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PAUL A. TAYLOR

THE ASTRONOMERS' MAGIC ENVELOPE

An Introduction to Astrophysics Emphasizing General Principles and Orders of Magnitude

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Preface

Each day since the middle of 1995, NASA's *Astronomy Picture of the Day* has drawn our attention to something other-worldly. Some of the APOD pictures, like the Blue Marble, are unusual views of places we know very well. Some, such as the Andromeda galaxy, are a beautiful detail on objects we can barely see on the sky with our unaided eyes or a small telescope. Some, like the microwave-background sky, are so far from our everyday experience that ordinary language has no words to convey their significance.

Visiting any of these worlds, beyond the very few within light-minutes or -hours, may be the stuff of science fiction. But understanding something of how they work is possible with ordinary science, sometimes quite simple science. That is the subject of this book—astrophysics, which is the part of astronomy dealing with physical explanations, or the branch of physics dealing with other worlds.

This book grew out of a course at the AIMS-SA, the South-African node of the African Institutes for Mathematical Sciences. Most of our students had not seen much astronomy before, and looking at the sky through a small telescope was a first for nearly everyone. But they were familiar with most of the mathematics needed, and enjoyed scientific computing. Above all, they were motivated to educate themselves, never embarrassed about asking for more explanations, and totally unafraid of strange new concepts. A way to build up an introduction to astrophysics then more or less suggested itself.

Working physicists generally, and astrophysicists especially, have great respect for back-of-the-envelope calculations. When, as students, we had first encountered the concept, we thought it meant a crude solution done by someone too lazy to work out something properly. But gradually we came to understand that is not at all what back-of-the-envelope means. It means an abstraction of what is really essential in a problem, to a form where general principles can be applied. The result may be quite close to correct, or it may be ten times too small or large—but the envelope calculation always provides insight, and hints at how a more detailed solution could be found. In the hands of the wisest among our colleagues, a scrap of paper (envelope, napkin) can seem almost magical. So, we thought, why not teach a course explaining how working astrophysicists do this?

Colleagues will recognize the influence of several well-known books. *Astrophysics in a Nutshell* (Maoz, 2016) set the example for us of an introduction to astrophysics today, based on undergraduate physics. The emphasis on general principles and orders of magnitude was inspired foremost by *First Principles of Cosmology* (Linder, 1997). Other texts we consulted for particular topics included *Principles of Stellar Evolution and Nucleosynthesis* (Clayton, 1984), *Black Holes, White Dwarfs and Neutron Stars: The Physics of Compact Objects* (Shapiro and Teukolsky, 1986), the classic *An Introduction to the Study of Stellar Structure* (Chandrasekhar, 1939), and the two books *Cosmology and Astrophysics through Problems* and *An Invitation to Astrophysicss* (Padmanabhan, 1996; Padmanabhan, 2006). *Galactic Dynamics* (Binney and Tremaine, 2008) was an indirect influence; we say little of galaxies in this book, but we have plenty of dynamics. Other enjoyable books are *Astrophysics for Physicists* (Choudhuri, 2010) and *Introduction to Cosmology* (Ryden, 2016).

Naturally, any new book should offer some distinctive features of its own. This one has four, which we hope readers will find interesting and useful.

First, this book is short! It is a tenth as long as *An Introduction to Modern Astrophysics* (Carroll and Ostlie, 2006) and it is minuscule compared to category astrophysics in Wikipedia. To be useful within this length, we have tried to focus on topics that best bring out general principles. As a result many fascinating topics in contemporary astrophysics are regretfully 'beyond the scope of this book'. In particular, we do not cover formation processes, whether of planets, stars, galaxies, or the periodic table, at all. For an initiation into those topics we recommend the only astrophysics book we know of that is even shorter than this one—*Astrophysics: A Very Short Introduction* (Binney, 2016). We assume that readers are familiar with the basic concepts and vocabulary of astronomy—such as for following the explanations in *Astronomy Picture of the Day*—and are comfortable with looking up unfamiliar terms online. For example, the actual definition of a parsec occurs late in the book, but it is assumed that readers basically already know what it means. We do, however, review some essential concepts (Euler angles, Hamiltonians, quantum statistics, and reaction cross-sections), either in the text or as an Appendix.

A second feature of this book is our opportunistic attitude to units. Working astrophysicists often slip in and out of SI units, and sometimes they even define units on the fly. This book will do both, as well. In particular, we introduce Planckian units in Chapter 4 and use them extensively thereafter. In this, we were largely inspired by Brandon Carter's provocative essay *The Significance of Numerical Coincidences in Nature* (Carter, 2007), which showed how Planckian units can bring out the underlying simplicity of many astrophysical processes. Fortunately, our students have also found these units to be useful and informative, overcoming the initial novelty factor with (as in all things) a bit of practice.

Third, it being the 21st century and all, scientific computing (at the level of numerically integrating differential equations and plotting the results) is incorporated alongside other mathematics. Our students worked with the Python ecosystem, but this book does not assume any particular programming language or software. We strongly urge readers to try the computing exercises; some of them will produce pretty pictures, and hopefully all of them will deliver some insight.

The fourth unusual feature of this book is the use of some historical narrative. We will not delve into Newton's love life or Chandrasekhar's religion, fascinating though such topics may be. We will, however, try to get some feeling for the great paradigm shifts that underlie the history of astrophysics, which have their own logic and illuminate where the subject stands today. In many cases these shifts were felt well beyond the scientific community, as well.

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Contents

1 Orbits

As far as we know, every ancient society practiced something recognisable as astronomy. While early astronomers would not have thought in terms of the Earth being in orbit around the Sun and of the Moon being in orbit around the Earth (though there were exceptions, notably Aristarchos of Samos *circa* 250 BCE), we can see from classical calendars that our ancient forbears did closely account for the facts that the Earth's orbital period is not exactly 365 times its spin period, nor is it exactly 12 times the Moon's orbital period (it's closer to $12\frac{7}{19}$, a ratio which was applied in ancient Babylonian and Metonic calendars). Some of the achievements of ancient astronomers are quite startling—for example, the *Antikythera mechanism*, or the observation of *SN 185.* The first was an intricately geared mechanical calculator for predicting the positions of the planets; it was lost in a shipwreck for 2000 years and is still the subject of reconstructive research. The second was the world's first recorded supernova, the impressive explosion of a star at the end of its life. Today, we can see the remnant of that event (for a picture, see[1] APOD `111110`), but modern attention was drawn to it by ancient Chinese astronomers who recorded the explosion as a 'guest star' in 185 CE.

The 17th century, however, brought two completely new developments to astronomy. The first was technological: telescopes were invented, initially for seeing distant things on Earth, but soon turned towards the sky by Galileo and his successors. Even the earliest telescopes had an aperture several times that of the human eye, so that it was as if astronomers' vision had suddenly become ten times sharper and a hundred times more sensitive; further improvements soon followed. The second development was cognitive: no longer satisfied with predicting what would be where on the sky when, astronomers wanted explanations with forces and accelerations. It was the beginning of astro*physics.* To the extent that one can associate the birth of astrophysics with any one individual and event, it would surely be the publication of Newton's *Philosophiæ Naturalis Principia Mathematica* in 1687. Newton's real contribution was not the laws of motion, which were already in use before him. (*Principia* credits the first two laws to Galileo, and the third law to Wren, Wallis, and Huygens.) The idea of a gravitational force varying as inverse distance-squared was also 'in the air' (Newton's frenemy Hooke may have considered it). Working out the consequences, however—using universal gravity to calculate the orbits of planets and moons, and also terrestrial tides—all begins with *Principia.*

[1]This is an abbreviation we will use through this book, referring to NASA's online *Astronomy Picture of the Day* collection. 'APOD 111110' means

`http://apod.nasa.gov/apod/ap111110.html`

which is the particular image for 2011, November 10.

The Astronomer's Magic Envelope. Prasenjit Saha and Paul A. Taylor.

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We therefore begin our introduction to astrophysics by revisiting a topic that *Principia* addresses, and looking at it with our 21st century eyes.[2]

1.1 The Apple and the Earth

In his later years, Newton recounted some of his early reasoning to his first biographer, William Stukeley. According to Stukeley, Newton remarked, 'Why should that apple always descend perpendicularly to the ground... Why should it not go sideways or upwards, but constantly to the earths centre? Assuredly, the reason is, that the earth draws it.' One can speculate about whether the young Newton was really thinking about apples, or whether the apple was just an explanatory device used by the much older Newton. But there is a subtle science question in the quote. According to Newtonian gravity, there is an attractive force pulling between every particle in the apple and every particle in the Earth. How does the apple know the location of the Earth's *centre* so exactly and why does it head *there,* particularly when there is so much other matter around?

It turns out that the integrated gravitational force due to all the particles in a spherical body is equivalent to concentrating the mass at the centre. That is, if you were blindfolded, you wouldn't be able to feel the difference between the pull of a huge rock, a hollow shell, or a tiny pin, as long as they were (1) centred at the same location, (2) of the same mass, and (3) spherically symmetric. When Newton was working, this was a very difficult theorem. But using mathematics developed long after Newton, we can prove it relatively concisely.

First, consider the point-by-point view. In modern notation[3] Newton postulated that, given a particle of mass M at the origin of a coordinate system, another particle at $\mathbf{r}$ and having velocity $\mathbf{v} = \dot{\mathbf{r}}$ will experience an acceleration,

$$\dot{\mathbf{v}} \equiv \ddot{\mathbf{r}} = -\frac{GM}{r^2}\hat{\mathbf{r}}\,, \tag{1.1}$$

which we may call the gravitational field. Here G is a constant of nature, which reflects how strong the change in motion is. The unit vector $\hat{\mathbf{r}}$ shows that the acceleration is directed along the line connecting M and the particle, with the negative sign meaning that the latter is pulled towards the former. Note that $\mathbf{v}$ does not appear on the right-hand side of the equation, so that the particle's acceleration (that is, its *change* in velocity) is independent of its velocity. In terms of particle properties, the size of the acceleration just depends on its distance r from the mass in an *inverse-square* relationship.

As a more general description, if the mass were not at the system's origin, but at some location $\mathbf{r}_1$, the acceleration would be given by

[2] *Principia,* in original or translation, can be found online, but the 17th-century style of writing equations in words makes the text pretty incomprehensible without expert commentary. *Newton's Principia for the Common Reader* (Chandrasekhar, 1995) explains what Newton wrote, in modern mathematical language.

[3] We will use boldface for vectors, hat symbols for unit vectors, and dots over variables for derivatives with respect to time.

$$\dot{\mathbf{v}} = -GM\frac{\mathbf{r} - \mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|^3}\,, \tag{1.2}$$

instead. Note that there is still an overall inverse-square relationship on the distance $|\mathbf{r} - \mathbf{r}_1|$. We can imagine lines of force emanating spherically outwards from the mass; the denser the lines, the stronger the gravitational field.

Let us now define an integral

$$\oint_S \dot{\mathbf{v}} \cdot d\mathbf{S}\,, \tag{1.3}$$

which we call the *flux*. This is an integral, over an arbitrary closed surface S, of the field component normal to the surface. We can also think of it as the net number of lines coming out through the surface. If the surface has a wiggly shape, a line of force could come out, go back in, and come out again. But ultimately, every line of force has to come out. If the mass is outside the surface, all lines of force going into the surface have to come out, so there are no net lines of force coming out. This picture suggests that the flux depends only on whether or not the mass is inside, and not on its precise location.

To show this more formally, we invoke Gauss's divergence theorem. The theorem (found in textbooks on calculus or mathematical physics) states that a surface integral of any smooth vector field $\mathbf{F}(\mathbf{r})$ can be replaced by a volume integral, thus

$$\oint_S \mathbf{F} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{F}\, d^3\mathbf{r}\,, \tag{1.4}$$

where S is a closed surface and V is the volume enclosed by it. The gravitational flux (1.3) can thus be written as

$$\oint_S \dot{\mathbf{v}} \cdot d\mathbf{S} = -GM \int_V \nabla \cdot (\mathbf{r}/r^3)\, d^3\mathbf{r}\,. \tag{1.5}$$

The origin can be moved to $\mathbf{r}_1$ if desired. Now, by expanding in Cartesian coordinates we can verify that

$$\nabla \cdot (\mathbf{r}/r^3) = 0, \quad \text{except for } r = 0, \tag{1.6}$$

where the divergence becomes singular. Hence, only an integrable singularity at the origin contributes to the volume integral. We conclude that the flux indeed depends only on whether the mass is inside the surface, and not on its precise location.

Now let us consider not one point mass but many masses, or a distribution of mass. Since the gravitational field Eq. (1.2) is linear in the masses, the flux through any closed surface S from a distributed mass will depend only on the mass inside S. Redistributing the enclosed mass will change the field at different points on S, but it will not change the flux. An analogous theorem applies to electric charges, and is known as Gauss's flux law.

With the gravitational version of Gauss's flux law in hand, let us specialize to a spherical mass distribution. The mass need not be homogeneous, just spherically symmetric. Let S be a spherical surface concentric with the mass. Spherical symmetry implies that there is no preferred direction other than radial. So the gravitational field