J. PIERRUS

SOLVED PROBLEMS IN CLASSICAL ELECTROMAGNETISM

Analytical and Numerical Solutions with Comments

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Preface

These days there are many excellent textbooks ranging from the introductory to the advanced, and which cover all the core parts of a traditional physics curriculum. The *Solved problems in ...* books (this being the second) were written to fill a gap for those students who prefer self-study. Hopefully, the format is sufficiently appealing to justify entering an already crowded space where there isn't much room for original insight and new points of view.

This book follows its predecessor[1] both in style and approach. It contains nearly 300 questions and solutions on a range of topics in classical electromagnetism that are usually encountered during the first four years of a university physics degree. Most questions end with a series of comments that emphasize important conclusions arising from the problem. Sometimes, possible extensions of the problem and additional aspects of interest are also mentioned. The book is aimed primarily at physics students, although it will be useful to engineering and other physical science majors as well. In addition, lecturers may find that some of the material can be readily adapted for examination purposes.

Wherever possible, an attempt has been made to develop the theme of each chapter from a few fundamental principles. These are outlined either in the introduction or in the first few questions of the chapter. Various applications then follow. Inevitably, the author's personal preferences are reflected in the choice of subject matter, although hopefully not at the expense of providing a balanced overview of the core material. Questions are arranged in a way which leads to a natural flow of the key concepts and ideas, rather than according to their 'degree of difficulty'. Those marked with a ** superscript indicate specialized material and are most likely suitable for postgraduate students. Questions without a superscript will invariably be encountered in middle to senior undergraduate-level courses. A * superscript denotes material which is on the borderline between the two categories mentioned above. In all cases, students are encouraged to attempt the questions on their own before looking at the solutions provided.

It is widely recognized that learning (and teaching!) electromagnetism is one of the most challenging parts of any physics curriculum. In the preface to his book *Modern electrodynamics*, Zangwill explains that 'another stumbling block is the non-algorithmic nature of electromagnetic problem-solving. There are many entry points to a typical electromagnetism problem, but it is rarely obvious which lead to a quick solution and which lead to frustrating complications'. These remarks rather clearly

[1] O. L. de Lange and J. Pierrus, *Solved problems in classical mechanics: Analytical and numerical solutions with comments.* Oxford: Oxford University Press, 2010.

outline the challenge. Certainly, it is my firm belief that students benefit from a high exposure to problem solving. Topics which require the use of a computer are especially valuable because one is forced to ask at each stage in the calculation: 'Is my answer reasonable?' For the most part, the computer cannot assist in this regard. Other considerations play a role. Experience definitely helps. So does that somewhat elusive yet much-prized attribute which we call 'physical intuition'.

All the computational work is carried out using *Mathematica*®, version 10.0. The relevant code (referred to as a notebook) is provided in a shadebox in the text. For easy reference, those questions involving computational work are listed in Appendix J. Readers who use different software for their computer algebra are nevertheless encouraged to read these notebooks and adapt the code—wherever necessary—to suit their own environment. That is to say, students using alternative programming packages should not be 'put off' by our exclusive use of *Mathematica*; this book will certainly be useful to them as well. Also, readers without prior knowledge of *Mathematica* can rapidly learn the basics from the online Help at www.Wolfram.com (or various other places; try a simple internet search). From my experience, students learn enough of the basic concepts to make a reasonable start after only a few hours of training. All graphs of numerical results have been drawn to scale using Gnuplot.

For a book like this there are, of course, certain prerequisites. First, it is assumed that readers have previously encountered the basic phenomena and laws of electricity and magnetism. Second, a working knowledge of standard vector analysis and calculus is required. This includes the ability to solve elementary ordinary differential equations. An acquaintance with some of the special functions of mathematical physics will also be useful. Because readers will have diverse mathematical backgrounds and skills, Chapter 1 is devoted to setting out the important analytical techniques on which the rest of the book depends. As a further aid, nine appendices containing some specialized material have been included. In keeping with the modern trend, SI units are adopted throughout. This has the distinct advantage of producing quantities which are familiar from our daily lives: volts, amps, ohms and watts.

Usually one of the first decisions the author of a physics book must face is the important matter of notation: which symbol to use for which quantity. A cursory look at several standard textbooks immediately reveals notable differences (Φ or V for electric potential, dv or $d\tau$ for a volume element, $\mathbf{S}$ or $\mathbf{N}$ for the Poynting vector, and so on). Because the choice of notation is somewhat subjective, colleagues in the same department often possess divergent opinions on this topic. So my own preferences and prejudices are reflected in the notation used in this book. For easy reference, a comprehensive glossary of symbols is appended.

Chapters 2–4 focus primarily on static electricity and magnetism. Then in Chapters 5 and 6 we begin the transition from quasi-static phenomena to the complete time-dependent Maxwell equations which appear from Chapter 7 onwards. For the most part this is a book that deals with the microscopic theory, except in Chapters 9 and 10, which touch on macroscopic electromagnetism. We end in Chapter 12 with a collection of questions which connect Maxwell's electrodynamics to Einstein's theory of special relativity.

Although the questions and solutions are reasonably self-contained, it may be necessary to consult a standard textbook from time to time. University libraries will usually have a wide selection of these. Some of my favourites, listed by their date of publication, are:

- ☞ *Classical electrodynamics*, J. D. Jackson, 3rd edition, John Wiley (1998).
- ☞ *Introduction to electrodynamics*, D. J. Griffiths, 3rd edition, Prentice Hall (1999).
- ☞ *Electricity and magnetism*, E. M. Purcell and D. J. Morin, Cambridge University Press (2013).
- ☞ *Modern electrodynamics*, A. Zangwill, Cambridge University Press (2013).

Without the help, guidance and assistance of many people this book would never have reached publication. In particular, I extend my sincere thanks to the following:

- ☞ Allard Welter for drawing the circuit diagrams of Chapter 6, for his advice on various *Mathematica* queries and for resolving (usually in a good-natured way!) some pedantic issues with LaTeX.
- ☞ Karl Penzhorn for attending to my other computer-related problems and also for helping with the CorelDRAW software which was used to produce many of the diagrams in this book.
- ☞ Professor Owen de Lange who conceived the format of these *Solved problems in ...* books, and with whom I co-authored Ref. [1]. Hopefully, at least some of Owen's professionalism and attention to detail has rubbed off onto me since we began collaborating in the early 1990s.
- ☞ Professor Roger Raab for his encouragement and advice. Roger's research interests have strongly influenced my career, and I still recall our first discussion on the use of Cartesian tensors and the importance of symmetry in problem solving. Indeed, most of Appendix A and several questions at the beginning of Chapter 1 are based on some of his original lecture material.
- ☞ Former lecturers and colleagues who, in one way or another, helped foster my continuing enjoyment of classical electromagnetic theory. In approximate chronological order they include: Peter Krumm, Dave Walker, Manfred Hellberg, Max Michaelis, Roger Raab, Clive Graham, Paul Jackson, Tony Eagle, Owen de Lange, Frank Nabarro and Assen Ilchev.
- ☞ Several generations of bright undergraduate and postgraduate students who have provided valuable feedback on lecture notes, tutorial problems and other material from which this book has gradually evolved.

Pietermaritzburg, South Africa J. Pierrus
December 2017

Contents

1
Some essential mathematics

Nearly all of the questions in this introductory chapter are designed to introduce the essential mathematics required for formulating the theory of electromagnetism. All of the techniques discussed here will be used repeatedly throughout this book, and readers will hopefully find it convenient to have the important mathematical material summarized in a single place. Topics covered include Cartesian tensors, standard vector algebra and calculus, the method of separation of variables, the Dirac delta function, time averaging and the concept of solid angle. Our primary emphasis in this chapter is not on physical content, although certain comments pertaining to electricity and magnetism are made whenever appropriate.

Although the scalar potential Φ, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are familiar quantities in electromagnetism, it is not always known that they are examples of a mathematical entity called a tensor. Furthermore, it is sometimes necessary to introduce more complicated tensors than these. This chapter begins with a series of questions involving the use of Cartesian tensors. We will find that the compact nature of tensor notation greatly facilitates the solution of many questions throughout this book. Readers who are unfamiliar with tensors and the associated terminology, or who need to revise the background material, are advised to consult Appendix A before proceeding. At the end of this appendix, we include a '*checklist for detecting errors when using tensor notation*'. This guide will be helpful for both the uninitiated and the experienced tensor user.

Question 1.1

Let $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ be the position vector of a point in space. Use Cartesian tensors to calculate:

(a) $\nabla_i r_j$,

(b) $\boldsymbol{\nabla} \cdot \mathbf{r}$,

(c) $\boldsymbol{\nabla} r$,

(d) $\boldsymbol{\nabla} r^k$ where k is rational,

(e) $\nabla_i (r_j / r^3)$,

(f) $\nabla_i \{(3r_j r_k - r^2 \delta_{jk})/r^5\}$ and,

(g) $\boldsymbol{\nabla} e^{i\mathbf{k}\cdot\mathbf{r}}$ where $\mathbf{k}$ is a constant vector.

Solved Problems in Classical Electromagnetism. J. Pierrus, Oxford University Press (2018).
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Solution

(a) The operation $\nabla_i r_j\,(=\partial r_j/\partial r_i)$ produces a tensor of rank two with nine components. Six of these components have $i \neq j$, and for them $\partial r_j/\partial r_i = 0$. The remaining three components for which $i = j$ all have the value one. Thus

$$\nabla_i r_j \;=\; \delta_{ij}\,, \tag{1}$$

where δ_{ij} is the Kronecker delta defined by (III) of Appendix A.

(b) Expressing $\boldsymbol{\nabla}\cdot\mathbf{r}$ in tensor notation and putting $i = j$ in (1) gives $\boldsymbol{\nabla}\cdot\mathbf{r} = \nabla_i r_i = \delta_{ii}$. Using the Einstein summation convention (see (I) of Appendix A) yields

$$\boldsymbol{\nabla}\cdot\mathbf{r} \;=\; \delta_{xx} + \delta_{yy} + \delta_{zz} \;=\; 3. \tag{2}$$

(c) Writing $r = \sqrt{\mathbf{r}\cdot\mathbf{r}} = \sqrt{r_j r_j}$ and differentiating give

$$\nabla_i r = \frac{\partial r}{\partial r_i} = \frac{\partial}{\partial r_i}(r_j r_j)^{1/2} = \tfrac{1}{2}(r_j r_j)^{-1/2}\left(\frac{\partial r_j}{\partial r_i}r_j + r_j\frac{\partial r_j}{\partial r_i}\right) = \frac{r_j}{r}\frac{\partial r_j}{\partial r_i} = \frac{r_j}{r}\,\delta_{ij}$$

because of (1). Using the contraction property of the Kronecker delta gives

$$\nabla_i r \;=\; \frac{r_j}{r}\,\delta_{ij} \;=\; \frac{r_i}{r}. \tag{3}$$

But (3) is true for $i = x, y$ and z, and so

$$\boldsymbol{\nabla} r \;=\; \frac{\mathbf{r}}{r} \;=\; \hat{\mathbf{r}}. \tag{4}$$

(d) Consider the ith component. Then $[\boldsymbol{\nabla} r^k]_i \;=\; \nabla_i r^k \;=\; \dfrac{\partial r^k}{\partial r_i} \;=\; \dfrac{\partial r^k}{\partial r}\dfrac{\partial r}{\partial r_i} \;=\; k\,r^{k-2}\,r_i$ because of (3). The result is

$$\boldsymbol{\nabla} r^k = k\,r^{k-2}\mathbf{r} \qquad \text{or} \qquad \boldsymbol{\nabla} r^k = k\,r^{k-1}\hat{\mathbf{r}}. \tag{5}$$

Putting $k = -1$ gives an important case

$$\boldsymbol{\nabla}\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3} \qquad \text{or} \qquad \boldsymbol{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{\mathbf{r}}}{r^2} \tag{6}$$

(see also Question 1.6).

(e)
$$\begin{aligned}\nabla_i(r_j/r^3) \;&=\; \frac{\nabla_i r_j}{r^3} + r_j\nabla_i r^{-3} \;=\; \frac{\nabla_i r_j}{r^3} + r_j\frac{\partial r^{-3}}{\partial r_i} \;=\; \frac{\nabla_i r_j}{r^3} + r_j\frac{\partial r^{-3}}{\partial r}\,\frac{\partial r}{\partial r_i}\\ &=\; \frac{r^2\delta_{ij} - 3r_i r_j}{r^5}\,,\end{aligned} \tag{7}$$

where in the last step we use (1) and (3).

(f) Similarly,

$$\begin{aligned}\nabla_i(3r_jr_kr^{-5}) - \nabla_i(r^{-3}\delta_{jk}) &= 3r_kr^{-5}\delta_{ij} + 3r_jr^{-5}\delta_{ik} - 15r_ir_jr_kr^{-7} + 3r_ir^{-5}\delta_{jk} \\ &= \frac{3r^2(r_i\delta_{jk} + r_j\delta_{ki} + r_k\delta_{ij}) - 15r_ir_jr_k}{r^7}\,. \end{aligned} \tag{8}$$

(g) $$\nabla_j e^{i\mathbf{k}\cdot\mathbf{r}} = e^{i\mathbf{k}\cdot\mathbf{r}}\frac{\partial(ik_lr_l)}{\partial r_j} = ie^{i\mathbf{k}\cdot\mathbf{r}}k_l\delta_{jl} = ie^{i\mathbf{k}\cdot\mathbf{r}}k_j, \text{ and so } \boldsymbol{\nabla}e^{i\mathbf{k}\cdot\mathbf{r}} = i\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{9}$$

Comments

(i) Since $\nabla_i r_j = \nabla_j r_i$ we can write $\delta_{ij} = \delta_{ji}$ (i.e. the Kronecker delta is symmetric in its subscripts). It possesses the following important property:

$$A_i\delta_{ij} = A_x\delta_{xj} + A_y\delta_{yj} + A_z\delta_{zj} = A_j\,. \tag{10}$$

In the final step leading to (10), j is either x, y or z. Of the three Kronecker deltas (δ_{xj}, δ_{yj} and δ_{zj}) two will always be zero, whilst the third will have the value one. Because of this, δ_{ij} is sometimes also known as the substitution tensor.

(ii) Subscripts that are repeated are said to be contracted. So in (10), i is contracted in $A_i\delta_{ij}$. Equivalently, one can say that $A_i\delta_{ij}$ is contracted with respect to i.

(iii) A tensor is said to be isotropic if its components retain the same values under a proper transformation.‡ δ_{ij} is an example of an isotropic tensor: any second-rank *isotropic* tensor T_{ij} can be expressed as a scalar multiple of δ_{ij} (i.e. $T_{ij} = \alpha\,\delta_{ij}$).[1]

Question 1.2

(a) Consider the cross-product $\mathbf{c} = \mathbf{a}\times\mathbf{b}$. Show that

$$c_i = \varepsilon_{ijk}a_jb_k\,, \tag{1}$$

where ε_{ijk} is the Levi-Civita tensor defined by

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is taken as any even permutation of } x,y,z \\ -1 & \text{if } ijk \text{ is taken as any odd permutation of } x,y,z \\ 0 & \text{if any two subscripts are equal.} \end{cases} \tag{2}$$

(b) Prove that

$$\boldsymbol{\nabla}\times\mathbf{r} = 0, \tag{3}$$

where $\mathbf{r} = (x,\,y,\,z)$.

‡Proper and improper transformations are described in Appendix A.

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[1] H. Jeffreys, *Cartesian tensors*, Chap. VII, pp. 66–8. Cambridge: Cambridge University Press, 1952.

Solution

(a) The Cartesian form $\mathbf{c} = \hat{\mathbf{x}}(a_y b_z - a_z b_y) + \hat{\mathbf{y}}(a_z b_x - a_x b_z) + \hat{\mathbf{z}}(a_x b_y - a_y b_x)$ has x-component $c_x = a_y b_z - a_z b_y = \varepsilon_{xyz} a_y b_z + \varepsilon_{xzy} a_z b_y$ as a result of the properties $(2)_1$ and $(2)_2$. Because repeated subscripts imply a summation over Cartesian components, we can write $c_x = \varepsilon_{xjk} a_j b_k$ using $(2)_3$. Similarly, $c_y = \varepsilon_{yjk} a_j b_k$ and $c_z = \varepsilon_{zjk} a_j b_k$. Now the ith component of $\mathbf{c}$ is $(\mathbf{a} \times \mathbf{b})_i$ which is (1).

(b) Following the solution of (a) we write $(\boldsymbol{\nabla} \times \mathbf{r})_i = \varepsilon_{ijk} \nabla_j r_k = \varepsilon_{ijk} \delta_{jk} = \varepsilon_{ijj} = 0$. Here we use the contraction $\varepsilon_{ijk}\delta_{jk} = \varepsilon_{ijj}$ and the property $\varepsilon_{ijj} = 0$ (the same conclusion also follows from (4) of Question 1.5). This result is true for $i = x, y$ and z. Hence (3).

Comments

(i) The Levi-Civita tensor is a third-rank tensor. It is clear from (2) that it is anti-symmetric in any pair of subscripts.

(ii) ε_{ijk} is also known as the alternating tensor or isotropic tensor of rank three: any third-rank *isotropic* tensor T_{ijk} can be expressed as a scalar multiple of ε_{ijk} (i.e. $T_{ijk} = \alpha\, \varepsilon_{ijk}$).[1]

Question 1.3

(a) Consider the product of two Levi-Civita tensors which have a subscript in common. Show that

$$\varepsilon_{ijk}\varepsilon_{\ell mk} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}. \tag{1}$$

Hint: The product $\varepsilon_{ijk}\varepsilon_{\ell mk}$ is an isotropic tensor of rank four. Prove (1) by making a linear combination of products of the Kronecker delta.

(b) Use (1) to prove the identity

$$A_i B_j - A_j B_i = \varepsilon_{ijk}(\mathbf{A} \times \mathbf{B})_k, \tag{2}$$

where $\mathbf{A}$ and $\mathbf{B}$ are arbitrary vectors.

Solution

(a) Because of the hint, $\varepsilon_{ijk}\varepsilon_{\ell mk} = a\, \delta_{ij}\delta_{\ell m} + b\, \delta_{i\ell}\delta_{jm} + c\, \delta_{im}\delta_{j\ell}$ where the constants a, b and c are determined as follows:

$$\begin{array}{llllcl} i = x, & j = x, & \ell = x, & m = x & : & \varepsilon_{xxk}\varepsilon_{xxk} = 0 = a + b + c. \\ i = x, & j = y, & \ell = x, & m = y & : & \varepsilon_{xyk}\varepsilon_{xyk} = \varepsilon_{xyz}\varepsilon_{xyz} = 1 = b. \\ i = x, & j = y, & \ell = y, & m = x & : & \varepsilon_{xyk}\varepsilon_{yxk} = \varepsilon_{xyz}\varepsilon_{yxz} = -1 = c. \end{array}$$

Thus $a = 0$ and we obtain (1).

(b) Equations (1) and (2) of Question 1.2 give $(\mathbf{A}\times\mathbf{B})_k = \varepsilon_{k\ell m}A_\ell B_m = \varepsilon_{\ell mk}A_\ell B_m$. Multiplying both sides of this equation by ε_{ijk} and using (1) yield $\varepsilon_{ijk}(\mathbf{A}\times\mathbf{B})_k = \varepsilon_{ijk}\varepsilon_{\ell mk}A_\ell B_m = (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})A_\ell B_m$. Contracting subscripts gives (2).

Comments

(i) Notice the following contractions that follow from (1):

$$\varepsilon_{ijk}\varepsilon_{ij\ell} = 2\,\delta_{k\ell} \qquad \text{and} \qquad \varepsilon_{ijk}\varepsilon_{ijk} = 6. \tag{3}$$

(ii) Making the replacements $\mathbf{A}\to\boldsymbol{\nabla}$; $\mathbf{B}\to\mathbf{F}$ in (2) gives

$$\nabla_i F_j - \nabla_j F_i = \varepsilon_{ijk}(\boldsymbol{\nabla}\times\mathbf{F})_k, \tag{4}$$

and if $\boldsymbol{\nabla}\times\mathbf{F} = 0$ then

$$\nabla_i F_j = \nabla_j F_i. \tag{5}$$

Question 1.4

Suppose $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are differentiable vector fields which are functions of the parameter t. Prove the following:

(a) $$\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{B}\cdot\frac{d\mathbf{A}}{dt} + \mathbf{A}\cdot\frac{d\mathbf{B}}{dt}, \tag{1}$$

(b) $$\frac{d}{dt}(\mathbf{A}\times\mathbf{B}) = \frac{d\mathbf{A}}{dt}\times\mathbf{B} + \mathbf{A}\times\frac{d\mathbf{B}}{dt}, \tag{2}$$

(c) $$\frac{d}{dt}\big[\alpha(t)\mathbf{A}\big] = \mathbf{A}\frac{d\alpha}{dt} + \alpha\frac{d\mathbf{A}}{dt}. \tag{3}$$

(Here $\alpha(t)$ is a differentiable scalar function of t.)

Solution

These results are all proved by applying the product rule of differentiation.

(a) $\dfrac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \dfrac{d}{dt}(A_iB_i) = B_i\dfrac{dA_i}{dt} + A_i\dfrac{dB_i}{dt}$ which is (1).

(b) From (1) of Question 1.2 it follows that $\dfrac{d}{dt}\big[(\mathbf{A}\times\mathbf{B})\big]_i = \dfrac{d}{dt}\big(\varepsilon_{ijk}A_jB_k\big)$. So

$$\frac{d}{dt}\big[(\mathbf{A}\times\mathbf{B})\big]_i = \varepsilon_{ijk}\frac{dA_j}{dt}B_k + \varepsilon_{ijk}A_j\frac{dB_k}{dt} = \left(\frac{d\mathbf{A}}{dt}\times\mathbf{B}\right)_i + \left(\mathbf{A}\times\frac{d\mathbf{B}}{dt}\right)_i.$$

Since this is true for $i = x$, y and z, equation (2) follows.

(c) The result is obvious by inspection.

Comment

The parameter t often represents time in physics. Thus $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are time-dependent fields, and accordingly the derivatives (1)–(3) represent their rates of change.

Question 1.5

Suppose s_{ij} and a_{ij} represent second-rank symmetric and antisymmetric tensors respectively. Using the definitions

$$s_{ij} = s_{ji} \qquad \text{and} \qquad a_{ij} = -a_{ji}, \tag{1}$$

prove that

$$s_{ij}a_{ij} = 0. \tag{2}$$

Solution

The subscript notation is arbitrary, and so

$$s_{ij}a_{ij} = s_{ji}a_{ji}. \tag{3}$$

Substituting (1) in (3) gives $s_{ij}a_{ij} = -s_{ij}a_{ij}$ or $2s_{ij}a_{ij} = 0$, which proves (2).

Comment

Equation (2) is a special case of a general property: the product of a tensor $s_{ijk\ell\ldots}$ symmetric in any two of its subscripts with another tensor $a_{mkni\ldots}$ that is antisymmetric in the *same* two subscripts is zero. That is,

$$s_{ijk\ell\ldots}\, a_{mkni\ldots} = 0. \tag{4}$$

Question 1.6

Suppose $\mathbf{r} = (x, y, z)$ and $\mathbf{r}' = (x', y', z')$ represent position vectors[‡] of points P and P′ respectively. Prove the following results:

$$\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \qquad \text{and} \qquad \boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}, \tag{1}$$

where $\boldsymbol{\nabla} = \hat{\mathbf{x}}\,\dfrac{\partial}{\partial x} + \hat{\mathbf{y}}\,\dfrac{\partial}{\partial y} + \hat{\mathbf{z}}\,\dfrac{\partial}{\partial z}$ and $\boldsymbol{\nabla}' = \hat{\mathbf{x}}\,\dfrac{\partial}{\partial x'} + \hat{\mathbf{y}}\,\dfrac{\partial}{\partial y'} + \hat{\mathbf{z}}\,\dfrac{\partial}{\partial z'}$ denote differentiation with respect to the unprimed and primed coordinates respectively.

[‡]The common origin O of these vectors is completely arbitrary.

Solution

It is convenient to let $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. Then

$$\nabla_i\left(\frac{1}{R}\right) = \frac{\partial R^{-1}}{\partial r_i} = \frac{\partial R^{-1}}{\partial R}\frac{\partial R}{\partial r_i} = -\frac{1}{R^2}\frac{\partial R}{\partial r_i}. \tag{2}$$

But $\dfrac{\partial R}{\partial r_i} = \dfrac{\partial}{\partial r_i}(r^2 + r'^2 - 2r_j r'_j)^{1/2} = \dfrac{2r_i - 2r'_j\delta_{ij}}{2R} = \dfrac{r_i - r'_i}{R} = \dfrac{R_i}{R}$ using (1) and (3) of Question 1.1. Substituting this last result in (2) gives $(1)_1$. Similarly, $(1)_2$ follows, since $\partial R/\partial r_i = -\partial R/\partial r'_i$.

Comment

In electromagnetism, it is important to distinguish between the unprimed coordinates of a field point P and the primed coordinates locating the sources[♯] of the field. As we have seen in the solution above, mathematical operations such as differentiation and integration can be with respect to coordinates of either type.

Question 1.7

Express the Taylor-series expansion of a function $f(x, y, z)$ about an origin O in the form

$$f(x,y,z) = [f(x,y,z)]_0 + [\nabla_i f(x,y,z)]_0\, r_i + \tfrac{1}{2}\,[\nabla_i\nabla_j f(x,y,z)]_0\, r_i r_j + \cdots. \tag{1}$$

Solution

The Taylor-series expansion of $f(x, y, z)$ about O is

$$\begin{aligned} f(x,y,z) = [f(x,y,z)]_0 &+ \left[\frac{\partial f(x,y,z)}{\partial x}\right]_0 x + \left[\frac{\partial f(x,y,z)}{\partial y}\right]_0 y + \left[\frac{\partial f(x,y,z)}{\partial z}\right]_0 z \; + \\ &\frac{1}{2}\left\{\left[\frac{\partial^2 f(x,y,z)}{\partial x^2}\right]_0 x^2 + \left[\frac{\partial^2 f(x,y,z)}{\partial x\partial y}\right]_0 xy + \left[\frac{\partial^2 f(x,y,z)}{\partial x\partial z}\right]_0 xz \; + \right. \\ &\left[\frac{\partial^2 f(x,y,z)}{\partial y\partial x}\right]_0 yx + \left[\frac{\partial^2 f(x,y,z)}{\partial y^2}\right]_0 y^2 + \left[\frac{\partial^2 f(x,y,z)}{\partial y\partial z}\right]_0 yz \; + \\ &\left.\left[\frac{\partial^2 f(x,y,z)}{\partial z\partial x}\right]_0 zx + \left[\frac{\partial^2 f(x,y,z)}{\partial z\partial y}\right]_0 zy + \left[\frac{\partial^2 f(x,y,z)}{\partial z^2}\right]_0 z^2\right\} + \cdots, \end{aligned} \tag{2}$$

which, in terms of the Einstein summation convention, is (1).

[♯]These being electric charges and currents.

Comments

(i) Note the compact form of the tensor equation (1), and compare this with (2).

(ii) Sometimes the function f is itself a component of a vector (say, the electric field y-component E_y). Then, using tensor notation to express the component of a vector, we have

$$E_i = [E_i]_0 + [\nabla_j E_i]_0 r_j + \tfrac{1}{2}[\nabla_j \nabla_k E_i]_0 r_j r_k + \cdots . \tag{3}$$

Question 1.8

Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, f and g represent continuous and differentiable‡ vector or scalar fields as appropriate. Use tensor notation to prove the following identities:

(a) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$ and all other cyclic permutations, (1)

(b) $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{B}) = A^2B^2 - (\mathbf{A}\cdot\mathbf{B})^2$, (2)

(c) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$, (3)

(d) $\boldsymbol{\nabla}(fg) = g\boldsymbol{\nabla}f + f\boldsymbol{\nabla}g$, (4)

(e) $\boldsymbol{\nabla}\cdot(f\mathbf{A}) = \mathbf{A}\cdot\boldsymbol{\nabla}f + f(\boldsymbol{\nabla}\cdot\mathbf{A})$, (5)

(f) $\boldsymbol{\nabla}\times(f\mathbf{A}) = \boldsymbol{\nabla}f\times\mathbf{A} + f(\boldsymbol{\nabla}\times\mathbf{A})$, (6)

(g) $\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = (\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathbf{B} - (\boldsymbol{\nabla}\times\mathbf{B})\cdot\mathbf{A}$, (7)

(h) $\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + \mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A})$, (8)

(i) $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = 0$, (9)

(j) $\boldsymbol{\nabla}\times\boldsymbol{\nabla}f = 0$, (10)

(k) $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) = -\nabla^2\mathbf{A} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})$, (11)

(l) $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}f\times\boldsymbol{\nabla}g) = 0$, (12)

(m) $\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A})$. (13)

Solution

(a) The various permutations in (1) may all be proved by invoking the cyclic nature of the subscripts of the Levi-Civita tensor. Consider, for example, $(1)_1$. Using tensor notation for a scalar product and (1) of Question 1.2 gives

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = A_i(\mathbf{B}\times\mathbf{C})_i = A_i\varepsilon_{ijk}B_jC_k = \varepsilon_{ijk}A_iB_jC_k .$$

Now $\varepsilon_{ijk} = \varepsilon_{kij}$, and so $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \varepsilon_{kij}A_iB_jC_k = (\mathbf{A}\times\mathbf{B})_kC_k$, which proves the result. The remaining cyclic permutations can be found in a similar way.

‡Suppose these fields have continuous second-order derivatives, so $\nabla_i\nabla_jA_k = \nabla_j\nabla_iA_k$, etc.

(b) Clearly,
$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{B}) &= (\mathbf{A}\times\mathbf{B})_i(\mathbf{A}\times\mathbf{B})_i\\ &= \varepsilon_{ijk}A_jB_k\,\varepsilon_{ilm}A_lB_m\\ &= (\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl})A_jA_lB_kB_m\\ &= A_iA_iB_jB_j - A_iB_iA_jB_j \quad \text{(subscripts are arbitrary)}\\ &= (\mathbf{A}\cdot\mathbf{A})(\mathbf{B}\cdot\mathbf{B})-(\mathbf{A}\cdot\mathbf{B})^2. \quad \text{Hence (2).}\end{aligned}$$

(c) It is sufficient to show that $[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})]_i = B_i(\mathbf{A}\cdot\mathbf{C}) - C_i(\mathbf{A}\cdot\mathbf{B})$. From (1) of Question 1.2
$$\begin{aligned}[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})]_i &= \varepsilon_{ijk}A_j(\mathbf{B}\times\mathbf{C})_k\\ &= \varepsilon_{ijk}A_j\varepsilon_{klm}B_lC_m = \varepsilon_{ijk}\varepsilon_{lmk}A_jB_lC_m = (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})A_jB_lC_m,\end{aligned}$$
using the cyclic property of ε_{klm} and (1) of Question 1.3. Contracting the right-hand side gives $A_mB_iC_m - A_lB_lC_i = B_i(\mathbf{A}\cdot\mathbf{C}) - C_i(\mathbf{A}\cdot\mathbf{B})$ as required.

(d) Consider the ith component. Then $\nabla_i(fg) = g\nabla_i f + f\nabla_i g$ by the product rule of differentiation and the result follows.

(e) $\boldsymbol{\nabla}\cdot(f\mathbf{A}) = \nabla_i(f\mathbf{A})_i = \nabla_i(fA_i) = A_i\nabla_i f + f\nabla_i A_i = \mathbf{A}\cdot\boldsymbol{\nabla}f + f(\boldsymbol{\nabla}\cdot\mathbf{A})$.

(f) Consider the ith component. Then
$$[\boldsymbol{\nabla}\times(f\mathbf{A})]_i = \varepsilon_{ijk}\nabla_j(fA_k) = \varepsilon_{ijk}(A_k\nabla_j f + f\nabla_j A_k) = (\boldsymbol{\nabla}f\times\mathbf{A})_i + f(\boldsymbol{\nabla}\times\mathbf{A})_i.$$

(g)
$$\begin{aligned}\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = \nabla_i(\mathbf{A}\times\mathbf{B})_i &= \nabla_i\varepsilon_{ijk}A_jB_k\\ &= \varepsilon_{ijk}(B_k\nabla_iA_j + A_j\nabla_iB_k)\\ &= (\varepsilon_{kij}\nabla_iA_j)B_k - (\varepsilon_{jik}\nabla_iB_k)A_j \qquad \text{(properties of } \varepsilon_{ijk})\\ &= (\boldsymbol{\nabla}\times\mathbf{A})_kB_k - (\boldsymbol{\nabla}\times\mathbf{B})_jA_j\\ &= (\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathbf{B} - (\boldsymbol{\nabla}\times\mathbf{B})\cdot\mathbf{A}.\end{aligned}$$

(h)
$$\begin{aligned}[\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B})]_i &= \varepsilon_{ijk}\nabla_j\varepsilon_{klm}A_lB_m\\ &= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\nabla_j(A_lB_m)\\ &= \nabla_m(A_iB_m) - \nabla_l(A_lB_i) \qquad \text{(contract subscripts)}\\ &= B_m\nabla_mA_i + A_i\nabla_mB_m - B_i\nabla_lA_l - A_l\nabla_lB_i \qquad \text{(product rule)}\\ &= (\mathbf{B}\cdot\boldsymbol{\nabla})A_i - (\mathbf{A}\cdot\boldsymbol{\nabla})B_i + A_i(\boldsymbol{\nabla}\cdot\mathbf{B}) - B_i(\boldsymbol{\nabla}\cdot\mathbf{A}),\end{aligned}$$
which proves the result.

(i) $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = \nabla_i(\boldsymbol{\nabla}\times\mathbf{A})_i = \nabla_i\varepsilon_{ijk}\nabla_jA_k = \varepsilon_{ijk}\nabla_i\nabla_jA_k = 0$,
since $\nabla_i\nabla_jA_k$ is symmetric in i and j, whereas ε_{ijk} is antisymmetric in these subscripts (see Question 1.5). Hence (9).

(j) $[\boldsymbol{\nabla}\times\boldsymbol{\nabla}f]_i = \varepsilon_{ijk}\nabla_j\nabla_k f = 0$ as in (i). Hence (10).

(k)
$$\begin{aligned}[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})]_i &= \varepsilon_{ijk}\nabla_j\varepsilon_{klm}\nabla_l A_m \\ &= \varepsilon_{ijk}\varepsilon_{lmk}\nabla_j\nabla_l A_m && \text{(cyclic property of } \varepsilon_{ijk}) \\ &= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\nabla_j\nabla_l A_m && \text{(contracting subscripts)} \\ &= (\nabla_i\nabla_m A_m - \nabla^2 A_i) = \nabla_i(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2 A_i\end{aligned}$$

as required.

(l) This result follows immediately from (7) and (10) above.

(m)
$$\begin{aligned}\nabla_i(\mathbf{A}\cdot\mathbf{B}) &= \nabla_i(A_jB_j) \\ &= A_j\nabla_i B_j + B_j\nabla_i A_j \\ &= A_j[\nabla_j B_i + \varepsilon_{ijk}(\boldsymbol{\nabla}\times\mathbf{B})_k] + B_j[\nabla_j A_i + \varepsilon_{ijk}(\boldsymbol{\nabla}\times\mathbf{A})_k],\end{aligned}$$

where in the last step we use (4) of Question 1.3. This proves the result.

Comments

(i) Equations (1) and (3) are the well-known scalar and vector triple products respectively. We note the following:

- ☞ In (1) the positions of the dot and cross may be interchanged, provided that the cyclic order of the vectors is maintained.
- ☞ The identity (3) is used often and is worth remembering. For easy recall, some textbooks call it the '**BAC–CAB** rule'. See, for example, Ref. [2].

(ii) Suppose **A**, **B** and **C** are polar vectors.‡ The transformation $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) \xrightarrow{\text{p}} -\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ results in the scalar triple product changing sign under inversion, and so it is a pseudoscalar.♯ If **A**, **B** and **C** are the spanning vectors of a crystal lattice, then $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ is the pseudovolume of the unit cell.†

(iii) In electromagnetism (1)–(13) are very useful identities. Although proved here for Cartesian coordinates, the results are valid in all coordinate systems.♭

Question 1.9

Consider the scalar functions $f(\mathbf{r})$ and $g\big(\mathbf{r}(t),t\big)$. Suppose $\mathbf{r}=\mathbf{r}(t)$ is a time-dependent position vector. Show that

$$\frac{df}{dt} = \left(\frac{d\mathbf{r}}{dt}\cdot\boldsymbol{\nabla}\right)f \qquad \text{and} \qquad \frac{dg}{dt} = \frac{\partial g}{\partial t} + \left(\frac{d\mathbf{r}}{dt}\cdot\boldsymbol{\nabla}\right)g. \tag{1}$$

‡The distinction between polar and axial vectors is described in Appendix A.

♯See also Appendix A. In the above, p is the parity operator described on p. 598.

†In this example, the *volume* of the unit cell is $|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|$.

♭This also applies to other results in this chapter, such as Gauss's theorem and Stokes's theorem.

[2] D. J. Griffiths, *Introduction to electrodynamics*, Chap. 1, p. 8. New York: Prentice Hall, 3 edn, 1999.

Solution

Since both proofs are similar, we consider that for $(1)_2$ only. The total differential of $g(x, y, z, t)$ is

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz + \frac{\partial g}{\partial t}dt.$$

Then

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \left(\frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z}\right)g,$$

which is $(1)_2$ since $d\mathbf{r}/dt = (dx/dt, dy/dt, dz/dt)$ and $\boldsymbol{\nabla} = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$.

Comments

(i) Equation $(1)_1$ is the chain rule of differentiation. Equation $(1)_2$ is often called the convective derivative. It is composed of two parts: the local or Eulerian derivative $\partial g/\partial t$ and the convective term $(\mathbf{v}\cdot\boldsymbol{\nabla})g$, where $\mathbf{v} = d\mathbf{r}/dt$ is the velocity of an element of charge or mass as it travels along its trajectory $\mathbf{r}(t)$.

(ii) Suppose $T(\mathbf{r}, t)$ represents a temperature field. The local derivative $\partial T/\partial t$ provides the change in temperature with time at a fixed point in space, whereas the convective term $(\mathbf{v}\cdot\boldsymbol{\nabla})T$ accounts for the rate at which the temperature changes in a fixed mass of air as it moves, for example, in a convection current.

(iii) For the vector fields $\mathbf{f}\big(\mathbf{r}(t)\big)$ and $\mathbf{g}\big(\mathbf{r}(t), t\big)$, these derivatives are

$$\frac{d\mathbf{f}}{dt} = \left(\frac{d\mathbf{r}}{dt}\cdot\boldsymbol{\nabla}\right)\mathbf{f} \qquad \text{and} \qquad \frac{d\mathbf{g}}{dt} = \frac{\partial\mathbf{g}}{\partial t} + \left(\frac{d\mathbf{r}}{dt}\cdot\boldsymbol{\nabla}\right)\mathbf{g}. \tag{2}$$

Question 1.10**

The flux ϕ of an arbitrary vector field $\mathbf{F}(\mathbf{r}, t)$ is

$$\phi = \int_s \mathbf{F}\cdot d\mathbf{a},$$

where s is any surface spanning an arbitrary contour c. Suppose the position, size and shape of c (and therefore s) change with time. Show that

$$\frac{d}{dt}\int_s \mathbf{F}\cdot d\mathbf{a} = \int_s \frac{d\mathbf{F}}{dt}\cdot d\mathbf{a}. \tag{1}$$

Hint: Let $\mathbf{r}\big(u(t), v(t)\big)$ be a parametric representation of s where $u_1 \leq u \leq u_2$ and $v_1 \leq v \leq v_2$ (see Appendix H). Then

$$\phi = \int_{u_1}^{u_2}\int_{v_1}^{v_2} \mathbf{F}\Big(\mathbf{r}\big(u(t), v(t)\big)\Big)\cdot\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right)du\,dv. \tag{2}$$

Solution

Differentiating (2) gives

$$\begin{aligned}
\frac{d\phi}{dt} &= \frac{d}{dt}\int_{u_1}^{u_2}\int_{v_1}^{v_2}\mathbf{F}\Big(\mathbf{r}\big(u(t),v(t)\big)\Big)\cdot\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right)du\,dv \\
&= \int_{u_1}^{u_2}du\int_{v_1}^{v_2}\left[\frac{d\mathbf{F}}{dt}\cdot\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right)+\mathbf{F}\cdot\frac{d}{dt}\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right)\right]dv. \qquad (3)
\end{aligned}$$

Consider the second term in square brackets in (3). Using $(1)_1$ of Question 1.9 yields

$$\frac{d}{dt}\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v} = \left(\frac{d\mathbf{r}}{dt}\cdot\boldsymbol{\nabla}\right)\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right). \qquad (4)$$

Using tensor notation, (4) can be written as

$$\begin{aligned}
\left[\left(\frac{d\mathbf{r}}{dt}\cdot\boldsymbol{\nabla}\right)\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right)\right]_i &= \varepsilon_{ijk}\frac{dr_l}{dt}\frac{\partial}{\partial r_l}\frac{\partial r_j}{\partial u}\frac{\partial r_k}{\partial v} && (5) \\
&= \varepsilon_{ikj}\frac{dr_l}{dt}\frac{\partial}{\partial r_l}\frac{\partial r_k}{\partial u}\frac{\partial r_j}{\partial v} && \text{(subscripts are arbitrary)} \\
&= -\varepsilon_{ijk}\frac{dr_l}{dt}\frac{\partial}{\partial r_l}\frac{\partial r_k}{\partial v}\frac{\partial v}{\partial u}\frac{\partial r_j}{\partial u}\frac{\partial u}{\partial v} && (\varepsilon_{ikj} = -\varepsilon_{ijk}) \\
&= -\varepsilon_{ijk}\frac{dr_l}{dt}\frac{\partial}{\partial r_l}\frac{\partial r_k}{\partial v}\frac{\partial r_j}{\partial u} \\
&= -\varepsilon_{ijk}\frac{dr_l}{dt}\frac{\partial}{\partial r_l}\frac{\partial r_j}{\partial u}\frac{\partial r_k}{\partial v} && \text{(rearranging terms).} \quad (6)
\end{aligned}$$

Comparing (5) and (6) shows that $\left[\left(\frac{d\mathbf{r}}{dt}\cdot\boldsymbol{\nabla}\right)\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right]_i = 0$, which is true for all components of this vector. Then (3) becomes

$$\begin{aligned}
\frac{d\phi}{dt} &= \int_{u_1}^{u_2}du\int_{v_1}^{v_2}\left[\frac{d\mathbf{F}}{dt}\cdot\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right)\right]dv \\
&= \int_s\frac{d\mathbf{F}}{dt}\cdot d\mathbf{a},
\end{aligned}$$

which is (1).

Comment

Equation (1) is a useful result for calculating emfs in non-stationary circuits or media. See Question 5.4.

Question 1.11

Use the relevant definition and the result that $\mathbf{r}$ is a polar vector to determine whether the following vectors are polar or axial: (a) velocity $\mathbf{u}$, (b) linear momentum $\mathbf{p}$, (c) force $\mathbf{F}$, (d) electric field $\mathbf{E}$, (e) magnetic field $\mathbf{B}$ and (f) $\mathbf{E} \times \mathbf{B}$.

(Assume that time, mass and charge are invariant quantities).

Solution

(a) $\mathbf{u} = d\mathbf{r}/dt$ is polar since t is invariant and $\mathbf{r}$ is polar.

(b) $\mathbf{p} = m\mathbf{u}$ is polar since m is invariant and $\mathbf{u}$ is polar.

(c) $\mathbf{F} = d\mathbf{p}/dt$ is polar since t is invariant and $\mathbf{p}$ is polar.

(d) $\mathbf{E} = \mathbf{F}/q$ is polar since q is invariant and $\mathbf{F}$ is polar.

(e) Apply the parity transformation to the force

$$\mathbf{F} \xrightarrow{\mathrm{p}} \mathbf{F}' = -\mathbf{F} = q\mathbf{u}' \times \mathbf{B}' = q(-\mathbf{u}) \times \mathbf{B}'.$$

Clearly, $\mathbf{F} = q\mathbf{u} \times \mathbf{B}'$ requires $\mathbf{B}' = \mathbf{B}$ which shows that $\mathbf{B}$ is axial.

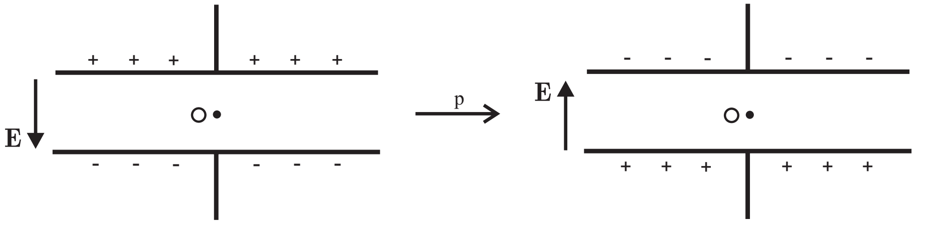
(f) $\mathbf{E} \times \mathbf{B} \xrightarrow{\mathrm{p}} (-\mathbf{E}) \times \mathbf{B} = -\mathbf{E} \times \mathbf{B}$ which is polar. This vector represents the energy flux per unit time‡ in the vacuum electromagnetic field (see (7) of Question 7.6).

Comments

(i) The polar (axial) nature of the electric (magnetic) field established in the above solution above can be confirmed by the following intuitive approach. We suppose uniform $\mathbf{E}$- and $\mathbf{B}$-fields are created by an ideal parallel-plate capacitor and an ideal solenoid respectively, and consider how these fields behave when their sources are inverted. This is illustrated in the figures below; notice that $\mathbf{E}$ reverses sign, whereas $\mathbf{B}$ does not.

E-field: cross-section through capacitor perpendicular to the plates

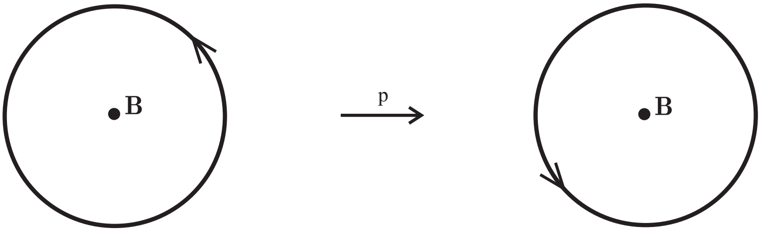
$\{\text{source } q \text{ at } \mathbf{r}\} \xrightarrow{\mathrm{p}} \{\text{source } q \text{ at } -\mathbf{r}\}$



‡Apart from a factor μ_0, which is a polar constant of proportionality. See Comment (ii) on p. 14.

B-field: cross-section through solenoid perpendicular to the symmetry axis

$\{\text{source } Id\boldsymbol{l} \text{ at } \mathbf{r}\} \xrightarrow{\text{p}} \{\text{source } -Id\boldsymbol{l} \text{ at } -\mathbf{r}\}$



(ii) Suppose $\mathbf{c} = \mathbf{a} \times \mathbf{b}$. Clearly, $\mathbf{c}$ is:

- ☞ polar if either **a** or **b** is polar and the other is axial (see the $\mathbf{E} \times \mathbf{B}$ example above).
- ☞ axial if **a** and **b** are either both polar or both axial.

(iii) Let **a** and **b** represent arbitrary vectors that satisfy laws of physics which we express algebraically as:

$$\mathbf{b} = \alpha\,\mathbf{a} \qquad \text{and} \qquad b_i = \beta_{ij} a_j. \tag{1}$$

Here **a** is taken to be the 'cause' and **b** the 'effect'. The constants of proportionality α and β_{ij} are tensors of rank zero and two respectively.♯ Under rotation of axes, they behave as follows:

- ☞ α and β_{ij} are polar if **a** and **b** are either both polar or both axial,
- ☞ α and β_{ij} are axial if either **a** or **b** is polar and the other is axial.

So, for example, in the Biot–Savart law (see $(7)_2$ of Question 4.4) $d\mathbf{B} = \dfrac{\mu_0}{4\pi}\dfrac{Id\boldsymbol{l} \times \boldsymbol{r}}{r^3}$, and we conclude that μ_0 is a polar scalar since both $d\mathbf{B}$ and $Id\boldsymbol{l} \times \mathbf{r}$ are axial vectors.

(iv) These results can be generalized to physical tensors and physical property tensors of any rank.[3]

(v) Considerations of symmetry and the spatial nature of tensors can sometimes be exploited to gain useful insight into a physical system. Consider, for example, a sphere of charge which is symmetric about its centre O. Suppose the sphere is spinning about an axis through O. Inversion through O obviously leaves the sphere unchanged as well as all its physical tensors and physical property tensors.

♯The following terminology is used in the literature (see, for example, Ref. [3]): **a**, **b** are called physical tensors (here they are physical vectors) and α, β_{ij} are physical property tensors (see also Comment (viii) of Question 2.26).

[3] R. E. Raab and O. L. de Lange, *Multipole theory in electromagnetism*, Chap. 3, pp. 59–72. Oxford: Clarendon Press, 2005.

Because polar vectors change sign under inversion it follows that the electric field at O is necessarily zero, whereas the magnetic field, being an axial vector, may have a finite value at the centre. Symmetry arguments alone cannot reveal the value of B at O; this can be determined either by solving the relevant Maxwell equation or by measurement.

(vi) In addition to characterizing physical tensors and physical property tensors by their spatial properties, it is also possible to consider how such quantities behave under a time-reversal transformation T. In classical physics, time reversal changes the sign of the time coordinate $t \xrightarrow{\mathrm{T}} t' = -t$. For motion in a conservative field the time-reversed trajectory is indistinguishable from the actual trajectory;[3] $\mathbf{r} \xrightarrow{\mathrm{T}} \mathbf{r}' = \mathbf{r}$. With this in mind, we consider the effect of a time-reversal transformation on the following first-rank tensors:

☞ $\mathbf{u} = d\mathbf{r}/dt \xrightarrow{\mathrm{T}} d\mathbf{r}/dt' = -\mathbf{u}$.

☞ $\mathbf{p} = m\mathbf{u} \xrightarrow{\mathrm{T}} -\mathbf{p}$.

☞ $\mathbf{F} = d\mathbf{p}/dt \xrightarrow{\mathrm{T}} \mathbf{F}$.

☞ $\mathbf{E} = \mathbf{F}/q \xrightarrow{\mathrm{T}} \mathbf{E}$.

☞ $q\mathbf{u} \times \mathbf{B} = \mathbf{F} \xrightarrow{\mathrm{T}} q(-\mathbf{u}) \times \mathbf{B}'$ requires $\mathbf{B} \xrightarrow{\mathrm{T}} \mathbf{B}' = -\mathbf{B}$.

Tensors which remain unchanged by time-reversal transformations are called time-even ($\mathbf{F}$ and $\mathbf{E}$ above), whilst those which change sign are time-odd ($\mathbf{u}$, $\mathbf{p}$ and $\mathbf{B}$ above). The space-time symmetry properties of these five vectors are thus:

☞ $\mathbf{u}$ and $\mathbf{p}$ are time-odd polar vectors,

☞ $\mathbf{E}$ and $\mathbf{F}$ are time-even polar vectors, and

☞ $\mathbf{B}$ is a time-odd axial vector.‡

(In the bulleted lists above, it has been assumed implicitly that m and q are time-even, polar scalars.[3]) Ref. [3] also provides interesting applications of these symmetry transformations to physical systems. For example, it is shown that the Faraday effect♯ in a fluid (whether optically active or inactive) is not vetoed by a space-time transformation, whereas the electric analogue of this effect, which has never been observed, is vetoed.[3]

(vii) The symmetries referred to in (vi) above are part of a much more general idea based on Neumann's principle which states that every physical property tensor of a system must possess the full space-time symmetry of the system. (This is quite apart from any intrinsic symmetry of the tensor subscripts themselves.)

‡An example of a time-even axial vector is torque, $\mathbf{\Gamma} = m\,\dfrac{d}{dt}(\mathbf{r} \times \mathbf{p})$.

♯In this effect, a magnetostatic field $\mathbf{B}$ applied parallel to the path of linearly polarized light in a fluid induces a rotation of the plane of polarization through an angle proportional to B.

Question 1.12

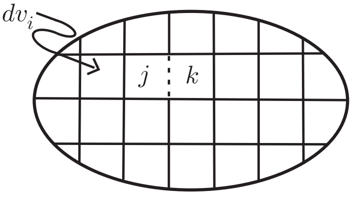
Consider a vector field $\mathbf{F}(\mathbf{r})$ with continuous first derivatives in some region of space having volume v bounded by the closed surface s. Use the definition of divergence‡ to prove that

$$\oint_s \mathbf{F}\cdot d\mathbf{a} = \int_v (\boldsymbol{\nabla}\cdot\mathbf{F})\,dv, \tag{1}$$

where $d\mathbf{a}$ is an element of area on s.

Solution

Imagine subdividing the macroscopic volume v into a large number n of infinitesimal elements having volume dv_i, where $i = 1, 2, \ldots, n$ (the elements might, for example, be cuboids with six faces). For the ith element the net outward flux is the sum over six faces. Using the definition of divergence we write



$$\sum_{\substack{\text{six}\\ \text{faces}}} \mathbf{F}_i\cdot d\mathbf{a}_i = (\boldsymbol{\nabla}\cdot\mathbf{F})_i\,dv_i. \tag{2}$$

The total flux through v is obtained by summing over all volume elements. In this summation the $\mathbf{F}_i\cdot d\mathbf{a}_i$ terms cancel in pairs for all interior surfaces.♯ The only terms which survive are those on the exterior surfaces for which no cancellation can occur and (2) becomes

$$\sum_{\substack{\text{exterior}\\ \text{faces}}} \mathbf{F}_i\cdot d\mathbf{a}_i = \sum_{\substack{\text{volume}\\ \text{elements}}} (\boldsymbol{\nabla}\cdot\mathbf{F})_i\,dv_i. \tag{3}$$

In the limit $n\to\infty$, the summation on the left-hand side of (3) becomes an integral over s and that on the right-hand side becomes an integral over v, which is (1).

‡The divergence of $\mathbf{F}$ at any point P is defined as follows:

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \lim_{v\to 0}\frac{1}{v}\oint_s \mathbf{F}\cdot d\mathbf{a}.$$

Here P lies within an arbitrary region of space having volume v and bounded by the closed surface s.

♯Consider the common face of the volume elements labelled j and k in the above figure (shown, in cross-section, as a dashed boundary line and assumed to be contained entirely within the interior of v). Then the outward flux through this face for element j equals the inward flux through this same face for element k. Since $d\mathbf{a}_j = -d\mathbf{a}_k$, then $\mathbf{F}_j\cdot d\mathbf{a}_j = -\mathbf{F}_k\cdot d\mathbf{a}_k$.

Comments

(i) This important result is known as Gauss's theorem (or sometimes the divergence theorem). It is a mathematical theorem, and should not be confused with Gauss's law which is a law of physics.

(ii) We mention two useful corollaries of the divergence theorem. They are Green's first and second identities:

$$\oint_s \big(f\boldsymbol{\nabla}g\big)\cdot d\mathbf{a} \;=\; \int_v \big(\boldsymbol{\nabla}f\cdot\boldsymbol{\nabla}g + f\nabla^2 g\big)\,dv, \tag{4}$$

and

$$\oint_s \big(g\boldsymbol{\nabla}f - f\boldsymbol{\nabla}g\big)\cdot d\mathbf{a} \;=\; \int_v \big(g\nabla^2 f - f\nabla^2 g\big)\,dv, \tag{5}$$

respectively. Here f and g are any two well-behaved scalar fields. Equation (4) is easily proved by substituting $\mathbf{F} = f\boldsymbol{\nabla}g$ in (1) and using (5) of Question 1.8. Equation (5) follows directly from (4).

(iii) Another useful identity, which follows from Gauss's theorem and (7) of Question 1.8, is

$$\oint_s \big(\mathbf{A}\times\mathbf{B}\big)\cdot d\mathbf{a} \;=\; \int_v \big[\big(\boldsymbol{\nabla}\times\mathbf{A}\big)\cdot\mathbf{B} - \big(\boldsymbol{\nabla}\times\mathbf{B}\big)\cdot\mathbf{A}\big]\,dv. \tag{6}$$

Question 1.13

Consider a vector field $\mathbf{F}(\mathbf{r})$ having continuous first derivatives in a region of space, in which c is an arbitrary closed contour and s any surface spanning c. Prove that

$$\oint_c \mathbf{F}\cdot d\boldsymbol{l} \;=\; \int_s \big(\boldsymbol{\nabla}\times\mathbf{F}\big)\cdot d\mathbf{a}, \tag{1}$$

where $d\mathbf{a}$ is an element of area on s.

Solution

To prove (1) we start by evaluating $\oint \mathbf{F}\cdot d\boldsymbol{l}$ around an infinitesimal rectangular path δc in the xy-plane:

$$(x,\,y,\,z) \to (x+dx,\,y,\,z) \to (x+dx,\,y+dy,\,z) \to (x,\,y+dy,\,z) \to (x,\,y,\,z).$$

If we label the corners of this rectangle 1, 2, 3 and 4, then

$$\begin{aligned}\oint_{\delta c} \mathbf{F}\cdot d\boldsymbol{l} \;&=\; \left(\int_{1\to 2} + \int_{2\to 3} - \left\{\int_{1\to 4} + \int_{4\to 3}\right\}\right)\mathbf{F}\cdot d\boldsymbol{l} \\ &=\; F_x(x,y,z)\,dx + F_y(x+dx,y,z)\,dy - F_y(x,y,z)\,dy - F_x(x,y+dy,z)\,dx\end{aligned}$$

$$= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dx\, dy$$

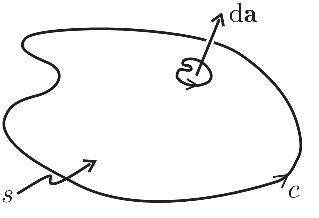
$$= (\nabla \times \mathbf{F})_z\, dx\, dy$$

$$= (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, da, \tag{2}$$

where $\mathbf{n}$ is a unit vector perpendicular to the rectangular element of area da. (There is a sign convention (a right-hand rule) implicit in (2), relating the sense in which δc is traversed and the direction of $\mathbf{n}$; see the figure below.) Equation (2) is independent of the choice of coordinates, and applies to an element of any orientation. An arbitrary finite surface s with boundary c can be subdivided into infinitesimal rectangular elements δc_i $(i = 1, 2, \ldots)$. Then

$$\int_c \mathbf{F} \cdot d\boldsymbol{l} = \sum_i \oint_{\delta c_i} \mathbf{F} \cdot d\boldsymbol{l}, \tag{3}$$

because on common segments of adjacent elements the $d\boldsymbol{l}$ point in opposite directions. So the contributions of $\mathbf{F} \cdot d\boldsymbol{l}$ to the sum in (3) cancel, whereas no such cancellation occurs on the boundary c. Equations (2) and (3) yield (1). The figure below illustrates the right-hand convention that is assumed here.



Comment

Equation (1) is known as Stokes's theorem (or sometimes the curl theorem), and it is another very important result.

Question 1.14

(This question and its solution are based on Questions 5.7 and 5.8 of Ref. [4].)

Use Stokes's theorem to prove that a necessary and sufficient condition for a vector field $\mathbf{F}(\mathbf{r})$ to be irrotational‡ (or conservative) is that $\nabla \times \mathbf{F} = 0$. Split the proof into two parts:

‡That is, $\mathbf{F}(\mathbf{r})$ is derivable from a single-valued scalar potential $V(\mathbf{r})$ as $\mathbf{F} = -\nabla V$.

[4] O. L. de Lange and J. Pierrus, *Solved problems in classical mechanics: Analytical and numerical solutions with comments.* Oxford: Oxford University Press, 2010.

necessary

Assume $\mathbf{F} = -\boldsymbol{\nabla} V$ then show that $\boldsymbol{\nabla} \times \mathbf{F} = 0$. (1)

sufficient

Assume $\boldsymbol{\nabla} \times \mathbf{F} = 0$ then show that $\mathbf{F} = -\boldsymbol{\nabla} V$. (2)

Solution

the necessary condition

If $\mathbf{F} = -\boldsymbol{\nabla} V$, then Stokes's theorem (see (1) of Question 1.13) yields

$$\int_s (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{a} = \oint_c \mathbf{F} \cdot d\boldsymbol{l} = -\oint_c \boldsymbol{\nabla} V \cdot d\boldsymbol{l} = -\oint dV(\mathbf{r}) = 0, \tag{3}$$

because $V(\mathbf{r})$ is a single-valued function. The surface s in (3) is arbitrary, and therefore it follows that $\boldsymbol{\nabla} \times \mathbf{F} = 0$ everywhere.

the sufficent condition

This part of the proof is less obvious than the preceding 'necessary' part because one has to prove the existence of the function $V(\mathbf{r})$. If $\boldsymbol{\nabla} \times \mathbf{F} = 0$ *everywhere*, it follows from Stokes's theorem that

$$\oint_c \mathbf{F} \cdot d\boldsymbol{l} = 0 \tag{4}$$

for *all* closed curves c. According to (4):

$$\int_1 \mathbf{F} \cdot d\boldsymbol{l} = \int_2 \mathbf{F} \cdot d\boldsymbol{l}, \tag{5}$$

where 1 and 2 are any two paths from point A to point B. Therefore, the line integral between any two such points is independent of the path followed from A to B: it depends only on the endpoints A and B. Thus, $\mathbf{F} \cdot d\boldsymbol{l}$ must be the differential of some single-valued scalar function $V(\mathbf{r})$, which we call a perfect differential:

$$\mathbf{F} \cdot d\boldsymbol{l} = -dV(\mathbf{r}), \tag{6}$$

where a minus sign has been inserted to conform with the standard convention. But

$$dV(\mathbf{r}) = \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz = (\boldsymbol{\nabla} V) \cdot d\boldsymbol{l}. \tag{7}$$

The line element $d\boldsymbol{l}$ in (6) and (7) is arbitrary, and therefore $\mathbf{F} = -\boldsymbol{\nabla} V$.

Question 1.15

(This question and its solution are based on Questions 5.22 and 5.23 of Ref. [4].)

Use both Stokes's theorem and Gauss's theorem to prove that a necessary and sufficient condition for a vector field $\mathbf{F}(\mathbf{r})$ to be solenoidal‡ is that $\boldsymbol{\nabla}\cdot\mathbf{F} = 0$. Split the proof into two parts:

necessary

Assume $\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{A}$ then show that $\boldsymbol{\nabla}\cdot\mathbf{F} = 0$. (1)

sufficient

Assume $\boldsymbol{\nabla}\cdot\mathbf{F} = 0$ then show that $\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{A}$. (2)

Solution

the necessary condition

Divide the closed surface s into two 'caps', s_1 and s_2, bounded by a common closed curve c, as shown in the figure on p. 21. According to Stokes's theorem

$$\int_{s_1}\mathbf{F}\cdot d\mathbf{a}_1 = \int_{s_1}(\boldsymbol{\nabla}\times\mathbf{A})\cdot d\mathbf{a}_1 = \oint_c \mathbf{A}\cdot d\boldsymbol{l} = \int_{s_2}(\boldsymbol{\nabla}\times\mathbf{A})\cdot d\mathbf{a}_2 = \int_{s_2}\mathbf{F}\cdot d\mathbf{a}_2,$$

where the sense in which c is traversed and the directions of $d\mathbf{a}_1$ and $d\mathbf{a}_2$ are fixed by the right-hand rule. Therefore♯

$$\oint_s \mathbf{F}\cdot d\mathbf{a} = \int_{s_1}\mathbf{F}\cdot d\mathbf{a}_1 + \int_{s_2}\mathbf{F}\cdot(-d\mathbf{a}_2) = 0,$$

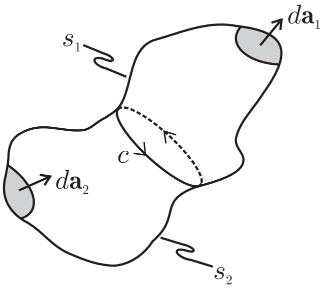
which by Gauss's theorem means that

$$\int_v (\boldsymbol{\nabla}\cdot\mathbf{F})\,dv = 0.$$

Because s_1 and s_2 are arbitrary, so is the volume v that they enclose. It therefore follows that $\boldsymbol{\nabla}\cdot\mathbf{F} = 0$.

‡That is, $\mathbf{F}(\mathbf{r})$ is derivable from a vector potential $\mathbf{A}(\mathbf{r})$ as $\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{A}$.

♯Note that $d\mathbf{a}_2$ is along an inward normal, as shown in the figure on p. 21, and therefore the element to be used in Gauss's theorem is $-d\mathbf{a}_2$.



the sufficient condition

The initial part of the proof involves the inverse of the reasoning used in the necessary condition above. If $\boldsymbol{\nabla}\cdot\mathbf{F} = 0$ *everywhere*, then it follows from Gauss's theorem that

$$\oint_s \mathbf{F}\cdot d\mathbf{a} = 0 \tag{3}$$

for *all* closed surfaces s. That is, for 'caps' s_1 and s_2 that share a common bounding curve c, as depicted in the above figure, we have

$$\int_{s_2} \mathbf{F}\cdot d\mathbf{a}_2 = \int_{s_1} \mathbf{F}\cdot d\mathbf{a}_1, \tag{4}$$

meaning that the flux of $\mathbf{F}$ through a cap is unchanged by any deformation of the cap that leaves the bounding curve c unaltered. Therefore, the fluxes in (4) can depend only on the curve c and not on other details of s_1 and s_2. These fluxes can be expressed as the line integral around c of some vector field $\mathbf{A}(\mathbf{r})$:

$$\int_{s_i} \mathbf{F}\cdot d\mathbf{a}_i = \oint_c \mathbf{A}\cdot d\boldsymbol{l} \qquad (i = 1, 2) \tag{5}$$

$$= \int_{s_i} (\boldsymbol{\nabla}\times\mathbf{A})\cdot d\mathbf{a}_i \qquad (i = 1, 2), \tag{6}$$

where Stokes's theorem is used in the last step. Since the surface s_i in (6) is arbitrary, we conclude that

$$\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{A}. \tag{7}$$

Question 1.16

(a) Consider the spherically symmetric vector field $\mathbf{F}(\mathbf{r}) = f(r)\,\hat{\mathbf{r}}/r^n$. Use the divergence operator for spherical polar coordinates (see $(\text{XI})_2$ of Appendix C) to prove that

$$\boldsymbol{\nabla}\cdot\mathbf{F} = 0 \Rightarrow f(r) = \alpha r^{n-2}, \tag{1}$$

where α is a constant.

(b) Consider the cylindrically symmetric vector field $\mathbf{G}(\mathbf{r}) = g(r)\,\hat{\mathbf{r}}/r^n$. Use the divergence operator for cylindrical polar coordinates (see $(\text{VIII})_2$ of Appendix D) to prove that

$$\boldsymbol{\nabla}\cdot\mathbf{G} = 0 \;\Rightarrow\; g(r) = \beta r^{n-1}, \tag{2}$$

where β is a constant.

Solution

(a) $\boldsymbol{\nabla}\cdot\mathbf{F} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\Big(r^2 F_r\Big) = \dfrac{1}{r^2}\dfrac{d}{dr}\Big(r^{2-n} f\Big) = 0$, implying that the term in brackets is a constant. Hence (1).

(b) $\boldsymbol{\nabla}\cdot\mathbf{G} = \dfrac{1}{r}\dfrac{\partial}{\partial r}\Big(r\,G_r\Big) = \dfrac{1}{r}\dfrac{d}{dr}\Big(r^{1-n} g\Big) = 0$, implying that the term in brackets is a constant. Hence (2).

Comments

(i) Because $\boldsymbol{\nabla} r^{-(n-1)} = -(n-1)r^{-n}$, it follows that $\dfrac{1}{r^n} = \dfrac{1}{1-n}\boldsymbol{\nabla}\Big(\dfrac{1}{r^{n-1}}\Big)$, assuming $n \neq 1$. Hence $\boldsymbol{\nabla}\times\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{G} = 0$, since the curl of any gradient is identically zero.

(ii) With $n = 2$ and $\alpha = q/4\pi\epsilon_0$, $\mathbf{F}(\mathbf{r})$ is the electric field $\mathbf{E}$ of a stationary point charge q. Here $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ (which is one of Maxwell's equations in a source-free vacuum) is valid everywhere except at the location of the charge.

(iii) With $n = 1$ and $\beta = \lambda/2\pi\epsilon_0$, $\mathbf{G}(\mathbf{r})$ is the electric field $\mathbf{E}$ of an infinite electric line charge having uniform density λ. As before, $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ is valid everywhere except at $r = 0$.

Question 1.17

Below we prove that magnetic fields are always zero. The 'proof' is based on two fundamental equations from electromagnetism (both of which are discussed in later chapters of this book): $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ and $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. Read the 'proof' and then explain where the (fatal) flaw lies.

'proof'

Maxwell's equation: $\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \quad\Rightarrow\quad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}.$ (1)

Gauss's theorem and $(1)_1$ give: $\displaystyle\int_v \boldsymbol{\nabla}\cdot\mathbf{B}\,dv = \int_s \mathbf{B}\cdot d\mathbf{a} = 0.$ (2)

Substituting $(1)_2$ in the surface integral (2) and using Stokes's theorem yield

$$\int_s (\boldsymbol{\nabla}\times \mathbf{A})\cdot d\mathbf{a} \;=\; \oint_c \mathbf{A}\cdot d\boldsymbol{l} \;=\; 0. \tag{3}$$

Now (3) implies that $\mathbf{A} = \boldsymbol{\nabla}\phi_{\mathrm{m}}$, (4)

where ϕ_{m} is a scalar potential. Equations $(1)_2$ and (4) then give $\mathbf{B} = \boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi_{\mathrm{m}}$. Since the curl of any gradient is identically zero it necessarily follows that $\mathbf{B}\equiv 0$.
Q.E.D.

Solution

In Gauss's theorem s must be a closed surface (see (2) where the theorem is applied incorrectly), but in Stokes's theorem s is open. Therefore, one cannot conclude that the circulation of $\mathbf{A}$ in (3), which follows from (2), is always zero.

Comment

Pay careful attention to all the details in every calculation. Do not assume that nuances in notation are simply a matter of pedantry. This question illustrates how careless execution can lead to incorrect physics (clearly, in this case, spectacularly incorrect).

Question 1.18

Laplace's equation $\nabla^2\Phi = 0$ is a second-order partial differential equation which arises in many branches of physics. Although there are no general techniques for solving this equation, the 'method of separation of variables' sometimes works. This method is based on a trial solution in which the variables of the problem are separated from one another (see, for example, (1) below). If this trial solution can be made to fit the boundary conditions of the problem (assuming that these have been suitably specified), then its uniqueness is guaranteed.[‡]

(a) Suppose $\Phi = \Phi(x,\, y,\, z)$. Attempt solutions to Laplace's equation of the form

$$\Phi(x,\, y,\, z) \;=\; X(x)Y(y)Z(z), \tag{1}$$

where X, Y and Z are all functions of a single variable. Show that (1) leads to

$$\Phi(x,\, y,\, z) \;=\; \Phi_0 e^{\pm k_1 x} e^{\pm k_2 y} e^{\pm k_3 z}, \tag{2}$$

where the k_i^2 are real constants.

(b) Find the form of (2) for k_1^2 and k_2^2 both negative.

(c) Find the form of (2) for k_1^2 and k_2^2 both positive.

[‡]See also Question 3.3(d).

Solution

(a) Substituting (1) in $\nabla^2\Phi = 0$ and dividing by XYZ give

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0. \tag{3}$$

The first term in (3) is independent of y and z, the second term is independent of x and z and the third term is independent of x and y. Now the sum of the terms in (3) is identically zero for *all* x, y and z. This requires that *each* term is independent of x, y and z and is therefore a constant. So

$$\frac{1}{X}\frac{d^2X}{dx^2} = k_1^2, \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = k_2^2 \qquad \text{and} \qquad \frac{1}{Z}\frac{d^2Z}{dz^2} = k_3^2, \tag{4}$$

where

$$k_1^2 + k_2^2 + k_3^2 = 0. \tag{5}$$

These three ordinary differential equations have the solutions

$$X = X_0 e^{\pm k_1 x}, \qquad Y = Y_0 e^{\pm k_2 y} \qquad \text{and} \qquad Z = Z_0 e^{\pm k_3 z},$$

and together with (1) they yield (2) where $\Phi_0 = X_0 Y_0 Z_0$.

(b) Let $k_1^2 = -\alpha^2$ and $k_2^2 = -\beta^2$ where α and β are real constants. Substituting $k_1 = i\alpha$ and $k_2 = i\beta$ in (5) gives $k_3^2 = -k_1^2 - k_2^2 = \alpha^2 + \beta^2 = \gamma^2$ say. Then $k_3 = \gamma$ is also clearly real, and (2) becomes

$$\Phi(x,\, y,\, z) = \Phi_0 e^{\pm i\alpha x} e^{\pm i\beta y} e^{\pm\gamma z}. \tag{6}$$

(c) Now we let $k_1^2 = \alpha^2$ and $k_2^2 = \beta^2$ (again the constants α, β are real). Then $k_1 = \alpha$, $k_2 = \beta$ and $k_3^2 = -k_1^2 - k_2^2 = -\alpha^2 - \beta^2 = -\gamma^2 \Rightarrow k_3 = i\gamma$ (as before γ is real). Substituting these k_i in (2) gives

$$\Phi(x,\, y,\, z) = \Phi_0 e^{\pm\alpha x} e^{\pm\beta y} e^{\pm i\gamma z}. \tag{7}$$

Comments

(i) The k_i^2 are known as the separation constants and they may be positive or negative. Their signs are determined by the physics of the problem via the boundary conditions. Choosing k_1^2 and k_2^2 with opposite signs reproduces solutions of the form (6) and (7), but with different permutations of the axes.

(ii) If any one of the various combinations in (6) and (7) is to be 'the' solution to a particular physical problem, then it must be made to satisfy all the boundary conditions of that problem. Furthermore, the boundary conditions can be used to select which (if any) of these possible combinations is a suitable solution. For example, if $\Phi \to 0$ as $z \to \pm\infty$ then a choice involving $e^{\mp\gamma z}$ must be made.

(iii) Linear combinations of these product solutions also satisfy Laplace's equation,$^\sharp$ and alternative forms of (6) and (7) are therefore

$$\Phi(x,\, y,\, z) \;=\; \Phi_0 \begin{cases} \cos\alpha x \\ \sin\alpha x \end{cases} \begin{cases} \cos\beta y \\ \sin\beta y \end{cases} \begin{cases} \cosh\gamma z \\ \sinh\gamma z \end{cases} \tag{8}$$

and

$$\Phi(x,\, y,\, z) \;=\; \Phi_0 \begin{cases} \cosh\alpha x \\ \sinh\alpha x \end{cases} \begin{cases} \cosh\beta y \\ \sinh\beta y \end{cases} \begin{cases} \cos\gamma z \\ \sin\gamma z\,, \end{cases} \tag{9}$$

respectively. Because the boundary conditions impose restrictions on the possible values of α, β and γ, they often have a further role in determining the value of the constant Φ_0.

(iv) Separable solutions of Laplace's equation can also be found for other coordinate systems, as for example in spherical polar coordinates. See Question 1.19.

(v) We end these comments with two descriptions of the method of separation of variables. The first, rather colourful, account describes the method 'as one of the most beautiful techniques in all of mathematical physics'.[5] The second description explains that

> the method of separation of variables is perhaps the oldest systematic method for solving partial differential equations. Its essential feature is to transform the partial differential equation by a set of ordinary differential equations. The required solution of the partial differential equation is then exposed as a product $u(x, y) = X(x)Y(y) \neq 0$, or as a sum $u(x, y) = X(x) + Y(y)$, where $X(x)$ and $Y(y)$ are functions of x and y, respectively. Many significant problems in partial differential equations can be solved by the method of separation of variables. This method has been considerably refined and generalized over the last two centuries and is one of the classical techniques of applied mathematics, mathematical physics and engineering science. ... In many cases, the partial differential equation reduces to two ordinary differential equations for X and Y. A similar treatment can be applied to equations in three or more independent variables. However, the question of separability of a partial differential equation into two or more ordinary differential equations is by no means a trivial one. In spite of this question, the method is widely used in finding solutions of a large class of initial boundary-value problems. This method of solution is also known as the Fourier method (or the method of eigenfunction expansion). Thus, the procedure outlined above leads to the important ideas of eigenvalues, eigenfunctions, and orthogonality, all of which are very general and powerful for dealing with linear problems.[6]

$^\sharp$This property is proved in Question 3.3(a).

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[5] Source unknown: possibly R. P. Feynman.

[6] L. Debnath, *Differential equations for scientists and engineers*, Chap. 2, pp. 51–2. Boston: Birkhäuser, 4 edn, 2007.

Question 1.19

Suppose Φ is an axially symmetric potential which satisfies Laplace's equation.

(a) Using ∇^2 for spherical polar coordinates, show that

$$\Phi(r,\theta) = R(r)\Theta(\theta) \tag{1}$$

accomplishes a separation of variables, and leads to the decoupled equations

$$\left.\begin{aligned} \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - kR &= 0 \\ \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) + k\Theta &= 0 \end{aligned}\right\}, \tag{2}$$

where k is a constant.

(b) Hence show that in spherical polar coordinates the general solution of Laplace's equation for boundary-value problems with axial symmetry is

$$\Phi(r,\theta) = \sum_{n=0}^{\infty}\left[A_n r^n + B_n r^{-(n+1)}\right]P_n(\cos\theta), \tag{3}$$

where A_n, B_n are constants and $P_n(\cos\theta)$ is the Legendre polynomial of order n in $\cos\theta$ (see Appendix F).

Hint: Begin with the substitution $R(r) = U(r)/r$ and assume that the separation constant $k = n(n+1)$ where n is a non-negative integer.

Solution

(a) Because of the axial symmetry $\partial\Phi/\partial\phi = 0$ in $\nabla^2\Phi$ (see $(\mathrm{XI})_4$ of Appendix C), and so

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\Phi}{\partial\theta}\right) = 0. \tag{4}$$

Substituting (1) in (4) and dividing by $R\Theta$ gives

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right). \tag{5}$$

Now the left-hand side of (5) is a function of r only, and the right-hand side is a function of θ only. So each must be equal to a constant, k say. Hence (2).

(b) Because of the hint, $(2)_1$ becomes

$$\frac{d^2U}{dr^2} - \frac{n(n+1)}{r^2}U = 0,$$

whose general solution is $U(r) = Ar^{n+1} + Br^{-n}$. Thus

$$R(r) = Ar^n + Br^{-(n+1)}. \tag{6}$$

Next we turn to equation $(2)_2$. Its solutions (as outlined in Appendix F) are of the form

$$\Theta(\theta) = P_n(\cos\theta), \tag{7}$$

apart from an overall constant (which can later be absorbed into other constants). Substituting (6) and (7) in (1), and recalling that the Legendre polynomials form a complete set of functions on the interval $0 \leq \theta \leq \pi$, it follows that Φ can be expanded as an infinite series

$$\Phi(r,\theta) = \sum_{n=0}^{\infty}\Big[A_n r^n P_n(\cos\theta) + B_n r^{-(n+1)} P_{-(n+1)}(\cos\theta)\Big].$$

Now

$$P_{-(n+1)}(\cos\theta) = P_n(\cos\theta)$$

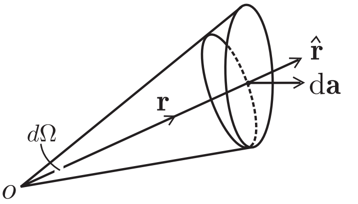
as we show below.‡ Hence (3).

Question 1.20

Let $d\mathbf{a}$ be an infinitesimal area element of some surface s and O any point. The solid angle $d\Omega$ subtended by $d\mathbf{a}$ at O is defined as

$$d\Omega = \frac{d\mathbf{a}\cdot\hat{\mathbf{r}}}{r^2}, \tag{1}$$

where $\mathbf{r}$ is the vector from O to $d\mathbf{a}$.



(a) Suppose s is the unit sphere centred at O. What is the solid angle Ω subtended by s at O?

(b) Suppose s is a closed surface of arbitrary shape. Show that

$$\Omega = \oint_s d\Omega = \begin{cases} 4\pi & \text{if } O \text{ lies inside } s \\ 0 & \text{if } O \text{ lies outside } s. \end{cases} \tag{2}$$

‡If integer n satisfies $k = n(n+1)$, then so does $n' = -(n+1)$, since $n'(n'+1) = -(n+1)(-n) = k$.

Solution

(a) $$\Omega = \frac{\text{surface area}}{\text{radius squared}} = 4\pi. \tag{3}$$

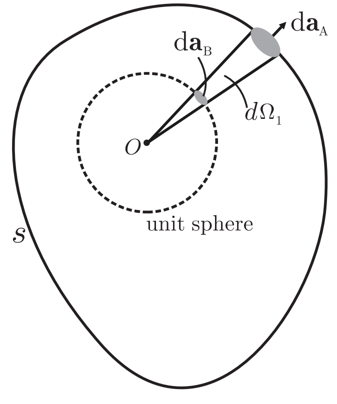
(b) The rays from O passing through the periphery of $d\mathbf{a}$ generate an infinitesimal cone with apex at O. Similar cones can be generated for all the surface elements of s (say N in total where $N \to \infty$).

<u>O inside s</u>

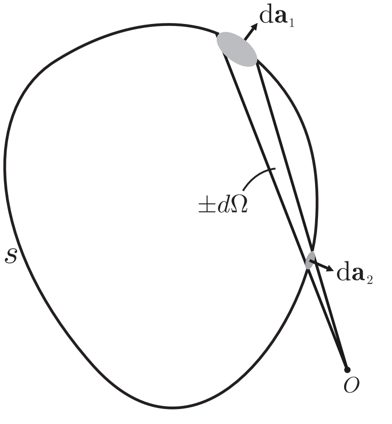
Fig. (I) shows origin O chosen arbitrarily inside s. Also shown is the unit sphere centred at O. The area element $d\mathbf{a}_\mathrm{A}$ of s subtends the same solid angle $d\Omega_1$ at O as the area element $d\mathbf{a}_\mathrm{B}$ of the unit sphere. This is true for all the other cones and $\Omega = d\Omega_1 + d\Omega_2 + \cdots + d\Omega_N = 4\pi$ because of (3).

<u>O outside s</u>

The cone shown in Fig. (II) intersects the surface twice. The solid angles subtended at O by the area elements $d\mathbf{a}_1$ and $d\mathbf{a}_2$ are $d\Omega$ and $-d\Omega$‡ respectively, and the sum of these two contributions is zero. This cancellation occurs in pairs for all the other cones and in this case $\Omega = 0$.



(I)



(II)

Comments

(i) In the SI system the unit of measure of solid angle is called the steradian which is a dimensionless quantity (compare with plane angles which are measured in radians and are also dimensionless).

(ii) Equation (2) can be conveniently expressed as

‡By convention the area element $d\mathbf{a}$ is directed along the *outward* normal.

$$\oint_s \frac{d\mathbf{a}\cdot\hat{\mathbf{r}}}{r^2} \;=\; 4\pi\int_v \delta(\mathbf{r})\,dv, \tag{4}$$

where $\delta(\mathbf{r})$ is the Dirac delta function. See (X) of Appendix E.

(iii) Consider the vector field $\mathbf{F}(\mathbf{r}) = F(r)\hat{\mathbf{r}}$ where $F(r) = k/r^2$.[♯] The flux ψ of $\mathbf{F}$ through any closed surface s follows directly from (1) and (2) and is

$$\psi \;=\; \oint_s \mathbf{F}\cdot d\mathbf{a} \;=\; k\,\Omega \;=\; \begin{cases} 4\pi k & \text{if the source point } O \text{ lies inside } s, \\ 0 & \text{if the source point } O \text{ lies outside } s. \end{cases} \tag{5}$$

This equation (known as Gauss's law[†]) is a very important law in physics. It relates the flux of $\mathbf{F}$ to its source(s). Familiar examples are:

- ☞ the gravitational acceleration of a planet having mass M (where $k = GM$), and
- ☞ the electric field of a point charge q in vacuum (where $k = q/4\pi\epsilon_0$).

(iv) The generality implied by (5) is the reason why Gauss's law is so useful. The flux through the closed surface is *independent* of the location of O. For the source point anywhere inside s we have $\psi = 4\pi k$, and $\psi = 0$ if the source point is anywhere outside s.

(v) The two features of $\mathbf{F}(\mathbf{r})$ upon which Gauss's law critically depends are:

- ☞ the inverse-square nature of the field, and
- ☞ the central nature of the field (i.e. $\mathbf{F}$ directed along $\hat{\mathbf{r}}$).

The spherical symmetry present in Newton's law of gravitation and Coulomb's law is not a necessary condition for (5). We show in Question 12.15 that Gauss's law also holds for the non-spherically symmetric, inverse-square, central electric field of a point charge moving relativistically at constant speed.

Question 1.21

Use Gauss's theorem and the definition of solid angle to prove that the Laplacian of r^{-1} (i.e. the divergence of the gradient of r^{-1}) is

$$\nabla^2\left(\frac{1}{r}\right) \;=\; -4\pi\delta(\mathbf{r}), \tag{1}$$

where $\delta(\mathbf{r})$ is the Dirac delta function (see Appendix E).

[♯] $\mathbf{F}$ is called 'spherically symmetric' because $F(r)$ depends only on the magnitude of $\mathbf{r}$ and not on the direction of $\mathbf{r}$.

[†] As previously mentioned (see Comment (i) of Question 1.12), Gauss's law and Gauss's theorem are separate entities and should not be confused.

Solution

Substituting $\mathbf{F} = -\boldsymbol{\nabla}\left(\frac{1}{r}\right) = \frac{\mathbf{r}}{r^3}$ in $\oint_s \mathbf{F}\cdot d\mathbf{a} = \int_v (\boldsymbol{\nabla}\cdot\mathbf{F})\, dv$ gives

$$\oint_s \frac{\mathbf{r}\cdot d\mathbf{a}}{r^3} = -\int_v \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\left(\frac{1}{r}\right) dv = -\int_v \nabla^2\left(\frac{1}{r}\right) dv. \tag{2}$$

Then from (4) of Question 1.20

$$\int_v \nabla^2\left(\frac{1}{r}\right) dv = -4\pi\int_v \delta(\mathbf{r})\, dv, \tag{3}$$

and hence (1) because the volume v is arbitrary.

Comments

(i) Shifting the singularity from $\mathbf{r}=0$ to $\mathbf{r}=\mathbf{r}'$ gives the more general form of (1):

$$\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -4\pi\delta(\mathbf{r}-\mathbf{r}'). \tag{4}$$

The proof given in the solution above is standard, but see also Ref. [7].

(ii) The form of (1) raises the question: will other derivatives such as $\nabla_i\nabla_j\big(1/r\big)^\ddagger$ also contain a delta function? They do.[8] For example,

$$\nabla_i\nabla_j\left(\frac{1}{r}\right) = \frac{3r_ir_j - r^2\delta_{ij}}{r^5} - \frac{4\pi}{3}\delta_{ij}\,\delta(\mathbf{r}) \tag{5}$$

(a non-rigorous proof of this result is provided in Ref. [9]). The first term on the right-hand side of (5) applies at points *away* from the origin, whilst the second term is zero everywhere except *at* the origin.$^\sharp$ For an application involving (5), see Comment (iii) of Question 2.11.

‡They should, because the Laplacian of r^{-1} is just a special case of $\nabla_i\nabla_j\big(1/r\big)$ when $i=j$.

$^\sharp$Apart from the minus sign, the first term of (5) is (7) of Question 1.1. Evidently, explicit differentiation fails to reveal the δ-function contribution.

[7] V. Hnizdo, 'On the Laplacian of $1/r$', *European Journal of Physics*, vol. 21, pp. L1–L3, 2000.
[8] See, for example, R. Estrada and R. P. Kanwal, 'The appearance of nonclassical terms in the analysis of point-source fields', *American Journal of Physics*, vol. 63, p. 278, 1995.
[9] C. P. Frahm, 'Some novel delta-function identities', *American Journal of Physics*, vol. 51, pp. 826–9, 1983.

Question 1.22

Suppose f and $\mathbf{F}$ represent suitably continuous and differentiable scalar and vector fields respectively. Let $\mathbf{b}$ be an arbitrary but constant vector. Using the hint provided alongside each, prove the following integral theorems:

(a) $$\oint_s f\,d\mathbf{a} = \int_v (\boldsymbol{\nabla} f)\,dv \qquad \text{(in Gauss's theorem let } \mathbf{F} = f\mathbf{b}\text{)}, \tag{1}$$

(b) $$\oint_s \mathbf{F}\times\,d\mathbf{a} = -\int_v (\boldsymbol{\nabla}\times\mathbf{F})\,dv \qquad \text{(in Gauss's theorem let } \mathbf{F} \to \mathbf{b}\times\mathbf{F}\text{)}, \tag{2}$$

where the region v is bounded by the closed surface s.

(c) $$\oint_c f\,d\boldsymbol{l} = -\int_s \boldsymbol{\nabla} f\times\,d\mathbf{a} \qquad \text{(in Stokes's theorem let } \mathbf{F} = f\mathbf{b}\text{)}, \tag{3}$$

(d) $$\oint_c \mathbf{F}\times\,d\boldsymbol{l} = -\int_s (d\mathbf{a}\times\boldsymbol{\nabla})\times\mathbf{F} \quad \text{(in Stokes's theorem let } \mathbf{F} \to \mathbf{b}\times\mathbf{F}\text{)}, \tag{4}$$

where the closed contour c is spanned by the surface s.

Solution

(a) Gauss's theorem becomes $\oint_s f\,\mathbf{b}\cdot d\mathbf{a} = \int_v (\mathbf{b}\cdot\boldsymbol{\nabla} f)\,dv$ which, because $\mathbf{b}$ is a constant vector, can be written as $\mathbf{b}\cdot\left[\oint_s f\,d\mathbf{a} - \int_v (\boldsymbol{\nabla} f)\,dv\right] = 0$. Now $|\mathbf{b}| \neq 0$ and since $\mathbf{b}$ is arbitrary, the cosine of the included angle is not always zero. This equation can only be satisfied if the term in brackets is zero, which proves (1).

(b) Gauss's theorem becomes $\oint_s (\mathbf{b}\times\mathbf{F})\cdot d\mathbf{a} = -\int_v \mathbf{b}\cdot(\boldsymbol{\nabla}\times\mathbf{F})\,dv$ by (7) of Question 1.8 and $\boldsymbol{\nabla}\times\mathbf{b} = 0$. Using the cyclic property of the scalar triple product $\big($see (1) of Question 1.8$\big)$, this can be written as $\mathbf{b}\cdot\left[\oint_s \mathbf{F}\times\,d\mathbf{a} + \int_v (\boldsymbol{\nabla}\times\mathbf{F})\,dv\right] = 0$. As before, this equation can only be satisfied if the term in square brackets is zero, which proves (2).

(c) Stokes's theorem becomes $\oint_c f\,\mathbf{b}\cdot d\boldsymbol{l} = \int_s (\boldsymbol{\nabla} f\times\mathbf{b})\cdot d\mathbf{a}$ because $\boldsymbol{\nabla}\times\mathbf{b} = 0$ $\big($see (6) of Question 1.8$\big)$. Use of the non-commutative property of the cross-product and the cyclic nature of the scalar triple product yields $\oint_c f\,\mathbf{b}\cdot d\boldsymbol{l} = -\int_s \mathbf{b}\cdot(\boldsymbol{\nabla} f\times d\mathbf{a})$, or $\mathbf{b}\cdot\left[\oint_c f\,d\boldsymbol{l} + \int_s \boldsymbol{\nabla} f\times\,d\mathbf{a}\right] = 0$. As in (a) and (b), this equation can only be satisfied if the term in square brackets is zero, which proves (3).

(d) Stokes's theorem becomes $\oint_c (\mathbf{b}\times\mathbf{F})\cdot d\boldsymbol{l} = -\int_s [\boldsymbol{\nabla}\times(\mathbf{b}\times\mathbf{F})]\cdot d\mathbf{a}$. Applying (1) and (8) of Question 1.8 to this result, and because $\mathbf{b}$ is a constant vector, we obtain

$$
\begin{aligned}
\mathbf{b}\cdot\oint_c \mathbf{F}\times d\boldsymbol{l} &= \int_s [\mathbf{b}(\boldsymbol{\nabla}\cdot\mathbf{F}) - (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{F}]\cdot d\mathbf{a} \;=\; \int_s b_i\nabla_k F_k da_i - \int_s (b_i\nabla_i F_j)\, da_j \\
&= b_i(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{kj})\int_s da_j\nabla_l F_k \;=\; b_i\,\varepsilon_{mik}\,\varepsilon_{mjl}\int_s da_j\nabla_l F_k \\
&= b_i\,\varepsilon_{mik}\int_s (d\mathbf{a}\times\boldsymbol{\nabla})_m F_k \;=\; -b_i\,\varepsilon_{imk}\int_s (d\mathbf{a}\times\boldsymbol{\nabla})_m F_k \\
&= b_i\int_s [(d\mathbf{a}\times\boldsymbol{\nabla})\times\mathbf{F}]_i \;=\; \mathbf{b}\cdot\int_s [(d\mathbf{a}\times\boldsymbol{\nabla})\times\mathbf{F}].
\end{aligned}
$$

Thus $\mathbf{b}\cdot\left[\oint_c \mathbf{F}\times d\boldsymbol{l} - \int_s (d\mathbf{a}\times\boldsymbol{\nabla})\times\mathbf{F}\right] = 0$, which leads to (4) in the usual way.

Comment

From (4) we can derive a useful result for the (vector) area $\mathbf{a}$ of a plane surface s:

$$
\mathbf{a} = \tfrac{1}{2}\oint_c (\mathbf{r}\times d\boldsymbol{l}). \tag{5}
$$

The proof of (5) is straightforward. Let $\mathbf{F} = \mathbf{r}$ in (4) and use Cartesian tensors to show that $\int_s (d\mathbf{a}\times\boldsymbol{\nabla})\times\mathbf{r} = -2\int_s d\mathbf{a}$. The details are left as an exercise for the reader.

Question 1.23

A vector field $\mathbf{F}(\mathbf{r})$ is continuous at all points inside a volume v and on a surface s bounding v, as are its divergence and curl

$$
\left.\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{F} &= S(\mathbf{r}) \\
\boldsymbol{\nabla}\times\mathbf{F} &= \mathbf{C}(\mathbf{r})
\end{aligned}\right\}. \tag{1}
$$

(Note that $\boldsymbol{\nabla}\cdot\mathbf{C} = 0$ for self-consistency.)

(a) Prove that $\mathbf{F}$ is a unique solution of (1) when it satisfies the boundary condition

$$
\mathbf{F}\cdot\hat{\mathbf{n}} \text{ is known everywhere on } s, \tag{2}
$$

where $\hat{\mathbf{n}}$ is a unit normal on s.

(b) Repeat (a) for the boundary condition

$$\mathbf{F} \times \hat{\mathbf{n}} \text{ is known everywhere on } s. \tag{3}$$

Hint: Construct a proof by contradiction. Begin by assuming that $\mathbf{F}_1(\mathbf{r})$ and $\mathbf{F}_2(\mathbf{r})$ are two different vector fields having the same divergence and curl and satisfying the same boundary condition. Then let $\mathbf{W} = \mathbf{F}_1 - \mathbf{F}_2$ and use the results of Question 1.12 (use Green's identity (4) for (a); use (6) for (b)) to prove that $\mathbf{W} = 0$.

Solution

Since $\mathbf{W} = \mathbf{F}_1 - \mathbf{F}_2$ and because of the hint, it is clear that

$$\left.\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{W} &= 0 \\ \boldsymbol{\nabla} \times \mathbf{W} &= 0 \end{aligned}\right\}. \tag{4}$$

We consider each of the two boundary conditions separately:

(a) Equation $(4)_2$ implies that (apart from a possible minus sign)

$$\mathbf{W} = \boldsymbol{\nabla} V, \tag{5}$$

where $V(\mathbf{r})$ is a scalar potential. It then follows immediately from $(4)_1$ that

$$\nabla^2 V = 0. \tag{6}$$

Substituting $f = g = V$ in Green's first identity (see (4) of Question 1.12) gives

$$\oint_s V \boldsymbol{\nabla} V \cdot d\mathbf{a} = \oint_s V \frac{\partial V}{\partial n}\, da = \int_v (\boldsymbol{\nabla} V \cdot \boldsymbol{\nabla} V + V \nabla^2 V)\, dv, \tag{7}$$

since in (7) $d\mathbf{a} = \hat{\mathbf{n}} da$ and $\boldsymbol{\nabla} V \cdot \hat{\mathbf{n}} = \partial V / \partial n$ is the normal derivative of V over the boundary surface s. Because of (6), this identity simplifies to

$$\oint_s V \frac{\partial V}{\partial n}\, da = \int_v |\boldsymbol{\nabla} V|^2\, dv. \tag{8}$$

Now $\mathbf{F}_1$ and $\mathbf{F}_2$ both satisfy the same boundary condition. Hence from (5)

$$(\mathbf{F}_1 - \mathbf{F}_2) \cdot \hat{\mathbf{n}} = \mathbf{W} \cdot \hat{\mathbf{n}} = \boldsymbol{\nabla} V \cdot \hat{\mathbf{n}} = \frac{\partial V}{\partial n} = 0,$$

and (8) becomes

$$\int_v |\boldsymbol{\nabla} V|^2 dv = 0. \tag{9}$$

The integrand in (9) is clearly non-negative, which requires that $\boldsymbol{\nabla} V = 0$ throughout v. Thus $\mathbf{W} = 0$ and $\mathbf{F}_1 = \mathbf{F}_2$ everywhere. A vector field $\mathbf{F}(\mathbf{r})$ which satisfies the boundary condition (2) is therefore a unique solution of (1).

(b) Equation $(4)_1$ implies that

$$\mathbf{W} = \boldsymbol{\nabla}\times\mathbf{A}, \tag{10}$$

where $\mathbf{A}(\mathbf{r})$ is a vector potential, and so from $(4)_2$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) = 0. \tag{11}$$

Substituting $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ in the identity (6) of Question 1.12 gives

$$\oint_s \big[\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{A})\big]\cdot d\mathbf{a} = \int_v \Big\{(\boldsymbol{\nabla}\times\mathbf{A})\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\cdot\big[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})\big]\Big\}dv. \tag{12}$$

Now $\big[\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{A})\big]\cdot d\mathbf{a} = -\big[\hat{\mathbf{n}}\times(\boldsymbol{\nabla}\times\mathbf{A})\big]\cdot\mathbf{A}\,da$ follows from the properties of the scalar triple product. Then (11) and (12) yield

$$-\oint_s \big[\hat{\mathbf{n}}\times(\boldsymbol{\nabla}\times\mathbf{A})\big]\cdot\mathbf{A}\,da = \int_v (\boldsymbol{\nabla}\times\mathbf{A})^2\,dv. \tag{13}$$

From (10), and since both $\mathbf{F}_1$ and $\mathbf{F}_2$ satisfy the same boundary condition, we have

$$-\hat{\mathbf{n}}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \hat{\mathbf{n}}\times(\mathbf{F}_2-\mathbf{F}_1) = 0.$$

Therefore, (13) becomes

$$\int_v (\boldsymbol{\nabla}\times\mathbf{A})^2\,dv = 0. \tag{14}$$

Using similar reasoning as in (a), we conclude that $\mathbf{W} = \boldsymbol{\nabla}\times\mathbf{A} = 0$. Therefore $\mathbf{F}_1 = \mathbf{F}_2$ everywhere, and any vector field $\mathbf{F}(\mathbf{r})$ satisfying the boundary condition (3) is a unique solution of (1).

Comments

(i) The quantities $S(\mathbf{r})$ and $\mathbf{C}(\mathbf{r})$ are known as the source and circulation densities respectively. In electromagnetism, $S(\mathbf{r})$ is the electric charge density and $\mathbf{C}(\mathbf{r})$ the current density.

(ii) Clearly, the boundary conditions (2) and (3) specify the normal component and the tangential component of $\mathbf{F}$ on s respectively.

Question 1.24*

Suppose $\mathbf{F}(\mathbf{r})$ is any continuous vector field which tends to zero at least as fast as $1/r^2$ as $r \to \infty$.

(a) Prove that $\mathbf{F}(\mathbf{r})$ can be decomposed as follows:

$$\mathbf{F}(\mathbf{r}) \;=\; -\boldsymbol{\nabla} V \;+\; \boldsymbol{\nabla}\times \mathbf{A}, \tag{1}$$

where $V = V(\mathbf{r})$ is a scalar potential and $\mathbf{A} = \mathbf{A}(\mathbf{r})$ is a vector potential.

Hint: Begin with $(\mathrm{XI})_2$ of Appendix E and make use of the vector identity (11) of Question 1.8: $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{w}) = -\nabla^2\mathbf{w} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{w})$.

(b) Hence show that

$$\left.\begin{aligned} V(\mathbf{r}) &= \frac{1}{4\pi}\int_v \frac{S(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv' \\ \mathbf{A}(\mathbf{r}) &= \frac{1}{4\pi}\int_v \frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv' \end{aligned}\right\}, \tag{2}$$

where $S(\mathbf{r}') = \boldsymbol{\nabla}'\cdot\mathbf{F}(\mathbf{r}')$ and $\mathbf{C}(\mathbf{r}') = \boldsymbol{\nabla}'\times\mathbf{F}(\mathbf{r}')$.

(c) Prove that the decomposition (1) is unique.

Solution

(a) Because of the hint,

$$\mathbf{F}(\mathbf{r}) \;=\; \int_v \mathbf{F}(\mathbf{r}')\,\delta(\mathbf{r}-\mathbf{r}')\,dv', \tag{3}$$

where v is any region that contains the point $\mathbf{r}$. But $\delta(|\mathbf{r}-\mathbf{r}'|) = -\dfrac{1}{4\pi}\nabla^2(|\mathbf{r}-\mathbf{r}'|)^{-1}$, and so (3) becomes

$$\begin{aligned} \mathbf{F}(\mathbf{r}) &= -\frac{1}{4\pi}\int_v \mathbf{F}(\mathbf{r}')\,\nabla^2(|\mathbf{r}-\mathbf{r}'|)^{-1}\,dv' \\ &= -\nabla^2\left[\frac{1}{4\pi}\int_v \frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv'\right], \end{aligned} \tag{4}$$

since $\nabla^2\mathbf{F}(\mathbf{r}')$ is zero.[‡] Applying the given identity to (4) yields

$$\mathbf{F}(\mathbf{r}) \;=\; -\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\frac{1}{4\pi}\int_v \frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv'\right) + \boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\frac{1}{4\pi}\int_v \frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv'\right),$$

[‡]Reason: $\mathbf{F}(\mathbf{r}')$ does not depend on the unprimed coordinates of ∇^2.

which is (1) with

$$\left.\begin{aligned} V(\mathbf{r}) &= \boldsymbol{\nabla}\cdot\frac{1}{4\pi}\int_v \frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv' \\ \mathbf{A}(\mathbf{r}) &= \boldsymbol{\nabla}\times\frac{1}{4\pi}\int_v \frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv' \end{aligned}\right\}. \tag{5}$$

(b) In the proofs below, we will use the following results repeatedly:

1. $\boldsymbol{\nabla}\cdot(f\mathbf{w}) = f\boldsymbol{\nabla}\cdot\mathbf{w} + \mathbf{w}\cdot\boldsymbol{\nabla}f$ or alternatively $\boldsymbol{\nabla}'\cdot(f\mathbf{w}) = f\boldsymbol{\nabla}'\cdot\mathbf{w} + \mathbf{w}\cdot\boldsymbol{\nabla}'f$,
2. $\boldsymbol{\nabla}(|\mathbf{r}-\mathbf{r}'|)^{-1} = -\boldsymbol{\nabla}'(|\mathbf{r}-\mathbf{r}'|)^{-1}$,
3. $\boldsymbol{\nabla}$ acts on a function of $\mathbf{r}$ only; $\boldsymbol{\nabla}'$ acts on a function of $\mathbf{r}'$ only.

☞ $V(\mathbf{r})$

From $(5)_1$ it follows that:

$$\begin{aligned} V(\mathbf{r}) &= \frac{1}{4\pi}\int_v \mathbf{F}(\mathbf{r}')\cdot\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dv' = -\frac{1}{4\pi}\int_v \mathbf{F}(\mathbf{r}')\cdot\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dv' \\ &= -\frac{1}{4\pi}\int_v \boldsymbol{\nabla}'\cdot\left(\frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)dv' + \frac{1}{4\pi}\int_v \frac{\boldsymbol{\nabla}'\cdot\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv' \\ &= -\frac{1}{4\pi}\oint_s \frac{\mathbf{F}(\mathbf{r}')\cdot d\mathbf{a}'}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi}\int_v \frac{\boldsymbol{\nabla}'\cdot\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv', \end{aligned}$$

where, in the last step, we use Gauss's theorem. Now if v is chosen over all space, s is a surface at infinity and the surface integral is zero.[♯] Hence $(2)_1$.

☞ $\mathbf{A}(\mathbf{r})$

From $(5)_2$ it follows that:

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \frac{1}{4\pi}\int_v \boldsymbol{\nabla}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\times\mathbf{F}(\mathbf{r}')\,dv' = -\frac{1}{4\pi}\int_v \boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\times\mathbf{F}(\mathbf{r}')\,dv' \\ &= -\frac{1}{4\pi}\int_v \boldsymbol{\nabla}'\times\left(\frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)dv' + \frac{1}{4\pi}\int_v \frac{\boldsymbol{\nabla}'\times\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv' \\ &= \frac{1}{4\pi}\oint_s \frac{\mathbf{F}(\mathbf{r}')\times d\mathbf{a}'}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi}\int_v \frac{\boldsymbol{\nabla}'\times\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv', \end{aligned}$$

[♯]Recall that F scales as $1/r^{2+\epsilon}$ (here the parameter $\epsilon \geq 0$) and da scales as r^2. So, for a distant surface, $\dfrac{\mathbf{F}\cdot d\mathbf{a}'}{|\mathbf{r}-\mathbf{r}'|} \simeq \dfrac{\mathbf{F}\cdot d\mathbf{a}'}{r}$ scales as $1/r^{1+\epsilon}$ which tends to zero as $r\to\infty$.

where, in the last step, we use the version of Gauss's theorem (2) of Question 1.22. The surface integral is zero as before and hence $(2)_2$.

(c) The same reasoning used in (a) of the previous question can be used to prove that the decomposition of $\mathbf{F}(\mathbf{r})$ is unique. We again assume two different solutions $\mathbf{F}_1$ and $\mathbf{F}_2$ and consider their difference $\mathbf{W} = \boldsymbol{\nabla} V = \mathbf{F}_1 - \mathbf{F}_2$, arriving at the result

$$\oint_s V \frac{\partial V}{\partial n}\, da \;=\; \int_v |\boldsymbol{\nabla} V|^2\, dv. \tag{6}$$

If the limit is taken in which v becomes infinite, the integral over the surface s at infinity vanishes.† Then (6) yields

$$\int_v |\boldsymbol{\nabla} V|^2 dv \;=\; 0.$$

Thus $\mathbf{W} = \boldsymbol{\nabla} V = 0$ everywhere so that $\mathbf{F}_1 = \mathbf{F}_2$ and $\mathbf{F}$ is unique.

Comments

(i) Equation (1) shows that any well-behaved vector field can be expressed as the sum of an irrotational field $-\boldsymbol{\nabla}\Phi(\mathbf{r})$ and a solenoidal field $\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r})$. This is a fundamental result of vector calculus and is known as Helmholtz's theorem. Its relevance to electromagnetism through Maxwell's equations, expressed as they are in terms of divergences and curls, is apparent.

(ii) The quantities $\boldsymbol{\nabla} \cdot \mathbf{F} = S(\mathbf{r})$ and $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{C}(\mathbf{r})$ serve as source functions which completely determine the field $\mathbf{F}(\mathbf{r})$. Ref. [10] explains that Helmholtz's theorem 'establishes that these serve as complete sources of the field, and that all continuous vector fields can be classified by the two mathematical types, the conservative and the solenoidal'.

Question 1.25

Prove that a uniform vector field $\mathbf{F}_0$ can be expressed as:

(a) an irrotational field

$$\mathbf{F}_0 \;=\; -\boldsymbol{\nabla} V, \tag{1}$$

where $V(\mathbf{r}) = -\mathbf{r} \cdot \mathbf{F}_0$.

† $\dfrac{\partial V}{\partial n}$, being the normal component of $\boldsymbol{\nabla} V$, scales as $1/r^{2+\epsilon}$ and therefore V scales as $1/r^{1+\epsilon}$. So $V \dfrac{\partial V}{\partial n}\, da$ scales as $1/r^{1+2\epsilon} \to 0$ as $r \to \infty$.

[10] B. P. Miller, 'Interpretations from Helmholtz' theorem in classical electromagnetism', *American Journal of Physics*, vol. 52, pp. 948–50, 1984.

(b) a solenoidal field

$$\mathbf{F}_0 = \boldsymbol{\nabla}\times \mathbf{A}, \tag{2}$$

where $\mathbf{A}(\mathbf{r}) = -\frac{1}{2}(\mathbf{r}\times\mathbf{F}_0)$.

Solution

Considering the ith component of $\boldsymbol{\nabla} V$ and $\boldsymbol{\nabla}\times\mathbf{A}$ respectively, and recalling that the F_{0i} are spatially constant, give:

(a) $\nabla_i(\mathbf{r}\cdot\mathbf{F}_0) = \nabla_i(r_jF_{0j}) = F_{0j}\nabla_i r_j = F_{0j}\,\delta_{ij}$, which contracts to F_{0i} as required.

(b) $(\boldsymbol{\nabla}\times\mathbf{A})_i = -\frac{1}{2}\varepsilon_{ijk}\nabla_j(\mathbf{r}\times\mathbf{F}_0)_k = -\frac{1}{2}\varepsilon_{ijk}\varepsilon_{klm}F_{0m}\nabla_j r_l = -\frac{1}{2}(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})F_{0m}\delta_{jl}$. Contracting subscripts gives F_{0i} as required.

Comments

(i) The results $V(\mathbf{r}) = -\mathbf{r}\cdot\mathbf{F}_0$ and $\mathbf{A}(\mathbf{r}) = -\frac{1}{2}(\mathbf{r}\times\mathbf{F}_0)$ are often convenient potentials for representing uniform electrostatic and magnetostatic fields respectively.

(ii) Because a uniform field does not satisfy the conditions of Helmholtz's theorem (it does not tend to zero at infinity), $\mathbf{F}_0$ has no unique representation. It is easily verified that $\mathbf{F}_0$ can be expressed as a linear combination of (1) and (2) in infinitely many ways.

Question 1.26

Consider a sphere having radius r_0 centred at an arbitrary origin O. Let $\mathbf{r}'$ be the position vector of any point P$'$ inside or on the surface of the sphere; let $\mathbf{r}$ be the position vector of a fixed field point P. Prove the following results:

$$\text{(a)}\quad \oint_s \frac{da'}{|\mathbf{r}-\mathbf{r}'|} = \begin{cases} \dfrac{4\pi r_0^2}{r} & r\geq r_0 \\ 4\pi r_0 & r\leq r_0, \end{cases} \tag{1}$$

$$\text{(b)}\quad \int_v \frac{dv'}{|\mathbf{r}-\mathbf{r}'|} = \begin{cases} \dfrac{4\pi}{3}\dfrac{r_0^3}{r} & r\geq r_0 \\ 2\pi(r_0^2-\frac{1}{3}r^2) & r\leq r_0, \end{cases} \tag{2}$$

$$\text{(c)}\quad \oint_s \frac{d\mathbf{a}'}{|\mathbf{r}-\mathbf{r}'|} = \begin{cases} \dfrac{4\pi}{3}\left(\dfrac{r_0}{r}\right)^3\mathbf{r} & r\geq r_0 \\ \dfrac{4\pi}{3}\mathbf{r} & r\leq r_0. \end{cases} \tag{3}$$

Solution

Orient Cartesian axes so that the point P is located on the z-axis. Then the integrand is axially symmetric about the z-axis and for the point P$'$ it is convenient to use the spherical polar coordinates (r', θ', ϕ').

(a) Here $r' = r_0$ and the element of area $da' = r_0^2 \sin\theta' d\theta' d\phi'$. By the cosine rule, $|\mathbf{r} - \mathbf{r}'| = \sqrt{r^2 + r_0^2 - 2rr_0\cos\theta'}$, and so

$$\oint_s \frac{da'}{|\mathbf{r}-\mathbf{r}'|} = \int_0^\pi\int_0^{2\pi} \frac{r_0^2 \sin\theta' d\theta' d\phi'}{\sqrt{r^2 + r_0^2 - 2rr_0\cos\theta'}} = \int_{-1}^1 \frac{2\pi r_0^2\, d\cos\theta'}{\sqrt{r^2 + r_0^2 - 2rr_0\cos\theta'}}\,. \tag{4}$$

With the substitution $u^2 = r^2 + r_0^2 - 2rr_0\cos\theta'$, equation (4) becomes

$$2\pi r_0^2 \frac{1}{r_0 r}\int_{\sqrt{(r-r_0)^2}}^{\sqrt{(r+r_0)^2}} du\,. \tag{5}$$

For $r > r_0$ the lower limit is $r - r_0$; for $r < r_0$ the lower limit is $r_0 - r$. These limits in (5) yield (1).

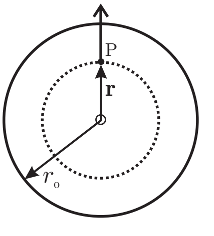
(b) We proceed as in (a). Taking the volume element $dv' = r'^2 \sin\theta' dr' d\theta' d\phi'$ and substituting $u^2 = r^2 + r'^2 - 2rr'\cos\theta'$ give

$$\int_v \frac{dv'}{|\mathbf{r}-\mathbf{r}'|} = \int_0^{r_0}\int_0^\pi\int_0^{2\pi} \frac{r'^2 \sin\theta' dr' d\theta' d\phi'}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta'}} = \int_0^{r_0} r'^2 \int_{\sqrt{(r-r')^2}}^{\sqrt{(r+r')^2}} \frac{2\pi}{rr'}\, du\, dr'. \tag{6}$$

$r \geq r_0$

The lower limit for the integration over u is $r - r'$ (see (a) above) and integration of (6) yields $(2)_1$.

$r \leq r_0$



The point P now lies inside the sphere, which we partition as shown in the figure alongside. The lower limit for the integration over u is either $r - r'$ for $r' < r$ or $r' - r$ for $r' > r$ (see the discussion in (a) above). Equation (6) becomes

$$\begin{aligned}
\int_v \frac{dv'}{|\mathbf{r}-\mathbf{r}'|} &= 2\pi\int_0^r r'^2 \int_{(r-r')}^{(r+r')} \frac{1}{rr'}\, du\, dr' + 2\pi\int_r^{r_0} r'^2 \int_{(r'-r)}^{(r+r')} \frac{1}{rr'}\, du\, dr' \\
&= \frac{4\pi}{r}\int_0^r r'^2 dr' + 4\pi\int_r^{r_0} r' dr' \\
&= \frac{4\pi}{3} r^2 + 2\pi(r_0^2 - r^2),
\end{aligned}$$

which is $(2)_2$.

(c) We use (1) of Question 1.22, $\oint_s f\, d\mathbf{a}' = \int_v (\boldsymbol{\nabla}' f)\, dv'$, and put $f = |\mathbf{r} - \mathbf{r}'|^{-1}$. Then

$$\oint_s \frac{d\mathbf{a}'}{|\mathbf{r} - \mathbf{r}'|} = \int_v \boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) dv' = -\boldsymbol{\nabla}\int_v \frac{dv'}{|\mathbf{r} - \mathbf{r}'|} \tag{7}$$

since $\boldsymbol{\nabla}' f = -\boldsymbol{\nabla} f$. Substituting (2) in (7) and differentiating give (3).

Comment

Alternative forms of (1)–(3) are sometimes required. For example, if $r \leq r_0$ and the integration is relative to unprimed coordinates, then

$$\oint_s \frac{d\mathbf{a}}{|\mathbf{r} - \mathbf{r}'|} = \begin{cases} \dfrac{4\pi}{3}\left(\dfrac{r_0}{r'}\right)^3 \mathbf{r}' & r' \geq r_0 \\[2ex] \dfrac{4\pi}{3}\,\mathbf{r}' & r' \leq r_0. \end{cases} \tag{8}$$

Question 1.27

The time average of a dynamical quantity $Q(t)$ is defined as

$$\langle Q(t)\rangle = \langle Q\rangle = \lim_{t\to\infty} \frac{1}{t}\int_0^t Q(t')\, dt'. \tag{1}$$

(a) Suppose $Q(t)$ is periodic having period T. That is, $Q(t + nT) = Q(t)$ where $n = 0, \pm 1, \pm 2, \ldots$. Prove that

$$\langle Q\rangle = \frac{1}{T}\int_0^T Q(t')\, dt'. \tag{2}$$

(b) Use (2) to prove the following:

$$\left.\begin{aligned} \langle \cos\omega t\rangle &= \langle \sin\omega t\rangle = 0 \\ \langle \cos^2\omega t\rangle &= \langle \sin^2\omega t\rangle = \tfrac{1}{2} \\ \langle \cos^2(kr - \omega t)\rangle &= \langle \sin^2(kr - \omega t)\rangle = \tfrac{1}{2} \end{aligned}\right\}. \tag{3}$$

Here $\omega = 2\pi/T$, k is a constant and r is the position vector of a point on a plane wavefront.

Solution

(a) From the definition (1) it follows that

$$\begin{aligned}
\langle Q\rangle &= \lim_{n\to\infty}\frac{1}{nT}\int_0^{nT} Q(t)\,dt \\
&= \lim_{n\to\infty}\frac{1}{nT}\left[\int_{(n-1)T}^{nT} Q(t)\,dt \;+\int_{(n-2)T}^{(n-1)T} Q(t)\,dt \;+\;\cdots\;+\int_0^{T} Q(t)\,dt\right] \\
&= \lim_{n\to\infty}\frac{1}{nT}\left[\int_{(n-1)T}^{nT} Q\big(t-(n-1)T\big)\,dt \;+\int_{(n-2)T}^{(n-1)T} Q\big(t-(n-2)T\big)\,dt\;+\right. \\
&\qquad\qquad\left.\cdots+\int_0^{T} Q(t)\,dt\right]. \qquad (4)
\end{aligned}$$

Making the substitutions $t' = t - mT$ where $m = n-1,\ n-2,\ \ldots,\ 0$ in (4) gives

$$\begin{aligned}
\langle Q\rangle &= \lim_{n\to\infty}\frac{1}{nT}\left[\int_0^{T} Q(t')\,dt' \;+\int_0^{T} Q(t')\,dt' \;+\;\cdots\;+\int_0^{T} Q(t')\,dt'\right] \\
&= \frac{1}{T}\int_0^{T} Q(t')\,dt',
\end{aligned}$$

as required.

(b) $\langle\cos\omega t\rangle$ and $\langle\sin\omega t\rangle$

These results are obvious by inspection since the definite integrals of $\cos\theta$ and $\sin\theta$ between 0 and 2π are zero.

$\langle\cos^2\omega t\rangle$ and $\langle\sin^2\omega t\rangle$

Substituting $Q(t) = \cos^2\omega t$ in (2) and using the identity $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ yield

$$\langle Q\rangle = \frac{1}{2\pi/\omega}\int_0^{2\pi/\omega}\cos^2\omega t\,dt = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\tfrac{1}{2}(1+\cos 2\omega t)\,dt.$$

Inserting the limits and cancelling terms give $(3)_2$. Similarly for $\langle\sin^2\omega t\rangle$.

$\langle\cos^2(kr-\omega t)\rangle$ and $\langle\sin^2(kr-\omega t)\rangle$

$\cos(kr-\omega t) = \cos kr\cos\omega t + \sin kr\sin\omega t$, and so

$$\begin{aligned}
\langle\cos^2(kr-\omega t)\rangle &= \langle(\cos kr\cos\omega t+\sin kr\sin\omega t)^2\rangle \\
&= \cos^2 kr\langle\cos^2\omega t\rangle + \sin^2 kr\langle\sin^2\omega t\rangle + \tfrac{1}{2}\sin 2kr\langle\sin 2\omega t\rangle \\
&= \tfrac{1}{2}(\cos^2 kr + \sin^2 kr) = \tfrac{1}{2},
\end{aligned}$$

where, in the penultimate step, we use $(3)_1$ and $(3)_2$. Similarly for $\langle\sin^2(kr-\omega t)\rangle$.

Comments

(i) This question provides a formal derivation of (3), although these results are really intuitively obvious if one thinks of a graph (that of $\cos^2\omega t$ vs t, say).

(ii) Averages like (3) are frequently encountered in electromagnetism, where the time average of a harmonically varying quantity $Q(t)$ is often of more interest than its instantaneous value (e.g. Poynting vectors and dissipated/radiated power).

Question 1.28

Suppose $\mathbf{A} = \mathbf{A}_0 e^{-i\omega t}$ and $\mathbf{B} = \mathbf{B}_0 e^{-i\omega t}$ are time-harmonic fields whose amplitudes $\mathbf{A}_0$ and $\mathbf{B}_0$ are in general complex. Show that

(a) $$\big\langle(\mathrm{Re}\,\mathbf{A})^2\big\rangle = \tfrac{1}{2}\mathbf{A}\cdot\mathbf{A}^*, \qquad (1)$$

(b) $$\big\langle(\mathrm{Re}\,\mathbf{A})\cdot(\mathrm{Re}\,\mathbf{B})\big\rangle = \tfrac{1}{2}\mathbf{A}\cdot\mathbf{B}^* = \tfrac{1}{2}\mathbf{A}^*\cdot\mathbf{B}. \qquad (2)$$

(Here $\mathbf{A}^*$ and $\mathbf{B}^*$ are complex conjugates.)

Solution

(a) Clearly $(\mathrm{Re}\,\mathbf{A}) = \frac{1}{2}(\mathbf{A}+\mathbf{A}^*)$, and so $(\mathrm{Re}\,\mathbf{A})^2 = \frac{1}{4}(A^2 + A^{*2} + 2\mathbf{A}\cdot\mathbf{A}^*)$. Now A^2 and A^{*2} are both time-harmonic functions of frequency 2ω, whereas $\mathbf{A}\cdot\mathbf{A}^*$ is time-independent. From (3) of the previous question we obtain $\langle A^2\rangle = \langle A^{*2}\rangle = 0$, and hence (1).

(b) Now $(\mathrm{Re}\,\mathbf{A})\cdot(\mathrm{Re}\,\mathbf{B}) = \frac{1}{2}(\mathbf{A}+\mathbf{A}^*)\cdot\frac{1}{2}(\mathbf{B}+\mathbf{B}^*) = \frac{1}{4}(\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{B}^*+\mathbf{A}^*\cdot\mathbf{B}+\mathbf{A}^*\cdot\mathbf{B}^*)$. Performing a time average of this last result and using the same reasoning as before give

$$\begin{aligned}\big\langle(\mathrm{Re}\,\mathbf{A})\cdot(\mathrm{Re}\,\mathbf{B})\big\rangle = \mathbf{A}\cdot\mathbf{B}^* + \mathbf{A}^*\cdot\mathbf{B} &= \mathbf{A}\cdot\mathbf{B}^* + (\mathbf{A}\cdot\mathbf{B}^*)^* \\ &= 2\mathrm{Re}\,(\mathbf{A}\cdot\mathbf{B}^*) = 2\mathrm{Re}\,(\mathbf{A}^*\cdot\mathbf{B}).\end{aligned}$$

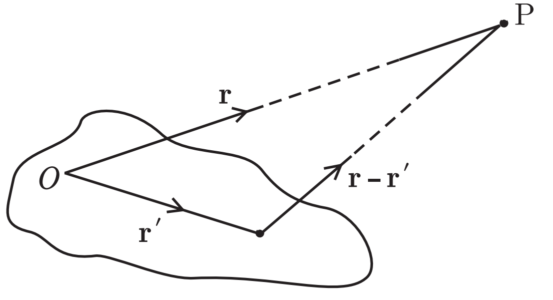
Hence (2).

Comment

It is often convenient to represent time-dependent quantities (e.g. voltages, currents, electric and magnetic fields, etc.) in terms of a complex exponential. This is done mainly for reasons of algebra in a complicated expression, since it is usually easier to manipulate exponential, rather than trigonometric, functions. It is understood that the physically meaningful quantity will always be recovered at the end of a calculation from either the real part (usually) or the imaginary part (less usually).

Question 1.29*

The figure below shows vectors $\mathbf{r}$ and $\mathbf{r}'$ both measured relative to an arbitrary origin O.



Suppose P is a distant field point and that $r'/r < 1$. Use Cartesian tensors and the binomial theorem‡ to derive the following expansions:

$$|\mathbf{r}-\mathbf{r}'| = r - \frac{r_i}{r}\,r_i' - \left[\frac{r_ir_j - r^2\delta_{ij}}{2r^3}\right] r_i'r_j' - \left[\frac{3r_ir_jr_k - r^2(r_i\delta_{jk}+r_j\delta_{ki}+r_k\delta_{ij})}{6r^5}\right] r_i'r_j'r_k' - \left[\frac{10r_ir_jr_kr_\ell - 2(r_ir_j\delta_{k\ell} + r_ir_k\delta_{j\ell} + r_ir_\ell\delta_{jk} + r_jr_k\delta_{i\ell} + r_jr_\ell\delta_{ik} + r_kr_\ell\delta_{ij})\,r^2 - (\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})\,r^4}{16r^7}\right] r_i'r_j'r_k'r_\ell' - \cdots, \tag{1}$$

$$|\mathbf{r}-\mathbf{r}'|^{-1} = \frac{1}{r} + \frac{r_i}{r^3}r_i' + \frac{3r_ir_j - r^2\delta_{ij}}{2r^5}r_i'r_j' + \frac{5r_ir_jr_k - r^2(r_i\delta_{jk} + r_j\delta_{ki} + r_k\delta_{ij})}{2r^7}r_i'r_j'r_k' + \left[\frac{35r_ir_jr_kr_\ell - 5(r_ir_j\delta_{k\ell} + r_ir_k\delta_{j\ell} + r_ir_\ell\delta_{jk} + r_jr_k\delta_{i\ell} + r_jr_\ell\delta_{ik} + r_kr_\ell\delta_{ij})\,r^2 + (\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})\,r^4}{8r^9}\right] r_i'r_j'r_k'r_\ell' - \cdots. \tag{2}$$

Hint: Exploit, wherever possible, the arbitrary nature of repeated subscripts to express a term in its most symmetric form. So, for example, the symmetric form of $3r^2r_i\delta_{jk}r_i'r_j'r_k'$ is $r^2(r_i\delta_{jk} + r_j\delta_{ki} + r_k\delta_{ij})r_i'r_j'r_k'$.

‡The binomial expansion with $x < 1$ is: $(1+x)^n = 1+nx+\frac{1}{2!}n(n-1)x^2+\frac{1}{3!}n(n-1)(n-2)x^3+\cdots$.

Solution

Applying the cosine rule to the triangle of vectors in the figure on p. 43 gives

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2+r'^2-2\mathbf{r}\cdot\mathbf{r}'} = r\sqrt{1+\frac{r'^2-2\mathbf{r}\cdot\mathbf{r}'}{r^2}}\,.$$

We now use this result for each of the following expansions:

☞ $|\mathbf{r}-\mathbf{r}'|$

Substituting $x = \dfrac{r'^2-2\mathbf{r}\cdot\mathbf{r}'}{r^2}$ in $(1+x)^{1/2} = 1+\dfrac{x}{2}-\dfrac{x^2}{8}+\dfrac{x^3}{16}-\dfrac{5x^4}{128}+\cdots$ and expanding in powers of r'/r yield:

$$\begin{aligned}|\mathbf{r}-\mathbf{r}'| &= (r^2+r'^2-2\mathbf{r}\cdot\mathbf{r}')^{1/2}\\ &= r\Bigg[1-\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}-\frac{(\mathbf{r}\cdot\mathbf{r}')^2-r^2r'^2}{2r^4}-\frac{(\mathbf{r}\cdot\mathbf{r}')^3-r^2r'^2(\mathbf{r}\cdot\mathbf{r}')}{2r^6}-\\ &\qquad\frac{5(\mathbf{r}\cdot\mathbf{r}')^4-6r^2r'^2(\mathbf{r}\cdot\mathbf{r}')^2+r^4r'^4}{8r^8}+\cdots\Bigg].\end{aligned} \tag{3}$$

Using tensor notation in (3) and remembering the hint give (1).

☞ $|\mathbf{r}-\mathbf{r}'|^{-1}$

Substituting $x = \dfrac{r'^2-2\mathbf{r}\cdot\mathbf{r}'}{r^2}$ in $(1+x)^{-1/2} = 1-\dfrac{x}{2}+\dfrac{3x^2}{8}-\dfrac{5x^3}{16}+\dfrac{35x^4}{128}-\cdots$ and expanding in powers of r'/r yield:

$$\begin{aligned}|\mathbf{r}-\mathbf{r}'|^{-1} &= (r^2+r'^2-2\mathbf{r}\cdot\mathbf{r}')^{-1/2}\\ &= \frac{1}{r}\Bigg[1+\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}+\frac{3(\mathbf{r}\cdot\mathbf{r}')^2-r^2r'^2}{2r^4}+\frac{5(\mathbf{r}\cdot\mathbf{r}')^3-3r^2r'^2(\mathbf{r}\cdot\mathbf{r}')}{2r^6}+\\ &\qquad\frac{35(\mathbf{r}\cdot\mathbf{r}')^4-30r^2r'^2(\mathbf{r}\cdot\mathbf{r}')^2+3r^4r'^4}{8r^8}+\cdots\Bigg].\end{aligned} \tag{4}$$

Using tensor notation in (4) and again applying the hint give (2).

Comment

The results (1) and (2) are required in multipole expansions of the electric scalar and magnetic vector potentials. See Chapters 2, 4, 8 and 11.

Question 1.30*

Consider a bounded distribution of time-dependent charge and current densities $\rho(\mathbf{r}', t)$ and $\mathbf{J}(\mathbf{r}', t)$ in vacuum‡ which satisfy the equation

$$\boldsymbol{\nabla} \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \tag{1}$$

Use (1) and Gauss's theorem to prove the following integral transforms:

(a) $$\int_v J_i \, dv' = \int_v r'_i \dot{\rho} \, dv', \tag{2}$$

(b) $$\int_v r'_j J_i \, dv' = -\tfrac{1}{2}\varepsilon_{ijk}\int_v (\mathbf{r}' \times \mathbf{J})_k \, dv' + \tfrac{1}{2}\int_v r'_i r'_j \dot{\rho} \, dv', \tag{3}$$

(c) $$\int_v r'_j r'_k J_i \, dv' = -\tfrac{1}{3}\varepsilon_{ij\ell}\int_v (\mathbf{r}' \times \mathbf{J})_\ell r'_k dv' - \tfrac{1}{3}\varepsilon_{ik\ell}\int_v (\mathbf{r}' \times \mathbf{J})_\ell r'_j \, dv' + \tfrac{1}{3}\int_v r'_i r'_j r'_k \dot{\rho} \, dv'. \tag{4}$$

Hint: The identity $r'_i J_j - r'_j J_i = \varepsilon_{ijk}(\mathbf{r}' \times \mathbf{J})_k$, derived in Question 1.3, is required in the proof of (3) and (4).

Solution

(a) Clearly $\boldsymbol{\nabla}' \cdot (r'_i \mathbf{J}) = \nabla_j (r'_i J_j) = (\delta_{ij} J_j + r'_i \nabla'_j J_j) = (J_i - r'_i \dot{\rho})$. So

$$\int_v \boldsymbol{\nabla}' \cdot (r'_i \mathbf{J}) \, dv' = \int_v (J_i - r'_i \dot{\rho}) \, dv'.$$

Now converting the left-hand side of this equation to a surface integral using Gauss's theorem gives

$$\oint_s r'_i \, \mathbf{J} \cdot d\mathbf{a}' = \int_v (J_i - r'_i \dot{\rho}) \, dv'.$$

But $\mathbf{J} \cdot d\mathbf{a}' = 0$ everywhere on s and (2) follows immediately because v is arbitrary.

‡The term 'bounded distribution' implies that all electric currents are confined to a finite volume v, and that $\mathbf{J} \cdot d\mathbf{a} = 0$ everywhere on the boundary surface s spanning v.

(b) As before, $\boldsymbol{\nabla}'\cdot(r'_i r'_j \mathbf{J}) = \nabla_k(r'_i r'_j J_k) = (\delta_{ik} r'_j J_k + \delta_{jk} r'_i J_k + r'_i r'_j \nabla'_k J_k)$. Integrating over v, applying Gauss's theorem and using $\mathbf{J}\cdot d\mathbf{a}' = 0$ yield

$$\int_v (r'_j J_i + r'_i J_j - r'_i r'_j \dot{\rho})\, dv' = 0,$$

or

$$\int_v r'_j J_i \, dv' = -\int_v r'_i J_j \, dv' + \int_v r'_i r'_j \dot{\rho}\, dv'.$$

Adding $\displaystyle\int_v r'_j J_i \, dv'$ to both sides of this last equation gives

$$2\int_v r'_j J_i \, dv' = \int_v (r'_j J_i - r'_i J_j)\, dv' + \int_v r'_i r'_j \dot{\rho}\, dv'.$$

Using the identity $r'_j J_i - r'_i J_j = \varepsilon_{jik}(\mathbf{r}'\times\mathbf{J})_k = -\varepsilon_{ijk}(\mathbf{r}'\times\mathbf{J})_k$ (see (2) of Question 1.3) yields (3).

(c) Apart from an additional term, this proof is identical to (b).

Comments

(i) In electromagnetism, (1) is an important result known as the continuity equation for electric charge. It is discussed further in later chapters. See, for example, Question 7.1.

(ii) The identities (2)–(4), and others like them, are used to transform multipole expansions of both the static and dynamic vector potentials. We consider such applications in Chapters 4 and 8.

(iii) The integrals (2)–(4) give the moments of $\mathbf{J}$ about an arbitrary origin. It is clear from the emerging trend that one can write down, by inspection, the moment of $\mathbf{J}$ in any order. So, for example, the next member of the series is:

$$\begin{aligned}\int_v r'_j r'_k r'_\ell J_i \, dv' = &- \tfrac{1}{4}\varepsilon_{ijm}\int_v (\mathbf{r}'\times\mathbf{J})_m r'_k r'_\ell \, dv' - \tfrac{1}{4}\varepsilon_{ikm}\int_v (\mathbf{r}'\times\mathbf{J})_m r'_j r'_\ell \, dv' - \\ &\tfrac{1}{4}\varepsilon_{i\ell m}\int_v (\mathbf{r}'\times\mathbf{J})_m r'_j r'_k \, dv' + \tfrac{1}{4}\int_v r'_i r'_j r'_k r'_\ell \, \dot{\rho}\, dv'.\end{aligned} \tag{5}$$

2
Static electric fields in vacuum

The first observations of an electrical nature can be traced back to the ancient Greek philosophers,[‡] but it took until the middle of the eighteenth century for the basic facts of electrostatics to be established: the presence in nature of two types of electric charge (which long ago were given the arbitrary labels 'positive' and 'negative'), the conservation of charge[♯] and the existence of conductors and insulators. During the ensuing fifty years, investigators set about the important task of determining the law of force between charges. Through a series of ingenious experiments involving torsion balances and charged spheres, Coulomb generalized the work of Priestley and others. The law of force that today bears Coulomb's name applies to both like and unlike charges. Formally, Coulomb's law can be stated as follows: suppose q_1 and q_2 represent two stationary point charges in vacuum having position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively (relative to some arbitrary origin O). The force $\mathbf{F}_{12}$ exerted by q_1 on q_2 is given by

$$\mathbf{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2). \tag{I}$$

Notice that (I) depends inversely on the square of the distance between q_1 and q_2, that it satisfies Newton's third law ($\mathbf{F}_{12} = -\mathbf{F}_{21}$) and that if the charges have the same (opposite) sign then the force is repulsive (attractive). This important law is central to the study of electrostatics.

After two preliminary questions, we begin this chapter with the derivation of Maxwell's electrostatic equations (in a vacuum) from Coulomb's law. The integral forms for the electric potential Φ and field $\mathbf{E}$ emerge naturally during this process. These results are then used to determine Φ and $\mathbf{E}$ for various distributions of charge where some inherent symmetry is usually present. Two important methods are used to illustrate this: (1) direct application of Gauss's law and (2) integrating a known charge density over a line, surface or volume. Problems which require computer algebra software (*Mathematica*) to facilitate their solutions are included. A series expansion of $\Phi(\mathbf{r})$ leads to the various electric multipole moments of a static charge distribution, and examples of calculating these moments are presented. Other important topics (like origin independence) are treated along the way.

[‡]For instance, it was discovered that a rubbed amber rod acquired the ability to attract a variety of very light objects like human hair, pieces of straw, etc.

[♯]Experiments which established that charge was also quantized and invariant came much later.

Solved Problems in Classical Electromagnetism. J. Pierrus, Oxford University Press (2018).
 DOI: 10.1093/oso/9780198821915.001.0001

Question 2.1

Consider a distribution of n stationary point charges $q_1, q_2, \ldots, q_n$ located in vacuum at positions $\mathbf{r}'_1, \mathbf{r}'_2, \ldots, \mathbf{r}'_n$ relative to an arbitrary origin O. Let P be a point in the field whose position vector (relative to O) is $\mathbf{r}$. Use Coulomb's law and the principle of superposition to show that the electric field‡ $\mathbf{E}$ at P is given by:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n} q_i \frac{(\mathbf{r}-\mathbf{r}'_i)}{|\mathbf{r}-\mathbf{r}'_i|^3}\,. \tag{1}$$

Solution

The force exerted on the positive test charge q_0 at P due to any one of these n charges (q_i, say) is given by Coulomb's law:♯ $\mathbf{F}_{i0} = \dfrac{q_i q_0}{4\pi\epsilon_0}\dfrac{(\mathbf{r}-\mathbf{r}'_i)}{|\mathbf{r}-\mathbf{r}'_i|^3}$. The net force $\mathbf{F}$ on the test charge is the sum of these n two-body forces. Thus

$$\mathbf{F} = \mathbf{F}_{10} + \mathbf{F}_{20} + \cdots + \mathbf{F}_{n0} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n} q_i q_0 \frac{(\mathbf{r}-\mathbf{r}'_i)}{|\mathbf{r}-\mathbf{r}'_i|^3}\,. \tag{2}$$

The definition $\mathbf{E} = \mathbf{F}/q_0$ yields (1) with $\mathbf{F}$ given by (2).

Comments

(i) It is a remarkable fact that the two-body interaction between the test charge q_0 and any other charge (q_i, say) is unaffected by the presence of the remaining $(n-1)$ charges. This is the crux of the principle of linear superposition which asserts that the net force $\mathbf{F}$ on q_0 is the vector sum of these n two-body forces.

(ii) Crucially, the superposition principle applies to time-dependent electric and magnetic fields as well, and classical electromagnetism, based on Maxwell's equations, is a *linear* theory. Ref. [1] explains that

> at the macroscopic and even at the atomic level, linear superposition is remarkably valid. It is in the subatomic domain that departures from linear superposition can be legitimately sought. As charged particles approach each other very closely, electric field strengths become enormous. . . . The final conclusion about linear superposition of fields *in vacuum* is that in the classical domain of sizes and attainable field strengths there is abundant evidence for the validity of linear superposition and no evidence against it. In the atomic and subatomic domain there are small quantum-mechanical nonlinear effects whose origins are in the coupling between charged particles and the electromagnetic field.

‡The electric field at P is defined as the force per unit stationary test charge placed at P. That is,

$$\mathbf{E}(\mathbf{r}) = \mathbf{F}(\text{on a test charge } q_0 \text{ at } \mathbf{r}) \div q_0\,.$$

♯In the presence of more than two charges, it is not obvious that Coulomb's law applies. It turns out that it does. See also Comment (i) above.

[1] J. D. Jackson, *Classical electrodynamics*, Chap. I, pp. 9–13. New York: Wiley, 3 edn, 1998.

(iii) Suppose n becomes so large that the charge is effectively distributed *continuously* over some region v of space. Replacing q_i in (1) by dq and converting the sum to an integral give

$$\mathbf{E}(\mathbf{r}) \;=\; \frac{1}{4\pi\epsilon_0}\int \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,dq\,. \tag{3}$$

(iv) Depending on the geometrical nature of the problem (e.g. three-, two- or one-dimensional), the infinitesimal charge dq in (3) may be written as $\rho\,dv'$, $\sigma\,da'$ or $\lambda\,dl'$.† The electric field can then be expressed in alternative forms, such as

$$\mathbf{E}(\mathbf{r}) = k\int_v \frac{\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,dv' \;\text{ or }\; k\int_s \frac{\sigma(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,da' \;\text{ or }\; k\int_c \frac{\lambda(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,dl', \tag{4}$$

where $k = (4\pi\epsilon_0)^{-1}$. Equation (4) provides a means of determining $\mathbf{E}$ for a known charge distribution, assuming that the relevant integral can be evaluated.

Question 2.2

Express the electric-charge density ρ for the following charge distributions in terms of delta functions (if necessary, review Appendix E now).

(a) Charge q is distributed uniformly along the z-axis of Cartesian coordinates from $-\frac{1}{2}L$ to $\frac{1}{2}L$.
(b) Charge q is distributed uniformly over the surface of a spherical shell of radius a centred on the origin of spherical polar coordinates.
(c) Charge q is distributed uniformly over the surface of a cylinder of length L and radius a aligned along the z-axis of cylindrical polar coordinates.
(d) Charge q is distributed uniformly around the circumference of a circle of radius a centred on the origin of spherical polar coordinates.
(e) Repeat (b) for cylindrical polar coordinates.
(f) Charge q is distributed uniformly over the surface of a circular disc of radius a centred on the origin of cylindrical polar coordinates.

Solution

(a) The charge density is zero everywhere except on the z-axis between $\pm\frac{1}{2}L$. So we let

$$\rho(x',\,y',\,z') \;=\; \alpha\,\delta(x')\,\delta(y')\,H(\tfrac{1}{2}L - |z'|), \tag{1}$$

†Suppose dq is the charge contained in an infinitesimal volume element dv' located at $\mathbf{r}'$. The charge per unit volume or charge density is defined as $\rho(\mathbf{r}') = dq/dv'$. Analogous definitions for the surface and line densities are $\sigma = dq/da'$ and $\lambda = dq/dl'$, where da' and dl' are elements of area and length respectively.

where α is a constant to be determined and $H(u)$ is the Heaviside function (see (VIII) of Appendix E).

Since $q = \int_v \rho \, dv'$ by definition, it follows that

$$q = \alpha \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x')\, \delta(y')\, H(\tfrac{1}{2}L - |z'|)\, dx'\, dy'\, dz'.$$

Now the two integrals involving x' and y' above each have the value one, and so

$$q = \alpha \int_{-\frac{1}{2}L(1-\epsilon)}^{\frac{1}{2}L(1-\epsilon)} dz' \qquad \Rightarrow \qquad \alpha = \frac{q}{L-\epsilon},$$

where ϵ is a parameter very much less than unity. In the limit $\epsilon \to 0$ we obtain $\alpha = q/L$. Substituting this result in (1) gives

$$\boxed{\rho(x',\, y',\, z') = \frac{q}{L}\, \delta(x')\, \delta(y')\, H(\tfrac{1}{2}L - |z'|)}\,. \tag{2}$$

(b) Proceeding as in (a) we let $\rho(r') = \alpha\, \delta(r' - a)$. Then

$$q = \alpha \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \delta(r' - a)\, r'^2 \sin\theta' dr' d\theta' d\phi' \qquad \Rightarrow \qquad \alpha = \frac{q}{4\pi a^2},$$

and so

$$\boxed{\rho(r') = \frac{q}{4\pi a^2}\, \delta(r' - a)}\,. \tag{3}$$

(c) We let $\rho(r') = \alpha\, \delta(r' - a)$ which gives

$$q = \alpha \int_0^{L} \int_0^{2\pi} \int_0^{\infty} \delta(r' - a)\, r' dr' d\theta' dz' \qquad \Rightarrow \qquad \alpha = \frac{q}{2\pi L a},$$

and hence

$$\boxed{\rho(r') = \frac{q}{2\pi L a}\, \delta(r' - a)}\,. \tag{4}$$

(d) Now $\rho(r',\, \theta') = \alpha\, \delta(r' - a)\, \delta(\theta' - \frac{1}{2}\pi)$, and so

$$\begin{aligned} q &= \alpha \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \delta(r' - a)\, \delta(\theta' - \tfrac{1}{2}\pi)\, r'^2 \sin\theta' dr' d\theta' d\phi' \\ &= 2\pi\alpha \int_{-1}^{1} \int_0^{\infty} \delta(r' - a)\, \delta(\cos\theta')\, r'^2 dr' d(\cos\theta') \qquad \Rightarrow \qquad \alpha = \frac{q}{2\pi a^2}. \end{aligned}$$

Therefore $\rho(r',\, \theta') = \dfrac{q}{2\pi a^2}\, \delta(r' - a)\, \delta(\theta' - \frac{1}{2}\pi)$, or

$$\boxed{\rho(r',\,\theta') \;=\; \frac{q}{2\pi a^2}\,\delta(r'-a)\,\delta(\cos\theta')}\,. \tag{5}$$

(e) As before, $\rho(r',\,z') = \alpha\,\delta(r'-a)\,\delta(z')$ which gives

$$q \;=\; \alpha\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}\delta(r'-a)\,\delta(z')\,r'dr'd\theta'dz' \qquad\Rightarrow\qquad \alpha = \frac{q}{2\pi a}\,,$$

and hence

$$\boxed{\rho(\mathbf{r}') \;=\; \frac{q}{2\pi a}\,\delta(r'-a)\,\delta(z')}\,. \tag{6}$$

(f) Now $\rho(r',\,z') = \alpha\,H(a-r')\,\delta(z')$, and so

$$\begin{aligned} q \;&=\; \alpha\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}H(a-r')\,\delta(z')\,r'dr'd\theta'dz' \\ &=\; \alpha\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{a}\delta(z')\,r'dr'd\theta'dz' \quad\Rightarrow\quad \alpha = \frac{q}{\pi a^2}\,. \end{aligned}$$

Hence

$$\boxed{\rho(r',\,z') \;=\; \frac{q}{\pi a^2}\,H(a-r')\,\delta(z')}\,. \tag{7}$$

Question 2.3*

Use (3) of Question 2.1 to derive the equations

$$\boldsymbol{\nabla}\cdot\mathbf{E} \;=\; \frac{\rho(\mathbf{r})}{\epsilon_0} \qquad\text{and}\qquad \boldsymbol{\nabla}\times\mathbf{E} \;=\; 0, \tag{1}$$

which apply at a point in vacuum.

Solution

Substituting $\boldsymbol{\nabla}\Big(\dfrac{1}{|\mathbf{r}-\mathbf{r}'|}\Big) \;=\; -\dfrac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$ (see (1) of Question 1.6) in (4) of Question 1.1 gives

$$\begin{aligned} \mathbf{E}(\mathbf{r}) \;&=\; -\frac{1}{4\pi\epsilon_0}\int_v \rho(\mathbf{r}')\,\boldsymbol{\nabla}\Big(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\Big)\,dv' \\ &=\; -\boldsymbol{\nabla}\bigg[\frac{1}{4\pi\epsilon_0}\int_v \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv'\bigg]. \end{aligned} \tag{2}$$

In the last step we use the fact that the operator $\boldsymbol{\nabla}$ differentiates the field (unprimed) coordinates only, and so $\boldsymbol{\nabla}\rho(\mathbf{r}') = 0$. Equation (2) can be written as

$$\mathbf{E}(\mathbf{r}) \;=\; -\boldsymbol{\nabla}\Phi(\mathbf{r}), \tag{3}$$

where

$$\Phi(\mathbf{r}) \;=\; \frac{1}{4\pi\epsilon_0}\int_v \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv'. \tag{4}$$

In (4), $\Phi(\mathbf{r})$ is the electrostatic potential at the field point $\mathbf{r}$ and here it is determined up to an arbitrary additive constant.

☞ Taking the divergence of (2) yields

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{E} &= -\nabla^2\left[\frac{1}{4\pi\epsilon_0}\int_v \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv'\right] \\
&= -\frac{1}{4\pi\epsilon_0}\int_v \rho(\mathbf{r}')\,\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dv' && \big(\text{since } \nabla^2\rho(\mathbf{r}')=0\big) \\
&= \frac{1}{4\pi\epsilon_0}\int_v 4\pi\rho(\mathbf{r}')\,\delta(\mathbf{r}-\mathbf{r}')\,dv' && \big(\text{using (4) of Question 1.21}\big) \\
&= \frac{\rho(\mathbf{r})}{\epsilon_0} && \big(\text{using (XI)}_2 \text{ of Appendix E}\big),
\end{aligned}$$

as required.

☞ Equation $(1)_2$ follows immediately from (3) because the curl of a gradient is identically zero (see (10) of Question 1.8).

Comments

(i) It follows from (3) that $\mathbf{E}\cdot d\boldsymbol{l} = -\boldsymbol{\nabla}\Phi\cdot d\boldsymbol{l} = -\left(\frac{\partial\Phi}{\partial x}\,dx + \frac{\partial\Phi}{\partial y}\,dy + \frac{\partial\Phi}{\partial z}\,dz\right) = -d\Phi$, or $\int d\Phi \;=\; -\int \mathbf{E}\cdot d\boldsymbol{l}$. Now if a and b represent two arbitrary points in the field, then

$$\Phi(\mathrm{b}) - \Phi(\mathrm{a}) \;=\; -\int_{\mathrm{a}}^{\mathrm{b}} \mathbf{E}\cdot d\boldsymbol{l}. \tag{5}$$

The difference in potential $\Phi(\mathrm{b})-\Phi(\mathrm{a})$ is the potential of b relative to a and we write it as Φ_{ab}.

(ii) Equation (1) reveals that electrostatic fields are not, in general, solenoidal but they are always conservative (see also Questions 1.14 and 1.15).

(iii) ☞ Integrating $(1)_1$ over an arbitrary volume gives $\displaystyle\int_v \boldsymbol{\nabla}\cdot\mathbf{E}\,dv' = \frac{1}{\epsilon_0}\int_v \rho(\mathbf{r}')\,dv'$. Because of Gauss's theorem this becomes

$$\oint_s \mathbf{E}\cdot d\mathbf{a}' \;=\; \frac{1}{\epsilon_0}\int_v \rho(\mathbf{r}')\,dv' \;=\; \frac{1}{\epsilon_0}\times q_{\text{net}}\,, \tag{6}$$

where $q_{\text{net}} = \displaystyle\int_v \rho(\mathbf{r}')\,dv'$ is the net charge enclosed by s. Known as Gauss's law, (6) is a fundamental result.‡

☞ It is evident from (5) that around any closed loop

$$\oint_c \mathbf{E}\cdot d\boldsymbol{l} \;=\; 0. \tag{7}$$

(iv) Sometimes (4) is required in the alternative forms

$$\Phi(\mathbf{r}) \;=\; \frac{1}{4\pi\epsilon_0}\int \frac{\sigma\,da'}{|\mathbf{r}-\mathbf{r}'|} \qquad \text{or} \qquad \Phi(\mathbf{r}) \;=\; \frac{1}{4\pi\epsilon_0}\int \frac{\lambda\,dl'}{|\mathbf{r}-\mathbf{r}'|}\,, \tag{8}$$

where σ and λ are surface- and line-charge densities respectively.

(v) The differential equations (1) apply at a point in vacuum, whereas the integral equations (5)–(8) apply over a finite region of space.

Question 2.4

A charge q is distributed uniformly over the surface of a spherical shell of radius a having negligible thickness and centred at the origin O. Calculate the electric field at an arbitrary point P in space using (a) Gauss's law and (b) direct integration.

Solution

(a) Clearly, $\mathbf{E}$ is a spherically symmetric♯ field. We therefore choose a spherical Gaussian surface G of radius r centred on O and passing through P. The electric flux through G is

$$\oint_s \mathbf{E}\cdot d\mathbf{a} \;=\; \oint_s E(r)\hat{\mathbf{r}}\cdot da\,\hat{\mathbf{r}} \;=\; \oint_s E(r)\,da \;=\; 4\pi r^2 E, \tag{1}$$

‡Stated in words: the outward electric flux ψ through any closed surface s lying in vacuum equals ${\epsilon_0}^{-1}\times$ (the net charge enclosed by s).

♯Meaning that $\mathbf{E}$ has the following properties. It is:

1. central (i.e. the field is directed towards or away from the origin).
2. dependent, in magnitude, only on the distance from the origin. That is,

$$\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}}.$$

where in the last step we use the formula for the surface area of a sphere. Two cases are of interest:

$r \geq a$

Here $q_{\text{net}} = q$ and Gauss's law and (1) give $4\pi r^2 E = q/\epsilon_0$, or

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}. \tag{2}$$

$r \leq a$

Here $q_{\text{net}} = 0$, and so

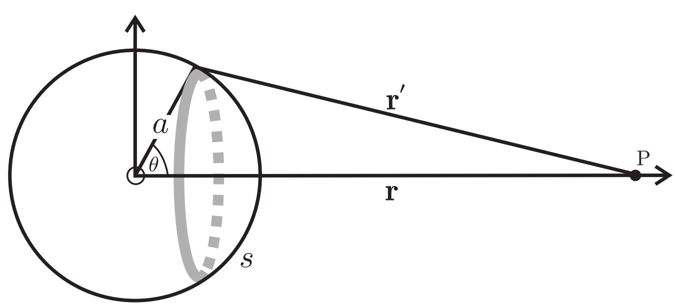
$$\mathbf{E}(\mathbf{r}) = 0. \tag{3}$$

(b) Two alternative solutions are provided:

Method 1

We begin by calculating the electric potential and then obtain $\mathbf{E}$ by differentiation. From $(8)_1$ of Question 2.3, $\Phi(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \frac{\sigma da'}{r'}$ where here r' is the distance to P from an infinitesimal band of charge (see the figure) and $\sigma da = \sigma(2\pi a \sin\theta)(a d\theta)$. Thus,

$$\Phi(\mathbf{r}) = \frac{2\pi a^2 \sigma}{4\pi\epsilon_0}\int_0^\pi \frac{\sin\theta}{r'}\,d\theta. \tag{4}$$



Now by the cosine rule, $r'^2 = r^2 + a^2 - 2ar\cos\theta$. So $2r'dr' = 2ar\sin\theta\, d\theta$ and

$$\int_0^\pi \frac{\sin\theta}{r'}\,d\theta = \frac{1}{ar}\int_{|r-a|}^{r+a} dr', \tag{5}$$

where the lower limit is either $r - a$ if P lies outside the sphere or $a - r$ if P is inside the sphere. Substituting (5) in (4) gives

$$\Phi(\mathbf{r}) = \frac{a\sigma}{2\epsilon_0 r}\int_{|r-a|}^{r+a} dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{2ar}\int_{|r-a|}^{r+a} dr', \tag{6}$$

because $\sigma = q/4\pi a^2$. As before, we consider the two cases separately: