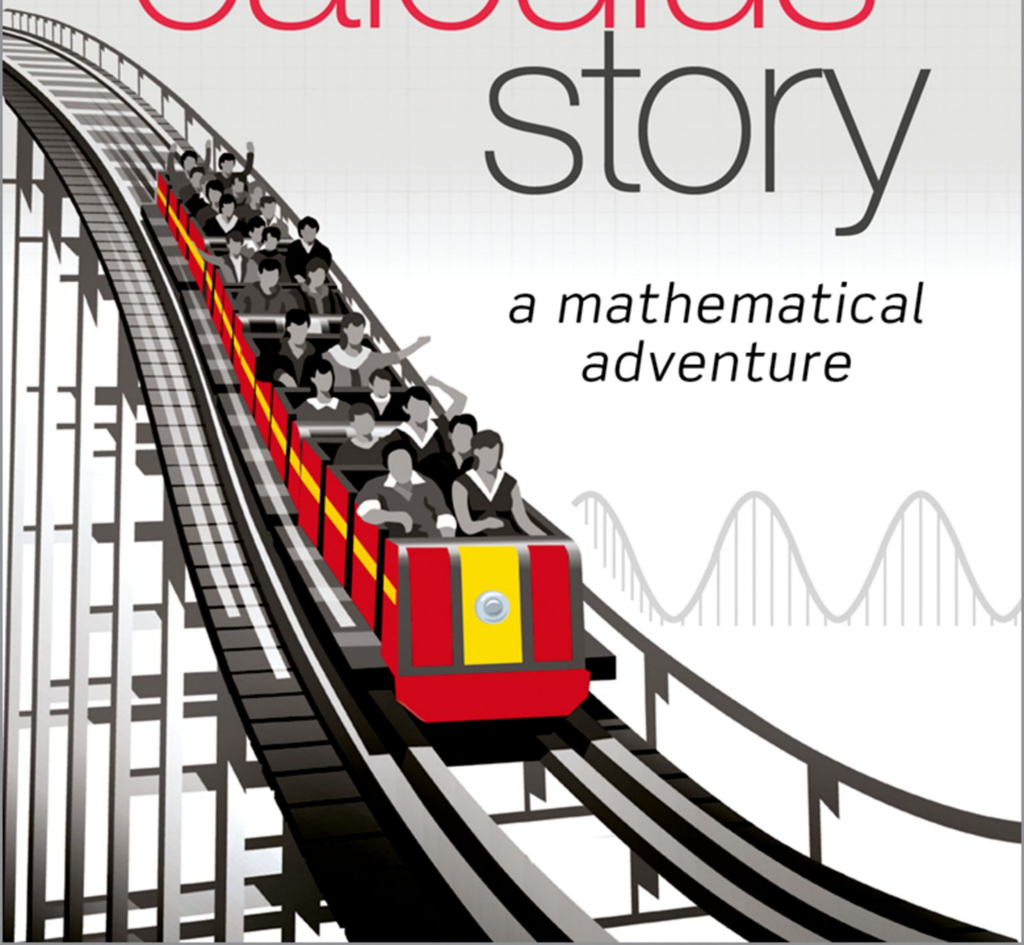
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

the calculus story

DAVID ACHESON

a mathematical adventure



The Calculus Story

The Calculus Story

A Mathematical Adventure

DAVID ACHESON

OXFORD
UNIVERSITY PRESS

OXFORD
UNIVERSITY PRESS

Great Clarendon Street, Oxford, OX2 6DP,
United Kingdom

Oxford University Press is a department of the University of Oxford.
It furthers the University's objective of excellence in research, scholarship,
and education by publishing worldwide. Oxford is a registered trade mark of
Oxford University Press in the UK and in certain other countries

First Edition published in 2017
Impression: 4

Published in the United States of America by Oxford University Press
198 Madison Avenue, New York, NY 10016, United States of America

British Library Cataloguing in Publication Data
Data available

Library of Congress Control Number: 2017935884

ISBN 978-0-19-880454-3

Printed in Great Britain by
Clays Ltd, St Ives plc

In memory of

Dr Janet Mills

(1954–2007)

who once claimed that she never quite understood calculus

CONTENTS

1

Introduction

In the summer of 1666, Isaac Newton saw an apple fall in his garden, and promptly invented the theory of gravity.

That, at least, is the story.

And, however oversimplified this version of events may be, it makes as good a starting point as any for an introduction to calculus.

Because the apple *speeds up* as it falls.

1. Newton and the apple.

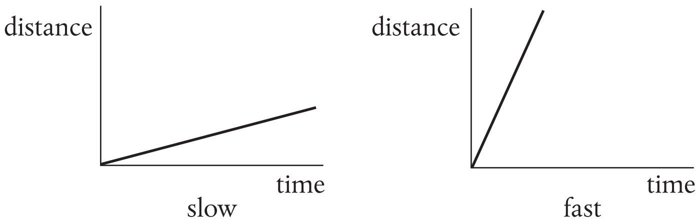
It even raises the whole question of what we mean, exactly, by the speed of the apple at any given moment.

This is because the well-known formula

$$speed = \frac{distance}{time}$$

only applies when the speed of motion is constant, i.e. when distance is proportional to time.

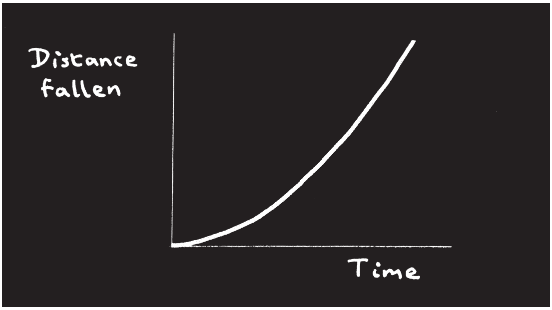
To put it another way, the formula only applies if the graph of distance against time is a straight line, the speed then being represented by the slope, or steepness of the line, as in Figure 2.



2. Motion at constant speed.

But, with a falling apple, distance isn't proportional to time. As Galileo discovered, the distance fallen in time t is proportional to t^2.

So, after a certain time the apple will have fallen a certain distance, but after twice as long it will have fallen not twice as far but four times as far, because $2^2 = 4$. And if we plot the distance fallen against time we get the curve in Figure 3, which bends upwards.

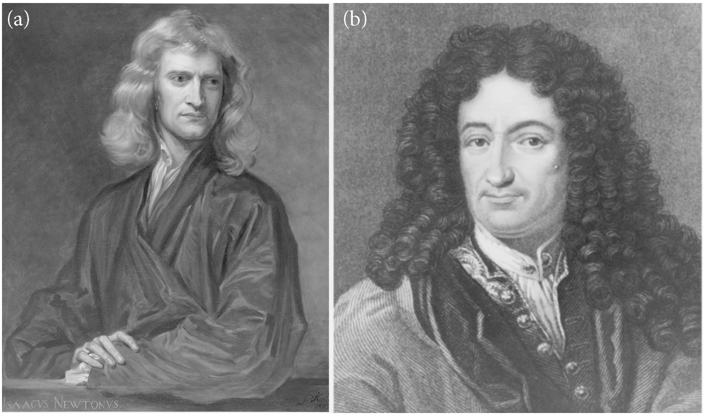


3. How an apple falls.

Plainly, the increasing steepness of the curve reflects, in some way, the increasing rate at which the apple falls, as time goes on.

And this idea of the rate at which something is changing with time is one of the most central ideas in the whole of calculus.

Calculus is sometimes said to be all about change, but a better description, arguably, is that it is all about the *rates* at which things change.



4. (a) Isaac Newton (1642–1727) (b) Gottfried Leibniz (1646–1716)

The subject came fully to life in the second half of the 17th century, largely through the work of Isaac Newton, in England, and Gottfried Leibniz, in Germany.

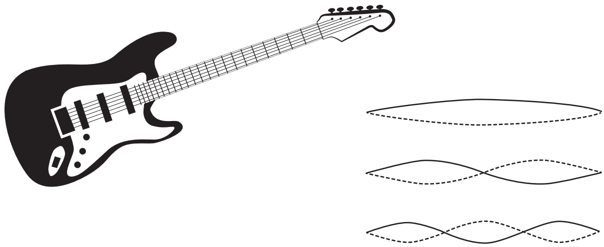
The two never met, but there was a certain amount of wary (and indirect) correspondence between them. At first, this was amicable and polite, but the relationship eventually deteriorated into a major row about who had 'invented' calculus.

While I will say more about this later, my main concern in this short book is with calculus itself.

Above all, I want to offer a 'big picture' of the subject as a whole, concentrating on the most important ideas, and something of their history.

We will see, also, how calculus is fundamental to physics and the other sciences.

One particular aim, for instance, will be to take the theory far enough that we can understand the vibrations of a guitar string (see Figure 5).



5. Guitar string vibrations.

But I will also stress, throughout the book, occasions on which results from calculus can be enjoyed purely for their own sake, regardless of any possible practical application.

Figure 6, for instance, shows an extraordinary connection between π—which is all about circles—and the odd numbers.

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

6. A surprising connection.

And, in due course, I will try to show just why this result is true.

In short, then, this little book is more ambitious than it looks.

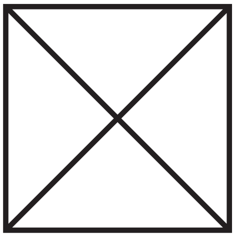
If all goes well, we will see not only what calculus is really about, *but how to actually start doing it.*

And to set about that, we need first to think a little about the very nature and spirit of mathematics itself.

2

The Spirit of Mathematics

In the Babylonian Collection at Yale University there is a famous clay tablet, known as YBC 7289. It dates from roughly 1700 BC, and has a simple geometrical figure on it (Figure 7).



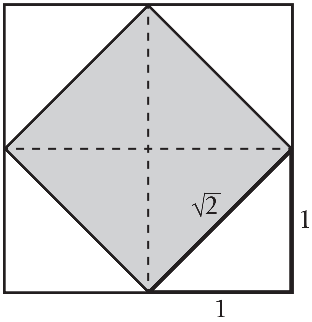
7. Square and diagonals.

The figure is accompanied by some cuneiform writing, and when that was deciphered it was found to be an approximation to the number $\sqrt{2}$—correct to better than 1 part in a million.

How, then, did the writer *know* that, for a square, the ratio of diagonal to side is $\sqrt{2}$?

We can only guess, I think, that they appealed to a diagram such as Figure 8.

The area of the large square is $2 \times 2 = 4$. The area of the shaded square is evidently half of this, and therefore 2. So the side of the shaded square must be $\sqrt{2}$.

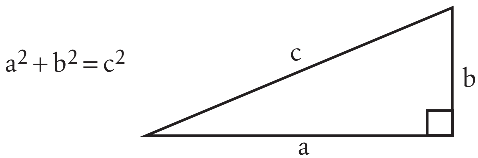


8. A simple deduction.

Today, this deductive aspect of mathematics is seen as central to the whole subject.

We continually ask not simply 'What is true?' but '*Why* is it true?'

Mathematicians also seek *generality* whenever possible, and Pythagoras' theorem is a famous example, for it provides an unexpectedly simple relationship between the three sides of *any* right-angled triangle – short and fat or long and thin.



9. Pythagoras' theorem.

And, as with much that is best in mathematics, it is this generality which gives the theorem its power.

Algebra

While geometry dates back to ancient Greece and beyond, algebra—at least as we know it today—is a much more recent development.

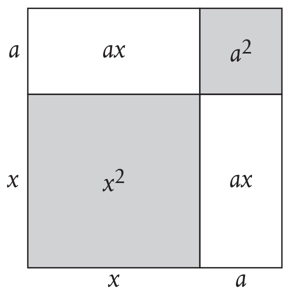
Even the familiar equals '=' sign only appeared in 1557, less than a century before Newton was born.

The main purpose of algebra is, again, to help us express and manipulate general ideas in mathematics, in a succinct manner.

And one such result, of great value in this book, is

$$(x+a)^2 = x^2 + 2ax + a^2.$$

This is true for any numbers x and a, positive or negative, by the rules of elementary algebra, but when x and a are both positive it can even be seen geometrically, using areas (Figure 10).

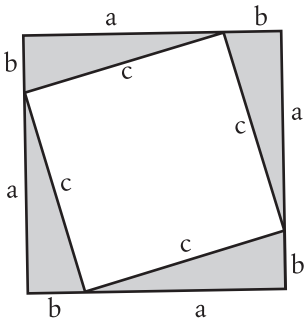


10. Algebra as geometry.

Proof

Sometimes in mathematics, the actual deductive arguments, or proofs, can be a source of pleasure in themselves.

Consider, for instance, the proof of Pythagoras' theorem in Figure 11.



11. Proving Pythagoras' theorem.

Here, we have placed four copies of our right-angled triangle inside a square of side $a + b$, leaving a square of area c^2 in the middle.

Each right-angled triangle has area $\frac{1}{2}ab$, so the area of the large square is $c^2 + 2ab$.

But it is also $(a + b)^2 = a^2 + 2ab + b^2$.

So $a^2 + b^2 = c^2$.

I would argue that this is one of the best proofs of Pythagoras' theorem, in fact, because it is so concise and elegant.

The way to the stars…

Throughout its history, mathematics has played a crucial part in our understanding of how the world really works.

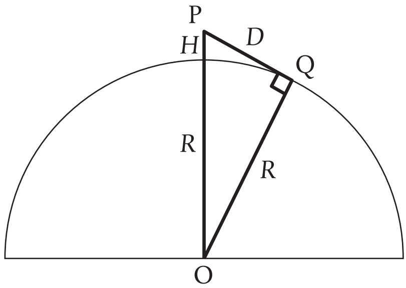
The nature of the Universe, in particular, has always been a source of wonder. Yet to study it, we must begin, inevitably, by measuring the Earth.

And one way of doing that is to climb a mountain of known height H and estimate the distance D to the horizon (Figure 12). As the line of sight PQ will be tangent to the Earth, it will be at right angles to the radius of the Earth, OQ, so OQP will be a right-angled triangle.

Applying Pythagoras' theorem, we have

$$(R+H)^2 = R^2 + D^2,$$

where R is the radius of the Earth. After rewriting the left-



12. Measuring the Earth.

hand side as $R^2 + 2RH + H^2$ and cancelling the R^2 terms we have $2RH + H^2 = D^2$.

In practice, H will be tiny compared to the radius of the Earth R, so that H^2 will be tiny compared to $2RH$. Thus, $2RH$ is approximately equal to D^2, and so

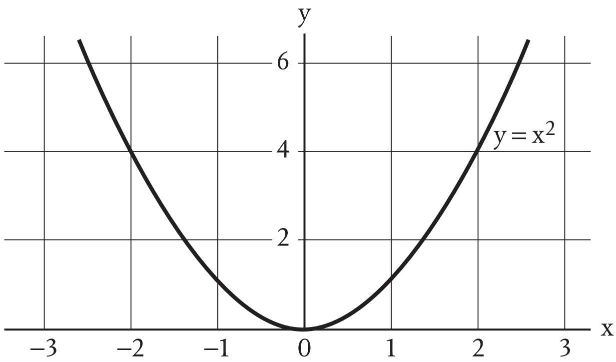
$$R \approx \frac{D^2}{2H}.$$

In about 1019, the scholar Al-Biruni used broadly similar ideas to estimate the radius R of the Earth, obtaining a result which differed from the currently accepted value by less than 1%. This was a quite extraordinary achievement for the time.

Equations and curves

I should like to end this chapter by pointing out one particularly powerful way in which geometry and algebra come together.

Today, if we have a relationship between two numbers—$y = x^2$, for example—we think nothing of using x and y as *coordinates* to plot a graph, as in Figure 13. Our equation is then represented by a curve. And, conversely, if some problem in geometry involves a certain curve, we can try and represent it by an equation.



13. Coordinate geometry.

But in Newton's time this was a very new idea indeed, largely due to two French mathematicians, Pierre de Fermat (1601–65) and René Descartes (1596–1650).

And while it takes us very close to calculus itself, we need, first, just one more key idea…