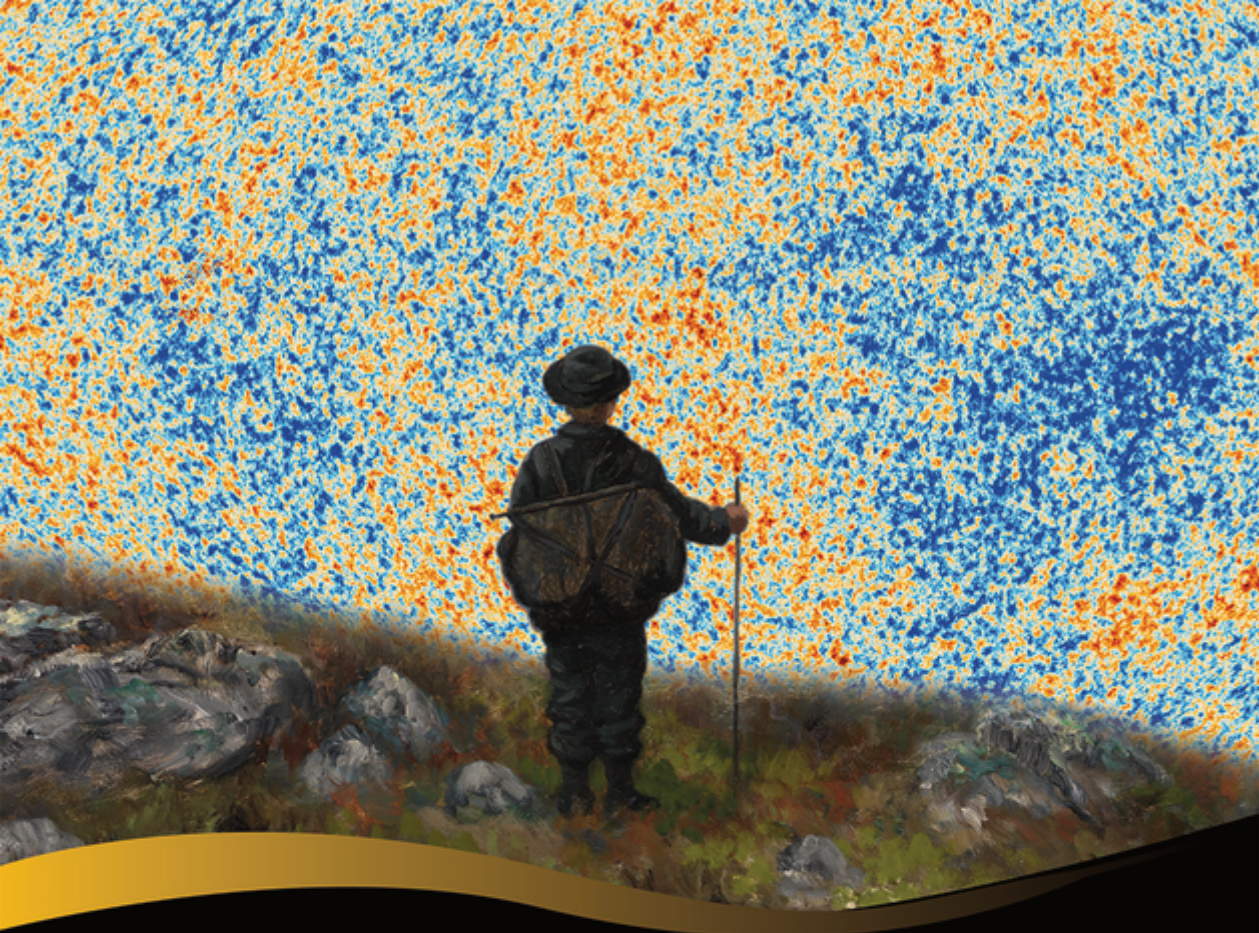

# Quantum Fields

*From the Hubble to the Planck Scale*

MICHAEL KACHELRIESS

OXFORD GRADUATE TEXTS

# QUANTUM FIELDS

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## From the Hubble to the Planck Scale

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# Preface

**Why this book?** The number of excellent introductory books on quantum field theory and on cosmology has grown much in the last years. Teaching a one-semester course on Gravitation and Cosmology and a one-year course on Quantum Field Theory (QFT) since 2009, I profited enormously from these textbooks. Working out my own lectures, I tried however to teach the two courses in a more unified manner than is usually done. One motivation for doing so was the belief that studying a subject in depth is only half the premise; the remaining—and not least—struggle is to put the pieces into a comprehensible picture. This is particularly true for students who aim to work at the interface between theoretical particle physics, cosmology and astroparticle physics. Thus I tried to stress the basic principles and methods with which rather dispersed phenomena in these fields can be analysed. Moreover, this approach saves also time and makes it thus possible to discuss additional applications within the restricted time for lectures.

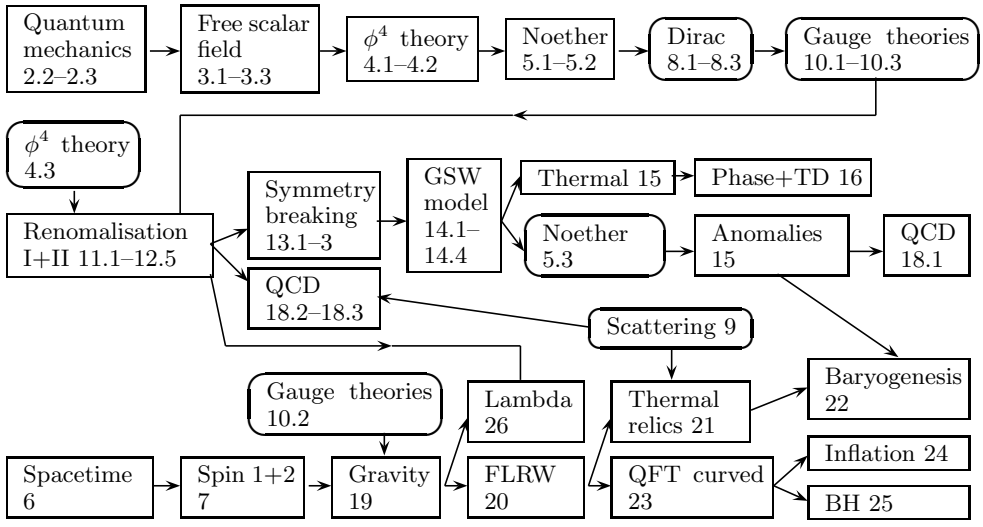
This book reflects this approach and aims to introduce QFT together with its most important applications to processes in our universe in a coherent framework. As in many modern textbooks, the more universal path-integral approach is used right from the beginning. Massless spin one and two fields are introduced on an equal footing, and gravity is presented as a gauge theory in close analogy with the Yang–Mills case. Concepts relevant to modern research as helicity methods, effective theories, decoupling, or the stability of the electroweak vacuum are introduced. Various applications as topological defects, dark matter, baryogenesis, processes in external gravitational fields, inflation and black holes help students to bridge the gap between undergraduate courses and the research literature.

**How to use this book.** I tried to present all derivations in such detail that the book can be used for self-studies. It should be accessible for students with a solid knowledge of calculus, classical mechanics, electrodynamics including special relativity and quantum mechanics. As always, it is indispensable to work through the text and the exercises to get a grip on the material. Although the book is written with the intention to be read from cover to cover, time constraints and special interests will typically push students to omit several topics in a first round. A chart showing the interdependence of the chapters is shown below.

Additionally to being suitable for self-study, the book may serve as basis for a course in quantum field theory or an advanced course in astroparticle physics and cosmology. For a standard two-semester course on QFT, one can use chapters 2–12 plus, depending on preferences and the time budget, material from chapters 13–18. For an advanced course in astroparticle physics and cosmology, one may select suitable chapters from the second half of the book. The order of some of the topics in the book may be reshuffled:

- Section 4.3 introduces some basic tools needed to perform loop calculations and applies them to three examples. If one prefers a more systematic approach, this section could be shifted to the end of section 11.4.
- Section 5.3 discusses symmetries on the quantum level. It could be postponed and used as introductory section to chapter 17.
- Chapter 8 and 10 on fermions and on gauge theories could be omitted in a first round, restricting the discussion of renormalisation to the scalar case.
- Chapter 9 on scattering is rather independent of the main text. While the prediction of scattering cross-sections is the main occupation of most theorists working in particle physics, it will be needed only rarely in the latter parts of the text. Section 9.1 introduces the optical theorem which will be applied in chapter 14 and 21. Section 9.4 is useful as preparation for chapter 18, and explains why we consider only fields with spin $s \leq 2$.

A minimal path through the QFT oriented chapters is shown graphically in the first two rows, with round boxes denoting material that could be omitted in a first round and shifted to latter places. The two lines at the bottom show a similar path collecting the chapters discussing gravitation and cosmology.




Some will miss important topics in this chart. For instance, grand unified theories or supersymmetry are two aspects of "beyond the standard model physics" (BSM) which are not only very attractive from a theoretical point of view but have also often been invoked to explain dark matter or baryogenesis. Having digested the material presented in this book, students may consult e.g. Dine (2016) as an entrée into the world of BSM. Moreover, I adapted from the field of astroparticle physics and cosmology only few topics directly relevant to the main theme of this book. Thus all more phenomenological aspects like, for example, neutrino oscillations or cosmic ray physics are omitted.

As this book is intended as an introduction, I suggest mainly review articles and textbooks in the "Further reading" sections at the end of each chapter. References to the original research literature are almost absent—I have to apologise to all those whose papers have been only indirectly referenced via these reviews. Moreover, even a minimal account of the historical development of this field is missing in this book. To compensate this deficit, I recommend the reader to consult some references specialised on the history of physics: Schweber (1994) gives a very readable account how QED was created, while O'Raifeartaigh (1997) reviews the development of the gauge principle. The story how the hot big bang model became the leading cosmological theory is told by Kragh (2013) and Peebles *et al.* (2009).

**Website.** A list of corrections, updates, solutions to more than 100 exercises as well as some software is available on the website of the book, `www.oup.co.uk/companion/quantumfields2018`. Comments and corrections are welcome and can be submitted via this website.

**Acknowledgements.** First of all, I would like to thank the students of my courses which had to digest various test versions of these lectures. I am grateful to Peder Galteland, Jonas Glesaaen and William Naylor for working out a first LaTeXversion of the lecture notes for FY3466, and to all the students of the following semesters who spotted errors and pointed out obscure passages in the draft versions of this book. Jens Andersen, Eugeny Babichev, Sergey Ostapchenko and Pasquale Serpico read parts of the book and made valuable comments which helped to improve the text. Last but not least I would like to thank all my collaborators for sharing their insights with me.

**Acknowledgements for the figures.**

- Figure 12.3 is courtesy of D. Kazakov (hep-ph/0012288) who adapted it from Fig. 1 in U. Amaldi, W. de Boer, W. and H. Fürstenau, "Comparison of grand unified theories with electroweak and strong coupling constants measured at LEP", Phys. Lett. **B260**, 447 (1991). It has been reproduced with permission of Elsevier.
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## Preface to the Paperback Edition

Various minor errors have been corrected in this paperback edition, and at a few places the text has been clarified. I am grateful to all readers who sent corrections and would like to thank in particular Rodrigo Alvarez, Howard Haber, Magnus Malmquist, Christian Thierfelder and Jonas Tjemsland.

# Notation and conventions

We use natural units with $\hbar = c = 1$, but mostly keep Newton's gravitational constant $G_\mathrm{N} \neq 1$. Then all units can be expressed as powers of a basic unit which we choose as mass or energy. Instead of $G_\mathrm{N}$, we use also $\kappa = 8\pi G_\mathrm{N}$, the Planck mass $M_\mathrm{Pl} = 1/\sqrt{G_\mathrm{N}}$ or the reduced Planck mass $\widetilde{M}_\mathrm{Pl} = 1/\sqrt{8\pi G_\mathrm{N}}$. Maxwell's equations are written in the Lorentz–Heaviside version of the cgs system. Thus there is a factor $4\pi$ in the Coulomb law, but not in Maxwell's equations. Sommerfeld's fine-structure constant is $\alpha = e^2/(4\pi) \simeq 1/137$.

We choose as signature of the metric $-2$, thence the metric tensor in Minkowski space is $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. If not otherwise specified, Einstein's summation convention is implied.

The d'Alembert or wave operator is $\Box \equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \Delta$, while the four-dimensional nabla operator has the components $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.

A boldface italic letter denotes the components of a three-vector $\boldsymbol{V} = \{V_x, V_y, V_z\} = \{V_i, i = 1, 2, 3\}$ or the three-dimensional part of a contravariant vector with components $V^\mu = \{V^0, V^1, V^2, V^3\} = \{V^0, \boldsymbol{V}\}$; a covariant vector has in Minkowski space the components $V_\mu = (V_0, -\boldsymbol{V})$. Scalar products of four-vectors are also denoted by $p_\mu q^\mu = p \cdot q$, of three-vectors by $\boldsymbol{p} \cdot \boldsymbol{q} = p_i q^i$. If there is no danger of confusion, the dot is omitted. Vectors and tensors in index free notation are denoted by boldface Roman letters, $\mathbf{V} = V^\mu \partial_\mu$ or $\mathbf{g} = g_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$.
Greek indices $\alpha, \beta, \ldots$ encompass the range $\alpha = \{0, 1, 2, \ldots d-1\}$, Latin indices $i, j, k, \ldots$ the range $i = \{1, 2, \ldots d-1\}$, where $d$ denotes the dimension of the space-time. In chapter 19, Latin indices $a, b, c, \ldots$ denote tensor components with respect to the vielbein field $e^a_\mu$.

Our convention for the Fourier transformation is asymmetric, putting the factor $1/(2\pi)^n$ into

$$f(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} f(k) \mathrm{e}^{-\mathrm{i}kx} \quad \text{and} \quad f(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{x}}.$$

If no borders are specified in definite integrals, integration from $-\infty$ to $\infty$ is assumed.

Our nomenclature for disconnected, connected and one-particle irreducible (1PI) $n$-point Green functions and their corresponding generating functionals is as follows:

| | Green function | generating functional |
|---|---|---|
| (dis-) connected | $\mathcal{G}(x_1, \ldots, x_n)$ | $Z[J, \ldots]$ |
| connected | $G(x_1, \ldots, x_n)$ | $W[J, \ldots]$ |
| 1PI | $\Gamma(x_1, \ldots, x_n)$ | $\Gamma[\phi, \ldots]$ |

Dirac spinors are normalised as $\bar{u}(p,s)u(p,s) = 2m$.

We use as covariant derivative $D_\mu = \partial_\mu + \mathrm{i}gA^a_\mu T^a$ with coupling $g > 0$, field strength $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$ and generators $T^a$ satisfying $[T^a, T^b] = \mathrm{i} f^{abc} T^c$ for all gauge groups. Special cases used in the SM are the groups $\mathrm{U_{em}}(1)$, $\mathrm{U_Y}(1)$, $\mathrm{SU_L}(2)$ and SU(3) with $g = \{q, g', g, g_s\}$ and $T^a = \{1, 1, \tau^a/2, \lambda^a/2\}$ in the fundamental representation. In particular, the electric charge of the positron is $q = e > 0$.

Employing dimensional regularisation (DR), we change the dimension of loop integrals from $d = 4$ to $d = 2\omega = 4 - 2\varepsilon$.

The results of problems marked by ♣ are used later in the text, those marked by ♥ require more efforts and time than average ones. Solutions to selected problems can be found on the webpage of this book. Commonly used symbols are

| | |
|---|---|
| $a$ | scale factor in FLRW metric |
| $\delta(x)$ | Dirac's delta function, $\int \mathrm{d}x\, f(x)\delta(x) = f(0)$ |
| $\varepsilon$ | infinitesimal quantity, slow-roll parameter |
| $\varepsilon_\mu, \varepsilon_{\mu\nu}$ | polarisation vector and tensor for spin $s = 1, 2$ |
| $\varepsilon_{\mu\nu\rho\sigma}$ | Levi-Civita tensor with $\epsilon_{0123} = -\epsilon^{0123} = 1$ |
| $\eta$ | boost parameter, conformal time, slow-roll parameter |
| $g = \det(g_{\mu\nu})$ | determinant of the metric tensor $g_{\mu\nu}$ |
| $g_*, g_{*,S}$ | relativistic degrees of freedom entering $\rho$, $S$ |
| $H(q,p)$, $\mathscr{H}(\phi,\pi)$ | Hamiltonian, Hamiltonian density |
| $H^\dagger$ | Hermitian conjugate (h.c.) with $M^\dagger = M^{*T}$ or $M^\dagger_{ij} = M^*_{ji}$ |
| $H = \dot{a}/a$, $\mathscr{H} = a'/a$ | Hubble parameter |
| $L(q,\dot{q})$, $\mathscr{L}(\phi, \partial_\mu\phi)$ | Lagrangian, Lagrangian density |
| $\Omega_i = \rho_i/\rho_{\mathrm{cr}}$ | fraction of critical energy density in component $i$ |
| $p^\mu; P; P_{ij}$ | four momentum $p^\mu = (E, \boldsymbol{p})$, pressure |
| $\bar{\psi}$ | adjoint spinor with $\bar{\psi} = \psi^\dagger\gamma^0$ |
| $R^\alpha{}_{\beta\rho\sigma} = [\partial_\rho \Gamma^\alpha{}_{\beta\sigma} - \ldots$ | Riemann or curvature tensor |
| $R_{\alpha\beta} = R^\rho{}_{\alpha\rho\beta}$ | Ricci tensor |
| $R = g_{\mu\nu}R^{\mu\nu}$ | curvature scalar $R$ |
| $S[\phi]$, $S[\phi, \partial_\mu\phi]$ | action functional |
| $T_{\mu\nu}$ | (energy–momentum) stress tensor |
| $\vartheta(x)$ | Heaviside step function, $\vartheta(x) = 1$ for $x > 0$, 0 for $x < 0$. |
| tr | trace of a matrix $\mathrm{tr}(A) = \sum_i A_{ii}$, of a tensor $\mathrm{tr}(T) = T_\mu{}^\mu$ |
| Tr | sum/integration over a complete set of quantum numbers |
| $u, v$ | solutions of Dirac equation, light-cone coordinates $t \pm x$ |
| $w = P/\rho$ | equation of state (EoS) parameter |
| $Y = n_X/s$ | abundance of particle type $X$ relative to entropy density |
| $X = n_X/n_B$ | abundance of particle type $X$ relative to baryon density |

# Contents

# 1
# Classical mechanics

To begin, in this chapter we review those concepts of classical mechanics which are essential for progressing towards quantum theory. First we recall briefly the Lagrangian and Hamiltonian formulation of classical mechanics and their derivation from an action principle. We also illustrate the Green function method using as example the driven harmonic oscillator and recall the action of a relativistic point particle.

## 1.1 Action principle

**Variational principles.** Fundamental laws of nature as Newton's axioms or Maxwell's equations were discovered in the form of differential equations. Starting from Leibniz and Euler, it was realised that one can re-express differential equations in the form of variational principles. In this approach, the evolution of a physical system is described by the extremum of an appropriately chosen functional. Various versions of such variational principles exist, but they have in common that the functionals used have the dimension of "energy $\times$ time"; that is, the functionals have the same dimension as Planck's constant $\hbar$. A quantity with this dimension is called action $S$. An advantage of using the action as main tool to describe dynamical systems is that this allows us to implement easily both spacetime and internal symmetries. For instance, choosing as ingredients of the action local functions that transform as scalars under Lorentz transformations leads automatically to relativistically invariant field equations. Moreover, the action $S$ economically summarises the information contained typically in a set of various coupled differential equations.

If the variational principle is formulated as an integral principle, then the functional $S$ will depend on the whole path $q(t)$ described by the system between the considered initial and final time. In the formulation of quantum theory we will pursue, we will look for a direct connection from the classical action $S[q]$ of the path $[q(t)\colon q'(t')]$ to the transition amplitude $\langle q', t'|q, t\rangle$. Thus the use of the action principle will not only simplify the discussion of symmetries of a physical system but it also lies at the heart of the approach to quantum theory we will follow.

### 1.1.1 Hamilton's principle and Lagrange's equations

A functional $F[f(x)]$ is a map from a certain space of functions $f(x)$ into the real or complex numbers. We will consider mainly functionals from the space of (at least) twice differentiable functions between fixed points $a$ and $b$. More specifically, Hamilton's principle uses as functional the action $S$ defined by

$$S[q^i] = \int_a^b \mathrm{d}t\, L(q^i, \dot{q}^i, t), \tag{1.1}$$

where $L$ is a function of the $2n$ independent functions $q^i$ and $\dot{q}^i = \mathrm{d}q^i/\mathrm{d}t$ as well as of the parameter $t$. In classical mechanics, we call $L$ the Lagrange function of the system, $q^i$ are its $n$ generalised coordinates, $\dot{q}^i$ the corresponding velocities and $t$ is the time. The extrema of this action give those paths $q(t)$ from $a$ to $b$ which are solutions of the equations of motion for the system described by $L$.

How do we find those paths that extremize the action $S$? First of all, we have to prescribe which variables are kept constant, which are varied and which constraints the variations have to obey. Depending on the variation principle we choose, these conditions and the functional form of the action will differ. Hamilton's principle corresponds to a smooth variation of the path,

$$q^i(t, \varepsilon) = q^i(t, 0) + \varepsilon \eta^i(t),$$

that keeps the endpoints fixed, $\eta^i(a) = \eta^i(b) = 0$ but is otherwise arbitrary. The scale factor $\varepsilon$ determines the magnitude of the variation for the one-parameter family of paths $\varepsilon\eta^i(t)$. The notation $S[q^i]$ stresses that we consider the action as a functional only of the coordinates $q^i$. The velocities $\dot{q}^i$ are not varied independently because $\varepsilon$ is time-independent. Since the time $t$ is not varied in Hamilton's principle, varying the path $q^i(t, \varepsilon)$ requires only to calculate the resulting change of the Lagrangian $L$. Following this prescription, the action has an extremum if

$$0 = \left.\frac{\partial S[q^i(t,\varepsilon)]}{\partial \varepsilon}\right|_{\varepsilon=0} = \int_a^b \mathrm{d}t \left(\frac{\partial L}{\partial q^i}\frac{\partial q^i}{\partial \varepsilon} + \frac{\partial L}{\partial \dot{q}^i}\frac{\partial \dot{q}^i}{\partial \varepsilon}\right) = \int_a^b \mathrm{d}t \left(\frac{\partial L}{\partial q^i}\eta^i + \frac{\partial L}{\partial \dot{q}^i}\dot{\eta}^i\right). \tag{1.2}$$

Here we applied—as always in the following—Einstein's convention to sum over a repeated index pair. Thus, for example, the first term in the bracket equals

$$\frac{\partial L}{\partial q^i}\,\eta^i \equiv \sum_{i=1}^{n} \frac{\partial L}{\partial q^i}\,\eta^i$$

for a system described by $n$ generalised coordinates. We can eliminate $\dot{\eta}^i$ in favour of $\eta^i$, integrating the second term by parts, arriving at

$$\left.\frac{\partial S[q^i(t,\varepsilon)]}{\partial \varepsilon}\right|_{\varepsilon=0} = \int_a^b \mathrm{d}t \left[\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\right]\eta^i + \left[\frac{\partial L}{\partial \dot{q}^i}\eta^i\right]_a^b. \tag{1.3}$$

The boundary term $[\ldots]_a^b$ vanishes because we required that the functions $\eta^i$ are zero at the endpoints $a$ and $b$. Since these functions are otherwise arbitrary, each individual term in the first bracket has to vanish for an extremal curve. The $n$ equations resulting from the condition $\partial S[q^i(t,\varepsilon)]/\partial\varepsilon = 0$ are called the (Euler–) Lagrange equations of the action $S$,

$$\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} = 0 \tag{1.4}$$

and give the equations of motion for the system specified by $L$. In the future, we will use a more concise notation, calling

$$\delta q^i \equiv \lim_{\varepsilon\to 0} \frac{q^i(t,\varepsilon) - q^i(t,0)}{\varepsilon} = \left.\frac{\partial q^i(t,\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} \tag{1.5}$$

the variation of $q^i$, and similarly for functions and functionals of $q^i$. Thus we can rewrite, for example, Eq. (1.2) in a more evident form as

$$0 = \delta S[q^i] = \int_a^b \mathrm{d}t\, \delta L(q^i, \dot{q}^i, t) = \int_a^b \mathrm{d}t\, \left(\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i\right). \tag{1.6}$$

We close this paragraph with three remarks. First, we note that Hamilton's principle is often called the principle of least action. This name is somewhat misleading, since the extremum of the action can be also a maximum or a saddle-point. Second, observe that the Lagrangian $L$ is not uniquely fixed. Adding a total time derivative, $L \to L' = L + \mathrm{d}f(q,t)/\mathrm{d}t$, does not change the resulting Lagrange equations,

$$S' = S + \int_a^b \mathrm{d}t\, \frac{\mathrm{d}f}{\mathrm{d}t} = S + f(q(b), t_b) - f(q(a), t_a), \tag{1.7}$$

since the last two terms vanish varying the action with the restriction of fixed endpoints $a$ and $b$. Finally, note that we used a Lagrangian that depends only on the coordinates and their *first* derivatives. Such a Lagrangian leads to second-order equations of motion and thus to a mechanical system specified by the $2n$ pieces of information $\{q_i, \dot{q}_i\}$. Ostrogradsky showed 1850 that a stable ground-state is impossible, if the Lagrangian contains higher derivatives $\ddot{q}, q^{(3)}, \ldots$, cf. problem 1.3. Therefore such theories contradict our experience that the vacuum is stable. Constructing Lagrangians for the fundamental theories describing Nature, we should restrict ourselves thus to Lagrangians that lead to second-order equations of motion.

**Lagrange function.** We illustrate now how one can use symmetries to constrain the possible form of a Lagrangian $L$. As example, we consider the case of a free non-relativistic particle with mass $m$ subject to the Galilean principle of relativity. More precisely, we use that the homogeneity of space and time forbids that $L$ depends on $\boldsymbol{x}$ and $t$, while the isotropy of space implies that $L$ depends only on the norm of the velocity vector $\boldsymbol{v}$, but not on its direction. Thus the Lagrange function of a free particle can be only a function of $v^2$, $L = L(v^2)$.

Let us consider two inertial frames moving with the infinitesimal velocity $\boldsymbol{\varepsilon}$ relative to each other. (Recall that an inertial frame is defined as a coordinate system where a force-free particle moves along a straight line.) Then a Galilean transformation connects the velocities measured in the two frames as $\boldsymbol{v}' = \boldsymbol{v} + \boldsymbol{\varepsilon}$. The Galilean principle of relativity requires that the laws of motion have the same form in both frames, and thus the Lagrangians can differ only by a total time derivative. Expanding the difference $\delta L$ in $\varepsilon$ gives with $\delta v^2 = 2\boldsymbol{v}\cdot\boldsymbol{\varepsilon} + \mathcal{O}(\varepsilon^2)$

$$\delta L = \frac{\partial L}{\partial v^2}\,\delta v^2 = 2\boldsymbol{v}\cdot\boldsymbol{\varepsilon}\frac{\partial L}{\partial v^2}. \tag{1.8}$$

Since $v^i = \mathrm{d}x^i/\mathrm{d}t$, the term $\partial L/\partial v^2$ has to be independent of $v$ such that the difference $\delta L$ is a total time derivative. Hence, the Lagrangian of a free particle has the form

$L = av^2 + b$. The constant $b$ drops out of the equations of motion, and we can set it therefore to zero. To be consistent with usual notation, we call the proportionality constant $m/2$, and the total expression kinetic energy $T$,

$$L = T = \frac{1}{2}mv^2. \tag{1.9}$$

For a system of non-interacting particles, the Lagrange function $L$ is additive, $L = \sum_a \frac{1}{2}m_a v_a^2$. If there are interactions (assumed for simplicity to depend only on the coordinates), then we subtract a function $V(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots)$ called potential energy. One confirms readily that this choice for $L$ reproduces Newton's law of motion.

**Energy.** The Lagrangian of a closed system does not depend on time because of the homogeneity of time. Its total time derivative is

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial q^i}\,\dot{q}^i + \frac{\partial L}{\partial \dot{q}^i}\,\ddot{q}^i. \tag{1.10}$$

Using the equations of motion and replacing $\partial L/\partial q^i$ by $(\mathrm{d}/\mathrm{d}t)\partial L/\partial \dot{q}^i$, it follows

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \dot{q}^i\,\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\partial L}{\partial \dot{q}^i} + \frac{\partial L}{\partial \dot{q}^i}\,\ddot{q}^i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}^i\,\frac{\partial L}{\partial \dot{q}^i}\right). \tag{1.11}$$

Hence the quantity

$$E \equiv \dot{q}^i\,\frac{\partial L}{\partial \dot{q}^i} - L \tag{1.12}$$

remains constant during the evolution of a closed system. This holds also more generally, for example in the presence of static external fields, as long as the Lagrangian is not time-dependent.

We have still to show that $E$ coincides indeed with the usual definition of energy. Using as Lagrange function $L = T(q, \dot{q}) - V(q)$, where the kinetic energy $T$ is quadratic in the velocities, we have

$$\dot{q}^i\,\frac{\partial L}{\partial \dot{q}^i} = \dot{q}^i\,\frac{\partial T}{\partial \dot{q}^i} = 2T \tag{1.13}$$

and thus $E = 2T - L = T + V$.

**Conservation laws.** In a general way, we can derive the connection between a symmetry of the Lagrangian and a corresponding conservation law as follows. Let us assume that under a change of coordinates $q^i \to q^i + \delta q^i$, the Lagrangian changes at most by a total time derivative,

$$L \to L + \delta L = L + \frac{\mathrm{d}\delta F}{\mathrm{d}t}. \tag{1.14}$$

In this case, the equations of motion are unchanged and the coordinate change $q^i \to q^i + \delta q^i$ is a symmetry of the Lagrangian. The change $\mathrm{d}\delta F/\mathrm{d}t$ has to equal $\delta L$ induced by the variation $\delta q^i$,

$$\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i - \frac{\mathrm{d}\delta F}{\mathrm{d}t} = 0. \tag{1.15}$$

Replacing again $\partial L/\partial q^i$ by $(\mathrm{d}/\mathrm{d}t)\partial L/\partial \dot{q}^i$ and applying the product rule gives as conserved quantity

$$Q = \frac{\partial L}{\partial \dot{q}^i}\delta q^i - \delta F. \tag{1.16}$$

Thus any continuous symmetry of a Lagrangian system results in a conserved quantity. In particular, energy conservation follows for a system invariant under time translations with $\delta q^i = \dot{q}^i \delta t$. Other conservation laws are discussed in problem 1.7.

### 1.1.2 Palatini's principle and Hamilton's equations

**Legendre transformation and the Hamilton function.** In the Lagrange formalism, we describe a system specifying its generalised coordinates and velocities using the Lagrangian, $L = L(q^i, \dot{q}^i, t)$. An alternative is to use generalised coordinates and their canonically conjugated momenta $p_i$ defined as

$$p_i = \frac{\partial L}{\partial \dot{q}^i}. \tag{1.17}$$

The passage from $\{q^i, \dot{q}^i\}$ to $\{q^i, p_i\}$ is a special case of a Legendre transformation:[1] Starting from the Lagrangian $L$ we define a new function $H(q^i, p_i, t)$ called Hamiltonian or Hamilton function via

$$H(q^i, p_i, t) = \frac{\partial L}{\partial \dot{q}^i}\,\dot{q}^i - L(q^i, \dot{q}^i, t) = p_i \dot{q}^i - L(q^i, \dot{q}^i, t). \tag{1.18}$$

Here we assume that we can invert the definition (1.17) and are thus able to substitute velocities $\dot{q}^i$ by momenta $p_i$ in the Lagrangian $L$.

The physical meaning of the Hamiltonian $H$ follows immediately comparing its defining equation with the one for the energy $E$. Thus the numerical value of the Hamiltonian equals the energy of a dynamical system; we insist, however, that $H$ is expressed as function of coordinates and their conjugated momenta. A coordinate $q_i$ that does not appear explicitly in $L$ is called cyclic. The Lagrange equations imply then $\partial L/\partial \dot{q}_i = \text{const.}$, so that the corresponding canonically conjugated momentum $p_i = \partial L/\partial \dot{q}^i$ is conserved.

**Palatini's formalism and Hamilton's equations.** Previously, we considered the action $S$ as a functional only of $q^i$. Then the variation of the velocities $\dot{q}^i$ is not independent and we arrive at $n$ second–order differential equations for the coordinates $q^i$. An alternative approach is to allow independent variations of the coordinates $q^i$ and of the velocities $\dot{q}^i$. We trade the latter against the momenta $p_i = \partial L/\partial \dot{q}^i$ and rewrite the action as

$$S[q^i, p_i] = \int_a^b \mathrm{d}t\ \left[p_i \dot{q}^i - H(q^i, p_i, t)\right]. \tag{1.19}$$

The independent variation of coordinates $q^i$ and momenta $p_i$ gives

[1] The concept of a Legendre transformation may be familiar from thermodynamics, where it is used to change between extensive variables (e.g. the entropy $S$) and their conjugate intensive variables (e.g. the temperature $T$).

$$\delta S[q^i, p_i] = \int_a^b \mathrm{d}t \left[ p_i \delta \dot{q}^i + \dot{q}^i \delta p_i - \frac{\partial H}{\partial q^i} \delta q^i - \frac{\partial H}{\partial p_i} \delta p_i \right]. \tag{1.20}$$

The first term can be integrated by parts, and the resulting boundary terms vanishes by assumption. Collecting then the $\delta q^i$ and $\delta p_i$ terms and requiring that the variation is zero, we obtain

$$0 = \delta S[q^i, p_i] = \int_a^b \mathrm{d}t \left[ - \left( \dot{p}_i + \frac{\partial H}{\partial q^i} \right) \delta q^i + \left( \dot{q}^i - \frac{\partial H}{\partial p_i} \right) \delta p_i \right]. \tag{1.21}$$

As the variations $\delta q^i$ and $\delta p_i$ are independent, their coefficients in the round brackets have to vanish separately. Thus we obtain in this formalism directly Hamilton's equations,

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{1.22}$$

Consider now an observable $O = O(q^i, p_i, t)$. Its time dependence is given by

$$\frac{\mathrm{d}O}{\mathrm{d}t} = \frac{\partial O}{\partial q^i} \dot{q}^i + \frac{\partial O}{\partial p_i} \dot{p}_i + \frac{\partial O}{\partial t} = \frac{\partial O}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial O}{\partial p_i} \frac{\partial H}{\partial q^i} + \frac{\partial O}{\partial t}, \tag{1.23}$$

where we used Hamilton's equations. If we define the Poisson brackets $\{A, B\}$ between two observables $A$ and $B$ as

$$\{A, B\} = \frac{\partial A}{\partial q^i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q^i}, \tag{1.24}$$

then we can rewrite Eq. (1.23) as

$$\frac{\mathrm{d}O}{\mathrm{d}t} = \{O, H\} + \frac{\partial O}{\partial t}. \tag{1.25}$$

This equations gives us a formal correspondence between classical and quantum mechanics. The time evolution of an operator $O$ in the Heisenberg picture is given by the same equation as in classical mechanics, if the Poisson bracket is changed to a commutator. Since the Poisson bracket is antisymmetric, we find

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t}. \tag{1.26}$$

Hence the Hamiltonian $H$ is a conserved quantity, if and only if $H$ is time-independent.

## 1.2 Green functions and the response method

We can test the internal properties of a physical system, if we impose an external force $J(t)$ on it and compare its measured to its calculated response. If the system is described by linear differential equations, then the superposition principle is valid. We can reconstruct the solution $x(t)$ for an arbitrary applied external force $J(t)$, if we know the response to a normalised delta function-like kick $J(t) = \delta(t - t')$. Mathematically, this corresponds to the knowledge of the Green function $G(t - t')$ for the differential

equation $D(t)x(t) = J(t)$ describing the system. Even if the system is described by a non-linear differential equation, we can often use a linear approximation in case of a sufficiently small external force $J(t)$. Therefore the Green function method is extremely useful and we will apply it extensively in discussing quantum field theories.

We illustrate this method with the example of the harmonic oscillator which is the prototype for a quadratic, and thus exactly solvable, action. In classical physics, causality implies that the knowledge of the external force $J(t')$ at times $t' < t$ is sufficient to determine the solution $x(t)$ at time $t$. We define therefore two Green functions $\widetilde{G}$ and $G_R$ by

$$x(t) = \int_{-\infty}^{t} \mathrm{d}t'\, \widetilde{G}(t-t')J(t') = \int_{-\infty}^{\infty} \mathrm{d}t' G_R(t-t')J(t'), \tag{1.27}$$

where the retarded Green function $G_R$ satisfies $G_R(t-t') = \widetilde{G}(t-t')\vartheta(t-t')$. The definition (1.27) is motivated by the trivial relation $J(t) = \int \mathrm{d}t'\, \delta(t-t')J(t')$: an arbitrary force $J(t)$ can be seen as a superposition of delta functions $\delta(t-t')$ with weight $J(t')$. If the Green function $G_R(t-t')$ determines the response of the system to a delta function-like force, then we should obtain the solution $x(t)$ integrating $G_R(t-t')$ with the weight $J(t')$.

We convert the equation of motion $m\ddot{x} + m\omega^2 x = J$ of a forced harmonic oscillator into the form $D(t)x(t) = J(t)$ by writing

$$D(t)x(t) \equiv m\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega^2\right) x(t) = J(t). \tag{1.28}$$

Inserting (1.27) into (1.28) gives

$$\int_{-\infty}^{\infty} \mathrm{d}t'\, D(t)G_R(t-t')J(t') = J(t). \tag{1.29}$$

For an arbitrary external force $J(t)$, this relation can be only valid if

$$D(t)G_R(t-t') = \delta(t-t'). \tag{1.30}$$

Thus a Green function $G(t-t')$ is the inverse of its defining differential operator $D(t)$. As we will see, Eq. (1.30) does not specify uniquely the Green function, and thus we will omit the index "$R$" for the moment. Performing a Fourier transformation,

$$G(t-t') = \int \frac{\mathrm{d}\Omega}{2\pi}\, G(\Omega)\mathrm{e}^{-\mathrm{i}\Omega(t-t')} \quad \text{and} \quad \delta(t-t') = \int \frac{\mathrm{d}\Omega}{2\pi}\, \mathrm{e}^{-\mathrm{i}\Omega(t-t')}, \tag{1.31}$$

we obtain

$$\int \frac{\mathrm{d}\Omega}{2\pi}\, G(\Omega)D(t)\mathrm{e}^{-\mathrm{i}\Omega(t-t')} = \int \frac{\mathrm{d}\Omega}{2\pi}\, \mathrm{e}^{-\mathrm{i}\Omega(t-t')}. \tag{1.32}$$

The action of $D(t)$ on the plane waves $\mathrm{e}^{-\mathrm{i}\Omega(t-t')}$ can be evaluated easily, since the differentiation has become equivalent with multiplication, $\mathrm{d}/\mathrm{d}t \to -\mathrm{i}\Omega$. Comparing

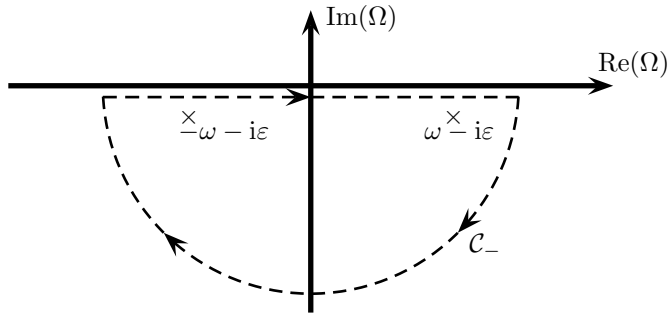


**Fig. 1.1** Poles and contour in the complex Ω plane used for the integration of the retarded Green function.

then the coefficients of the plane waves on both sides of this equation, we have to invert only an algebraic equation, arriving at

$$G(\Omega) = \frac{1}{m}\frac{1}{\omega^2 - \Omega^2}. \tag{1.33}$$

For the back-transformation with $\tau = t - t'$,

$$G(\tau) = \int \frac{\mathrm{d}\Omega}{2\pi m}\,\frac{\mathrm{e}^{-\mathrm{i}\Omega\tau}}{\omega^2 - \Omega^2}, \tag{1.34}$$

we have to specify how the poles at $\Omega^2 = \omega^2$ are avoided. It is this choice by which we select the appropriate Green function. In classical physics, we implement causality ("cause always precedes its effect") selecting the retarded Green function.

We will use Cauchy's residue theorem, $\oint \mathrm{d}z\, f(z) = 2\pi\mathrm{i}\sum \mathrm{res}_{z_0} f(z)$, to calculate the integral. Its application requires to close the integration contour adding a path which gives a vanishing contribution to the integral. This is achieved, when the integrand $G(\Omega)\mathrm{e}^{-\mathrm{i}\Omega\tau}$ vanishes fast enough along the added path. Thus we have to choose for positive $\tau$ the contour $\mathcal{C}_-$ in the lower plane, $\mathrm{e}^{-\mathrm{i}\Omega\tau} = \mathrm{e}^{-|\Im(\Omega)|\tau} \to 0$ for $\Im(\Omega) \to -\infty$, while we have to close the contour in the upper plane for negative $\tau$. If we want to obtain the retarded Green function $G_R(\tau)$ which vanishes for $\tau < 0$, we therefore have to shift the poles $\Omega_{1/2} = \pm\omega$ into the lower plane as shown in Fig. 1.1 by adding a small negative imaginary part, $\Omega_{1/2} \to \Omega_{1/2} = \pm\omega - \mathrm{i}\varepsilon$, or

$$G_R(\tau) = -\frac{1}{2\pi m}\int \mathrm{d}\Omega\,\frac{\mathrm{e}^{-\mathrm{i}\Omega\tau}}{(\Omega - \omega + \mathrm{i}\varepsilon)(\Omega + \omega + \mathrm{i}\varepsilon)}. \tag{1.35}$$

The residue $\mathrm{res}_{z_0} f(z)$ of a function $f$ with a single pole at $z_0$ is given by

$$\mathrm{res}_{z_0}\, f(z) = \lim_{z \to z_0} (z - z_0) f(z). \tag{1.36}$$

Thus we pick up at $\Omega_1 = -\omega - \mathrm{i}\varepsilon$ the contribution $2\pi\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\omega\tau}/(-2\omega)$, while we obtain $2\pi\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\omega\tau}/(2\omega)$ from $\Omega_2 = \omega - \mathrm{i}\varepsilon$. Combining both contributions and adding a minus sign because the contour is clockwise, we arrive at

$$G_R(\tau) = \frac{\mathrm{i}}{2m\omega}\left[\mathrm{e}^{-\mathrm{i}\omega\tau} - \mathrm{e}^{\mathrm{i}\omega\tau}\right]\vartheta(\tau) = \frac{1}{m}\,\frac{\sin(\omega\tau)}{\omega}\,\vartheta(\tau) \tag{1.37}$$

as result for the retarded Green function of the forced harmonic oscillator.

We can now obtain a particular solution solving (1.27). For instance, choosing $J(t') = \delta(t - t')$, results in

$$x(t) = \frac{1}{m}\,\frac{\sin(\omega t)}{\omega}\,\vartheta(t). \tag{1.38}$$

Thus the oscillator was at rest for $t < 0$, got a kick at $t = 0$, and oscillates according $x(t)$ afterwards. Note the following two points: first, the fact that the kick proceeds the movement is the result of our choice of the retarded (or causal) Green function. Second, the particular solution (1.38) for an oscillator initially at rest can be generalised by adding the solution to the homogeneous equation $\ddot{x} + \omega^2\, x = 0$.

## 1.3 Relativistic particle

In special relativity, we replace the Galilean transformations as symmetry group of space and time by Lorentz transformations. The latter are all those coordinate transformations $x^\mu \to \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu$ that keep the squared distance

$$s_{12}^2 \equiv (t_1 - t_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2 \tag{1.39}$$

between two spacetime events $x_1^\mu$ and $x_2^\mu$ invariant. The distance of two infinitesimally close spacetime events is called the line element $\mathrm{d}s$ of the spacetime. In Minkowski space, it is given by

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \tag{1.40}$$

using a Cartesian inertial frame. We can interpret the line element $\mathrm{d}s^2$ as a scalar product, if we introduce the metric tensor $\eta_{\mu\nu}$ with elements

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.41}$$

and a scalar product of two four-vectors as

$$a \cdot b \equiv \eta_{\mu\nu} a^\mu b^\nu = a_\mu b^\mu = a^\mu b_\mu\,. \tag{1.42}$$

In Minkowski space, we call a four-vector any four-tuple $V^\mu$ that transforms as $\tilde{V}^\mu = \Lambda^\mu{}_\nu V^\nu$. By convention, we associate three-vectors with the spatial part of four-vectors with upper indices, for example we set $x^\mu = \{t, x, y, z\}$ or $A^\mu = \{\phi, \boldsymbol{A}\}$. Lowering then the index by contraction with the metric tensor result in a minus sign of the spatial components of a four-vector, $x_\mu = \eta_{\mu\nu} x^\mu = \{t, -x, -y, -z\}$ or $A_\mu = \{\phi, -\boldsymbol{A}\}$. Summing over an index pair, one index occurs in an upper and one in a lower position. Note that in the denominator, an upper index counts as a lower index and vice versa; cf. for example with Eqs. (1.18) and (1.17). Additionally to four-vectors, we will meet tensors $T^{\mu_1 \cdots \mu_n}$ of rank $n$ which transform as $\tilde{T}^{\mu_1 \cdots \mu_n} = \Lambda^{\mu_1}{}_{\nu_1} \cdots \Lambda^{\mu_n}{}_{\nu_n} T^{\nu_1 \cdots \nu_n}$.

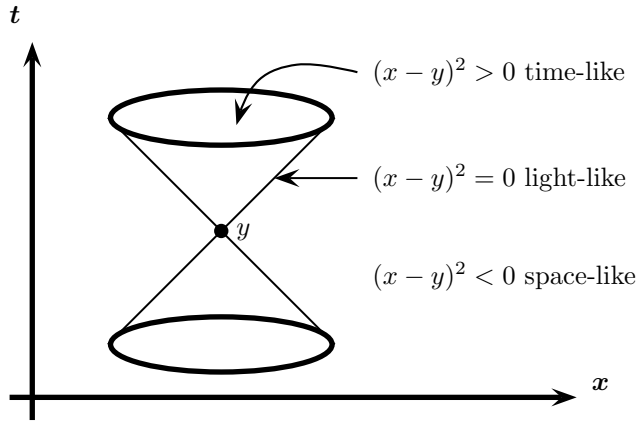


**Fig. 1.2** Light-cone at the point $P(y^\mu)$ generated by light-like vectors. Contained in the light-cone are the time-like vectors, outside the space-like ones.

Since the metric $\eta_{\mu\nu}$ is indefinite, the norm of a vector $a^\mu$ can be

$$a_\mu a^\mu > 0, \quad \text{time-like}, \tag{1.43a}$$
$$a_\mu a^\mu = 0, \quad \text{light-like or null-vector}, \tag{1.43b}$$
$$a_\mu a^\mu < 0, \quad \text{space-like}. \tag{1.43c}$$

The cone of all light-like vectors starting from a point $P$ is called light-cone, cf. Fig. 1.2. The time-like region inside the light-cone consists of two parts, past and future. Only events inside the past light-cone can influence the physics at point $P$, while $P$ can influence only the interior of its future light-cone. The proper-time $\tau$ is the time displayed by a clock moving with the observer. With our conventions—negative signature of the metric and $c = 1$—the proper-time elapsed between two spacetime events equals the integrated line element between them,

$$\tau_{12} = \int_1^2 \mathrm{d}s = \int_1^2 [\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu]^{1/2} = \int_1^2 \mathrm{d}t[1-v^2]^{1/2} < t_2 - t_1. \tag{1.44}$$

The last part of this equation, where we introduced the three-velocity $v^i = \mathrm{d}x^i/\mathrm{d}t$ of the clock, shows explicitly the relativistic effect of time dilation, as well as the connection between coordinate time $t$ and the proper-time $\tau$ of a moving clock, $\mathrm{d}\tau = (1-v^2)^{1/2}\mathrm{d}t \equiv \mathrm{d}t/\gamma$. The line describing the position of an observer is called world-line. Parameterising the world-line by the parameter $\sigma$, $x = x(\sigma)$, the proper-time is given by

$$\tau = \int \mathrm{d}\sigma \left[\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}\right]^{1/2}. \tag{1.45}$$

Note that $\tau$ is invariant under a reparameterisation $\tilde{\sigma} = f(\sigma)$.

The only invariant differential we have at our disposal to form an action for a free point-like particle is the line element, or equivalently the proper-time,

$$S_0 = \alpha \int_a^b \mathrm{d}s = \alpha \int_a^b \mathrm{d}\sigma \, \frac{\mathrm{d}s}{\mathrm{d}\sigma} \tag{1.46}$$

with $L = \alpha \mathrm{d}s/\mathrm{d}\sigma = \alpha \mathrm{d}\tau/\mathrm{d}\sigma$. We check now if this choice which implies the Lagrangian

$$L = \alpha \frac{\mathrm{d}\tau}{\mathrm{d}\sigma} = \alpha \left[ \eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} \right]^{1/2} \tag{1.47}$$

for a free particle is sensible. The action has the correct non-relativistic limit,

$$S_0 = \alpha \int_a^b \mathrm{d}s = \alpha \int_a^b \mathrm{d}t \, \sqrt{1 - v^2} = \int_a^b \mathrm{d}t \, \left( -m + \frac{1}{2} m v^2 + \mathcal{O}(v^4) \right), \tag{1.48}$$

if we set $\alpha = -m$. The mass $m$ corresponds to a potential energy in the non-relativistic limit and has therefore a negative sign in the Lagrangian. Moreover, a constant drops out of the equations of motion, and thus the term $-m$ can be omitted in the non-relativistic limit. The time $t$ enters the relativistic Lagrangian in a Lorentz invariant way as one of the dynamical variables, $x^\mu = (t, \boldsymbol{x})$, while $\sigma$ assumes now $t$'s purpose to parameterise the trajectory, $x^\mu(\sigma)$. Since a moving clock goes slower than a clock at rest, solutions of this Lagrangian maximise the action.

**Example 1.1:** Relativistic dispersion relation. We extend the non-relativistic definition of the momentum, $p_i = \partial L/\partial \dot{x}^i$, to four dimensions setting $p_\alpha = -\partial L/\partial \dot{x}^\alpha$. Note the minus sign that reflects the minus in the spatial components of a covariant vector, $p_\alpha = (E, -\boldsymbol{p})$. Then

$$p_\alpha = -\frac{\partial L}{\partial \dot{x}^\alpha} = m \, \frac{\mathrm{d}x_\alpha/\mathrm{d}\sigma}{\mathrm{d}\tau/\mathrm{d}\sigma} = m \, \frac{\mathrm{d}x_\alpha}{\mathrm{d}\tau} \equiv m u_\alpha. \tag{1.49}$$

In the last step, we defined the four-velocity $u^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\tau$. Using $\mathrm{d}t = \gamma \mathrm{d}\tau$, it follows $u_\alpha u^\alpha = 1$ and $p_\alpha p^\alpha = m^2$. The last relation expresses the relativistic dispersion relation $E^2 = m^2 + \boldsymbol{p}^2$.

The Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} \frac{\partial L}{\partial(\mathrm{d}x^\alpha/\mathrm{d}\sigma)} = \frac{\partial L}{\partial x^\alpha}. \tag{1.50}$$

Consider, for example, the $x^1$ component, then

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} \frac{\partial L}{\partial(\mathrm{d}x^1/\mathrm{d}\sigma)} = \frac{\mathrm{d}}{\mathrm{d}\sigma} \left( \frac{m^2}{L} \frac{\mathrm{d}x^1}{\mathrm{d}\sigma} \right) = 0. \tag{1.51}$$

Since $L = -m \mathrm{d}\tau/\mathrm{d}\sigma$, Newton's law follows for the $x^1$ coordinate after multiplication with $\mathrm{d}\sigma/\mathrm{d}\tau$,

$$\frac{\mathrm{d}^2 x^1}{\mathrm{d}\tau^2} = 0, \tag{1.52}$$

and similar for the other coordinates.

An equivalent but often more convenient form for the Lagrangian of a free particle is

$$L = -m\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu, \tag{1.53}$$

where we set $\dot{x}^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$. If there are no interactions (except gravity), we can neglect the mass $m$ of the particle and one often sets $m \to -1$.

Next we want to add an interaction term $S_{\mathrm{em}}$ between a particle with charge $q$ and an electromagnetic field. The simplest possible action is to integrate the potential $A_\mu$ along the world-line $x^\mu(\sigma)$ of the particle,

$$S_{\mathrm{em}} = -q \int \mathrm{d}x^\mu \, A_\mu(x) = -q \int \mathrm{d}\sigma \, \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \, A_\mu(x). \tag{1.54}$$

Using the choice $\sigma = \tau$, we can view $q\dot{x}^\mu$ as the current $j^\mu$ induced by the particle and thus the interaction has the form $L_{\mathrm{em}} = -j^\mu A_\mu$. Any candidate for $S_{\mathrm{em}}$ should be invariant under a gauge transformation of the potential,

$$A_\mu(x) \to A_\mu(x) - \partial_\mu \Lambda(x). \tag{1.55}$$

This is the case, since the induced change in the action,

$$\delta_\Lambda S_{\mathrm{em}} = q \int_1^2 \mathrm{d}\sigma \, \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \, \frac{\partial \Lambda(x)}{\partial x^\mu} = q \int_1^2 \mathrm{d}\Lambda = q[\Lambda(2) - \Lambda(1)], \tag{1.56}$$

depends only on the endpoints. Thus $\delta_\Lambda S_{\mathrm{em}}$ vanishes keeping the endpoints fixed. Assuming that the Lagrangian is additive,

$$L = L_0 + L_{\mathrm{em}} = -m \left[ \eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} \right]^{1/2} - q \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \, A_\mu(x) \tag{1.57}$$

the Lagrange equations give now

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} \left[ \frac{m \mathrm{d}x_\alpha/\mathrm{d}\sigma}{[\eta_{\mu\nu} \mathrm{d}x^\mu/\mathrm{d}\sigma \, \mathrm{d}x^\nu/\mathrm{d}\sigma]^{1/2}} + q A_\alpha \right] = q \, \frac{\mathrm{d}x^\lambda}{\mathrm{d}\sigma} \, \frac{\partial A_\lambda(x)}{\partial x^\alpha}. \tag{1.58}$$

Performing then the differentiation of $A(x(\sigma))$ with respect to $\sigma$ and moving it to the RHS, we find

$$m \frac{\mathrm{d}}{\mathrm{d}\sigma} \left[ \frac{\mathrm{d}x_\alpha/\mathrm{d}\sigma}{\mathrm{d}\tau/\mathrm{d}\sigma} \right] = q \left( \frac{\mathrm{d}x^\lambda}{\mathrm{d}\sigma} \, \frac{\partial A_\lambda}{\partial x^\alpha} - \frac{\mathrm{d}x^\lambda}{\mathrm{d}\sigma} \, \frac{\partial A_\alpha}{\partial x^\lambda} \right) = q \, \frac{\mathrm{d}x^\lambda}{\mathrm{d}\sigma} \, F_{\alpha\lambda}, \tag{1.59}$$

where we introduced the electromagnetic field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Choosing $\sigma = \tau$ we obtain the covariant version of the Lorentz equation,

$$m \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} = q \, F^\alpha{}_\lambda \, u^\lambda. \tag{1.60}$$

You should work through problem 1.9, if this equation and the covariant formulation of the Maxwell equations are not familiar to you.

## Summary

The Lagrange and Hamilton function are connected by a Legendre transformation, $L(q^i, \dot{q}^i, t) = p_i \dot{q}^i - H(q^i, p_i, t)$. Lagrange's and Hamilton's equations follow extremizing the action $S[q^i] = \int_a^b \mathrm{d}t\; L(q^i, \dot{q}^i, t)$ and $S[q^i, p_i] = \int_a^b \mathrm{d}t\; \left[p_i \dot{q}^i - H(q^i, p_i, t)\right]$, respectively, keeping the endpoints $a$ and $b$ in coordinate space fixed. Knowing the Green function $G(t-t')$ of a linear system, we can find the solution $x(t)$ for an arbitrary external force $J(t)$ by integrating $G(t-t')$ with the weight $J(t)$.


**Further reading.** The series of Landau and Lifshitz on theoretical physics is a timeless resource for everybody studying and working in this field; its Volume 1 (Landau and Lifshitz, 1976) presents a succinct treatment of classical mechanics.

## Problems

*1.1 Units.* ♣ a.) The four fundamental constants $\hbar$ (Planck's constant), $c$ (velocity of light), $G_N$ (gravitational constant) and $k$ (Boltzmann constant) can be combined to obtain the dimension of a length, time, mass, energy and temperature. Find the relations and calculate the numerical values of two of them. What is the physical meaning of these "Planck units"? b.) Find the connection between a cross-section $\sigma$ expressed in units of $\mathrm{cm}^2$, mbarn and $\mathrm{GeV}^{-2}$.

*1.2* $\mathrm{d}\delta = \delta\mathrm{d}$. Use the definition (1.5) to show that variation and differentiation commute, i.e. that "$\mathrm{d}\delta = \delta\mathrm{d}$".

*1.3 Higher derivatives.* a.) Find the Lagrange equation for a Lagrangian containing higher derivatives, $L = L(q, \dot{q}, \ddot{q}, \ldots)$. b.) Consider $L = L(q, \dot{q}, \ddot{q})$ choosing as canonical variables $Q_1 = q$, $Q_2 = \dot{q}$, $P_1 = \frac{\partial L}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}}$ and $P_2 = \frac{\partial L}{\partial \ddot{q}}$ and defining as Hamiltonian $H(Q_1, Q_2, P_1, P_2) = \sum_{i=1}^{2} P_i q^{(i)} - L$. Show that the resulting Hamilton equations give the correct time evolution and that $H$ corresponds to the energy. Why does $H$ describe a unstable system?

*1.4 Oscillator with friction.* Consider a one-dimensional system described by the Lagrangian $L = \exp(2\alpha t) L_0$ and $L_0 = \frac{1}{2} m \dot{q}^2 - V(q)$. a.) Show that the equation of motion corresponds to an oscillator with friction term. b). Derive the energy lost per time $\mathrm{d}E/\mathrm{d}t$ of the oscillator, with $E = \frac{1}{2} m \dot{q}^2 + V(q)$. c.) Show that the result in b.) agrees with the one obtained from the Lagrange equations of the first kind, $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q$, where the generalised force $Q$ perform the work $\delta A = Q \delta q$.

*1.5 Classical driven oscillator.* Consider an harmonic oscillator satisfying $\ddot{q}(t) - \Omega^2 q(t) = 0$ for $0 < t < T$ and $\ddot{q}(t) + \omega^2 q(t) = 0$ otherwise, with $\omega$ and $\Omega$ as real constants. a.) Show that for $q(t) = A_1 \sin(\omega t)$ for $t < 0$ and $\Omega T \gg 1$, the solution $q(t) = A_2 \sin(\omega_0 t + \alpha)$ with $\alpha = \text{const.}$ satisfies $A_2 \approx \frac{1}{2}(1 + \omega^2/\Omega^2)^{1/2} \exp(\Omega T)$. b.) If the oscillator was in the ground-state at $t < 0$, how many quanta are created?

*1.6 Functional derivative.* ♣ We define the derivative of a functional $F[\phi]$ by

$$\int \mathrm{d}x\, \eta(x) \frac{\delta F[\phi]}{\delta \phi(x)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{F[\phi + \varepsilon \eta] - F[\phi]\right\}.$$

a.) Find the functional derivative of $F[\phi] = \int \mathrm{d}x\, \phi(x)$ and show thereby that $\delta\phi(x)/\delta\phi(x') = \delta(x - x')$. b.) Re-derive the Lagrange equations.

*1.7 Conservation laws.* Discuss the symmetries of the Galilean transformations and the resulting conservation laws, following the example of time-translation invariance and energy conservation.

*1.8 Step function.* Heaviside's step function $\vartheta(\tau)$ is defined by $\vartheta(\tau) = 0$ for $\tau < 0$ and $\vartheta(\tau) = 1$ for $\tau > 0$. a.) Use Chauchy's residuum theorem to show that the integral representation

$$\vartheta(\tau) = -\frac{1}{2\pi \mathrm{i}} \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\mathrm{e}^{-\mathrm{i}\omega\tau}}{\omega + \mathrm{i}\varepsilon}$$

is valid. b.) Show that $\mathrm{d}\vartheta(\tau)/d\tau = \delta(\tau)$.

*1.9 Electrodynamics.* Compare Eq. (1.60) to the three-dimensional version of the Lorentz force and derive thereby the elements of the field-strength tensor $F_{\mu\nu}$. Find the Lorentz invariants that can be formed out of $F_{\mu\nu}$ and express them through $\boldsymbol{E}$ and $\boldsymbol{B}$. What is the meaning of the zero component of the Lorentz force?

*1.10 Transformation between inertial frames.* ♣ Consider two inertial frames $K$ and $K'$ with parallel axes at $t = t' = 0$ that are moving with the relative velocity $v$ in the $x$ direction. a.) Show that the linear transformation between the coordinates in $K$ and $K'$ can be written as $t' = At + Bx$, $x' = A(x - vt)$, $y' = y$, and $z' = z$. b.) Show that requiring (1.39) leads to Lorentz transformations. c.) What is the condition leading to Galilean transformations?

*1.11 Relativity of simultaneity.* ♣ Draw a spacetime diagram (in $d = 2$) for two inertial frames connected by a boost with velocity $\beta$. What are the angles between the axes $t$ and $t'$, $x$ and $x'$? Draw lines of constant $t$ and $t'$ and convince yourself that the time order of two space-like events is not invariant.

*1.12 Wave equation for a string.* Consider a string of length $L$, mass density $\rho$ and tension $\kappa$ in one spatial dimension. Denoting its deviation from its equilibrium position $x_0$ with $\phi(x,t) \equiv x(t) - x_0$, write down its kinetic and potential energy (density) and the corresponding action. Derive its equation of motion. [Note: $\phi(x,t)$ depends on $t$ and $x$, and the Lagrange equation for $L(\phi, \partial_t\phi, \partial_x\phi)$ will contain $\mathrm{d}/\mathrm{d}t$ and $\mathrm{d}/\mathrm{d}x$ terms.]

# 2
# Quantum mechanics

The main purpose of this chapter is to introduce Feynman's path integral as an alternative to the standard operator approach to quantum mechanics. Most of our discussion of quantum fields will be based on this approach and thus becoming familiar with this technique using the simpler case of quantum mechanics is of central importance. Instead of employing the path integral directly, we will use as a basic tool the vacuum persistence amplitude $\langle 0, \infty | 0, -\infty \rangle_J$. This quantity is the probability amplitude that a system under the influence of an external force $J$ stays in its ground-state. Since we can apply an arbitrary force $J$, the amplitude $\langle 0, \infty | 0, -\infty \rangle_J$ contains all information about the system. Moreover, it serves as a convenient tool to calculate Green functions which will become our main target studying quantum field theories.

## 2.1 Reminder of the operator approach

A classical system described by a Hamiltonian $H(q^i, p_i, t)$ can be quantised promoting $q^i$ and $p_i$ to operators[1] $\hat{q}^i$ and $\hat{p}_i$ which satisfy the canonical commutation relations $\left[\hat{q}^i, \hat{p}_j\right] = \mathrm{i}\delta^i_j$. The latter are the formal expression of Heisenberg's uncertainty relation. Apart from ordering ambiguities, the Hamilton operator $H(\hat{q}^i, \hat{p}_i, t)$ can be directly read from the Hamiltonian $H(q^i, p_i, t)$. The basic features of any quantum theory can be synthesised into a few principles.

**General principles.** A physical system in a pure state is fully described by a probability amplitude

$$\psi(a, t) = \langle a | \psi(t) \rangle \in \mathbb{C}, \tag{2.1}$$

where $\{a\}$ is a set of quantum numbers specifying the system and the states $|\psi(t)\rangle$ form a complex Hilbert space. The probability $p$ to find the specific values $a_*$ in a measurement is given by $p(a_*) = |\psi(a_*, t)|^2$. The possible values $a_*$ are the eigenvalues of Hermitian operators $\hat{A}$ whose eigenvectors $|a\rangle$ form an orthogonal, complete basis. In Dirac's bra-ket notation, we can express these statements by

$$\hat{A}|a\rangle = a|a\rangle\,, \qquad \langle a|a'\rangle = \delta\left(a - a'\right), \qquad \int \mathrm{d}a\, |a\rangle\langle a| = 1. \tag{2.2}$$

In general, operators do not commute. Their commutation relations can be obtained by the replacement $\{A, B\} \to \mathrm{i}[\hat{A}, \hat{B}]$ in the definition (1.24) of the Poisson brackets.

[1] When there is the danger of an ambiguity, operators will be written with a "hat"; otherwise we drop it.

The state of a particle moving in one dimension in a potential $V(q)$ can be described either by the eigenstates of the position operator $\hat{q}$ or of the momentum operator $\hat{p}$. Both eigenstates form a complete, orthonormal basis, and they are connected by a Fourier transformation which we choose to be asymmetric,

$$\psi(q) = \langle q|\,\psi\rangle = \int \frac{\mathrm{d}p}{2\pi}\,\mathrm{e}^{\mathrm{i}pq}\,\psi(p) = \int \frac{\mathrm{d}p}{2\pi}\,\langle q|\,p\rangle\,\langle p|\,\psi\rangle \tag{2.3a}$$

$$\psi(p) = \langle p|\,\psi\rangle = \int \mathrm{d}q\,\mathrm{e}^{-\mathrm{i}pq}\,\psi(q) = \int \mathrm{d}q\,\langle p|\,q\rangle\,\langle q|\,\psi\rangle. \tag{2.3b}$$

Choosing this normalisation has the advantage that the factor $1/(2\pi)$ in the Fourier integral over momenta is the same as in the density of free states, $L\,\mathrm{d}p/(2\pi)$, which will enter quantities like decay rates or cross-sections. From Eq. (2.3), it follows that the asymmetry in the Fourier transformation is reflected in the completeness relation of the states,

$$\int \mathrm{d}q\,|q\rangle\,\langle q| = 1 \quad \text{and} \quad \int \frac{\mathrm{d}p}{2\pi}\,|p\rangle\,\langle p| = 1. \tag{2.4}$$

**Time evolution.** Since the states $|\psi(t)\rangle$ form a complex Hilbert space, the superposition principle is valid: If $\psi_1$ and $\psi_2$ are possible states of the system, then also

$$\psi(t) = c_1\psi_1(t) + c_2\psi_2(t)\,, \qquad c_i \in \mathbb{C}. \tag{2.5}$$

In quantum mechanics, a stronger version of this principle holds which states that if $\psi_1(t)$ and $\psi_2(t)$ describe the possible time evolution of the system, then so does also the superposed state $\psi(t)$. This implies that the time evolution is described by a linear, homogeneous differential equation. Choosing it as first order in time, we can write the evolution equation as

$$\mathrm{i}\partial_t|\psi(t)\rangle = D|\psi(t)\rangle, \tag{2.6}$$

where the differential operator $D$ on the RHS has to be still determined.

We call the operator describing the evolution of a state from $\psi(t)$ to $\psi(t')$ the time-evolution operator $U(t',t)$. This operator is unitary, $U^{-1} = U^\dagger$, in order to conserve probability and forms a group, $U(t_3,t_1) = U(t_3,t_2)U(t_2,t_1)$ with $U(t,t) = 1$. For an infinitesimal time step $\delta t$,

$$|\psi(t+\delta t)\rangle = U(t+\delta t,t)\,|\psi(t)\rangle\,, \tag{2.7}$$

we can set with $U(t,t) = 1$

$$U(t+\delta t,t) = 1 - \mathrm{i}H\delta t. \tag{2.8}$$

Here we introduced the generator of infinitesimal time-translations $H$. The analogy to classical mechanics suggests that $H$ is the operator version of the classical Hamilton function $H(q,p)$. Inserting Eq. (2.8) into (2.7) results in

$$\frac{|\psi(t+\delta t)\rangle - |\psi(t)\rangle}{\delta t} = -\mathrm{i}H\,|\psi(t)\rangle\,. \tag{2.9}$$

Comparing then Eqs. (2.6) and (2.9) reveals that the operator $D$ on the RHS of Eq. (2.6) coincides with the Hamiltonian $H$. We call a time-evolution equation of this type for arbitrary $H$ Schrödinger equation.

Next we want to determine the connection between $H$ and $U$. Plugging $\psi(t) = U(t,0)\psi(0)$ in the Schrödinger equation gives

$$\left[\mathrm{i}\frac{\partial U(t,0)}{\partial t} - HU(t,0)\right]\psi(0) = 0. \tag{2.10}$$

Since this equation is valid for an arbitrary state $\psi(0)$, we can rewrite it as an operator equation,

$$\mathrm{i}\partial_{t'}U(t',t) = HU(t',t). \tag{2.11}$$

Integrating it, we find as formal solution

$$U(t',t) = 1 - \mathrm{i}\int_t^{t'} \mathrm{d}t''H(t'')U(t'',t) \tag{2.12}$$

or, if $H$ is time-independent,

$$U(t,t') = \exp(-\mathrm{i}H(t-t')). \tag{2.13}$$

Up to now, we have considered the Schrödinger picture where operators are constant and the time evolution is given by the change in the state vectors $|\psi(t)\rangle$. In the Heisenberg picture, the time evolution is driven completely by the one of the operators. States and operators in the two pictures are connected by

$$O_{\mathrm{S}}(t) = U(t,t_0)O_{\mathrm{H}}(t)U^{\dagger}(t,t_0), \tag{2.14a}$$
$$|\psi_{\mathrm{S}}(t)\rangle = U(t,t_0)\,|\psi_{\mathrm{H}}\rangle\,, \tag{2.14b}$$

if they agree at the time $t_0$.

**Propagator.** We insert the solution of $U$ for a time-independent $H$ into $|\psi(t')\rangle = U(t',t)|\psi(t)\rangle$ and multiply from the left with $\langle q'|$,

$$\psi(q',t') = \langle q'|\psi(t')\rangle = \langle q'|\exp[-\mathrm{i}H(t'-t)]|\psi(t)\rangle. \tag{2.15}$$

Then we insert $1 = \int \mathrm{d}^3q|q\rangle\langle q|$,

$$\psi(q',t') = \int \mathrm{d}^3q\,\langle q'|\exp[-\mathrm{i}H(t'-t)]|q\rangle\langle q|\psi(t)\rangle = \int \mathrm{d}^3q\,K(q',t';q,t)\psi(q,t). \tag{2.16}$$

In the last step we introduced the propagator or Green function $K$ in its coordinate representation,

$$K(q',t';q,t) = \langle q'|\exp[-\mathrm{i}H(t'-t)]|q\rangle. \tag{2.17}$$

The Green function $K$ equals the probability amplitude for the propagation between two spacetime points; $K(q',t';q,t)$ is therefore also called more specifically two-point

Green function. We can express the propagator $K$ by the solutions of the Schrödinger equation, $\psi_n(q,t) = \langle q|n(t)\rangle = \langle q|n\rangle \exp(-\mathrm{i}E_n t)$ as

$$\begin{aligned} K(q',t';q,t) &= \sum_{n,n'} \langle q'|n\rangle \underbrace{\langle n|\exp(-\mathrm{i}H(t'-t))|n'\rangle}_{\delta_{n,n'}\exp(-\mathrm{i}E_n(t'-t))} \langle n'|q\rangle \\ &= \sum_n \psi_n(q')\psi_n^*(q)\exp(-\mathrm{i}E_n(t'-t)), \end{aligned} \tag{2.18}$$

where $n$ represents the complete set of quantum numbers specifying the energy eigenvalues of the system. Note that this result is very general and holds for any time-independent Hamiltonian.

Let us compute the propagator of a free particle in one dimension, described by the Hamiltonian $H = p^2/2m$. We write with $\tau = t' - t$

$$\begin{aligned} K(q',t';q,t) &= \langle q'|\,\mathrm{e}^{-\mathrm{i}H\tau}\,|q\rangle = \langle q'|\,\mathrm{e}^{-\mathrm{i}\tau\hat{p}^2/2m}\int\frac{\mathrm{d}p}{2\pi}\,|p\rangle\,\langle p|\,q\rangle \\ &= \int\frac{\mathrm{d}p}{2\pi}\,\mathrm{e}^{-\mathrm{i}\tau p^2/2m}\,\langle q'|\,p\rangle\,\langle p|\,q\rangle = \int\frac{\mathrm{d}p}{2\pi}\,\mathrm{e}^{-\mathrm{i}\tau p^2/2m+\mathrm{i}(q'-q)p}\,, \end{aligned} \tag{2.19}$$

where we used $\langle q'|\,p\rangle = \exp(\mathrm{i}q'p)$ in the last step. The integral is Gaussian if we add an infinitesimal factor $\exp(-\varepsilon p^2)$ to the integrand in order to ensure the convergence of the integral. Thus the physical value of the energy $E = p^2/(2m)$ seen as a complex variable is approached from the negative imaginary plane, $E \to E - \mathrm{i}\varepsilon$. Taking afterwards the limit $\varepsilon \to 0$, we obtain

$$K(q',t';q,t) = \left(\frac{m}{2\pi\mathrm{i}\tau}\right)^{1/2}\mathrm{e}^{\mathrm{i}m(q'-q)^2/2\tau}. \tag{2.20}$$

Knowing the propagator, we can calculate the solution $\psi(t')$ at any time $t'$ for a given initial state $\psi(t)$ via Eq. (2.16).

**Example 2.1:** Calculate the integrals $A = \int \mathrm{d}x \exp(-x^2/2)$, $B = \int \mathrm{d}x \exp(-ax^2/2 + bx)$, and $C = \int \mathrm{d}x \cdots \mathrm{d}x_n \exp(-x^T Ax/2 + J^T x)$ for a symmetric $n \times n$ matrix $A$.
a.) We square the integral and calculate then $A^2$ introducing polar coordinates, $r^2 = x^2 + y^2$,

$$A^2 = \int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\exp(-(x^2+y^2)/2) = 2\pi\int_0^{\infty}\mathrm{d}r\,r\mathrm{e}^{-r^2/2} = 2\pi\int_0^{\infty}\mathrm{d}t\,\mathrm{e}^{-t} = 2\pi\,,$$

where we substituted $t = r^2/2$. Thus the result for the basic Gaussian integral is $A = \sqrt{2\pi}$. All other solvable variants of Gaussian integrals can be reduced to this result.
b.) We complete the square in the exponent,

$$-\frac{a}{2}\left(x^2 - \frac{2b}{a}x\right) = -\frac{a}{2}\left(x - \frac{b}{a}\right)^2 + \frac{b^2}{2a}\,,$$

and shift then the integration variable to $x' = x - b/a$. The result is

$$B = \int_{-\infty}^{\infty} \mathrm{d}x \exp(-ax^2/2 + bx) = \mathrm{e}^{b^2/2a} \int_{-\infty}^{\infty} \mathrm{d}x' \exp(-ax'^2/2) = \sqrt{\frac{2\pi}{a}}\, \mathrm{e}^{b^2/2a}\,. \quad (2.21)$$

c.) We should complete again the square and try $x' = x - A^{-1}J$. With

$$\begin{aligned}(x - A^{-1}J)^T A(x - A^{-1}J) &= x^T Ax - x^T AA^{-1}J - J^T A^{-1}Ax + J^T A^{-1}AA^{-1}J \\ &= x^T Ax - 2J^T x + J^T A^{-1}J\,,\end{aligned}$$

we obtain after shifting the integration vector,

$$C = \exp\left(J^T A^{-1}J/2\right) \int \mathrm{d}x'_1 \cdots \mathrm{d}x'_n \exp\left(-x'^T Ax'/2\right)\,. \quad (2.22)$$

Since the matrix $A$ is symmetric, we can diagonalise $A$ via an orthogonal transformation $D = OAO^T$. This corresponds to a rotation of the integration variables, $y = Ox'$. The Jacobian of this transformation is one, and thus the result is

$$C = \exp(J^T A^{-1}J/2) \prod_{i=1}^{n} \int \mathrm{d}y_i \exp(-a_i y_i^2/2) = \sqrt{\frac{(2\pi)^n}{\det A}}\, \exp\left(\frac{1}{2}J^T A^{-1}J\right). \quad (2.23)$$

In the last step we expressed the product of eigenvalues $a_i$ as the determinant of $A$.

## 2.2 Path integrals in quantum mechanics

In problem 2.1 you are asked to calculate the classical action of a free particle and of a harmonic oscillator and to compare them to the corresponding propagators found in quantum mechanics. Surprisingly, you will find that in both cases the propagator can be written as $K(q', t'; q, t) = N \exp(\mathrm{i}S)$ where $S$ is the classical action along the path $[q(t) : q'(t')]$ and $N$ a normalisation constant. This suggests that we can reformulate quantum mechanics, replacing the standard operator formalism used to evaluate the propagator (2.17) "somehow" by the classical action.

To get an idea how to proceed, we look at the famous double-slit experiment sketched in the left panel of Fig. 2.1: According to the superposition principle, the amplitude $A$ for a particle to move from the source at $q_1$ to the detector at $q_2$ is the sum of the amplitudes $A_i$ for the two possible paths,

$$A = K(q_2, t_2; q_1, t_1) = \sum_{\text{paths}} A_i. \quad (2.24)$$

Clearly, we could add in a gedankenexperiment more and more screens between $q_1$ and $q_2$, increasing at the same time the number of holes. Although in this way we replace continuous spacetime by a discrete lattice, the differences between these two descriptions should vanish for sufficiently small spacing $\tau$. Moreover, for $\tau \to 0$, we can expand $U(\tau) = \exp(-\mathrm{i}H\tau) \simeq 1 - \mathrm{i}H\tau$. Applying then $H = \hat{p}^2/(2m) + V(\hat{q})$ to eigenfunctions $|q\rangle$ of $V(\hat{q})$ and $|p\rangle$ of $\hat{p}^2$, we can replace the operator $H$ by its eigenvalues. In this way, we hope to express the propagator as a sum over paths, where the individual amplitudes $A_i$ contain only classical quantities.

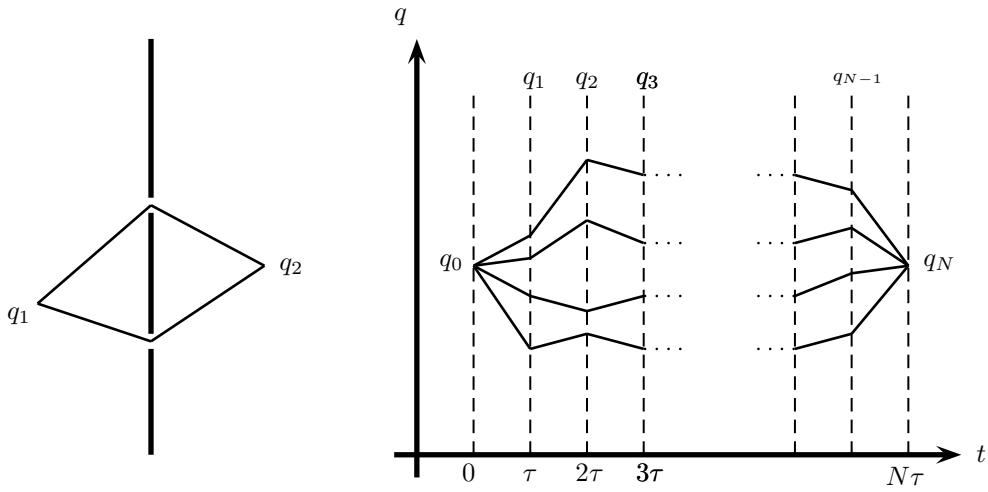


**Fig. 2.1** Left: The double slit experiment. Right: The propagator $K(q_N, \tau; q_0, 0)$ expressed as a sum over all $N$-legged continuous paths.

We apply now this idea to a particle moving in one dimension in a potential $V(q)$. The transition amplitude $A$ for the evolution from the state $|q, 0\rangle$ to the state $|q', t'\rangle$ is

$$A \equiv K(q', t'; q, 0) = \langle q' | \, \mathrm{e}^{-\mathrm{i}Ht'} \, |q\rangle \, . \tag{2.25}$$

This amplitude equals the matrix element of the propagator $K$ for the evolution from the initial point $q(0)$ to the final point $q'(t')$. Let us split the time evolution into two smaller steps, writing $\mathrm{e}^{-\mathrm{i}Ht'} = \mathrm{e}^{-\mathrm{i}H(t'-t_1)}\mathrm{e}^{-\mathrm{i}Ht_1}$. Inserting also $\int \mathrm{d}q_1 \, |q_1\rangle \, \langle q_1| = 1$, the amplitude becomes

$$A = \int \mathrm{d}q_1 \, \langle q' | \, e^{-\mathrm{i}H(t'-t_1)} \, |q_1\rangle \, \langle q_1 | \, \mathrm{e}^{-\mathrm{i}Ht_1} \, |q\rangle = \int \mathrm{d}q_1 \, K(q', t'; q_1, t_1) K(q_1, t_1; q, 0). \tag{2.26}$$

This formula expresses simply the group property, $U(t', 0) = U(t', t_1)U(t_1, 0)$, of the time evolution operator $U$ evaluated in the basis of the continuous variable $q$. More physically, we can view this equation as an expression of the quantum mechanical rule for combining amplitudes. If the same initial and final states can be connected by various ways, the amplitudes for each of these processes should be added. A particle propagating from $q$ to $q'$ must be somewhere at the intermediate time $t_1$. Labelling this intermediate position as $q_1$, we compute the amplitude for propagation via the point $q_1$ as the product of the two propagators in Eq. (2.26) and integrate over all possible intermediate positions $q_1$.

We continue to divide the time interval $t'$ into a large number $N$ of time intervals of duration $\tau = t'/N$. Then the propagator becomes

$$A = \langle q' | \underbrace{\mathrm{e}^{-\mathrm{i}H\tau}\mathrm{e}^{-\mathrm{i}H\tau} \cdots \mathrm{e}^{-\mathrm{i}H\tau}}_{N \text{ times}} |q\rangle \, . \tag{2.27}$$

We insert again a complete set of states $|q_i\rangle$ between each exponential, obtaining

$$\begin{aligned} A &= \int \mathrm{d}q_1 \cdots \mathrm{d}q_{N-1} \, \langle q' | \, \mathrm{e}^{-\mathrm{i}H\tau} \, | q_{N-1} \rangle \, \langle q_{N-1} | \, \mathrm{e}^{-\mathrm{i}H\tau} \, | q_{N-2} \rangle \cdots \langle q_1 | \, \mathrm{e}^{-\mathrm{i}H\tau} \, | q \rangle \\ &\equiv \int \mathrm{d}q_1 \cdots \mathrm{d}q_{N-1} K_{q_N, q_{N-1}} K_{q_{N-1}, q_{N-2}} \cdots K_{q_2, q_1} K_{q_1, q_0}, \end{aligned} \tag{2.28}$$

where we have defined $q_0 = q$ and $q_N = q'$. Note that these initial and final positions are fixed and therefore are not integrated. Figure 2.1 illustrates that we can view the amplitude $A$ as the integral over the partial amplitudes $A_{\mathrm{path}}$ of the individual $N$-legged continuous paths.

We ignore the problem of defining properly the limit $N \to \infty$, keeping $N$ large but finite. We rewrite the amplitude as sum over the amplitudes for all possible paths, $A = \sum_{\mathrm{paths}} A_{\mathrm{path}}$, with

$$\sum_{\mathrm{paths}} = \int dq_1 \cdots dq_{N-1}, \qquad A_{\mathrm{path}} = K_{q_N, q_{N-1}} K_{q_{N-1}, q_{N-2}} \cdots K_{q_2, q_1} K_{q_1, q_0}.$$

Let us look at the last expression in detail. We can expand the exponential in each propagator $K_{q_{j+1}, q_j} = \langle q_{j+1} | \, \mathrm{e}^{-\mathrm{i}H\tau} \, | q_j \rangle$ for a single sub-interval, because $\tau$ is small,

$$\begin{aligned} K_{q_{j+1}, q_j} &= \langle q_{j+1} | \left( 1 - \mathrm{i}H\tau - \frac{1}{2} H^2 \tau^2 + \cdots \right) | q_j \rangle \\ &= \langle q_{j+1} | \, q_j \rangle - \mathrm{i}\tau \, \langle q_{j+1} | \, H \, | q_j \rangle + \mathcal{O}(\tau^2). \end{aligned} \tag{2.29}$$

In the second term of (2.29), we insert a complete set of momentum eigenstates between $H$ and $|q_j\rangle$. This gives

$$\begin{aligned} &- \mathrm{i}\tau \, \langle q_{j+1} | \left( \frac{\hat{p}^2}{2m} + V(\hat{q}) \right) \int \frac{\mathrm{d}p_j}{2\pi} \, | p_j \rangle \, \langle p_j | \, q_j \rangle \\ &= -\mathrm{i}\tau \int \frac{\mathrm{d}p_j}{2\pi} \left( \frac{p_j^2}{2m} + V(q_{j+1}) \right) \langle q_{j+1} | \, p_j \rangle \, \langle p_j | \, q_j \rangle \\ &= -\mathrm{i}\tau \int \frac{\mathrm{d}p_j}{2\pi} \left( \frac{p_j^2}{2m} + V(q_{j+1}) \right) \mathrm{e}^{\mathrm{i}p_j (q_{j+1} - q_j)}. \end{aligned} \tag{2.30}$$

The expression (2.30) is not symmetric in $q_j$ and $q_{j+1}$. The reason for this asymmetry is that we could have inserted the factor 1 either to the right or to the left of the Hamiltonian $H$. In the latter case, we would have obtained $p_{j+1}$ and $V(q_j)$ in (2.30). Since the difference $[V(q_{j+1}) - V(q_j)]\tau \simeq V'(q_j)(q_{j+1} - q_j)\tau \simeq V'(q_j)\dot{q}_j \tau^2$ is of order $\tau^2$, the ordering problem should not matter in the continuum limit which we will take eventually; we set therefore $V(q_{j+1}) \simeq V(q_j)$.

The first term of (2.29) gives a delta function, which we can express as

$$\langle q_{j+1} | \, q_j \rangle = \delta(q_{j+1} - q_j) = \int \frac{\mathrm{d}p_j}{2\pi} \, \mathrm{e}^{\mathrm{i}p_j (q_{j+1} - q_j)}. \tag{2.31}$$

Now we can combine the two terms, obtaining as propagator for the step $q_j \to q_{j+1}$

$$K_{q_{j+1},q_j} = \int \frac{\mathrm{d}p_j}{2\pi}\, \mathrm{e}^{\mathrm{i}p_j(q_{j+1}-q_j)} \left[1 - \mathrm{i}\tau \left(\frac{{p_j}^2}{2m} + V(q_j)\right) + \mathcal{O}(\tau^2)\right]. \tag{2.32}$$

Since we work at $\mathcal{O}(\tau)$, we can exponentiate the factor in the square bracket,

$$1 - \mathrm{i}\tau\ H(p_j, q_j) + \mathcal{O}(\tau^2) = \mathrm{e}^{-\mathrm{i}\tau H(p_j,q_j)}. \tag{2.33}$$

Next we rewrite the exponent in the first factor of Eq. (2.32) using $\dot{q}_j = (q_{j+1} - q_j)/\tau$, such that we can factor out the time-interval $\tau$. The amplitude $A_{\mathrm{path}}$ consists of $N$ such factors. Combining them, we obtain

$$A_{\mathrm{path}} = \left(\prod_{j=0}^{N-1} \int \frac{\mathrm{d}p_j}{2\pi}\right) \exp \mathrm{i}\tau \sum_{j=0}^{N-1} [p_j\dot{q}_j - H(p_j, q_j)]. \tag{2.34}$$

We recognise the argument of the exponential as the discrete approximation of the action $S[q,p]$ in the Palatini form of a path passing through the points $q_0 = q, q_1, \cdots, q_{N-1}, q_N = q'$. The propagator $K = \int \mathrm{d}q_1 \cdots \mathrm{d}q_{N-1} A_{\mathrm{path}}$ becomes then

$$K = \left(\prod_{j=1}^{N-1} \int \mathrm{d}q_j\right) \left(\prod_{j=0}^{N-1} \int \frac{\mathrm{d}p_j}{2\pi}\right) \exp \mathrm{i}\tau \sum_{j=0}^{N-1} [p_j\dot{q}_j - H(p_j, q_j)]. \tag{2.35}$$

For $N \to \infty$, this expression approximates an integral over all functions $p(t)$, $q(t)$ consistent with the boundary conditions $q(0) = q$, $q(t') = q'$. We adopt the notation $\mathcal{D}p\mathcal{D}q$ for the *functional* or *path integral* over all functions $p(t)$ and $q(t)$,

$$K \equiv \int \mathcal{D}p(t)\mathcal{D}q(t)\mathrm{e}^{\mathrm{i}S[q,p]} = \int \mathcal{D}p(t)\mathcal{D}q(t) \exp\left(\mathrm{i}\int_0^{t'} \mathrm{d}t\, (p\dot{q} - H(p,q))\right). \tag{2.36}$$

This result expresses the propagator as a path integral in phase space. It allows us to obtain for any classical system which can be described by a Hamiltonian the corresponding quantum dynamics.

If the Hamiltonian is of the form $H = p^2/2m + V$, as we have assumed[2] in our derivation, we can carry out the quadratic momentum integrals in (2.35). We can rewrite this expression as

$$K = \left(\prod_{j=1}^{N-1} \int \mathrm{d}q_j\right) \exp -\mathrm{i}\tau \sum_{j=0}^{N-1} V(q_j) \left(\prod_{j=0}^{N-1} \int \frac{\mathrm{d}p_j}{2\pi}\right) \exp \mathrm{i}\tau \sum_{j=0}^{N-1} \left(p_j\dot{q}_j - p_j^2/2m\right). \tag{2.37}$$

The $p$ integrals are all uncoupled Gaussians. One such integral gives

$$\int \frac{\mathrm{d}p}{2\pi}\, \mathrm{e}^{\mathrm{i}\tau(p\dot{q} - p^2/2m)} = \sqrt{\frac{m}{2\pi\mathrm{i}\tau}}\, \mathrm{e}^{\mathrm{i}\tau m\dot{q}^2/2}, \tag{2.38}$$

[2]Since we evaluated $\exp(-\mathrm{i}H\tau)$ for infinitesimal $\tau$, the result (2.36) holds also for a time-dependent potential $V(q,t)$.

where we added again an infinitesimal factor $\exp(-\varepsilon p^2)$ to the integrand. Thence the propagator becomes

$$K = \left(\frac{m}{2\pi \mathrm{i}\tau}\right)^{N/2} \left(\prod_{j=1}^{N-1} \int \mathrm{d}q_j\right) \exp \mathrm{i}\tau \sum_{j=0}^{N-1} \left(\frac{m\dot{q}_j^2}{2} - V(q_j)\right). \tag{2.39}$$

The argument of the exponential is again a discrete approximation of the action $S[q]$ of a path passing through the points $q_0 = q, q_1, \cdots, q_{N-1}, q_N = q'$, but now seen as functional of only the coordinate $q$. As earlier, we can write this in a more compact form as

$$K = \langle q_f, t_f | q_i, t_i \rangle = \int \mathcal{D}q(t) \mathrm{e}^{\mathrm{i}S[q]} = \int \mathcal{D}q(t) \exp\left(\mathrm{i} \int_{t_i}^{t_f} \mathrm{d}t\, L(q, \dot{q})\right), \tag{2.40}$$

where the integration includes all paths satisfying the boundary condition $q(t_i) = q_i$ and $q(t_f) = q_f$. This is the main result of this section, and is known as the *path integral in configuration space*. It will serve us as starting point discussing quantum field theories of bosonic fields.

Knowing the path integral and thus the propagator is sufficient to solve scattering problems in quantum mechanics. In a relativistic theory, the particle number during the course of a scattering process is however not fixed, since energy can be converted into matter. In order to prepare us for such more complex problems, in the next section we will generalise the path integral to a *generating functional* for *n-point Green functions*. In this formalism, the usual propagator giving the probability amplitude that a single particle moves from $q_i(t_i)$ to $q_f(t_f)$ becomes the special case of a two-point Green function, while Green functions with $n > 2$ describe processes involving more points. For instance, the four-point Green function will be the essential ingredient to calculate $2 \to 2$ scattering processes in a quantum field theory (QFT). The corresponding generating functional is the quantity which $n$.th derivative returns the $n$-point Green functions.

## 2.3 Generating functional for Green functions

Having re-expressed the transition amplitude $\langle q_f, t_f | q_i, t_i \rangle$ of a quantum mechanical system as a path integral, we first want to generalise this result to the matrix elements of an arbitrary potential $V(q)$ between the states $|q_i, t_i\rangle$ and $|q_f, t_f\rangle$. For all practical purposes, we can assume that we can expand $V(q)$ as a power series in $q$; thus it is sufficient to consider the matrix elements $\langle q_f, t_f | q^m | q_i, t_i \rangle$. In a QFT, the initial and final states are generally free particles which are described mathematically as harmonic oscillators. In this case, we are able to reconstruct all excited states $|n\rangle$ from the ground-state,

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle .$$

Therefore it will be sufficient to study matrix elements between the ground-state $|0\rangle$. With this choice, we are able to extend the integration limit in the path integral (2.40) to $t = \pm\infty$. This will not only simplify its evaluation but also avoid the need to choose a specific inertial frame. As a result, the generating functional will have an obviously Lorentz invariant form in a relativistic theory.

**Time-ordered products of operators and the path integral.** In a first step, we try to include the operator $q^m$ into the transition amplitude $\langle q_f, t_f | q_i, t_i \rangle$. We can reinterpret our result for the path integral as follows,

$$\langle q_f, t_f | \mathbf{1} | q_i, t_i \rangle = \int \mathcal{D}q(t)\, 1\, \mathrm{e}^{\mathrm{i}S[q]}. \tag{2.41}$$

Thus we can see the LHS as matrix element of the unit operator $\mathbf{1}$, while the RHS corresponds to the path-integral average of the classical function $f(q, \dot{q}) = 1$. Now we want to generalise this rather trivial statement to two operators $\hat{A}(t_a)$ and $\hat{B}(t_b)$ given in the Heisenberg picture. In evaluating the unknown function $f$ on the RHS of

$$\int \mathcal{D}q(t)\, A(t_a) B(t_b)\, \mathrm{e}^{\mathrm{i}S[q(t)]} = \langle q_f, t_f |\, f\{A(t_a)B(t_b)\}\, | q_i, t_i \rangle, \tag{2.42}$$

we go back to Eq. (2.28) and insert $\hat{A}(t_a)$ and $\hat{B}(t_b)$ at the correct intermediate times,

$$= \begin{cases} \int \mathrm{d}q_1 \cdots \mathrm{d}q_{N-1} \ldots \langle q_{a+1}, t_{a+1} |\, \hat{A}\, | q_a, t_a \rangle \cdots \langle q_{b+1}, t_{b+1} |\, \hat{B}\, | q_b, t_b \rangle \ldots & \text{for} \quad t_a > t_b, \\ \int \mathrm{d}q_1 \cdots \mathrm{d}q_{N-1} \ldots \langle q_{b+1}, t_{b+1} |\, \hat{B}\, | q_b, t_b \rangle \cdots \langle q_{a+1}, t_{a+1} |\, \hat{A}\, | q_a, t_a \rangle \ldots & \text{for} \quad t_a < t_b. \end{cases} \tag{2.43}$$

Since the time along a classical path increases, the matrix elements of the operators $\hat{A}(t_a)$ and $\hat{B}(t_b)$ are also ordered with time increasing from the right to the left. If we define the *time-ordered product* of two operators as

$$T\{\hat{A}(t_a)\hat{B}(t_b)\} = \hat{A}(t_a)\hat{B}(t_b)\vartheta(t_a - t_b) + \hat{B}(t_b)\hat{A}(t_a)\vartheta(t_b - t_a), \tag{2.44}$$

then the path-integral average of the classical quantities $A(t_a)$ and $B(t_b)$ corresponds to the matrix element of the time-ordered product of these two operators,

$$\langle q_f, t_f | T\{\hat{A}(t_a)\hat{B}(t_b)\} | q_i, t_i \rangle = \int \mathcal{D}q(t)\, A(t_a) B(t_b)\, \mathrm{e}^{\mathrm{i}S[q(t)]}, \tag{2.45}$$

and similar for more than two operators.

**External sources.** Next in our formalism, we want to include the possibility that we can change the state of our system by applying an external driving force or source term $J(t)$. In quantum mechanics, we could imagine, for example, a harmonic oscillator in the ground-state $|0\rangle$, making a transition under the influence of an external force $J$ to the state $|n\rangle$ at the time $t$ and back to the ground-state $|0\rangle$ at the time $t' > t$. Including such transitions, we can mimic the relativistic process of particle creation and annihilation as follows. We identify the vacuum (i.e. the state containing zero real particles) with the ground-state of the quantum mechanical system, and the creation and annihilation of particles with the (de-) excitation of states that have a higher energy than the ground-state by an external source $J$. Schwinger realised that adding a linear coupling to an external source,

$$L \to L + J(t)q(t), \tag{2.46}$$

also leads to an efficient way to calculate the matrix elements of an arbitrary polynomial of operators $q(t_n) \cdots q(t_1)$. If the source $J(t)$ would be a simple number instead of a time-dependent function in the augmented path integral

$$\langle q_f, t_f | q_i, t_i \rangle_J \equiv \int \mathcal{D}q(t) \mathrm{e}^{\mathrm{i} \int_{t_i}^{t_f} \mathrm{d}t (L + Jq)}, \tag{2.47}$$

then we could obtain $\langle q_f, t_f | q^m | q_i, t_i \rangle_J$ simply by differentiating $\langle q_f, t_f | q_i, t_i \rangle_J$ $m$-times with respect to $J$. However, the LHS is a functional of $J(t)$ and thus we need to perform instead functional derivatives with respect to $J(t)$. By analogy with the rules for the differentiation of functions, e.g. $\partial x^l / \partial x^k = \delta^l_k$, we define a functional derivative as

$$\frac{\delta}{\delta J(x)} 1 = 0 \quad \text{and} \qquad \frac{\delta J(x)}{\delta J(x')} = \delta(x - x'). \tag{2.48}$$

Thus we replace for a continuous index the Kronecker delta by a delta function. Moreover, we assume that the Leibniz and the chain rule holds for sufficiently nice functions $J(x)$. As the notation suggest, the variation of a functional defined in Eq. (1.5) is the special case of a *directional* functional derivative, cf. problem 1.6

Now we are able to differentiate $\langle q_f, t_f | q_i, t_i \rangle_J$ with respect to the source $J$. Starting from

$$\frac{\delta}{\delta J(t_1)} \int \mathcal{D}q(t)\, \mathrm{e}^{\mathrm{i} \int_{t_i}^{t_f} \mathrm{d}t J(t) q(t)} = \mathrm{i} \int \mathcal{D}q(t)\, q(t_1) \mathrm{e}^{\mathrm{i} \int_{t_i}^{t_f} \mathrm{d}t J(t) q(t)}, \tag{2.49}$$

we obtain

$$\langle q_f, t_f | T\{\hat{q}(t_1) \cdots \hat{q}(t_n)\} | q_i, t_i \rangle = (-\mathrm{i})^n \frac{\delta^n}{\delta J(t_1) \cdots \delta J(t_n)} \langle q_f, t_f | q_i, t_i \rangle_J \Big|_{J=0}. \tag{2.50}$$

Thus the source $J(t)$ is a convenient tool to obtain the functions $q(t_1) \cdots q(t_n)$ in front of $\exp(\mathrm{i}S)$. Having performed the functional derivatives, we set the source $J(t)$ to zero, coming back to the usual path integral. Physically, the expression (2.50) corresponds to the probability amplitude that a particle moves from $q_i(t_i)$ to $q_f(t_f)$, having the intermediate positions $q(t_1), \ldots, q(t_n)$.

**Vacuum persistence amplitude.** As a last step, we want to eliminate the initial and final states $|q_i, t_i\rangle_J$ and $|q_f, t_f\rangle$ in favour of the ground-state or vacuum, $|0\rangle$. In this way, we convert the transition amplitude $\langle q_f, t_f | q_i, t_i \rangle_J$ into the probability amplitude that a system which was in the ground-state $|0\rangle$ at $t_i \to -\infty$ remains in this state at $t_f \to \infty$ despite the action of the source $J(t)$. Inserting a complete set of energy eigenstates, $1 = \sum_n |n\rangle\langle n|$, into the propagator, we obtain

$$\langle q', t' | q, t \rangle = \sum_n \psi_n(q') \psi_n^*(q) \exp(-\mathrm{i} E_n (t' - t)) \,. \tag{2.51}$$

We can isolate the ground-state $n = 0$ by adding either to the energies $E_n$ or to the time difference $\tau = (t' - t)$ a small negative imaginary part. In this case, all terms are exponentially damped in the limit $\tau \to \infty$, and the ground-state as the state with the smallest energy dominates more and more the sum. Alternatively, we can add a term $+\mathrm{i}\varepsilon q^2$ to the Lagrangian.

---

**Remark 2.1:** Wick rotation and Euclidean action. Instead of adding the infinitesimally small term $\mathrm{i}\varepsilon q^2$ to the Lagrangian, we can do a more drastic change, rotating in the action

the time axis clockwise by 90° in the complex plane. Inserting $t_E = \mathrm{i}t$ into $x_\mu x^\mu$, we see that this procedure called Wick rotation corresponds to the transition from Minkowski to Euclidean space,

$$x^2 = t^2 - \boldsymbol{x}^2 = (-\mathrm{i}t_E)^2 - \boldsymbol{x}^2 = -[t_E^2 + \boldsymbol{x}^2] = -x_E^2 \,.$$

Performing the changes $t = -\mathrm{i}t_E$ and $\mathrm{d}t = -\mathrm{i}\mathrm{d}t_E$ in the action of a particle moving in an one-dimensional potential gives

$$S = -\mathrm{i}\int \mathrm{d}t_E \left(-\frac{1}{2}m\dot{q}_E^2 - V(q)\right) \equiv \mathrm{i}S_E \,. \tag{2.52}$$

Note that the Euclidean action $S_E = T + V$ is bounded from below. The phase factor in the path integral transforms as $\mathrm{e}^{\mathrm{i}S} = \mathrm{e}^{-S_E}$, and thus contributions with large $S_E$ are exponentially damped in the Euclidean path integral.

---

Finally, we have only to connect the results we obtained so far. Adding a coupling to an external source $J(t)$ and a damping factor $+\mathrm{i}\varepsilon q^2$ to the Lagrangian gives us the ground-state or vacuum-persistence amplitude

$$Z[J] \equiv \langle 0,\infty|0,-\infty\rangle_J = \int \mathcal{D}q(t)\, \mathrm{e}^{\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t(L+Jq+\mathrm{i}\varepsilon q^2)} \tag{2.53}$$

in the presence of a classical source $J$. This amplitude is a functional of $J$ which we denote by $Z[J]$. Taking derivatives w.r.t. the external sources $J$, and setting them afterwards to zero, we obtain

$$\left.\frac{\delta^n Z[J]}{\delta J(t_1)\cdots\delta J(t_n)}\right|_{J=0} = \mathrm{i}^n \int \mathcal{D}q(t)\, q(t_1)\cdots q(t_n)\mathrm{e}^{\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t(L+\mathrm{i}\varepsilon q^2)}. \tag{2.54}$$

The RHS corresponds to the path integral in Eqs. (2.45), augmented by the factor $\mathrm{i}\varepsilon q^2$. This factor damps in the limit of large $t$ everything except the ground-state. Thus we found that $Z[J]$ is the generating functional for the vacuum expectation value of the time-ordered product of operators $\hat{q}(t_i)$,

$$(-\mathrm{i})^n \left.\frac{\delta^n Z[J]}{\delta J(t_1)\cdots\delta J(t_n)}\right|_{J=0} = \langle 0,\infty|T\{\hat{q}(t_1)\cdots\hat{q}(t_n)\}|0,-\infty\rangle = G(t_1,\ldots,t_n). \tag{2.55}$$

In the last step, we defined also the $n$-point Green function $G(t_1,\ldots,t_n)$. These functions will be the main building block we will use to perform calculations in quantum field theory, and the formula corresponding to Eq. (2.55) will be our master formula in field theory. For the special case $n = 2$, we will see that the $n$-point Green function coincides up to a phase with the Feynman propagator $K_F(t',t)$ for the system described by $L$: The $\mathrm{i}\varepsilon q^2$ prescription selects from the set of possible propagators (retarded, advanced, ...) the one suggested by Feynman.

## 2.4 Oscillator as a one-dimensional field theory

**Canonical quantisation.** A one-dimensional harmonic oscillator can be viewed as a free quantum field theory in one time and zero space dimensions. In order to exhibit this equivalence clearer, we rescale the usual Lagrangian

$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2, \tag{2.56}$$

where $m$ is the mass of the oscillator and $\omega$ its frequency as

$$\phi(t) \equiv \sqrt{m}x(t). \tag{2.57}$$

We call the variable $\phi(t)$ a "scalar field", and the Lagrangian now reads

$$L(\phi,\dot{\phi}) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\omega^2\phi^2. \tag{2.58}$$

After the rescaling, the kinetic term $\dot{\phi}^2$ has the dimensionless coefficient $1/2$. This choice is standard in field theory and therefore such a field is called "canonically normalised".

We derive the corresponding Hamiltonian, determining first the conjugate momentum $\pi$ as $\pi(t) = \partial L/\partial\dot{\phi} = \dot{\phi}(t)$. Thus the classical Hamiltonian follows as

$$H(\phi,\pi) = \frac{1}{2}\pi^2 + \frac{1}{2}\omega^2\phi^2. \tag{2.59}$$

The transition to quantum mechanics is performed by promoting $\phi$ and $\pi$ to operators which satisfy the canonical commutation relations $[\phi,\pi] = \mathrm{i}$. The harmonic oscillator is solved most efficiently introducing creation and annihilation operators, $a^\dagger$ and $a$. They are defined by

$$\phi = \frac{1}{\sqrt{2\omega}}\left(a^\dagger + a\right) \quad \text{and} \quad \pi = \mathrm{i}\sqrt{\frac{\omega}{2}}\left(a^\dagger - a\right), \tag{2.60}$$

and satisfy $\left[a, a^\dagger\right] = 1$. The Hamiltonian follows as

$$H = \frac{\omega}{2}\left(aa^\dagger + a^\dagger a\right) = \left(a^\dagger a + \frac{1}{2}\right)\omega. \tag{2.61}$$

We interpret $N \equiv a^\dagger a$ as the number operator, counting the number $n$ of quanta with energy $\omega$ in the state $|n\rangle$.

We now work in the Heisenberg picture where operators are time-dependent. The time evolution of the operator $a(t)$ can be found from the Heisenberg equation,

$$\mathrm{i}\frac{\mathrm{d}a}{\mathrm{d}t} = [a,H] = \omega a, \tag{2.62}$$

from which we deduce that

$$a(t) = a(0)\mathrm{e}^{-\mathrm{i}\omega t} = a_0\mathrm{e}^{-\mathrm{i}\omega t}. \tag{2.63}$$

As a consequence, the field operator $\phi(t)$ can be expressed in terms of the creation and annihilation operators as

$$\phi(t) = \frac{1}{\sqrt{2\omega}}\left(a_0\mathrm{e}^{-\mathrm{i}\omega t} + a_0^\dagger\mathrm{e}^{\mathrm{i}\omega t}\right). \tag{2.64}$$

If we look at $\phi(t)$ as a classical variable, then $a_0$ and $a_0^\dagger$ have to satisfy $a_0 = a_0^\dagger \equiv a_0^*$ in order to make $\phi$ real. Thus they are simply the Fourier coefficients of the single

eigenmode $\sin(\omega t)$. This suggests that we can shortcut the quantisation procedure as follows. We write down the field as sum over its eigenmodes $i = 1, \ldots, k$. Then we reinterpret the Fourier coefficients as creation and annihilation operators, requiring $[a_i, a_j^\dagger] = \delta_{ij}$.

**Path-integral approach.** We solve now the same problem, the rescaled Lagrangian (2.58), in the path-integral approach. Using this method, we have argued that it is convenient to include a coupling to an external force $J$. Let us define therefore the effective action $S_{\text{eff}}$ as the sum of the classical action $S$, the coupling to the external force $J$ and a small imaginary part $\mathrm{i}\varepsilon\phi^2$ to make the path integral well-defined,

$$S_{\text{eff}} = S + \int_{-\infty}^{\infty} \mathrm{d}t \left( J\phi + \mathrm{i}\varepsilon\phi^2 \right) = \int_{-\infty}^{\infty} \mathrm{d}t \left[ \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\omega^2\phi^2 + J\phi + \mathrm{i}\varepsilon\phi^2 \right]. \tag{2.65}$$

The function $\mathrm{e}^{\mathrm{i}S_{\text{eff}}}$ is the integrand of the path integral. We start our work by massaging $S_{\text{eff}}$ into a form such that the path integral can be easily performed. The first two terms in $S_{\text{eff}}$ can be viewed as the action of a differential operator $D(t)$ on $\phi(t)$, writing

$$\frac{1}{2}\left( \dot{\phi}^2 - \omega^2\phi^2 \right) = -\frac{1}{2}\phi(t) \left( \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega^2 \right) \phi(t) = \frac{1}{2}\phi(t) D(t) \phi(t). \tag{2.66}$$

Here we performed a partial integration and dropped the boundary term. This is admissible, because boundary terms vanish varying the action.

We can evaluate this operator going to Fourier space,

$$\phi(t) = \int \frac{\mathrm{d}E}{2\pi}\, \mathrm{e}^{-\mathrm{i}Et}\, \phi(E) \quad \text{and} \quad J(t) = \int \frac{\mathrm{d}E}{2\pi}\, \mathrm{e}^{-\mathrm{i}Et}\, J(E). \tag{2.67}$$

To keep the action real, we have to write all bilinear quantities as $\phi(E)\phi^*(E') = \phi(E)\phi(-E')$, etc. Since only the phases depend on time, the time integration gives a factor $2\pi\delta(E - E')$, expressing energy conservation,

$$S_{\text{eff}} = \frac{1}{2} \int \frac{\mathrm{d}E}{2\pi} \left[ \phi(E)(E^2 - \omega^2 + \mathrm{i}\varepsilon)\phi(-E) + J(E)\phi(-E) + J(-E)\phi(E) \right]. \tag{2.68}$$

In the path integral, this expression corresponds to a Gaussian integral of the type (2.21), where we should "complete the square". Shifting the integration variable to

$$\tilde{\phi}(E) = \phi(E) + \frac{J(E)}{E^2 - \omega^2 + \mathrm{i}\varepsilon},$$

we obtain

$$S_{\text{eff}} = \frac{1}{2} \int \frac{\mathrm{d}E}{2\pi} \left[ \tilde{\phi}(E)(E^2 - \omega^2 + \mathrm{i}\varepsilon)\tilde{\phi}(-E) - J(E)\, \frac{1}{E^2 - \omega^2 + \mathrm{i}\varepsilon}\, J(-E) \right]. \tag{2.69}$$

Here we see that the "damping rule" for the path integral makes also the integral over the energy denominator well-defined. The physical interpretation of this way of shifting the poles—which differs from our treatment of the retarded Green function in

the classical case—will be postponed to the next chapter, where we will discuss this issue in detail.

We are now in the position to evaluate the generating functional $Z[J]$. The path-integral measure is invariant under a simple shift of the integration variable, $\mathcal{D}\tilde{\phi} = \mathcal{D}\phi$, and we omit the tilde from now on. Furthermore, the second term in $S_{\text{eff}}$ does not depend on $\phi$ and can be factored out,

$$\begin{aligned} Z[J] = \exp\left(-\frac{\mathrm{i}}{2}\int \frac{\mathrm{d}E}{2\pi} J(E)\,\frac{1}{E^2-\omega^2+\mathrm{i}\varepsilon}\,J(-E)\right) \\ \times \int \mathcal{D}\phi\,\exp\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}E}{2\pi}\,\left[\phi(E)(E^2-\omega^2+\mathrm{i}\varepsilon)\phi(-E)\right]. \end{aligned} \tag{2.70}$$

Setting the external force to zero, $J = 0$, the first factor becomes one and the generating functional $Z[0]$ becomes equal to the path integral in the second line. For $J = 0$, however, the oscillator remains in the ground-state and thus $Z[0] = \langle 0, \infty | 0, -\infty\rangle = 1$. Therefore

$$Z[J] = \exp\left(-\frac{\mathrm{i}}{2}\int \frac{\mathrm{d}E}{2\pi} J(E)\,\frac{1}{E^2-\omega^2+\mathrm{i}\varepsilon}\,J(-E)\right). \tag{2.71}$$

Inserting the Fourier transformed quantities, we arrive at

$$Z[J] = \exp\left(-\frac{\mathrm{i}}{2}\int \mathrm{d}t'\,\mathrm{d}t\; J(t')K_F(t'-t)J(t)\right), \tag{2.72}$$

where we introduced also the Feynman propagator

$$K_F(t-t') = \int \frac{\mathrm{d}E}{2\pi}\,\mathrm{e}^{-\mathrm{i}E(t-t')}\,\frac{1}{E^2-\omega^2+\mathrm{i}\varepsilon}. \tag{2.73}$$

This Green function differs from the retarded propagator $G_R$ defined in Eq. (1.35) by the position of its poles.

The generating functional $Z[J]$ given by (2.72) is in the form most suitable for deriving arbitrary $n$-point Green functions using our master formula (2.55). Thus finding $Z[J]$ for a general quadratic action requires only to determine the inverse of the differential operator $D(t)$, accounting for the right boundary conditions induced by the $\mathrm{i}\varepsilon\phi^2$ term. This inverse is the Feynman propagator or two-point function $K_F(t'-t)$ which we can determine directly solving

$$D(t)K_F(t'-t) = \delta(t'-t). \tag{2.74}$$

Going to Fourier space, we find immediately

$$K_F(E) = \frac{1}{E^2-\omega^2+\mathrm{i}\varepsilon}. \tag{2.75}$$

Hence we can short-cut the calculation of $Z[J]$ by determining the Feynman propagator and using then directly Eqs. (2.71) or (2.72).

These results allow us also to calculate arbitrary matrix elements between oscillator states. For instance, we obtain the expectation value $\langle 0|\,\phi^2\,|0\rangle$ from

$$\langle 0|\,T\{\phi(t')\phi(t)\}\,|0\rangle = (-\mathrm{i})^2 \frac{\delta^2 Z[J]}{\delta J(t')\delta J(t)}\bigg|_{J=0} = \mathrm{i}K_F(t'-t) = \frac{1}{2\omega}\,\mathrm{e}^{\mathrm{i}\omega|t-t'|}. \qquad (2.76)$$

Here, we used in the last step the explicit expression for $K_F$ which you should check in problem 2.6. Taking the limit $t' \searrow t$ and replacing $\phi^2 \to mx^2$, we reproduce the standard result $\langle 0|\,x^2\,|0\rangle = 1/(2m\omega)$. Matrix elements between excited states $|n\rangle = (n!)^{-1/2}(a^\dagger)^n\,|0\rangle$ are obtained by expressing the creation operator $a^\dagger$ using $\pi(t) = \dot{\phi}(t)$ as

$$a^\dagger = \sqrt{\frac{\omega}{2}}\left(1 - \frac{\mathrm{i}}{\omega}\frac{\mathrm{d}}{\mathrm{d}t}\right)\phi(t). \qquad (2.77)$$

## 2.5 The need for quantum fields

We have already argued that any relativistic quantum theory has to be a many-particle theory. Such a theory has to include infinitely many degrees of freedom—as field theories like electrodynamics do. Before we move on to introduce the simplest quantum field theory in the next chapter, we present an argument that relativity and the single particle picture are incompatible.

In classical mechanics, the principle of relativity implies that all trajectories of massive particles are time-like, while massless particles move along light-like trajectories. This implements causality, that is, the requirement that no signal can be transmitted faster than light. How should we translate this principle into a quantum theory? Causality would be clearly satisfied, if the relativistic propagator $K(\boldsymbol{x}', t'; \boldsymbol{x}, t)$ vanishes for space-like distances. Another, less restrictive translation of the principle of relativity would be to ask that measurements performed at space-like separated points do not influence each other. This is achieved if all observables $O(x)$ commute for space-like distances,

$$[\hat{O}(\boldsymbol{x}, t), \hat{O}(\boldsymbol{x}', t')] = 0 \qquad \text{for} \quad (t-t')^2 < (\boldsymbol{x}-\boldsymbol{x}')^2\,. \qquad (2.78)$$

In quantum mechanics, the Heisenberg operators $\hat{\boldsymbol{x}}(t)$ and $\hat{\boldsymbol{p}}(t)$ depend, however, only on time. Therefore we cannot implement the condition (2.78) in such a framework.

The only rescue for causality in relativistic quantum mechanics is therefore the vanishing of the propagator $K(t', \boldsymbol{x}'; t, \boldsymbol{x})$ outside the light-cone. We evaluate the propagator as in the non-relativistic case,

$$K(\boldsymbol{x}', t'; \boldsymbol{x}, t) = \langle \boldsymbol{x}'|\,\mathrm{e}^{-\mathrm{i}H(t'-t)}\,|\boldsymbol{x}\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\,\langle \boldsymbol{x}'|\,\mathrm{e}^{-\mathrm{i}E_{\boldsymbol{p}}(t'-t)}\,|\boldsymbol{p}\rangle\,\langle \boldsymbol{p}|\,\boldsymbol{x}\rangle \qquad (2.79)$$

adapting, however, the relativistic dispersion relation, $E_{\boldsymbol{p}} = \sqrt{m^2 + \boldsymbol{p}^2}$. Next we use that the momentum operator $\hat{\boldsymbol{p}}$ generates space translations, $\exp(-\mathrm{i}\hat{\boldsymbol{p}}\boldsymbol{x})|\boldsymbol{0}\rangle = |\boldsymbol{x}\rangle$, to obtain

$$K(\boldsymbol{x}', t'; \boldsymbol{x}, t) = K(x'-x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\,|\langle \boldsymbol{0}|\boldsymbol{p}\rangle|^2\,\mathrm{e}^{-\mathrm{i}p(x'-x)}. \qquad (2.80)$$

Here we introduced also the four-vector $p^\mu = (E_p, \boldsymbol{p})$, rewriting the plane wave thereby in a Lorentz-invariant way. In order that the complete propagator is invariant, we

have to choose as integration measure $\propto \mathrm{d}^3p/E_{\boldsymbol{p}}$, cf. problem 2.8, and we set therefore $|\langle \mathbf{0}|\boldsymbol{p}\rangle|^2 = 1/(2E_{\boldsymbol{p}})$. Knowing its explicit expression, it is a straight-forward exercise to show that the propagator does not vanish outside the light-cone, but goes only exponentially to zero, $K(\boldsymbol{x}', 0; \boldsymbol{x}, 0) \propto \exp(-m|\boldsymbol{x}' - \boldsymbol{x}|)$. Thus we failed to implement both versions of causality into relativistic quantum mechanics. Instead, we will develop quantum field theory with the aim to implement causality via the condition (2.78).

Before starting this endeavour, we can draw still some important conclusion from Eq. (2.80). For space-like distances, $(x - x')^2 < 0$, a Lorentz boost can change the time order of two spacetime events, cf. problem 1.11. Consistency requires thus to include both time-orderings: if a particle is created at $t$ and absorbed at $t' > t$, then it can be created necessarily also at $t'$ and absorbed at $t > t'$. We extend therefore the propagator as

$$K(x' - x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}} \left[\vartheta(t' - t)\mathrm{e}^{-\mathrm{i}p(x'-x)} + \vartheta(t - t')\mathrm{e}^{\mathrm{i}p(x'-x)}\right], \tag{2.81}$$

where we chose the opposite sign for the plane wave in the second factor. In this way, the phase of the plane waves observed in both frames agree, $-E_{\boldsymbol{p}}\tau\vartheta(\tau) < 0$ and $+E_{\boldsymbol{p}}\tau\vartheta(-\tau) < 0$, and similarly for the momenta. If we imagine that the propagating particle carries a conserved charge, then we can associate the positive frequencies to the propagation of a particle (with charge $q$) and the negative frequencies to the propagation of an antiparticle (with charge $-q$). Then the resulting current is frame-independent, if the antiparticle has the same mass but the opposite additive charges. This prediction of relativistic quantum field theory is experimentally confirmed with extreme accuracy. For instance, the limits on the mass and charge difference of electrons and positrons are smaller than $8 \times 10^{-9}$ and $4 \times 10^{-8}$, respectively. The best experimental limit is currently the relative mass difference of the $K^0$ and $\bar{K}^0$ mesons, which is bounded by $10^{-18}$ (Olive *et al.*, 2014).

Finally, we should mention an alternative way to implement causality. Instead of defining quantum fields $\hat{\phi}(x^\mu)$ on classical spacetime, we could promote time $t$ to an operator, parameterising the world-line $\hat{x}^\mu(\tau)$ of a particle, for example, by its proper-time $\tau$. Considering then the surface $\hat{x}^\mu(\tau, \sigma)$ generated by a set of world-lines is the starting point of string theory.

## Summary

Using Feynman's path-integral approach, we can express a transition amplitude as a sum over all paths weighted by a phase which is determined by the classical action, $\langle q_f, t_f|q_i, t_i\rangle = \int \mathcal{D}q(t) \exp(\mathrm{i}S[q])$. Adding a linear coupling to an external source $J$ and a damping term to the Lagrangian, we obtain the ground-state persistence amplitude $\langle 0, \infty|0, -\infty\rangle_J$. This quantity serves as the generating functional $Z[J]$ for $n$-point Green functions $G(t_1, \ldots, t_n)$ which are the time-ordered vacuum expectation values of the operators $\hat{q}(t_1), \ldots, \hat{q}(t_n)$.

**Further reading.** For additional examples for the use of the path integral and Green functions in quantum mechanics see, for example, MacKenzie (2000) or Das (2006). Schweber (2005) sketches the historical development that lead to Schwinger's Green functions, including his quantum action principle.

## Problems

*2.1 Classical action.* Calculate the classical action $S[q]$ for a free particle and an harmonic oscillator. Compare the results with the expression for the propagator $K = \langle x', t'|x, t\rangle = N \exp(\mathrm{i}\phi)$ of the corresponding quantum mechanical system and express both $\phi$ and $N$ through the action $S$.

*2.2 Propagator as Green function.* Show that the Green function or propagator $K(x', t'; x, t) = \langle x'| \exp[-\mathrm{i}H(t' - t)]|x\rangle$ of the Schrödinger equation is the inverse of the differential operator $(\mathrm{i}\partial_t - H)$.

*2.3 Classical limit.* Sketch (without detailed calculation) why in the path integral the allowed paths dominate in the classical limit.

*2.4 Commutation relations.* Show that the commutation relations for the field, $[\phi, \pi] = \mathrm{i}$, imply $[a, a^\dagger] = 1$. What happens, if we change the normalisation (2.57)?

*2.5 Mode functions.* Consider the generalisation of (2.64) to $\phi(t) = ua_0\mathrm{e}^{-\mathrm{i}\omega t} + u^* a_0^\dagger \mathrm{e}^{\mathrm{i}\omega t}$, where the functions $u(t)$ are called mode functions. a.) Show that the usual commutation relations are valid, if $\Im(u\dot{u}) = 1$. b.) Show that the standard choice $u = 1/\sqrt{2\omega}$ minimises the energy of the ground-state.

*2.6 Feynman propagator.* Find the explicit expression for the Feynman propagator used in (2.76) from a.) its definition as time-ordered product of fields $\phi$, and b.) evaluating (2.73) using Cauchy's theorem.

*2.7 Matrix elements from $Z[J]$.* Evaluate the matrix element $\langle 0|\, \phi^2\, |1\rangle$ of an harmonic oscillator from $Z[J]$.

*2.8 Lorentz invariant integration measure.* ♣ Show that $\mathrm{d}^3k/(2\omega_k)$ is a Lorentz invariant integration measure by a) calculating the Jacobian of a Lorentz transformation, b) showing that

$$\int \mathrm{d}^4k\, \delta(k^2 - m^2)\vartheta(k^0) f(k^0, \boldsymbol{k}) = \\ = \int \frac{\mathrm{d}^3k}{2\omega_k}\, f(\omega_k, \boldsymbol{k}) \tag{2.82}$$

holds for any function $f$.

*2.9 Propagator at large $|\boldsymbol{x}|$.* Show that the propagator $K(\boldsymbol{x}, 0; 0, 0)$ defined in Eq. (2.81) decays exponentially outside the light-cone for $m > 0$. Find the propagator for $m = 0$.

*2.10 Statistical mechanics.* Derive the connection between the partition function $Z = \mathrm{tr}\ \mathrm{e}^{-\beta H}$ of statistical mechanics and the path integral of quantum mechanics in Euclidean time $t_E = -\mathrm{i}t$. (Hint: compare to remark 2.1.)

*2.11 Scattering at short-range potential.* ♣ Consider in $d = 1$ the scattering of modes with large wavelengths $\lambda$ on a short-range potential, $V(x) = 0$ for $|x| > a$ and $\lambda \gg a$. i) Show that the potential $V$ can be approximated by $V(x) = c_0\delta(x) + c_1\delta'(x) + \mathcal{O}(Va^2/\lambda^2)$. ii) Find the transmission and reflection coefficients setting $V(x) = c_0\delta(x)$. Argue that $T \simeq \mathrm{i}p/c_0$ holds for any short-range potential in the limit $p \ll 1/a$. iii) Show that no consistent solution exists setting $c_0 = 0$.

# 3
# Free scalar field

In this chapter we extend the path-integral approach from quantum mechanics to the simplest field theory, containing a single real scalar field $\phi(x)$. Such a field may either represent an elementary particle like the Higgs boson, a bound-state like a scalar meson, or a scalar parameter describing a specific property of a more complex theory. Proceeding similar to our approach in quantum mechanics, we will introduce the generating functional $Z[J] = \langle 0 + |0-\rangle_J$ of $n$-point Green functions as our main tool to calculate the time-ordered vacuum expectation value of a product of fields $\phi(x_1)\cdots\phi(x_n)$. Calculating the vacuum energy of the scalar field, we will encounter for the first time the fact that many calculations in quantum field theories return a formally infinite result. In order to extract sensible predictions, we therefore have to introduce the concepts of regularisation and renormalisation.

## 3.1 Lagrange formalism and path integrals for fields

A field is a map which associates to each spacetime point $x$ a $k$-tuple of values $\phi_a(x)$, $a = 1, \ldots, k$. We require that the fields $\phi_a(x)$ transform under a definite representation of the Poincaré group which is the group combining Lorentz transformations and translations[1]. For massive particles, these representations are labelled by the mass $m$ and the spin $s$ of one-particle states. Thus this condition guarantees that observers in all inertial frames agree what, for example, a spin $1/2$ particle with mass $m$ is. Additionally, particles can be characterised by their transformation properties under internal symmetry groups. These internal symmetries may lead to conserved quantum numbers like the electric charge $q$, by which we can distinguish further various particles types.

Except for a real scalar field $\phi$, a field has several components. Thus we have to generalise Hamilton's principle to a collection of fields $\phi_a(x)$, where the index $a$ includes all internal as well as Lorentz indices. Moreover, the Lagrangian for a field $\phi_a(x)$ will contain not only time but also space derivatives. To ensure Lorentz invariance, we consider a scalar Lagrange density $\mathscr{L}(x)$ that depends as a local function on the fields $\phi_a(x)$ and their derivatives $\partial_\mu\phi_a(x)$. By analogy to $L(q, \dot{q})$, we restrict ourselves to first derivatives. We include no explicit time-dependence, since "everything" should be explained by the fields and their interactions. The Lagrangian $L(\phi_a, \partial_\mu\phi_a)$ is obtained by integrating the density $\mathscr{L}$ over a given space volume $V$. The action $S$ is thus the four-dimensional integral

[1]The basic properties of these two groups are collected in the appendices B.3 and B.4.

$$S[\phi_a] = \int_{t_a}^{t_b} \mathrm{d}t\, L(\phi_a, \partial_\mu \phi_a) = \int_\Omega \mathrm{d}^4x\, \mathscr{L}(\phi_a, \partial_\mu \phi_a) \tag{3.1}$$

with $\Omega = V \times [t_a : t_b]$. A variation $\delta\phi_a(x)$ of the fields leads to a variation of the action,

$$\delta S = \int_\Omega \mathrm{d}^4x \left[ \frac{\partial \mathscr{L}}{\partial \phi_a}\, \delta\phi_a + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)}\, \delta(\partial_\mu \phi_a) \right], \tag{3.2}$$

where we have to sum over field components ($a = 1, \ldots, k$) and the Lorentz index $\mu = 0, \ldots, 3$. The correspondence $q(t) \to \phi(x^\mu)$ implies that the scale factor $\varepsilon$ parameterising the variations $\phi_a(x^\mu, \varepsilon)$ depends not on $x^\mu$. We can therefore eliminate again the variation of the field gradients $\partial_\mu \phi_a$ by a partial integration using Gauss' theorem,

$$\delta S = \int_\Omega \mathrm{d}^4x \left[ \frac{\partial \mathscr{L}}{\partial \phi_a} - \partial_\mu \left( \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \right) \right] \delta\phi_a = 0. \tag{3.3}$$

The surface term vanishes, since we require that the variation is zero on the boundary $\partial\Omega$. Thus the Lagrange equations for the fields $\phi_a$ are

$$\frac{\partial \mathscr{L}}{\partial \phi_a} - \partial_\mu \left( \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \right) = 0. \tag{3.4}$$

If the Lagrange density $\mathscr{L}$ is changed by a four-dimensional divergence, $\delta\mathscr{L} = \partial_\mu K^\mu$, and surface terms can be dropped, the same equations of motion result. Note also that it is often more efficient to perform directly the variation $\delta\phi_a$ in the action $S[\phi_a]$ than to use the Lagrange equations.

The path integral in configuration space becomes now a functional integral over the $k$ fields $\phi_a$,

$$K = \int \mathcal{D}\phi_1 \cdots \mathcal{D}\phi_k\, \mathrm{e}^{\mathrm{i}S[\phi_a]} = \int \mathcal{D}\phi_1 \cdots \mathcal{D}\phi_k\, \mathrm{e}^{\mathrm{i}\int_\Omega \mathrm{d}^4x\, \mathscr{L}(\phi_a, \partial_\mu \phi_a)}. \tag{3.5}$$

A major problem we have to address later is that the $k$ fields $\phi_a$ are often not independent. For instance, in electrodynamics all potentials $A^\mu$ connected by a gauge transformation describe the same physics. This redundancy makes the path integral (3.5) ill-defined. We therefore start with the simplest case of a single, real scalar field $\phi$ where such problems are absent. Moreover, we restrict ourselves in this chapter to a free field without interactions.

## 3.2 Generating functional for a scalar field

**Lagrangian.** The (free) Schrödinger equation $\mathrm{i}\partial_t \psi = H_0 \psi$ can be obtained substituting $\omega \to \mathrm{i}\partial_t$ and $\boldsymbol{k} \to -\mathrm{i}\boldsymbol{\nabla}_x$ into the non-relativistic energy–momentum relation $\omega = \boldsymbol{k}^2/(2m)$. With the same replacements, the relativistic $\omega^2 = m^2 + \boldsymbol{k}^2$ becomes the Klein–Gordon equation

$$(\Box + m^2)\phi = 0 \qquad \text{with} \quad \Box = \eta_{\mu\nu}\partial^\mu\partial^\nu = \partial_\mu\partial^\mu. \tag{3.6}$$

The relativistic energy–momentum relation implies that the solutions to the free Klein–Gordon equation consist of plane waves with positive and negative energies

$\pm\sqrt{\boldsymbol{k}^2 + m^2}$. For the stability of a quantum system it is, however, essential that its energy eigenvalues are bounded from below. Otherwise, we could generate, for example, in a scattering process $\phi + \phi \to n\phi$, an arbitrarily high number of $\phi$ particles with sufficiently low energy, and no stable form of matter could exist. Interpreting the Klein–Gordon equation as a relativistic wave equation for a single particle cannot therefore be fully satisfactory, since the energy of its solutions is not bounded from below.

How do we guess the correct Lagrange density $\mathscr{L}$? Plane waves can be seen as a collection of coupled harmonic oscillators at each spacetime point. The correspondence $\dot{q} \to \partial_\mu \phi$ means that the kinetic field energy is quadratic in the field derivatives. Relativistic invariance implies that the Lagrange density is a scalar, leaving as the only two possible terms containing derivatives

$$\eta_{\mu\nu}(\partial^\mu \phi)(\partial^\nu \phi) \quad \text{and} \quad \phi \Box \phi.$$

Using the action principle to derive the equation of motion, we can however drop boundary terms performing partial integrations. Thus these two terms are equivalent, up to a minus sign. The Klein–Gordon equation $\Box\phi = -m^2\phi$ suggests that the mass term is also quadratic in the field $\phi$. Therefore we try as Lagrange density

$$\mathscr{L} = \frac{1}{2}\eta_{\mu\nu}\left(\partial^\mu \phi\right)\left(\partial^\nu \phi\right) - \frac{1}{2}m^2\phi^2 \equiv \frac{1}{2}\eta_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - \frac{1}{2}m^2\phi^2. \tag{3.7}$$

From now on, we will drop the parenthesis around $\partial^\mu\phi$ and it should be understood from the context that the derivative $\partial^\mu$ acts only on the first field $\phi$. Even shorter alternative notations are $(\partial_\mu\phi)^2$ and the concise $(\partial\phi)^2$. Swapping the indices in the Lagrangian (3.7), we obtain for the second part of the Lagrange equation

$$\frac{\partial}{\partial(\partial_\alpha\phi)}\left(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) = \eta^{\mu\nu}\left(\delta^\alpha_\mu\partial_\nu\phi + \delta^\alpha_\nu\partial_\mu\phi\right) = \eta^{\alpha\nu}\partial_\nu\phi + \eta^{\mu\alpha}\partial_\mu\phi = 2\partial^\alpha\phi. \tag{3.8}$$

Hence the Lagrange equation becomes

$$\frac{\partial\mathscr{L}}{\partial\phi} - \partial_\alpha\left(\frac{\partial\mathscr{L}}{\partial(\partial_\alpha\phi)}\right) = -m^2\phi - \partial_\alpha\partial^\alpha\phi = 0, \tag{3.9}$$

and the Lagrange density (3.7) leads indeed to the Klein–Gordon equation. We can understand the relative sign in the Lagrangian splitting the relativistic kinetic energy into the "proper" kinetic energy $(\partial_t\phi)^2/2$ and the gradient energy density $(\boldsymbol{\nabla}\phi)^2/2$,

$$\mathscr{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2. \tag{3.10}$$

The last two terms correspond to a potential energy and therefore carry the opposite sign of the first one.

Instead of guessing, we can derive the correct Lagrangian $\mathscr{L}$ as follows: we multiply the free field equation for $\phi$ by a variation $\delta\phi$ that vanishes on $\partial\Omega$. Then we integrate

over $\Omega$, perform a partial integration of the kinetic term, use $\partial_\mu \delta = \delta \partial_\mu$, the Leibniz rule and ask that the variation vanishes,

$$A \int_\Omega \mathrm{d}^4x\, \delta\phi\, (\Box + m^2)\phi = A \int_\Omega \mathrm{d}^4x\, \left[ -\delta(\partial_\mu \phi)\partial^\mu \phi + \delta\phi\phi m^2 \right] = \tag{3.11a}$$

$$= A \int_\Omega \mathrm{d}^4x\, \delta \left[ -\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}\phi^2 m^2 \right] = 0. \tag{3.11b}$$

The term in the square brackets agrees with our guess (3.7), taking into account that the source-free field equation fixes the Lagrangian only up to the overall factor $A$. In analogy with a quantum mechanical oscillator, we want the coefficients of the two terms to be $\pm 1/2$ and thus we set $|A| = 1$.

We can determine the correct overall sign of $\mathscr{L}$ by calculating the energy density $\rho$ of the scalar field and requiring that it is bounded from below and stable against small perturbations. We identify the energy density $\rho$ of the scalar field with its Hamiltonian density $\mathscr{H}$, and use the connection between the Lagrangian and the Hamiltonian known from classical mechanics. The transition from a system with a finite number of degrees of freedom to one with an infinite number of degrees of freedom proceeds as follows,

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \qquad \Rightarrow \qquad \pi_a = \frac{\partial \mathscr{L}}{\partial \dot{\phi}_a}, \tag{3.12a}$$

$$H = p_i \dot{q}^i - L \qquad \Rightarrow \qquad \mathscr{H} = \sum_a \pi_a \dot{\phi}_a - \mathscr{L}. \tag{3.12b}$$

The canonically conjugated momentum $\pi$ of a real scalar field is

$$\pi = \frac{\partial \mathscr{L}}{\partial \dot{\phi}} = \dot{\phi}. \tag{3.13}$$

Thus the Hamilton density is

$$\mathscr{H} = \pi\dot{\phi} - \mathscr{L} = \pi^2 - \mathscr{L} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \geq 0 \tag{3.14}$$

and thus obviously positive definite. Moreover, generating fluctuations $\delta\phi$ costs energy and thus the system is stable against small perturbations. Hence the transition from a single-particle interpretation of the Klein–Gordon equation to a field theory has been sufficient to cure the problem of the negative energy solutions.

Note that we could subtract a constant $\rho_0$ from the Lagrangian which would drop out of the equation of motion. From Eq. (3.14) we see that such a constant corresponds to a uniform energy density of empty space. Such a term would act as an additional source of the gravitational field, but would be otherwise unobservable. Next we generalise the Lagrangian by subtracting a polynomial in the fields, $V(\phi)$, subject to the stability constraint discussed above. Hence the potential $V$ should be bounded from below, and we can expand it around its minimum at $\phi \equiv v$,

$$\left.\frac{\mathrm{d}V}{\mathrm{d}\phi}\right|_{\phi=v} = 0, \qquad \left.\frac{\mathrm{d}^2V}{\mathrm{d}\phi^2}\right|_{\phi=v} \equiv m^2 > 0. \tag{3.15}$$

The term $V''(v)$ acts as mass term for the field $\phi$. We will see soon that terms $\phi^n$ with $n \geq 3$ generate interactions between $n$ particles, as expected from the analogy of a quantum field to coupled quantum mechanical oscillators. The field $\phi$ has the non-zero value $\phi = v$ everywhere, if the minimum $v$ of $V(\phi)$ is not at zero, $v \neq 0$. If the value of $V(\phi)$ at the minimum $v$ is not zero, $V(v) \neq 0$, then the non-zero potential implies a non-zero uniform energy density $\rho = V(v)$.

**Generating functional.** Now we move on to the quantum theory of a scalar field, which we define by the path-integral over $\exp \mathrm{i}S[\phi]$. Since the Hamiltonian (3.14) is quadratic in the momentum, we can start directly from the path integral in configuration space. Then the Green functions which encode all information about this theory can be obtained from the generating functional

$$Z[J] = \langle 0+|0-\rangle_J = \mathcal{N} \int \mathcal{D}\phi \exp \mathrm{i} \int \mathrm{d}^4x \, \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + J\phi \right), \tag{3.16}$$

where we appended to the action a linear coupling between the field and an external source. To ensure the convergence of the integral, we included an infinitesimal small imaginary part into the squared mass of the particle, $m^2 \to m^2 - \mathrm{i}\varepsilon$. Moreover, we added a normalisation factor $\mathcal{N}$ which we will have to determine. We start performing an integration by part of the first term, where we exploit the fact that the boundary term vanishes,

$$Z[J] = \mathcal{N} \int \mathcal{D}\phi \exp \mathrm{i} \int \mathrm{d}^4x \, \left( -\frac{1}{2} \phi (\Box + m^2) \phi + J\phi \right). \tag{3.17}$$

The first two terms, $\phi A \phi = -\phi(\Box + m^2)\phi$, are quadratic and symmetric in the field $\phi$,

$$-\frac{1}{2} \int \mathrm{d}^4x \, \phi(x)(\Box_x + m^2)\phi(x) = \frac{1}{2} \int \mathrm{d}^4x \mathrm{d}^4x' \, \phi(x) A(x,x') \delta(x-x') \phi(x'). \tag{3.18}$$

Note that the operator $A$ is local, $A(x) \propto A(x,x')\delta(x-x')$. Since special relativity forbids action at a distance, non-local terms like $\phi(x')A(x,x')\phi(x)$ should not appear in a relativistic Lagrangian.

The expression on the RHS of Eq. (3.18) is the continuous version of the matrix equation $\phi_i A_{ij} \phi_j$. If we discretise continuous spacetime $x^\mu$ into a lattice, we can use Eq. (2.23) to perform the path integral,

$$Z[J] = \mathcal{N} \left( \frac{(2\pi \mathrm{i})^N}{\det[A]} \right)^{1/2} \exp \left( -\frac{1}{2} \, \mathrm{i} \, J A^{-1} J \right) \equiv \mathcal{N} Z[0] \exp(\mathrm{i} W[J]). \tag{3.19}$$

The pre-factor of the exponential function does not depend on $J$ and is thus given by $\mathcal{N}Z[0] = \langle 0+|0-\rangle$. The vacuum should be stable and normalised to one in the absence of sources, $\langle 0+|0-\rangle = 1$. Therefore the proper normalisation of $Z[J]$ implies that $\mathcal{N}^{-1} = Z[0]$. Thus we can omit the normalisation factor, if we normalise the path-integral measure $\mathcal{D}\phi$ such that the Gaussian integral over a free field is one. In the last step of Eq. (3.19), we defined a new functional $\mathrm{i}W[J] \equiv \ln(Z[J])$ that depends only quadratically on the source $J$; therefore it should be easier to handle than $Z[J]$.

Going for $N \to \infty$ back to continuous spacetime, the matrix multiplications become integrations,

$$Z[J] = \exp(\mathrm{i}W[J]) = \exp\left(-\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\mathrm{d}^4x' J(x)A^{-1}(x,x')J(x')\right) \tag{3.20}$$

and

$$W[J] = -\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4x' J(x)A^{-1}(x,x')J(x'). \tag{3.21}$$

**Propagator.** In order to evaluate the functional $W[J]$ we have to find the inverse $\Delta(x,x') \equiv A^{-1}(x,x')$ of the differential operator $A$, defined by

$$-(\Box + m^2)\Delta(x,x') = \delta(x-x'). \tag{3.22}$$

Because of translation invariance, the Green function $\Delta(x,x')$ can depend only on the difference $x-x'$. Therefore it is advantageous to perform a Fourier transformation and to go to momentum space,

$$-\int \frac{\mathrm{d}^4k}{(2\pi)^4}(\Box + m^2)\Delta(k)\mathrm{e}^{-\mathrm{i}k(x-x')} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}k(x-x')}, \tag{3.23}$$

or

$$\Delta_F(k) = \frac{1}{k^2 - m^2 + \mathrm{i}\varepsilon}, \tag{3.24}$$

where the pole at $k^2 = m^2$ is avoided by the $\mathrm{i}\varepsilon$. Thus the $m^2 \to m^2 - \mathrm{i}\varepsilon$ prescription introduced to ensure the convergence of the path integral tells us also how to handle the poles of the Green function. The index $F$ specifies that the propagator $\Delta_F$ is the Green function obtained with the $m^2 - \mathrm{i}\varepsilon$ prescription proposed by Feynman. (Some authors use instead $D_F$ for the propagator of massive bosons and $\Delta_F$ for the propagator of massless bosons.)

Note that the four momentum components $k^\mu$ are independent. Therefore $\Delta_F(k)$ describes the propagation of a *virtual* particle that has—in contrast to a real or external particle—not to be on "mass-shell": in general

$$k_0 \neq \pm\omega_k \equiv \pm\sqrt{\boldsymbol{k}^2 + m^2}.$$

We can evaluate the $k_0$ integral in the coordinate representation of $\Delta_F(x-x')$ explicitly,

$$\Delta_F(x-x') = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{e^{-\mathrm{i}k(x-x')}}{k_0^2 - \boldsymbol{k}^2 - m^2 + \mathrm{i}\varepsilon} \tag{3.25a}$$

$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\int \frac{\mathrm{d}k_0}{2\pi}\frac{\mathrm{e}^{-\mathrm{i}k_0(t-t')}\mathrm{e}^{\mathrm{i}\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')}}{(k_0 - \omega_k + \mathrm{i}\varepsilon)(k_0 + \omega_k - \mathrm{i}\varepsilon)}, \tag{3.25b}$$

using Cauchy's theorem.[2] The integrand has two simple poles at $+\omega_k - \mathrm{i}\varepsilon$ and $-\omega_k + \mathrm{i}\varepsilon$, cf. Fig. 3.1. For negative $\tau = t - t'$, we can close the integration contour $\mathcal{C}_+$ on the

[2]Since $\varepsilon$ is infinitesimal and $\omega_k > 0$, we can set $2\mathrm{i}\omega_k\varepsilon + \varepsilon^2 \to \mathrm{i}\varepsilon$.

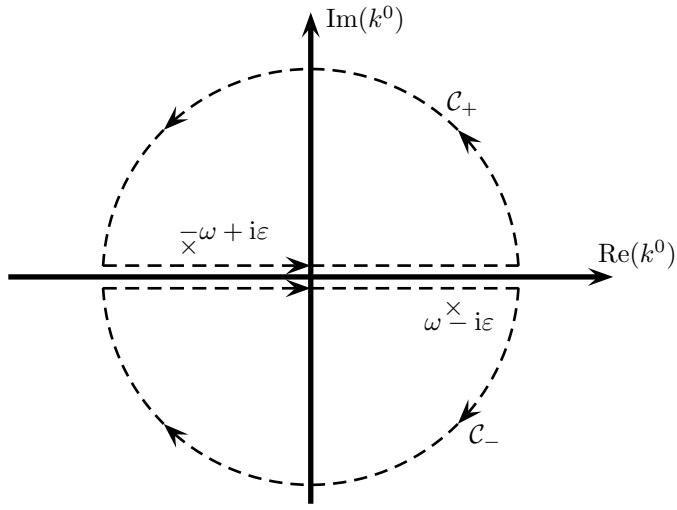


**Fig. 3.1** Poles and contours in the complex $k^0$ plane used for the integration of the Feynman propagator.

upper half-plane, including the pole at $-\omega_k$,

$$\int \mathrm{d}k_0 \frac{\mathrm{e}^{-\mathrm{i}k_0\tau}}{(k_0-\omega_k+\mathrm{i}\varepsilon)(k_0+\omega_k-\mathrm{i}\varepsilon)} = 2\pi\mathrm{i}\ \mathrm{res}_{-\omega_k} = 2\pi\mathrm{i}\ \frac{\mathrm{e}^{\mathrm{i}\omega_k\tau}}{-2\omega_k} \qquad \text{for} \quad \tau<0. \tag{3.26}$$

For positive $\tau$, we have to choose the contour $\mathcal{C}_-$ in the lower plane, picking up $2\pi\mathrm{i}\ \mathrm{e}^{-\mathrm{i}\omega_k\tau}/(2\omega_k)$ and an additional minus sign since the contour is clockwise. Combining both results, we obtain

$$\mathrm{i}\Delta_F(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\left[\mathrm{e}^{-\mathrm{i}\omega_k t}\vartheta(t) + \mathrm{e}^{\mathrm{i}\omega_k t}\vartheta(-t)\right]\mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{x}}, \tag{3.27}$$

or after shifting the integration variable $\boldsymbol{k}\to-\boldsymbol{k}$ in the second term

$$\mathrm{i}\Delta_F(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\left[\mathrm{e}^{-\mathrm{i}(\omega_k t-\boldsymbol{k}\boldsymbol{x})}\vartheta(t) + \mathrm{e}^{\mathrm{i}(\omega_k t-\boldsymbol{k}\boldsymbol{x})}\vartheta(-t)\right]. \tag{3.28}$$

Comparing this expression to our guess (2.81) at the end of the last chapter, we see that our intuitive arguments about the structure of a Lorentz-invariant propagator in a quantum theory were correct. We stress once again the salient features of the Feynman propagator: first, the propagator contains positive and negative frequencies, as expected from the existence of solutions to the Klein–Gordon equation with positive and negative energies. Second, positive frequencies propagate forward in time, while negative frequencies propagate backward. This implies the existence of antiparticles. Third, the relativistic normalisation of (on-shell) plane waves includes a factor $1/\sqrt{2\omega_k}$, or

$$\langle k|k'\rangle = 2\omega_k(2\pi)^3\delta(\boldsymbol{k}-\boldsymbol{k}'), \tag{3.29}$$

while the non-relativistic normalisation uses $\langle\boldsymbol{k}|\boldsymbol{k}'\rangle = \delta(\boldsymbol{k}-\boldsymbol{k}')$.