} \tag{1.71}
\end{equation*}
$$

where $\mathrm{e}^{-i P}$ is the shift operator that translates the lattice sites one unit to the right. Equivalently, $P$ can be interpreted as the momentum operator.

We know from the path-integral formalism that the transfer matrix (the time evolution operator) is the exponential of the quantum Hamiltonian. To make things completely precise, note that to first order in $u$, one may omit one of the factors $\sin (\gamma-u) I$ in (1.65) and take $\sin (u) E$ instead. The correct development in the limit $u \rightarrow 0$ therefore reads

$$
\begin{equation*}
t(u) \simeq t(0) \exp \left[-\frac{u}{\sin \gamma} H\right], \tag{1.72}
\end{equation*}
$$

where $H$ is the Hamiltonian of the spin chain. Equivalently

$$
\begin{equation*}
H=-\left.\sin \gamma \frac{\partial}{\partial u} \log t(u)\right|_{u \rightarrow 0}=-\sin \gamma t(0)^{-1} t^{\prime}(0) \tag{1.73}
\end{equation*}
$$

Here the inverse $t(0)^{-1}=(\sin \gamma)^{-L} \mathrm{e}^{i P}$ is just the shift in the opposite (left) direction. The derivative $t^{\prime}(0)$ gives $L$ terms, one for each of the factors in the product (1.45). Using (1.65) we have $\tilde{R}^{\prime}(0)=-\cos \gamma I+E$. Therefore

$$
\begin{equation*}
H=L \cos \gamma I-\sum_{m=1}^{L} E_{m} \tag{1.74}
\end{equation*}
$$

Inserting the expression (1.69) for the TL generators in terms of Pauli matrices, the piece in $i \sin \gamma\left(\sigma_{m}^{z}-\sigma_{m+1}^{z}\right)$ simplifies by telescopy. With open boundary condition it would become a surface magnetic field acting on the first and last spins. We consider instead periodic boundary conditions, so this term vanishes altogether. One is left with

$$
\begin{equation*}
H=-\frac{1}{2} \sum_{m=1}^{L}\left[\sigma_{m}^{x} \sigma_{m+1}^{x}+\sigma_{m}^{y} \sigma_{m+1}^{y}+\Delta\left(\sigma_{m}^{z} \sigma_{m+1}^{z}+I\right)\right] \tag{1.75}
\end{equation*}
$$

where we recall that $\Delta=-\cos \gamma$.
We thus arrive at the Hamiltonian of a Heisenberg-type spin chain, where however the interaction is anisotropic along the $z$-direction. For that reason, this is called the $X X Z$ spin chain with anisotropy parameter $\Delta .{ }^{5}$

Let us emphasize that due to the commutativity of transfer matrices, the eigenvectors of the six-vertex model transfer matrix and of the XXZ spin chain Hamiltonian are identical. It is thus equivalent to diagonalize one or the other, and in that sense the two models are equivalent.

### 1.3 Algebraic Bethe ansatz

We have seen that the existence of an integrable $\check{R}$-matrix entails an infinite number of conserved quantities. This makes us suspect that the corresponding statistical modelour main example being the selfdual Potts model on a square lattice, or the equivalent XXZ quantum spin chain-may be exactly solvable in some sense.

The algebraic Bethe ansatz [27] provides the fulfilment of these expectations. It provides a formalism in which the partition function and correlation functions can be exactly computed, at least in the thermodynamic limit. Recent years have also seen much

[^4]progress on the computation of certain correlation functions in finite size and/or at finite separation of the points, but this topic is beyond the scope of these lectures.

### 1.3.1 Monodromy matrix

We define the monodromy matrix $T(u)$ as the same product over $R$-matrices that was used in defining the transfer matrix (1.44), but without the trace over the auxiliary space,

$$
\begin{equation*}
T(u)=R_{\mathrm{a} L} R_{\mathrm{a} L-1} \cdots R_{\mathrm{a} 2} R_{\mathrm{a} 1} . \tag{1.76}
\end{equation*}
$$

Thus $T(u)$ is an endomorphism on the auxiliary space $V_{\mathrm{a}}$ and we have

$$
\begin{equation*}
t(u)=\operatorname{Tr}_{\mathrm{a}} T(u) . \tag{1.77}
\end{equation*}
$$

When several auxiliary spaces are involved we shall sometimes use the notation $T_{\mathrm{a}}(u)$ to make clear what space is involved. We shall also denote matrix elements of $T(u)$ using the same convention as for the $R$-matrix, and sometimes represent them graphically as

$$
\begin{equation*}
T_{i}^{j}(u)=i \not \| j \tag{1.78}
\end{equation*}
$$

These matrix elements are operators acting in the quantum spaces, here shown symbolically as a double line.

We can repeat the reasoning of (1.51),

to establish that

$$
\begin{equation*}
R_{\mathrm{ab}}(u-v)\left(T_{\mathrm{a}}(u) T_{\mathrm{b}}(v)\right)=\left(T_{\mathrm{b}}(v) T_{\mathrm{a}}(u)\right) R_{\mathrm{ab}}(u-v) . \tag{1.80}
\end{equation*}
$$

This identity is known popularly as the RTT relation. Using the double line convention of (1.78) it can also be written pictorially


### 1.3.2 Co-product and Yang-Baxter algebra

A Yang-Baxter algebra $\mathcal{A}$ is a couple $(R, T)$ satisfying the RTT relation (1.80). Its generators are the matrix elements $T_{i}^{j}(u)$. It is equipped with a product, obtained graphically by stacking two monodromy matrices along a common quantum space (represented as a double line).

In addition to this product, $\mathcal{A}$ is also equipped with a co-product $\Delta,{ }^{6}$ obtained graphically by gluing together two monodromy matrices along a common auxiliary space (represented as a single line). We have

The co-product thus serves to map the algebra $\mathcal{A}$ into the tensor product $\mathcal{A} \otimes \mathcal{A}$ :

$$
\begin{align*}
\Delta & : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A} \\
T_{i}^{j}(u) & \mapsto \sum_{k} T_{i}^{k}(u) \otimes T_{k}^{j}(u) \tag{1.83}
\end{align*}
$$

while preserving the algebraic relations of $\mathcal{A}$.

In particular, the co-product $\Delta T_{i}^{j}$ must again satisfy the RTT relation (1.80). It is a nice exercise to understand what this means and to prove it.

An algebra equipped with a product and a co-product is called a bi-algebra. To be precise, we need a little more structure (co-associativity, existence of a co-unit, ...). If in addition we have an antipode (and if various diagrams commute) one arrives at a Hopf algebra.

[^5]
### 1.3.3 Six-vertex model

When the auxiliary space is $\mathbb{C}^{2}$, the matrix elements of the monodromy matrix are usually denoted as follows:

$$
\begin{equation*}
T_{0}^{0}(u)=A(u), \quad T_{1}^{0}(u)=B(u), \quad T_{0}^{1}(u)=C(u), \quad T_{1}^{1}(u)=D(u) . \tag{1.84}
\end{equation*}
$$

Recall that the structure constants of a Lie algebra provide a representation, known as the adjoint. In the same way, the $R$-matrix provides a representation of dimension 2 of the Yang-Baxter algebra. Indeed, in the special case where the double line is just a single line, the monodromy matrix reduces to the $R$-matrix,

$$
\begin{equation*}
\left(T_{i}^{j}(u)\right)_{l}^{k}=R_{i l}^{j k}(u) . \tag{1.85}
\end{equation*}
$$

The RTT relation is then nothing but the Yang-Baxter relation for the $R$-matrix.
The notation (1.84) just amounts to reading the $R$-matrix as a $2 \times 2$ matrix of blocks of size $2 \times 2$. According to (1.37) we have

$$
R=\left[\begin{array}{cc|cc}
\omega_{1} & 0 & 0 & 0  \tag{1.86}\\
0 & \omega_{4} & \omega_{5} & 0 \\
\hline 0 & \omega_{6} & \omega_{3} & 0 \\
0 & 0 & 0 & \omega_{2}
\end{array}\right]=\left[\begin{array}{cc}
A(u) & B(u) \\
C(u) & D(u)
\end{array}\right] .
$$

We recall the usual weights $a, b, c$ (which now depend on the spectral parameter $u$ ), and we take the gauge $\eta=u$ in (1.53):

$$
\begin{equation*}
a(u)=\sin (\gamma-u), \quad b(u)=\sin u, \quad c(u)=\sin \gamma . \tag{1.87}
\end{equation*}
$$

We have then in explicit notation, and in terms of Pauli matrices,

$$
\begin{align*}
& A(u)=\left[\begin{array}{cc}
a(u) & 0 \\
0 & b(u)
\end{array}\right]=\frac{a(u)+b(u)}{2} I+\frac{a(u)-b(u)}{2} \sigma^{z}, \\
& B(u)=\left[\begin{array}{cc}
0 & 0 \\
c(u) & 0
\end{array}\right]=\frac{c(u)}{2}\left(\sigma^{x}-i \sigma^{y}\right)=c(u) \sigma^{-}, \\
& C(u)=\left[\begin{array}{cc}
0 & c(u) \\
0 & 0
\end{array}\right]=\frac{c(u)}{2}\left(\sigma^{x}+i \sigma^{y}\right)=c(u) \sigma^{+}, \\
& D(u)=\left[\begin{array}{cc}
b(u) & 0 \\
0 & a(u)
\end{array}\right]=\frac{a(u)+b(u)}{2} I-\frac{a(u)-b(u)}{2} \sigma^{z} . \tag{1.88}
\end{align*}
$$

Note that $B(u)$ (resp. $C(u)$ ) acts as a creation (resp. annihilation) operator on the quantum space, with respect to the pseudo-vacuum in which all spins are up. We shall
see later that this interpretation remains valid when taking co-products: $B(u)$ transforms $n$ particle states into $n+1$ particle states (and vice versa for $C(u)$ ).

### 1.3.3.1 Co-product

Establishing how the co-product acts on the operators $A(u), B(u), C(u)$, and $D(u)$ will turn out to be an important ingredient in the sequel. In more formal terms, we wish to obtain a representation of the six-vertex Yang-Baxter algebra $\mathcal{A}$ on the space $V^{\otimes L}$.

Let us begin by examining the case $L=2$ in details. Consider for instance the construction of $\Delta B(u)$. We have


Here the left and right indices define $B(u)=T_{1}^{0}(u)$, and the co-multiplication implies a sum over the middle index. The bottom (resp. top) indices define the in-state (resp. out-state) of the quantum spaces, here denoted as kets.

Proceeding in the same way for the three other in-states, we find that $\Delta B(u)$ can be written in the basis $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$ as

$$
\begin{align*}
\Delta B(u) & =\left[\begin{array}{cc|cc}
0 & 0 & 0 & 0 \\
b(u) c(u) & 0 & 0 & 0 \\
\hline c(u) a(u) & 0 & 0 & 0 \\
0 & c(u) b(u) & a(u) c(u) & 0
\end{array}\right] \\
& =\left[\begin{array}{cc|cc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline c(u) a(u) & 0 & 0 & 0 \\
0 & c(u) b(u) & 0 & 0
\end{array}\right]+\left[\begin{array}{cc|cc}
0 & 0 & 0 & 0 \\
b(u) c(u) & 0 & 0 & 0 \\
\hline 0 & 0 & 0 & 0 \\
0 & 0 & a(u) c(u) & 0
\end{array}\right] \\
& =B(u) \otimes A(u)+D(u) \otimes B(u) . \tag{1.90}
\end{align*}
$$

It is actually simpler to avoid specifying the states of the quantum spaces altogether. Applying (1.82) directly one then obtains

$$
\begin{equation*}
\Delta^{L-1} B(u)=\underbrace{1-0-1}_{\Delta^{L_{1}-1} B(u) \otimes \Delta^{L_{2}-1} A(u)} 0 \quad \underbrace{1-1-}_{\Delta^{L_{1}-1} D(u) \otimes \Delta^{L_{2}-1} B(u)} . \tag{1.91}
\end{equation*}
$$

Note that this derivation applies for any bipartition $L_{1}+L_{2}=L$, and not only for $L_{1}=L_{2}=1$.

Repeating the working for the three other operators, the complete co-multi-plication table reads

$$
\begin{align*}
& \Delta A(u)=A(u) \otimes A(u)+C(u) \otimes B(u), \\
& \Delta B(u)=B(u) \otimes A(u)+D(u) \otimes B(u), \\
& \Delta C(u)=C(u) \otimes D(u)+A(u) \otimes C(u), \\
& \Delta D(u)=D(u) \otimes D(u)+B(u) \otimes C(u) \tag{1.92}
\end{align*}
$$

To generalize this construction from $L=2$ to arbitrary $L$ it suffices to use the associativity of the co-multiplication. Indeed for $L \geq 2$ we have

$$
\begin{align*}
& \Delta^{L-1}: \mathcal{A} \rightarrow \mathcal{A}^{\otimes L} \\
& \Delta^{L-1} \mapsto\left(I^{\otimes L-2} \otimes \Delta\right) \Delta^{L-2} \tag{1.93}
\end{align*}
$$

When making this definition, we have chosen to insert new tensorands from the right. Inserting them from the left would make no difference to the result, since in any case it can also be computed directly along the lines of (1.89). In the latter case, one has to sum over all $L-1$ intermediate indices, of the type $k$ in (1.82). It is a useful exercise to compute $\Delta^{2} B(u)$ for $L=3$ in all three ways and check that one obtains identical results.

In the following we shall simplify the notation and write, for example, $B(u)$ instead of $\Delta^{L-1} B(u)$. Thus $B(u)$ is an operator that acts on all $L$ spaces in the tensor product $V^{\otimes L}$. Using (1.92)-(1.93) repeatedly it can be expanded in fully tensorized form, as an expression with $2^{L-1}$ terms. This expanded form is (1.92) for $L=2$, and the expression for $L=3$ is contained in the above exercise. The factors entering each term in the expanded form act on a single space $V$.

### 1.3.3.2 Commutation relations

The operators $A(u), B(u), C(u)$, and $D(u)$ satisfy a set of commutation relations which follow as a direct consequence of the RTT relation (1.80).

To see in details how this works, we first write out the RTT relation in component form,

$$
\begin{equation*}
\sum_{j_{1}, j_{2}} R_{j_{1} j_{2}}^{k_{1} k_{2}}(u-v) T_{i_{1}}^{j_{1}}(u) T_{i_{2}}^{j_{2}}(v)=\sum_{j_{1}, j_{2}} T_{j_{2}}^{k_{2}}(v) T_{j_{1}}^{k_{1}}(u) R_{i_{1} i_{2}}^{j_{1} j_{2}}(u-v) . \tag{1.94}
\end{equation*}
$$

This gives a relation for each choice of $\left(k_{1}, k_{2}, i_{1}, i_{2}\right)$. Consider for instance the choice ( $0,0,1,0$ ):

$$
\begin{equation*}
R_{00}^{00}(u-v) T_{1}^{0}(u) T_{0}^{0}(v)=T_{1}^{0}(v) T_{0}^{0}(u) R_{10}^{01}(u-v)+T_{0}^{0}(v) T_{1}^{0}(u) R_{10}^{10}(u-v) \tag{1.95}
\end{equation*}
$$

Insert now the $R$-matrix elements from (1.86)-(1.87) and the monodromy matrix elements from (1.84), recalling that the former are just scalars, whereas the latter are (non-commuting) operators. This gives

$$
\begin{equation*}
a(u-v) B(u) A(v)=c(u-v) B(v) A(u)+b(u-v) A(v) B(u) \tag{1.96}
\end{equation*}
$$

Among all the possible commutation relations we shall actually only need a few. Firstly, for two operators of the same type we have simply

$$
\begin{align*}
& A(u) A(v)=A(v) A(u), \quad B(u) B(v)=B(v) B(u), \\
& C(u) C(v)=C(v) C(u), \quad D(u) D(v)=D(v) D(u) \tag{1.97}
\end{align*}
$$

Secondly, to push an $A$ or a $D$ past a $B$ we have

$$
\begin{align*}
A(u) B(v) & =\frac{a(v-u)}{b(v-u)} B(v) A(u)-\frac{c(v-u)}{b(v-u)} B(u) A(v), \\
D(u) B(v) & =\frac{a(u-v)}{b(u-v)} B(v) D(u)-\frac{c(u-v)}{b(u-v)} B(u) D(v) . \tag{1.98}
\end{align*}
$$

The first of these relations follows from (1.96) after a relabelling $u \leftrightarrow v$ and some rearrangement. The second relation is obtained from a similar computation.

### 1.3.3.3 Algebraic Bethe ansatz

We now have all necessary ingredients to treat the six-vertex model using the algebraic Bethe ansatz.

As in the coordinate Bethe ansatz, one starts from the pseudo-vacuum, or reference state, in which all spins point up and no particle world-lines are present. We denote this state as

$$
\begin{equation*}
|\Uparrow\rangle=|\uparrow \uparrow \cdots \uparrow\rangle=|00 \cdots 0\rangle \tag{1.99}
\end{equation*}
$$

Recall that $B(u)$ creates a particle (or equivalently, flips down one spin), whereas $C(u)$ annihilates a particle. Thus, an $n$-particle state (i.e. with $n$ down spins) can be constructed as follows:

$$
\begin{equation*}
\left|\Psi_{n}\right\rangle=\prod_{i=1}^{n} B\left(u_{i}\right)|\Uparrow\rangle \tag{1.100}
\end{equation*}
$$

The states (1.100) are called algebraic Bethe ansatz states.
Our goal is to diagonalize the transfer matrix

$$
\begin{equation*}
t(u)=\operatorname{Tr}_{\mathrm{a}} T(u)=A(u)+D(u) \tag{1.101}
\end{equation*}
$$

This means solving the eigenvalue equation

$$
\begin{equation*}
t(u)\left|\Psi_{n}\right\rangle=[A(u)+D(u)] \prod_{i=1}^{n} B\left(u_{i}\right)|\Uparrow\rangle=\Lambda_{n}\left(u ;\left\{u_{i}\right\}\right) \prod_{i=1}^{n} B\left(u_{i}\right)|\Uparrow\rangle . \tag{1.102}
\end{equation*}
$$

This can obviously only be done if the parameters $\left\{u_{i}\right\}$ satisfy certain conditions, called the Bethe ansatz equations, that we shall derived shortly.

To compute $[A(u)+D(u)] \prod_{i=1}^{n} B\left(u_{i}\right)|\Uparrow\rangle$ we use the commutation relations (1.98) to push $A(u)$ and $D(u)$ to the right, past the string of $B$ 's. When they have been pushed completely to the right, one applies the relations

$$
\begin{equation*}
A(u)|\Uparrow\rangle=a(u)^{L}|\Uparrow\rangle, \quad D(u)|\Uparrow\rangle=b(u)^{L}|\Uparrow\rangle . \tag{1.103}
\end{equation*}
$$

Note that (1.103) follows from the first and last lines of (1.92), generalized for $L=2$ to arbitrary $L$. Consider for instance $\Delta A(u)$. It is easy to see that the right-hand side will contain a single term $A(u)^{\otimes L}$, and all remaining terms will contain at least one factor $C(u)$ in the tensor product. But this $C(u)$ will annihilate $|\Uparrow\rangle$, so the only contribution is $a(u)^{L}|\Uparrow\rangle$ indeed.

Each time we push $A(u)$ one position towards the right, we obtain two contributions from the right-hand side of (1.98). The unique term obtained by always choosing the first contribution is a wanted $A$-term. In this term, the arguments of the $B\left(u_{i}\right)$ remain unchanged, and $A(u)$ simply 'goes through'. The remaining $2^{n}-1$ terms are unwanted $A$-terms. In those terms, at least one of the arguments $u_{i}$ of the $B$ 's has been changed into $u$, and so the state is not of the form (1.100). Similarly, there is one wanted and $2^{n}-1$ unwanted $D$-terms.

The wanted $A$-term and the wanted $D$-term produce the expression for the eigenvalue of the transfer matrix,

$$
\begin{equation*}
\Lambda_{n}\left(u ;\left\{u_{i}\right\}\right)=a(u)^{L} \prod_{i=1}^{n} \frac{a\left(u_{i}-u\right)}{b\left(u_{i}-u\right)}+b(u)^{L} \prod_{i=1}^{n} \frac{a\left(u-u_{i}\right)}{b\left(u-u_{i}\right)} . \tag{1.104}
\end{equation*}
$$

The condition that the unwanted $A$-terms cancel the unwanted $D$-terms leads to the Bethe ansatz equations (BAE),

$$
\begin{equation*}
\left(\frac{a\left(u_{i}\right)}{b\left(u_{i}\right)}\right)^{L}=\prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{a\left(u_{i}-u_{j}\right) b\left(u_{j}-u_{i}\right)}{a\left(u_{j}-u_{i}\right) b\left(u_{i}-u_{j}\right)} . \tag{1.105}
\end{equation*}
$$

Proof of (1.105). Let us abbreviate $A_{0} \equiv A(u)$ and $A_{i} \equiv A\left(u_{i}\right)$ for $i=1,2, \ldots, n$, and similarly for the other types of operators. We also set

$$
\begin{equation*}
\alpha_{i j} \equiv \frac{a\left(u_{j}-u_{i}\right)}{b\left(u_{j}-u_{i}\right)}, \quad \beta_{i j} \equiv-\frac{c\left(u_{j}-u_{i}\right)}{b\left(u_{j}-u_{i}\right)}, \tag{1.106}
\end{equation*}
$$

so that the commutation relations (1.98) can be rewritten

$$
\begin{align*}
A_{i} B_{j} & =\alpha_{i j} B_{j} A_{i}+\beta_{i j} B_{i} A_{j}, \\
D_{i} B_{j} & =\alpha_{j i} B_{j} D_{i}+\beta_{j i} B_{i} D_{j} . \tag{1.107}
\end{align*}
$$

The unwanted $A$-terms (resp. $D$-terms) are those where $A_{0}$ (resp. $D_{0}$ ) exchanges its spectral parameter with one or more of the $B$ 's and hence becomes some $A_{i}$ (resp. $D_{i}$ ) with $i \geq 1$ as it is pushed to the right of $\prod_{i=1}^{n} B_{i}$. The sum of these unwanted $A$-terms is

$$
\begin{equation*}
\sum_{i=1}^{n} \bar{a}_{i}\left(\prod_{\substack{j=0 \\ j \neq i}}^{n} B_{j}\right) A_{i}|\Uparrow\rangle=\sum_{i=1}^{n} \bar{a}_{i} a\left(u_{i}\right)^{L}\left(\prod_{\substack{j=0 \\ j \neq i}}^{n} B_{j}\right)|\Uparrow\rangle \tag{1.108}
\end{equation*}
$$

At first sight, it may appear complicated to compute the coefficients $\bar{a}_{i}$, since the $A$-operator might exchange its rapidity with the $B$ 's several times, as it is moved through the product. However, we can simplify the computation of $\bar{a}_{i}$ dramatically by using (1.97) to rewrite (1.100) as

$$
\begin{equation*}
\left|\Psi_{n}\right\rangle=B_{i} \prod_{\substack{j=1 \\ j \neq i}}^{n} B_{j}|\Uparrow\rangle \tag{1.109}
\end{equation*}
$$

The action of $A_{0}$ on this can then only produce an $A_{i}$ on the right if the exchange of spectral parameter happens when $A_{0}$ is commuted through the very first factor $B_{i}$ in (1.109). Therefore,

$$
\begin{equation*}
\bar{a}_{i}=\beta_{0 i} \prod_{\substack{k=1 \\ k \neq i}}^{n} \alpha_{i k} . \tag{1.110}
\end{equation*}
$$

By this simple trick, the total number of unwanted $A$-terms has been reduced from $2^{n}-1$ to just $n$.

Similarly, the unwanted $D$-terms read

$$
\begin{equation*}
\sum_{i=1}^{n} \bar{d}_{i}\left(\prod_{\substack{j=0 \\ j \neq i}}^{n} B_{j}\right) D_{i}|\Uparrow\rangle=\sum_{i=1}^{n} \bar{d}_{i} b\left(u_{i}\right)^{L}\left(\prod_{\substack{j=0 \\ j \neq i}}^{n} B_{j}\right)|\Uparrow\rangle \tag{1.111}
\end{equation*}
$$

with

$$
\begin{equation*}
\bar{d}_{i}=\beta_{i 0} \prod_{\substack{k=1 \\ k \neq i}}^{n} \alpha_{k i} . \tag{1.112}
\end{equation*}
$$

Because of (1.87) we have $\beta_{0 i}=-\beta_{i 0}$. Therefore the sum of (1.108) and (1.111) vanishes provided that

$$
\begin{equation*}
\left(\frac{a\left(u_{i}\right)}{b\left(u_{i}\right)}\right)^{L}=\prod_{\substack{k=1 \\ k \neq i}}^{n} \frac{\alpha_{k i}}{\alpha_{i k}} . \tag{1.113}
\end{equation*}
$$

Plugging back (1.106) we arrive at (1.105).

Alternatively (1.105) follows also from the form (1.104) of the eigenvalue, as we now argue. We set $z=\mathrm{e}^{2 i u}$ and $q=\mathrm{e}^{i \gamma}$, and we define the shifted eigenvalue $\tilde{\Lambda}=$ $\left(2 i \mathrm{e}^{i u}\right)^{L} \Lambda_{n}\left(u,\left\{u_{i}\right\}\right)$. Elementary computations then bring (1.105) into the form

$$
\begin{equation*}
\tilde{\Lambda}=\left(q-q^{-1} z\right)^{L} \prod_{i=1}^{n} \frac{q^{-1} z_{i}-q z}{z-z_{i}}+(z-1)^{L} \prod_{i=1}^{n} \frac{q z_{i}-q^{-1} z}{z-z_{i}} . \tag{1.114}
\end{equation*}
$$

Defining the polynomials

$$
\begin{equation*}
Q(z)=\prod_{i=1}^{n}\left(z-z_{i}\right), \quad \phi_{L}(z)=(z-1)^{L}, \tag{1.115}
\end{equation*}
$$

we obtain

$$
\begin{equation*}
\tilde{\Lambda} Q(z)=(-q)^{L-n} \phi_{L}\left(q^{-2} z\right) Q\left(q^{2} z\right)+(-q)^{N} \phi_{L}(z) Q\left(q^{-2} z\right) . \tag{1.116}
\end{equation*}
$$

Whenever the spectral parameter equals one of the Bethe roots, we have $Q\left(z_{j}\right)=0$ on the left-hand side of (1.116). Therefore the right-hand side must also vanish. Working backwards through the change of variables then produces the BAE (1.105).

### 1.3.4 Coordinate Bethe ansatz

A more direct approach consists in considering the action of transfer matrix $t$ on $n$-particle states giving the positions of the world-lines of $|1\rangle$ states within one row of the lattice. Such states can be written $\left|x_{1}, x_{2}, \ldots, x_{n}\right\rangle$, where we have assumed $x_{1}<x_{2}<$ $\cdots<x_{n}$, and the world-lines are depicted in the bottom part of Figure 1.3.

This coordinate Bethe ansatz (CBA) approach [28] is often the first strategy one would try out on a new problem, but it lacks some of the power and elegance of the algebraic Bethe ansatz (ABA). On the other hand, in some cases, problems which are solvable by CBA do not admit the ABA approach. ${ }^{7}$ This might be due, e.g., to the failure to make

[^6]sense of the spectral parameter $u$, or because a proper pseudo vacuum $|\Uparrow\rangle$ cannot be identified.

In the CBA, one wishes to construct $n$-particle states

$$
\begin{equation*}
\left|\Psi_{n}\right\rangle=\sum_{1 \leq x_{1}<\cdots<x_{n} \leq L} g\left(x_{1}, \ldots, x_{n}\right)\left|x_{1}, \ldots, x_{n}\right\rangle \tag{1.117}
\end{equation*}
$$

which are eigenvectors of $t$,

$$
\begin{equation*}
t\left|\Psi_{n}\right\rangle=\Lambda_{n}\left|\Psi_{n}\right\rangle \tag{1.118}
\end{equation*}
$$

To this end one tries an ansatz of the form

$$
\begin{equation*}
g\left(x_{1}, \ldots, x_{n}\right)=\sum_{p \in \mathfrak{S}_{n}} A_{p} z_{p(1)}^{x_{1}} z_{p(2)}^{x_{2}} \cdots z_{p(n)}^{x_{n}} \tag{1.119}
\end{equation*}
$$

where the sum runs over all permutations $p \in \mathfrak{S}_{n}$ of the particle labels $\{1,2, \ldots, n\}$. The complex numbers $z_{j}$ are related to the so-called quasi-momenta $k_{j}$ through the relation $z_{j}=\exp \left(i k_{j}\right)$. Loosely speaking, the ansatz (1.119) can be interpreted as coupled plane waves.

The link between the two approaches is provided by relating the Bethe roots $\left\{u_{j}\right\}$ with the quasi-momenta $\left\{z_{j}\right\}$. This can be done by comparing the one-particle states, which read in the CBA

$$
\begin{equation*}
\left|\Psi_{1}\right\rangle=\sum_{x=1}^{L} z^{x}|x\rangle \tag{1.120}
\end{equation*}
$$

while in the ABA we have

$$
\begin{equation*}
\left|\Psi_{1}\right\rangle=\Delta^{L-1} B(u)|\Uparrow\rangle \tag{1.121}
\end{equation*}
$$

Expanding out $\Delta^{L-1} B(u)$, using (1.92)-(1.93), and using that any $C(u)$ tensorand annihilates $|\Uparrow\rangle$, we see that the ABA expression for $\left|\Psi_{1}\right\rangle$ also has precisely $L$ nonzero terms (rather than $2^{L}$ ), each one characterized by one flipped spin. Identifying the position of the flipped spin with $|x\rangle$ and matching coefficients, we arrive at

$$
\begin{equation*}
z_{j}=\mathrm{e}^{i k_{j}}=\frac{a\left(u_{j}\right)}{b\left(u_{j}\right)} \tag{1.122}
\end{equation*}
$$

The Bethe ansatz equations (1.105) can the be written in the suggestive form

$$
\begin{equation*}
z_{j}^{L}=\prod_{\substack{l=1 \\ l \neq j}}^{n} \hat{S}_{l j}\left(z_{l}, z_{j}\right) \quad \text { for } j=1,2, \ldots, n, \tag{1.123}
\end{equation*}
$$

where we have introduced the scattering phases

$$
\begin{equation*}
\hat{S}_{j i}=\frac{a\left(u_{j}-u_{i}\right) b\left(u_{i}-u_{j}\right)}{a\left(u_{i}-u_{j}\right) b\left(u_{j}-u_{i}\right)}=-\frac{1-2 \Delta z_{i}+z_{i} z_{j}}{1-2 \Delta z_{j}+z_{i} z_{j}} . \tag{1.124}
\end{equation*}
$$

We stress that the BAE depend only on the six-vertex weights $a, b, c$ via the combination $\Delta$. One consequence of this is that the universality class (critical exponents) will depend on $\Delta$, but not on the anisotropy given by (1.40).

The form (1.123) of the BAE can be interpreted physically as follows. When the particle $j$ is taken around the periodic direction and back to its original position, it picks up a scattering phase $\hat{S}_{l j}$ each time it crosses another particle $l$.

### 1.3.5 Energy and momentum

We can compute the energy $E$ of the Bethe ansatz state (1.100). To this end we just need to recall the link (1.73) between the transfer matrix $t(u)$ and the Hamiltonian $H$. Taking expectation values with respect to the state (1.100) the operator $H$ gets replaced by its expectation value $E$, and $t(u)$ gets replaced by the eigenvalue $\Lambda\left(u ;\left\{u_{i}\right\}\right)$. Therefore,

$$
\begin{equation*}
E_{n}\left(\left\{u_{i}\right\}\right)=-\left.\sin \gamma \frac{\partial}{\partial u} \log \Lambda_{n}\left(u ;\left\{u_{i}\right\}\right)\right|_{u \rightarrow 0} \tag{1.125}
\end{equation*}
$$

In (1.104) only the first term contributes in the $u \rightarrow 0$ limit:

$$
\begin{equation*}
\Lambda_{n} \simeq \sin ^{L}(\gamma) \prod_{i=1}^{n} \frac{a\left(u_{i}-u\right)}{b\left(u_{i}-u\right)} \tag{1.126}
\end{equation*}
$$

Taking the derivative we arrive at

$$
\begin{equation*}
E_{n}\left(\left\{u_{i}\right\}\right)=L \cos \gamma+\sum_{i=1}^{n} \epsilon\left(u_{i}\right) \tag{1.127}
\end{equation*}
$$

where the energy of a single particle with quasi-momentum (1.122) is

$$
\begin{equation*}
\epsilon\left(u_{i}\right)=-\frac{\sin ^{2}(\gamma)}{\sin \left(u_{i}\right) \sin \left(\gamma-u_{i}\right)} . \tag{1.128}
\end{equation*}
$$


[^0]:    1 The number of independent cycles-also known as the circuit rank, or the cyclomatic number-is the smallest number of edges to be removed from a graph in order that no graph cycle remains.

[^1]:    ${ }^{2}$ The term staggered means that the weights alternate between sublattices.

[^2]:    ${ }^{3}$ It is however possible to solve it also at the antiferromagnetic transition [21-23].

[^3]:    4 The transfer matrices depend also on the spectral parameters $v_{1}, v_{2}, \ldots, v_{L}$ of the quantum spaces, but we omit this dependence for notational convenience.

[^4]:    5 We are here referring to an anisotropy between the different components of the interaction in the space direction. This should not be confused with the space-time anisotropy linked with the spectral parameter $u$.

[^5]:    ${ }^{6}$ This $\Delta$ should not be confused with the anisotropy parameter of the six-vertex model (XXZ spin chain).

[^6]:    ${ }^{7}$ For example, the ABA version of the biquadratic model [29] is presently unknown.

