GEOMETRY AND PHYSICS

A Festschrift in Honour of Nigel Hitchin

VOLUME II

EDITED BY

JØRGEN ELLEGAARD ANDERSEN,

ANDREW DANCER,

OSCAR GARCÍA-PRADA

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Brauer Group of Moduli of Higgs Bundles and Connections

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Abstract. Given a compact Riemann surface X and a semi-simple affine algebraic group G defined over $\mathbb{C}$, there are moduli spaces of Higgs bundles and of connections associated to (X, G). We compute the Brauer group of the smooth locus of these varieties.

To Nigel Hitchin, on the occasion of his seventieth birthday.

1. Introduction

We dedicate this paper to the study of the Brauer group of the moduli spaces of certain Higgs bundles and of holomorphic connections on a Riemann surface. Recall that, given a complex quasiprojective variety Z, its Brauer group $\mathrm{Br}(Z)$ consists of the Morita equivalence classes of Azumaya algebras over Z. This group coincides with the equivalence classes of principal PGL-bundles over Z, where two principal PGL-bundles P and Q are identified if there are vector bundles V and W over Z such that the two principal PGL-bundles $P \otimes \mathbb{P}(V)$ and $Q \otimes \mathbb{P}(W)$ are isomorphic. The cohomological Brauer group $\mathrm{Br}'(Z)$ of the

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variety Z is the torsion part of the étale cohomology group $H^2(Z, \mathbb{G}_m)$. There is a natural injective homomorphism $\mathrm{Br}(Z) \longrightarrow \mathrm{Br}'(Z)$ which is in fact an isomorphism by a theorem of Gabber [dJ], [Ho].

Consider now a compact connected Riemann surface X of genus $g \geq 3$. Given a fixed base point x_0 and two integers $r \geq 2$ and δ, we let $\mathcal{M}_C$ denote the moduli space of all logarithmic connections (E, D) on X, singular over x_0, satisfying the following four natural conditions:

I. E is a holomorphic vector bundle on X of rank r with $\bigwedge^r E = \mathcal{O}_X(\delta x_0)$,
II. the logarithmic connection on $\bigwedge^r E = \mathcal{O}_X(\delta x_0)$ induced by D coincides with the connection on $\mathcal{O}_X(\delta x_0)$ defined by the de Rham differential,
III. the residue of D at x_0 is $-\frac{\delta}{r}\mathrm{Id}_{E_{x_0}}$ and
IV. there is no holomorphic subbundle $F \subset E$ with $1 \leq \mathrm{rank}(F) < r$ such that D preserves F.

This moduli space $\mathcal{M}_C$ has a natural projective bundle once we fix a point of X,

$$\mathbb{P}_C \longrightarrow \mathcal{M}_C,$$

through which in Section 2 we study the Brauer group $\mathrm{Br}(\mathcal{M}_C)$:

Theorem 2.4 *The Brauer group* $\mathrm{Br}(\mathcal{M}_C)$ *is isomorphic to* $\mathbb{Z}/\tau\mathbb{Z}$*, where* $\tau = \mathrm{g.c.d.}(r, \delta)$*. The group* $\mathrm{Br}(\mathcal{M}_C)$ *is generated by the class of* $\mathbb{P}_C$*.*

Fixing the compact connected Riemann surface X and the invariant δ, one can also compute the analytic Brauer group of the $\mathrm{SL}(r, \mathbb{C})$–character variety $\mathcal{R}$ associated to the pair (X, δ).

Theorem 2.5 *The analytic cohomological Brauer group* $\mathrm{Br}'_{\mathrm{an}}(\mathcal{R})$ *is isomorphic to a quotient of the cyclic group* $\mathbb{Z}/\tau\mathbb{Z}$*, where* $\tau = \mathrm{g.c.d.}(r, \delta)$*. The group* $\mathrm{Br}_{\mathrm{an}}(\mathcal{R})$ *is generated by the class of a naturally associated projective bundle* $\mathbb{P}_R$*.*

Over the compact connected Riemann surface X, one may also consider the moduli space $\mathcal{M}_H$ of stable Higgs bundles on X of the form (E, Φ), where

- E is a holomorphic vector bundle of rank r with $\bigwedge^r E = \mathcal{O}_X(\delta x_0)$ and
- Φ is a Higgs field on X with $\mathrm{trace}(\Phi) = 0$.

The moduli space $\mathcal{M}_H$ is a smooth quasiprojective variety which also has a natural projective bundle

$$\mathbb{P}_H \longrightarrow \mathcal{M}_H$$

once we fix a point of X. In Section 2.3, we study the Brauer group of $\mathcal{M}_H$ and prove the following:

Proposition 2.7. *The group* $\mathrm{Br}(\mathcal{M}_H)$ *is isomorphic to the cyclic group* $\mathbb{Z}/\tau\mathbb{Z}$*, and it is generated by the class of* $\mathbb{P}_H$*.*

One should note that, as seen in Section 3, the results of Section 2 extend to the context of principal bundles. We shall conclude this paper by looking at our results in the context of Langlands duality in Section 4.

2. Brauer Group of Some Moduli Spaces

As in the introduction, we shall consider a compact connected Riemann surface X of genus $g \geq 3$ with a fixed base point $x_0 \in X$, and denote by K_X its canonical bundle.

2.1. Brauer Group of Moduli Spaces of Connections

A *logarithmic connection* on X singular over x_0 is a pair of the form (E, D), where E is a holomorphic vector bundle on X and

$$D : E \longrightarrow E \otimes K_X \otimes \mathcal{O}_X(x_0)$$

is a holomorphic differential operator of order 1 satisfying the Leibniz identity

$$D(fs) = f \cdot D(s) + s \otimes (df), \tag{2.1}$$

for all locally defined holomorphic functions f on X and all locally defined holomorphic sections s of E. Note that the fibre $(K_X \otimes \mathcal{O}_X(x_0))_{x_0}$ is canonically identified with $\mathbb{C}$ by sending any $c \in \mathbb{C}$ to the evaluation at x_0 of the locally defined section $c\frac{dz}{z}$ of $K_X \otimes \mathcal{O}_X(x_0)$, where z is any holomorphic function defined around x_0 with $z(x_0) = 0$ and $(dz)(x_0) \neq 0$. Moreover, the evaluation of $\frac{dz}{z}$ at x_0 does not depend on the choice of the function z. Using this identification of $(K_X \otimes \mathcal{O}_X(x_0))_{x_0}$ with $\mathbb{C}$, for any logarithmic connection D as above, we have the linear endomorphism of the fibre E_{x_0} given by the composition

$$E \xrightarrow{D} E \otimes K_X \otimes \mathcal{O}_X(x_0) \longrightarrow (E \otimes K_X \otimes \mathcal{O}_X(x_0))_{x_0} = E_{x_0}.$$

This element of $\mathrm{End}(E_{x_0}) = E_{x_0} \otimes E_{x_0}^*$ is called the *residue* of D (see [De, p. 53]), which we shall denote by $\mathrm{Res}(D, x_0)$. Then, from [Oh, pp. 16–17, Theorem 3], and [De], one has

$$\mathrm{degree}(E) + \mathrm{trace}(\mathrm{Res}(D, x_0)) = 0. \tag{2.2}$$

For notational convenience, we shall let $\mathbb{K} := K_X \otimes \mathcal{O}_X(x_0)$.

Definition 2.1 A logarithmic connection (E, D) as above is called *semi-stable* (respectively, *stable*) if, for every holomorphic subbundle $0 \neq F \subsetneq E$ with $D(F) \subset F \otimes \mathbb{K}$, the following inequality holds:

$$\frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} \leq \frac{\mathrm{degree}(E)}{\mathrm{rank}(E)} \quad \left(\text{respectively, } \frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} < \frac{\mathrm{degree}(E)}{\mathrm{rank}(E)}\right).$$

As done in Section 1, fix two integers $r \geq 2$ and δ and, if $g = 3$, then assume that $r > 2$. The holomorphic line bundle $\mathcal{O}_X(\delta x_0)$ on X is equipped with the logarithmic connection given by the de Rham differential d. This logarithmic connection on $\mathcal{O}_X(\delta x_0)$ will be denoted by D_0. From (2.2) it follows that the residue of D_0 is $-\delta$.

In views of the notation introduced in Section 1, let $\overline{\mathcal{M}}_C$ denote the moduli space of logarithmic connections (E, D) on X, singular over x_0, satisfying the following three conditions:

I. E is a holomorphic vector bundle on X of rank r with $\bigwedge^r E = \mathcal{O}_X(\delta x_0)$,
II. the logarithmic connection on $\bigwedge^r E = \mathcal{O}_X(\delta x_0)$ induced by D coincides with D_0 defined above and
III. $\mathrm{Res}(D, x_0) = -\frac{\delta}{r}\mathrm{Id}_{E_{x_0}}$.

Note that, from (2.2), the above condition on $\mathrm{Res}(D, x_0)$ implies that (E, D) is automatically semi-stable. Moreover, if δ is coprime to r, then (E, D) is, in fact, stable. Since the residue $\mathrm{Res}(D, x_0)$ is a constant multiple of $\mathrm{Id}_{E_{x_0}}$, the logarithmic connection on the projective bundle $\mathbb{P}(E)$ induced by D is actually regular at x_0.

The above defined moduli space $\overline{\mathcal{M}}_C$ is a quasiprojective irreducible normal variety, defined over $\mathbb{C}$, of dimension $2(r^2-1)(g-1)$. Let

$$\mathcal{M}_C \subset \overline{\mathcal{M}}_C \tag{2.3}$$

be the Zariski open subset parametrizing the stable logarithmic connections. We note that $\mathcal{M}_C$ is contained in the smooth locus of $\overline{\mathcal{M}}_C$ (in fact, $\mathcal{M}_C$ is the smooth locus of the space $\overline{\mathcal{M}}_C$). We shall denote by $\mathrm{Br}(\mathcal{M}_C)$ the Brauer group of the smooth variety $\mathcal{M}_C$ which, as mentioned in Section 1, consists of the Morita equivalence classes of Azumaya algebras over $\mathcal{M}_C$. The reader should refer to [Gr1], [Gr2], [Gr3], [Mi] for the definition as well as some properties of the Brauer group.

For any $(E, D) \in \mathcal{M}_C$, consider any $T \in H^0(X, \mathrm{End}(E))$ which is flat with respect to the connection on $\mathrm{End}(E)$ induced by D or, equivalently, such that $D \circ T = (T \otimes \mathrm{Id}_{\mathbb{K}}) \circ D$. Then, for any $c \in \mathbb{C}$, the kernel of $T - c \cdot \mathrm{Id}_E \in H^0(X, \mathrm{End}(E))$ is preserved by D. Since $\mathrm{kernel}(T - c \cdot \mathrm{Id}_E)$ is either E or 0, it follows that either $T = c \cdot \mathrm{Id}_E$ or the endomorphism $T - c \cdot \mathrm{Id}_E$ is invertible. Now, taking c to be an eigenvalue of $T(x_0)$, it follows that $T = c \cdot \mathrm{Id}_E$. Consequently, there is a universal projective bundle

$$\widetilde{\mathbb{P}} \longrightarrow X \times \mathcal{M}_C \tag{2.4}$$

of relative dimension $(r-1)$ which is equipped with a relative holomorphic connection in the direction of X.

Definition 2.2 Given a point $x \in X$, let

$$\mathbb{P} := \widetilde{\mathbb{P}}|_{\{x\}\times\mathcal{M}_C} \longrightarrow \mathcal{M}_C \tag{2.5}$$

be the projective bundle obtained by restricting $\widetilde{\mathbb{P}}$, and denote its class by

$$\beta \in \mathrm{Br}(\mathcal{M}_C). \tag{2.6}$$

In order to study the Brauer group $\mathrm{Br}(\mathcal{M}_C)$, we shall first prove the following.

Lemma 2.3 *Let Y be a simply connected smooth quasiprojective variety defined over the complex numbers, W an algebraic vector bundle on Y and*

$$q : \mathcal{W} \longrightarrow Y$$

a torsor on Y for W. Then the pullback homomorphism

$$q^* : \mathrm{Br}(Y) \longrightarrow \mathrm{Br}(\mathcal{W})$$

is an isomorphism.

Proof Let $c \in H^1(Y, W)$ be the class of $\mathcal{W}$. Consider the extension of $\mathcal{O}_Y$ by W,

$$0 \longrightarrow W \longrightarrow \widehat{W} \xrightarrow{\xi} \mathcal{O}_Y \longrightarrow 0, \tag{2.7}$$

associated to the cohomology class c. We shall denote by 1_Y the image of the section of $\mathcal{O}_Y$ defined by the constant function 1 on Y. Then, the inverse image $\xi^{-1}(1_Y) \subset \widehat{W}$ under the projection ξ in (2.7) is a torsor on Y for the vector bundle W. This W torsor is isomorphic to the W torsor $\mathcal{W}$.

Let

$$\widehat{\eta} : P(\widehat{W}) \longrightarrow Y \quad \text{and} \quad \eta : P(W) \longrightarrow Y$$

be the projective bundles on Y parametrizing the lines in the fibres of $\widehat{W}$ and W, respectively. Note that $P(W) \subset P(\widehat{W})$, and

$$\mathcal{W} = \xi^{-1}(1_Y) = P(\widehat{W}) \setminus P(W),$$

by sending any element of $\xi^{-1}(1_Y)$ to the line in $\widehat{W}$ generated by it. Now, from [Fo, p. 365, Lemma 0.1] and [Fo, p. 367, Theorem 1.1], we conclude that there is an exact sequence

$$0 \longrightarrow \mathrm{Br}(P(\widehat{W})) \longrightarrow \mathrm{Br}(\mathcal{W}) \longrightarrow H^1(P(W), \mathbb{Q}/\mathbb{Z}) \longrightarrow \cdots . \tag{2.8}$$

Consider the long exact sequence of homotopy groups for the fibre bundle η. The fibres of $P(W)$ are projective spaces and hence are simply connected. Since Y is also simply connected, from the homotopy exact sequence for η it follows that $P(W)$ is simply connected as well. Hence, $H_1(P(W), \mathbb{Z}) = 0$, which implies that $H^1(P(W), \mathbb{Q}/\mathbb{Z}) = 0$ (universal coefficient theorem for cohomology; see [Ha, p. 195, Theorem 3.2]). Consequently, using (2.8), we conclude that

$$\mathrm{Br}(P(\widehat{W})) = \mathrm{Br}(\mathcal{W}), \tag{2.9}$$

with the isomorphism being induced by the inclusion of $\mathcal{W}$ in $P(\widehat{W})$.

The homomorphism $\widehat{\eta}^* : \mathrm{Br}(Y) \longrightarrow \mathrm{Br}(P(\widehat{W})$ induced by $\widehat{\eta}$ is an isomorphism [Ga, p. 193, Theorem 2], and the lemma follows from (2.9). □

Theorem 2.4 *The Brauer group* $\mathrm{Br}(\mathcal{M}_C)$ *is isomorphic to the cyclic group* $\mathbb{Z}/\tau\mathbb{Z}$*, where* $\tau =$ g.c.d.(r, δ)*, and it is generated by the element* β *in* (2.6).

Proof Let $\mathcal{N}$ denote the moduli space of stable vector bundles V on X of rank r such that $\bigwedge^r V = \mathcal{O}_X(\delta x_0)$, which is a smooth quasiprojective irreducible complex variety of dimension $(r^2-1)(g-1)$. Moreover, let $\mathcal{U} \subset \mathcal{M}_C$ be the locus of all (E, D) such that the underlying holomorphic vector bundle E is stable. Considering

$$p : \mathcal{U} \longrightarrow \mathcal{N}, \tag{2.10}$$

the forgetful morphism that sends any (E, D) to E, from the openness of the stability condition (see [Ma, p. 635, Theorem 2.8(B)]) it follows that $\mathcal{U}$ is a Zariski open subset of $\mathcal{M}_C$. Note that p is surjective because from [NS] one has that any $V \in \mathcal{N}$ admits a unique logarithmic connection D_V singular at x_0 such that

(a) the residue of D_V at x_0 is $-\frac{\delta}{r}\mathrm{Id}_{V_{x_0}}$ and
(b) the monodromy of D_V lies in $\mathrm{SU}(r)$.

Moreover, a pair (V, D_V) as above lies in $\mathcal{U}$. In fact, if D' is a logarithmic connection on V singular at x_0 such that $\mathrm{Res}(D', x_0) = -\frac{\delta}{r}\mathrm{Id}_{V_{x_0}}$, and the logarithmic connection on $\bigwedge^r V = \mathcal{O}_X(\delta x_0)$ induced by D' coincides with D_0, then clearly $(V, D') \in \mathcal{U}$. The space of all logarithmic connections D' on V satisfying the conditions (a) and (b) is an affine space for the vector space $H^0(X, \mathrm{ad}(V) \otimes K_X)$, where $\mathrm{ad}(V) \subset \mathrm{End}(V)$ is the subbundle of co-rank 1 defined by the sheaf of endomorphisms of trace zero. Furthermore, $H^0(X, \mathrm{ad}(V) \otimes K_X)$ is the fibre of the cotangent bundle $\Omega^1_{\mathcal{N}}$ over the point $V \in \mathcal{N}$. Therefore, the morphism p in (2.10) makes $\mathcal{U}$ a torsor over $\mathcal{N}$ for the vector bundle $\Omega^1_{\mathcal{N}}$.

From [BM1, p. 301, Lemma 3.1] and [BM2, Lemma 3.1] the complement $\mathcal{M}_C \setminus \mathcal{U} \subset \mathcal{M}_C$ is of codimension at least 2 and thus the inclusion map $\iota : \mathcal{U} \hookrightarrow \mathcal{M}_C$ produces an isomorphism of Brauer groups

$$\iota^* : \mathrm{Br}(\mathcal{M}_C) \xrightarrow{\sim} \mathrm{Br}(\mathcal{U}); \tag{2.11}$$

this follows from 'Cohomological purity' [Mi, p. 241, Theorem VI.5.1] (it also follows from [Gr2, p. 292–293]). Since p in (2.10) is a torsor on $\mathcal{U}$ for a vector bundle, and $\mathcal{U}$ is simply connected [BBGN, p. 266, Proposition 1.2(b)], from Lemma 2.3 it follows that the map p induces an isomorphism

$$p^* : \mathrm{Br}(\mathcal{N}) \xrightarrow{\sim} \mathrm{Br}(\mathcal{U}).$$

Combining this with (2.11) we get an isomorphism

$$(\iota^*)^{-1} \circ p^* : \mathrm{Br}(\mathcal{N}) \xrightarrow{\sim} \mathrm{Br}(\mathcal{M}_C). \tag{2.12}$$

We know that $\mathrm{Br}(\mathcal{N})$ is cyclic of order $\tau =$ g.c.d.(r, δ) [BBGN, p. 267, Theorem 1.8]. Therefore, from (2.12) we conclude that $\mathrm{Br}(\mathcal{M}_C)$ is isomorphic to $\mathbb{Z}/\tau\mathbb{Z}$.

Finally, in order to find a generator of $\mathrm{Br}(\mathcal{M}_C)$, let $\widehat{\mathbb{P}} \longrightarrow X \times \mathcal{N}$ be the universal projective bundle and

$$\mathbb{P}' := \widehat{\mathbb{P}}|_{\{x\}\times\mathcal{N}} \longrightarrow \mathcal{N}$$

be the projective bundle obtained by restricting $\widehat{\mathbb{P}}$, where x is the point of X in (2.5). The Brauer group $\mathrm{Br}(\mathcal{N})$ is generated by the class of $\mathbb{P}'$ [BBGN, p. 267, Theorem 1.8]. The pulled back projective bundle $(\mathrm{Id}_X \times p)^*\widehat{\mathbb{P}}$ is identified with the restriction $\widetilde{\mathbb{P}}|_{X\times\mathcal{U}}$, where $\widetilde{\mathbb{P}}$ is the projective bundle in (2.4). This implies that $p^*\mathbb{P}'$ is identified with the restriction $\mathbb{P}|_{\mathcal{U}}$. Since the class of $\mathbb{P}'$ generates $\mathrm{Br}(\mathcal{N})$, from the isomorphism $(\iota^*)^{-1} \circ p^*$ in (2.12) it follows that the class of $\mathbb{P}$ generates $\mathrm{Br}(\mathcal{M}_C)$. □

2.2. *Analytic Brauer Group of Representation Spaces*

Consider now the free group Γ generated by $2g$ elements $\{a_i, b_i\}_{i=1}^{g}$, and let

$$\gamma := \prod_{i=1}^{g}[a_i, b_i] = \prod_{i=1}^{g}(a_i b_i a_i^{-1} b_i^{-1}) \in \Gamma \tag{2.13}$$

be the product of the commutators. Then, one may consider the space of all homomorphisms $\mathrm{Hom}(\Gamma, \mathrm{SL}(r,\mathbb{C}))$ from the group Γ to $\mathrm{SL}(r,\mathbb{C})$. Let

$$\mathrm{Hom}^{\delta}(\Gamma, \mathrm{SL}(r,\mathbb{C})) \subset \mathrm{Hom}(\Gamma, \mathrm{SL}(r,\mathbb{C}))$$

be all such homomorphisms ρ satisfying the condition that

$$\rho(\gamma) = \exp(2\pi\sqrt{-1}\delta/r)\cdot I_{r\times r},$$

where $I_{r\times r}$ is the $r \times r$ identity matrix. The conjugation action of $\mathrm{SL}(r,\mathbb{C})$ on itself produces an action of $\mathrm{SL}(r,\mathbb{C})$ on the variety $\mathrm{Hom}(\Gamma, \mathrm{SL}(r,\mathbb{C}))$. Moreover, this action of $\mathrm{SL}(r,\mathbb{C})$ on $\mathrm{Hom}(\Gamma, \mathrm{SL}(r,\mathbb{C}))$ preserves the above subvariety $\mathrm{Hom}^{\delta}(\Gamma, \mathrm{SL}(r,\mathbb{C}))$. We shall denote by $\overline{\mathcal{R}}$ the geometric invariant theoretic quotient

$$\overline{\mathcal{R}} := \mathrm{Hom}^{\delta}(\Gamma, \mathrm{SL}(r,\mathbb{C}))//\mathrm{SL}(r,\mathbb{C}),$$

which parametrizes all the closed orbits of $\mathrm{SL}(r,\mathbb{C})$ in $\mathrm{Hom}^{\delta}(\Gamma, \mathrm{SL}(r,\mathbb{C}))$.

The moduli space $\overline{\mathcal{M}}_C$ defined in Section 2.1 is biholomorphic to $\overline{\mathcal{R}}$. After fixing a presentation of $\pi_1(X \setminus \{x_0\}, x)$, we have a map $\overline{\mathcal{M}}_C \longrightarrow \overline{\mathcal{R}}$ that sends a flat connection to its monodromy representation, and which is a biholomorphism. Indeed, it is the inverse of the map that associates a flat bundle on $X \setminus \{x_0\}$ of rank r to a representation of $\pi_1(X \setminus \{x_0\}, x)$. Note that although both $\overline{\mathcal{M}}_C$ and $\overline{\mathcal{R}}$ have natural algebraic structures, the above biholomorphism between them is not an algebraic map.

A homomorphism $\rho : \Gamma \longrightarrow \mathrm{SL}(r,\mathbb{C})$ is called *reducible* if $\rho(\Gamma)$ is contained in some proper parabolic subgroup of $\mathrm{SL}(r,\mathbb{C})$, equivalently, if $\rho(\Gamma)$ preserves some proper non-zero subspace of $\mathbb{C}^r$ under the standard action of $\mathrm{SL}(r,\mathbb{C})$. If ρ is not reducible, then it is called *irreducible*.

We shall denote by

$$\mathcal{R} \subset \overline{\mathcal{R}}$$

the locus of irreducible homomorphisms in $\overline{\mathcal{R}}$. It is easy to see that $\mathcal{R}$ is contained in the smooth locus of $\overline{\mathcal{R}}$ (in fact, $\mathcal{R}$ coincides with the smooth locus of $\overline{\mathcal{R}}$). From the definitions of $\mathcal{M}_C$ and $\mathcal{R}$, it follows immediately that the above biholomorphism $\overline{\mathcal{M}_C} \xrightarrow{\sim} \overline{\mathcal{R}}$ produces a biholomorphism

$$\mathcal{M}_C \xrightarrow{\sim} \mathcal{R}. \tag{2.14}$$

Let $\mathcal{H} \subset \mathrm{Hom}^{\delta}(\Gamma, \mathrm{SL}(r,\mathbb{C}))$ be the inverse image of $\mathcal{R}$; in other words, $\mathcal{H}$ is the locus of all elements of $\mathrm{Hom}^{\delta}(\Gamma, \mathrm{SL}(r,\mathbb{C}))$ that are irreducible homomorphisms. The quotient map

$$\mathcal{H} \longrightarrow \mathcal{H}//\mathrm{SL}(r,\mathbb{C}) = \mathcal{R}$$

makes $\mathcal{H}$ an algebraic principal $\mathrm{PSL}(r,\mathbb{C})$-bundle over $\mathcal{R}$. We shall denote by

$$\mathbb{P}_R := \mathcal{H} \times^{\mathrm{PSL}(r,\mathbb{C})} \mathbb{CP}^{r-1} \longrightarrow \mathcal{R} \tag{2.15}$$

the fibre bundle associated to the principal $\mathrm{PSL}(r,\mathbb{C})$-bundle $\mathcal{H} \longrightarrow \mathcal{R}$ for the standard action of $\mathrm{PSL}(r,\mathbb{C})$ on $\mathbb{CP}^{r-1}$.

The analytic Brauer group of $\mathcal{R}$ is defined to be the equivalence classes of holomorphic principal PGL-bundles on $\mathcal{R}$ where two principal PGL-bundles P and Q are equivalent if there are holomorphic vector bundles V and W on $\mathcal{R}$ such that the two principal PGL-bundles $P \otimes \mathbb{P}(V)$ and $Q \otimes \mathbb{P}(W)$ are isomorphic. Moreover, the analytic cohomological Brauer group of $\mathcal{R}$ is the torsion part of $H^2(\mathcal{R}, \mathcal{O}^*_{\mathcal{R}})$ (see [Sc]). Let $\mathrm{Br}_{\mathrm{an}}(\mathcal{R})$ (respectively, $\mathrm{Br}'_{\mathrm{an}}(\mathcal{R})$) denote the analytic Brauer group (respectively, analytic cohomological Brauer group) of $\mathcal{R}$. Then, from [Sc, p. 878] one has that

$$\mathrm{Br}_{\mathrm{an}}(\mathcal{R}) \subset \mathrm{Br}'_{\mathrm{an}}(\mathcal{R}).$$

Theorem 2.5 *The analytic cohomological Brauer group* $\mathrm{Br}'_{\mathrm{an}}(\mathcal{R})$ *is isomorphic to a quotient of the cyclic group* $\mathbb{Z}/\tau\mathbb{Z}$*, where* $\tau = \mathrm{g.c.d.}(r,\delta)$*. Moreover, the group* $\mathrm{Br}_{\mathrm{an}}(\mathcal{R})$ *is generated by the class of the projective bundle* $\mathbb{P}_R$ *in* (2.15).

Proof From the biholomorphism in (2.14), the group $\mathrm{Br}'_{\mathrm{an}}(\mathcal{R})$ coincides with the analytic Brauer group $\mathrm{Br}'_{\mathrm{an}}(\mathcal{M}_C)$ of the stable moduli space $\mathcal{M}_C$. Moreover, the forgetful map $\mathrm{Br}'(\mathcal{M}_C) \longrightarrow \mathrm{Br}'_{\mathrm{an}}(\mathcal{M}_C)$ is surjective [Sc, p. 879, Proposition 1.3]. Then, since $\mathrm{Br}'(\mathcal{M}_C) = \mathrm{Br}(\mathcal{M}_C)$, we conclude that $\mathrm{Br}'_{\mathrm{an}}(\mathcal{R})$ is a quotient of $\mathrm{Br}'(\mathcal{M}_C)$. Therefore, from the first part of Theorem 2.4, it follows that $\mathrm{Br}'_{\mathrm{an}}(\mathcal{R})$ is a quotient of the cyclic group $\mathbb{Z}/\tau\mathbb{Z}$.

The isomorphism in (2.14) takes the projective bundle $\mathbb{P}_R$ constructed in (2.15) holomorphically to the projective bundle $\mathbb{P}$ of (2.5). Therefore, from the second part of Theorem 2.4, it follows that $\mathrm{Br}'_{\mathrm{an}}(\mathcal{R})$ is generated by the class of $\mathbb{P}_R$. □

Note that while the natural homomorphism $\mathrm{Br}_{\mathrm{an}}(\mathcal{R}) \longrightarrow \mathrm{Br}'_{\mathrm{an}}(\mathcal{R})$ is injective [Sc, p. 878], Theorem 2.5 implies that this homomorphism is surjective and thus the following corollary is proved.

Corollary 2.6 *The analytic Brauer group* $\mathrm{Br}_{\mathrm{an}}(\mathcal{R})$ *coincides with* $\mathrm{Br}'_{\mathrm{an}}(\mathcal{R})$.

Regarding the above Corollary 2.6, it should be clarified that the analogue of Gabber's theorem, which would say that the Brauer group coincides with the cohomological Brauer group, is not available in the analytic category.

2.3. *Brauer Group of Moduli Spaces of Higgs Bundles*

We shall now consider Higgs bundles on a compact Riemann surface. As in Section 2.1, consider a compact connected Riemann surface X of genus $g \geq 3$, and $x_0 \in X$ a base point. Let $\mathcal{M}_H$ denote the moduli space of stable Higgs bundles on X of the form (E, Φ), where E is a holomorphic vector bundle of rank r with $\bigwedge^r E = \mathcal{O}_X(\delta x_0)$, and Φ is a Higgs field on X with $\mathrm{Tr}(\Phi) = 0$. The moduli space $\mathcal{M}_H$ is an irreducible smooth complex quasiprojective variety of dimension $2(r^2 - 1)(g - 1)$.

Consider the moduli space $\mathcal{N}$ from (2.10), for which the total space of the algebraic cotangent bundle $T^*\mathcal{N}$ is embedded in in $\mathcal{M}_H$ as a Zariski open subset. The codimension of the complement $\mathcal{M}_H \setminus T^*\mathcal{N}$ is at least two [Hi]. Therefore, by purity of cohomology, and Lemma 2.3, one has that

$$\mathrm{Br}(\mathcal{N}) = \mathrm{Br}(T^*\mathcal{N}) = \mathrm{Br}(\mathcal{M}_H);$$

as before, we use that $\mathcal{N}$ is simply connected.

Hence, we have the following:

Proposition 2.7 *The Brauer group* $\mathrm{Br}(\mathcal{M}_H)$ *is the cyclic group of order* $\mathrm{g.c.d.}(r, \delta)$. *Furthermore,* $\mathrm{Br}(\mathcal{M}_H)$ *is generated by the class of the projective bundle on* $\mathcal{M}_H$ *obtained by restricting to* $\{x\} \times \mathcal{M}_H$ *the universal projective bundle on* $X \times \mathcal{M}_H$.

3. Generalizations to Principal Bundles

Let G be a semi-simple simply connected affine algebraic group defined over $\mathbb{C}$. The topological types of principal G-bundles on X are parametrized by $\pi_1(G)$, which is a finite abelian group. Let $\overline{\mathcal{M}_C}(G)$ denote the moduli space of pairs of the form (E_G, D), where E_G is a topologically trivial holomorphic principal G-bundle on X, and D is a holomorphic connection on E_G. Following the notation from the previous sections, let

$$\mathcal{M}_C(G) \subset \overline{\mathcal{M}}_C(G)$$

be the smooth locus of $\overline{\mathcal{M}}_C(G)$.

The centre of G will be denoted by $Z(G)$. A stable principal G-bundle is called *regularly stable* if its automorphism group coincides with $Z(G)$. We shall denote by $\mathcal{N}_G$ the moduli space of regularly stable principal G-bundles on X that are topologically trivial. Recall from [BHol] that the Brauer group $\mathrm{Br}(\mathcal{N}_G)$ can be identified with the dual group $Z(G)^\vee = \mathrm{Hom}(Z(G), \mathbb{C}^*)$, and $\mathrm{Br}(\mathcal{N}_G)$ is generated by the tautological $Z(G)$-gerbe on $\mathcal{N}_G$ defined by the moduli stack of regularly stable topologically trivial principal G-bundles on X. Note that, given any homomorphism $Z(G)^\vee \longrightarrow \mathbb{G}_m$, using extension of the structure group, the above $Z(G)$-gerbe produces a $\mathbb{G}_m$-gerbe on $\mathcal{N}_G$.

Proposition 3.1 *The Brauer group* $\mathrm{Br}(\mathcal{M}_C(G))$ *is isomorphic to the dual group* $Z(G)^\vee$ *and is generated by the tautological* $Z(G)$*-gerbe on* $\mathcal{M}_C(G)$.

Proof A straightforward generalization of the proof of Theorem 2.4 proves the proposition. We note that $\mathcal{N}_G$ is simply connected [BLR, p. 416, Theorem 1.1]; hence, Lemma 2.3 is applicable. □

Similarly, the (analytic) Brauer group computations in Theorem 2.5 and Section 2.3 extend to G.

4. Langlands Duality and Brauer Groups

As previously, suppose that G is simply connected and let ${}^LG$ denote the Langlands dual group. There is a canonical isomorphism $\pi_1({}^LG) \cong Z(G)^\vee$, which can be seen from the root-theoretic construction of the Langlands dual. We shall denote by $\mathcal{M}_H(G)$ and $\mathcal{M}_H({}^LG)$ the moduli spaces of Higgs bundles for the groups G and ${}^LG$, respectively.

It is known that the connected components of $\mathcal{M}_H({}^LG)$ correspond to $\pi_1({}^LG)$, by taking the topological class of the underlying principal bundle. Recall that the moduli spaces $\mathcal{M}_H(G)$ and $\mathcal{M}_H({}^LG)$ satisfy SYZ mirror symmetry, that is, they are dual special Lagrangian torus fibrations over a common base [DoPa]. Under this duality, the choice of a connected component of $\mathcal{M}_H({}^LG)$ corresponds to fixing a $\mathbb{C}^*$-gerbe on $\mathcal{M}_H(G)$. Namely, given a class in $\pi_1({}^LG) \cong Z(G)^\vee$, we obtain from the universal $G/Z(G)$-bundle on $\mathcal{M}_H(G)$ the corresponding $\mathbb{C}^*$-gerbe. Our computations show that every class in $\mathrm{Br}(\mathcal{M}_H(G))$ is accounted for by this correspondence.

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15

Classification of Boundary Lefschetz Fibrations over the Disc

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Dedicated to Professor Nigel Hitchin on the occasion of his seventieth birthday

Abstract. Stable generalized complex structures can be constructed out of boundary Lefschetz fibrations. On 4-manifolds, these are essentially genus one Lefschetz fibrations over surfaces, except that generic fibres can collapse to circles over a codimension 1 submanifold, which is often the boundary of the surface. We show that a 4-manifold admits a boundary Lefschetz fibration over the disc degenerating over its boundary if and only if it is diffeomorphic to $S^1 \times S^3 \# n\overline{\mathbb{C}P^2}$, $\# m\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$ or $\# m(S^2 \times S^2)$. We conclude that the 4-manifolds $S^1 \times S^3 \# n\overline{\mathbb{C}P^2}$, $\#(2m+1)\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$ and $\#(2m+1)S^2 \times S^2$ admit stable generalized complex structures whose type change locus has a single component.

1. Introduction

Generalized complex structures, introduced by Hitchin [14] and Gualtieri [11] in 2003, are geometric structures which generalize simultaneously complex and symplectic structures

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while at the same time providing the mathematical background for string theory. One feature of generalized complex geometry is that the structure is not homogenous. In fact, a single connected generalized complex manifold may have complex and symplectic points. This lack of homogeneity is governed by the *type* of the structure, an integer-valued upper semicontinuous function on the given manifold which tells 'how many complex directions' the structure has at the given point. In particular, on a $2n$-dimensional manifold, points of type 0 are symplectic points, while points of type n are complex.

Among all type-changing generalized complex structures, one kind seems to deserve special attention: stable generalized complex structures. These are the structures whose canonical section of the anticanonical bundle vanishes transversally along a codimension 2 submanifold, $\mathcal{D}$, endowing it with the structure of an elliptic divisor in the language of [8]. Consequently, the type of such a structure is 0 on $X \setminus \mathcal{D}$, while on $\mathcal{D}$ it is equal to 2. Many examples of stable generalized complex structures were produced in dimension 4 [7, 10, 15, 16] and a careful study was carried out in [8]. One of the outcomes of that study was that it related stable generalized complex structures to symplectic structures on a certain Lie algebroid.

Theorem ([8, Theorem 3.7]) *Let $\mathcal{D}$ be a co-orientable elliptic divisor on X. Then there is a correspondence between gauge equivalence classes of stable generalized complex structures on X which induce the divisor $\mathcal{D}$, and zero-residue symplectic structures on $(X, \mathcal{D})$.*

This result paves the way for the use of symplectic techniques to study stable structures. One result, due to the last named authors [5], that exemplifies that use is the following relation between stable generalized complex structures and boundary Lefschetz fibrations in dimension 4. The latter are essentially genus one Lefschetz fibrations over surfaces whose generic fibres can collapse to circles over a codimension 1 submanifold, which is often the boundary of the surface (see Section 2 for details).

Theorem ([5, Theorem 7.1]) *Let X^4 be a closed connected and orientable 4-manifold and let Σ be a compact connected and orientable 2-manifold with boundary $Z = \partial\Sigma$. Let $f\colon X^4 \to \Sigma^2$ be a boundary Lefschetz fibration for which $\mathcal{D} = f^{-1}(\partial\Sigma)$ is a co-orientable submanifold of X, and with $0 \neq [f^{-1}(p)] \in H_2(X \setminus \mathcal{D}; \mathbb{R})$, where $p \in \Sigma$ is a regular value of f. Then X admits a stable generalized complex structure whose degeneracy locus is $\mathcal{D}$.*

This result is reminiscent of Gompf's original one [9], showing that Lefschetz fibrations give rise to symplectic structures. It is also is similar in content to a number of other results relating structures which are close to being symplectic to maps which are close to being Lefschetz fibrations. These include the relations between near-symplectic structures and broken Lefschetz fibrations [1], and between folded symplectic structures and real log-symplectic structures and achiral Lefschetz fibrations [2, 4, 6].

The upshot of these results is that they at the same time furnish (at least theoretically) a large number of examples of manifolds admitting the desired geometric structure, and provide us with a better grip on those structures. With this in mind, our aim here is to classify all 4-manifolds which admit boundary Lefschetz fibrations over the disc degenerating over its boundary. Our main result is the following (Theorem 3.13).

Theorem *Let $f\colon X^4 \to D^2$ be a relatively minimal boundary Lefschetz fibration and $\mathcal{D} = f^{-1}(\partial D^2)$. Then X is diffeomorphic to one of the following manifolds:*

(1) $S^1 \times S^3$;

(2) $\#m(S^2 \times S^2)$*, including* S^4 *for* $m = 0$;

(3) $\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ *with* $m > n \geq 0$.

In all cases, the generic fibre is non-trivial in $H_2(X \setminus \mathcal{D};\mathbb{R})$*. In case* (1)*,* $\mathcal{D}$ *is co-orientable while, in cases* (2) *and* (3)*,* $\mathcal{D}$ *is co-orientable if and only if* m *is odd.*

These last two theorems equip the manifolds $S^1 \times S^3\#n\overline{\mathbb{C}P^2}$, $\#(2m+1)\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ and $\#(2m+1)S^2 \times S^2$ with a stable generalized complex structure whose type change locus has a single component. Further, they provide a complete list of 4-manifolds whose stable generalized complex structures are obtained from boundary Lefschetz fibrations over the disc degenerating over its boundary. Note that the previous theorem does not address cases where $\mathcal{D}$ has multiple components.

We use essentially the same methods that were used by the first author in [3] and Hayano in [12, 13]. We translate the problem into combinatorics in the mapping class group of the torus, and then translate combinatorial results back into geometry using handle decompositions and Kirby calculus. Hayano's work turns out to be particularly relevant. In his classification of so-called genus one simplified broken Lefschetz fibrations, he was led to study monodromy factorizations of Lefschetz fibrations over the disc whose monodromy around the boundary is a signed power of a Dehn twist. It turns out that the same problem appears for boundary Lefschetz fibrations.

Organization of the paper

This paper is organised as follows. In Section 2 we introduce boundary fibrations and boundary Lefschetz fibrations, and summarize their basic properties. In Section 3 we start by studying boundary Lefschetz fibrations over $(D^2, \partial D^2)$ with few Lefschetz singularities. We then prove the main theorem using induction, by showing how to reduce the number of Lefschetz singularities.

2. Boundary Lefschetz Fibrations

In view of our interest in stable generalized complex structures and the results mentioned in the introduction, the basic object with which we will be dealing in this paper are boundary (Lefschetz) fibrations. In this section, we review the relevant definitions and basic results regarding them. We will use the following language. A *pair* $(X,\mathcal{D})$ consists of a manifold X and a submanifold $\mathcal{D} \subseteq X$. A *map of pairs* $f\colon (X,\mathcal{D}) \to (\Sigma, Z)$ is a map $f\colon X \to \Sigma$ for which $f(\mathcal{D}) \subseteq Z$. A *strong map of pairs* is a map of pairs $f\colon (X,\mathcal{D}) \to (\Sigma, Z)$ for which $f^{-1}(Z) = \mathcal{D}$.

Definition 2.1 (Boundary Lefschetz fibrations) Let $f\colon (X^{2n}, \mathcal{D}^{2n-2}) \to (\Sigma^2, Z^1)$ be a strong map of pairs which is proper and for which $\mathcal{D}$ and Z are compact.

- The map f is a *boundary map* if the normal Hessian of f along $\mathcal{D}$ is definite.
- The map f is a *boundary fibration* if it is a boundary map and the following two maps are submersions:
  (a) $f|_{X\setminus\mathcal{D}}\colon X\setminus\mathcal{D}\to\Sigma\setminus Z$, and
  (b) $f\colon\mathcal{D}\to Z$.

  The condition that f is a boundary fibration (in a neighbourhood of $\mathcal{D}$) is equivalent to the condition that, for every $x\in\mathcal{D}$, there are coordinates $(x_1,\ldots,x_{2n})$ centred at x, and (y_1,y_2) centred at $f(x)$, such that f takes the form

$$f(x_1,\ldots,x_{2n})=(x_1^2+x_2^2,x_3),\tag{2.1}$$

  where $\mathcal{D}$ corresponds to the locus $\{x_1=x_2=0\}$ and Z to the locus $\{y_1=0\}$;
- The map f is a *boundary Lefschetz fibration* if X and Σ are oriented, f is a boundary fibration from a neighbourhood of $\mathcal{D}$ to a neighbourhood of Z and $f|_{X^{2n}\setminus\mathcal{D}}\colon X\setminus\mathcal{D}\to\Sigma\setminus Z$ is a proper Lefschetz fibration, that is, for each critical point $x\in X\setminus\mathcal{D}$ and corresponding singular value $y\in\Sigma\setminus Z$, there are complex coordinates centred at x and y compatible with the orientations for which f acquires the form

$$f(z_1,\ldots,z_n)=z_1^2+\cdots+z_n^2.\tag{2.2}$$

Example 2.2 ($S^1\times S^3$) In this example we provide $X=S^1\times S^3$ with the structure of a boundary fibration over the disc, as described in [5, Example 8.3]. The map $f\colon S^1\times S^3\to D^2$ is a composition of maps, namely

$$S^1\times S^3\to S^3\to D^2,$$

where the first map is a projection onto the second factor and the last is the projection from $\mathbb{C}^2$ to $\mathbb{C}$, $(z_1,z_2)\mapsto z_1$, restricted to the sphere. In Theorem 3.1 we will see that this is, in fact, the only example of a boundary fibration over $(D^2,\partial D^2)$.

A few relevant facts about boundary Lefschetz fibrations were established in [5]. Beyond the local normal form (2.1) for the map f around points in $\mathcal{D}$, there is also a semi-global form for f in a neighbourhood of $\mathcal{D}$:

Theorem 2.3 (**[5, Proposition 5.15]**) *Let $f\colon(X^{2n},\mathcal{D}^{2n-2})\to(\Sigma^2,Z^1)$ be a boundary map which is a boundary fibration on neighbourhoods of $\mathcal{D}$ and Z and for which Z is co-orientable. Then there are*

- *neighbourhoods U of $\mathcal{D}$ and V of Z and diffeomorphisms between these sets and neighbourhoods of the zero sections of the corresponding normal bundles, $\Phi_{\mathcal{D}}\colon U\to N_{\mathcal{D}}$ and $\Phi_Z\colon V\to\mathbb{R}\times Z$, and*
- *a bundle metric g on $N_{\mathcal{D}}$,*

such that the following diagram commutes, where $\pi : N_{\mathcal{D}} \to \mathcal{D}$ is the bundle projection:

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\Phi_{\mathcal{D}}"] & V \arrow[d, "\Phi_Z"] \\ N_{\mathcal{D}} \arrow[r, "(\|\cdot\|_g^2, f|_{\mathcal{D}}\circ\pi)"] & \mathbb{R} \times Z. \end{tikzcd}$$

The most obvious consequence of this theorem is that, in the description above, the image of f lies on one side of Z, namely in $\mathbb{R}_+ \times Z$. At this stage, this is a local statement but, if Z is separating (i.e. represents the trivial homology class), this becomes a global statement: the image of f lies in closure of one component of $\Sigma \setminus Z$ and hence we can equally deal with f as a map between X and a manifold with boundary, Σ, whose boundary is Z. In this paper, we will be concerned with the case when Σ is the two-dimensional disc and Z is its boundary.

Corollary 2.4 *Let $f\colon (X^4, \mathcal{D}^2) \to (\Sigma^2, Z^1)$ be a boundary fibration with connected fibres, where Z is co-orientable and X is connected and orientable. Then its generic fibres are tori.*

Proof From Theorem 2.3 we see that the level set $f^{-1} \circ \Phi_Z^{-1}(\varepsilon, y)$ with $\varepsilon > 0$ is a surface which fibres over the level set of $f^{-1} \circ \Phi_Z^{-1}(0, y)$, which is a circle; hence, $f^{-1} \circ \Phi_Z^{-1}(\varepsilon, y)$ must be a torus or a Klein bottle. If X is orientable, $N_{\mathcal{D}} \setminus \mathcal{D}$ is also orientable and, due to Theorem 2.3, $\Phi_Z \circ f \circ \Phi_{\mathcal{D}}^{-1} : U \subset N_{\mathcal{D}} \setminus \mathcal{D} \to \mathbb{R} \times Z \setminus \{0\} \times Z$ is a fibration, where U is a neighbourhood of $\mathcal{D}$; hence, the fibres must be orientable. □

Remark 2.5 In the case when X is connected, Σ is a surface with boundary $Z = \partial\Sigma$, and $f\colon X \to \Sigma$ is surjective, we can lift f to a cover of Σ so that the fibres of the boundary Lefschetz fibration become connected. That is, this particular hypothesis is not really a restriction on the fibration (see [5, Proposition 5.23]). In what follows, we will always assume this is the case.

Remark 2.6 As shown in [5, Proposition 6.8], a boundary Lefschetz fibration $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ satisfies $\chi(X) = \mu$, where μ is the number of Lefschetz singular fibres.

2.1. Vanishing Cycles and Monodromy

Lefschetz fibrations on 4-manifolds can be described combinatorially in terms of their monodromy representations and vanishing cycles. We now extend this approach to boundary Lefschetz fibrations. For simplicity, we focus on fibrations over the disc (degenerating over its boundary) and assume they are injective on their Lefschetz singularities. The latter condition can always be achieved by a small perturbation, and the generalization to general base surfaces is similar to the Lefschetz case.

Definition 2.7 (Hurwitz systems) Let $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ be a boundary Lefschetz fibration with ℓ Lefschetz singularities, and let $y \in D^2$ be a regular value. A *Hurwitz system* for f based at y is a collection of embedded arcs $\eta_0, \eta_1, \ldots, \eta_\ell \subset D^2$ such that

(1) η_0 connects y to ∂D^2 and is transverse to ∂D^2,

(2) η_i connects y to a critical value y_i,

(3) the arcs intersect pairwise transversely in y and are otherwise disjoint, and

(4) the order of the arcs is counterclockwise around y.

Given a Hurwitz system, we obtain a collection of simple closed curves in the regular fibre $F_y = f^{-1}(y)$ as follows. For $i > 0$, we have the classical construction of *Lefschetz vanishing cycles*: as we move from y along η_i towards y_i, a curve $\lambda_i \subset F_y$ shrinks and eventually collapses into Lefschetz singularity over y_i, leading to a nodal singularity in F_{y_i}. For later reference, we also recall that the monodromy along a counterclockwise loop around y_i contained in a neighbourhood of η_i is given by a right-handed Dehn twist about λ_i. Along η_0 we see a slightly different degeneration: the boundary of a solid torus degenerates the core circle. Indeed, using the local model for f near $\mathcal{D}$ and the transversality of η_0 to ∂D^2 we can find a diffeomorphism $f^{-1}(\eta_0) \cong D^2 \times S^1$ and a parametrization of η_0 that takes f into the function $D^2 \times S^1 \to \mathbb{R} \times Z$ given by $(x_1, x_2, \theta) \mapsto (x_1^2 + x_2^2, z_0)$, where $z_0 = \eta_0(1)$. To summarize, $f^{-1}(\eta_0)$ is a solid torus whose boundary is F_y. Further F_y contains a well-defined isotopy class of meridional circles, represented in the model by $\partial D^2 \times \{\theta\}$ for arbitrary $\theta \in S^1$. We will henceforth refer to this isotopy class as the *boundary vanishing cycle* associated to η_0 and denote it by δ.

To make things even more concrete, we can fix an identification of the reference fibre F_y with T^2 and consider the vanishing cycles in the standard torus. To make a notational distinction, we denote the images in T^2 by $(a; b_1, \ldots, b_\ell)$.

Definition 2.8 (Cycle systems) A collection of curves $(a; b_1, \ldots, b_\ell)$ in T^2 associated to f by a choices of a Hurwitz system together with an identification of the reference fibre with T^2 is called a *cycle system* for f.

It is well known that the Lefschetz part of f can be recovered from the Lefschetz vanishing cycles. In the next section, we will explain how this statement extends to boundary Lefschetz fibrations. Just as in the Lefschetz case, the cycle system is not unique but the ambiguities are easy to understand and provide some flexibility to find particularly nice cycle systems representing a given boundary Lefschetz fibration. The following is a straightforward generalization of the analogous statement for Lefschetz fibrations (see also Figure 1).

Proposition 2.9 *Let $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ be a boundary Lefschetz fibration with ℓ Lefschetz singularities. Any two cycle systems for f are related by a finite sequence of the following modifications:*

$$
\begin{aligned}
\big(a; b_1, \ldots, b_\ell\big), &\sim \big(a; b_2, B_2(b_1), b_3, \ldots, b_\ell\big),\\
&\sim \big(a; B_1^{-1}(b_2), b_1, b_3, \ldots, b_\ell\big),\\
&\sim \big(B_1(a); b_2, \ldots, b_\ell, b_1\big),\\
&\sim \big(B_\ell^{-1}(a); b_\ell, b_1, \ldots, b_{\ell-1}\big),\\
&\sim \big(h(a); h(b_1), \ldots, h(b_\ell)\big).
\end{aligned}
$$

Here $B_i = \tau_{b_i}$ is a right-handed Dehn twist about b_i and h is any diffeomorphism of T^2.

Definition 2.10 (Hurwitz equivalence) If two cycle systems are related by the modifications listed in Theorem 2.9, we say that they are *(Hurwitz) equivalent.*

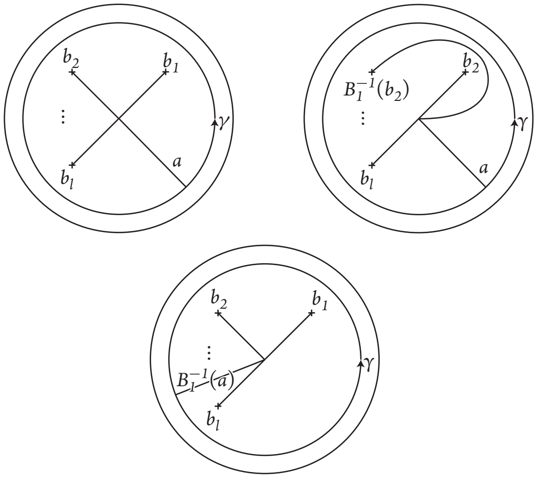

Figure 1 The origin of Hurwitz equivalence: here we illustrate how the equivalences $\left(a;b_1,\ldots,b_\ell\right) \sim (a;B_1^{-1}(b_2),b_1,b_3,\ldots,b_\ell) \sim \left(B_\ell^{-1}(a);b_\ell,b_1,\ldots,b_{\ell-1}\right)$ arise.

It turns out that the curves in a cycle system are not completely arbitrary. Let $S_r^1 \subset D^2$ be the circle of radius $r < 1$ such that all the Lefschetz singularities of f map to the interior of D_r^2. Fix a reference point let $y \in S_r^1$ and let

$$\mu(f) \in \mathcal{M}(F_y) = \pi_0 \mathrm{Diff}^+(F_y)$$

be the counterclockwise monodromy of f around S_r^1 as measured in the mapping class group of F_y. Then, for any cycle system for f derived from a Hurwitz system based at y, the anticlockwise monodromy of f around S_r^1 measured in F_y is given by the product of Dehn twists about the Lefschetz vanishing cycles $\lambda_i \subset F_y$,

$$\mu(f) = \tau_{\lambda_\ell} \circ \cdots \circ \tau_{\lambda_1} \in \mathcal{M}(F_y) = \pi_0 \mathrm{Diff}^+(F_y). \tag{2.3}$$

On the other hand, we can also describe the monodromy using the boundary part of the fibration. Recall that $f^{-1}(S_r^1)$ is the boundary of a tubular neighbourhood $N\mathcal{D}$ of $\mathcal{D}$ and that the fibration structure over S_r^1 essentially factors through the projection $N\mathcal{D} \to \mathcal{D}$. This exhibits $f^{-1}(S_r^1)$ as a circle bundle over $\mathcal{D}$, which is itself a circle bundle over S^1. It follows that the monodromy of f around S_r^1 must fix the circle fibres of $f^{-1}(S_r^1) \to \mathcal{D}$, and the circle fibre contained in F_y is precisely the boundary vanishing cycle δ of the Hurwitz system. To conclude, $\mu(f)$ fixes δ as a set, but not necessarily pointwise. Indeed, it can (and does) happen that $\mu(f)$ reverses the orientation of δ.

Remark 2.11 At this point, it is worthwhile to point out some perks of working on a torus. First, there is the fact that any diffeomorphism of T^2 is determined up to isotopy by its action on $H_1(T^2)$. Given any pair of oriented simple closed curves $a, b \subset T^2$

with (algebraic) intersection number $\langle a,b\rangle = 1$—called *dual pairs* from now on—we get an identification $\mathcal{M}(T^2) \cong SL(2,\mathbb{Z})$. Moreover, the right-handed Dehn twists $A,B \in \mathcal{M}(T^2)$ about a and b are the generators in a finite presentation with relations $ABA = BAB$ and $(AB)^6 = 1$. In particular, we have that $(AB)^3$ maps to $-1 \in SL(2,\mathbb{Z})$, which we will also denote by writing $-1 = (AB)^3 \in \mathcal{M}(T^2)$. Second, in a similar fashion, simple closed curves up to ambient isotopies are uniquely determined by their (integral) homology classes. Note that this involves a choice of orientation, since simple closed curves are a priori unoriented objects. However, it is true that essential simple closed curves in T^2 correspond bijectively with primitive elements of $H_1(T^2)$ up to sign. In what follows, we adopt the common bad habit of identifying simple closed curves with elements of $H_1(T^2)$ without explicitly mentioning orientation. In particular, we will freely use the homological expression for a Dehn twist, i.e. write

$$\tau_c(d) = d + \langle c,d\rangle c \in H_1(T^2). \tag{2.4}$$

We record two facts that are important for our purposes:

(1) If $h \in \mathcal{M}(T^2)$ satisfies $h(a) = a$ for some essential curve a, then $h = \pm\tau_a^k$ for some k with a negative sign if and only if h is orientation reversing on a.

(2) If oriented curves $a,b,c \subset T^2$ satisfy $\langle a,b\rangle = \langle a,c\rangle = 1$, then $c = \tau_a^k(b) = b + ka$ for some k.

Returning to the discussion of the monodromy $\mu(f)$, we can conclude that the vanishing cycles have to satisfy the condition

$$\mu(f) = \tau_{\lambda_\ell} \circ \cdots \circ \tau_{\lambda_1} = \pm\tau_\delta^k \in \mathcal{M}(F_y), \qquad k \in \mathbb{Z}.$$

It is easy to see from the above discussion that a negative sign appears if and only if $\mathcal{D}$ fails to be co-orientable. Moreover, the integer k is precisely the Euler number of the normal bundle of $\mathcal{D}$ in X. Here we remark that a vector bundle $E \to M$ with M compact has a well-defined integer Euler number if the total space of E is orientable, even if M is not orientable itself.

For practical purposes, it is more convenient to work with cycle systems in the model T^2. Here is the upshot of the above discussion:

Proposition 2.12 *Let $f\colon (X^4,\mathcal{D}^2) \to (D^2,\partial D^2)$ be a boundary Lefschetz fibration. If $(a;b_1,\ldots,b_\ell)$ is any cycle system for f, then*

$$B_\ell \circ \cdots \circ B_1 = \pm A^k \in \mathcal{M}(T^2) \tag{2.5}$$

for some $k \in \mathbb{Z}$, where the sign is positive if and only if $\mathcal{D}$ is co-orientable. The integer k agrees with the Euler number of the normal bundle of $\mathcal{D}$ in X.

This motivates an abstract definition without reference to boundary Lefschetz fibrations.

Definition 2.13 (Abstract cycle systems) An ordered collection of curves $(a;b_1,\ldots,b_\ell)$ in T^2 is called an *abstract cycle system* if it satisfies the condition in (2.5). *Hurwitz equivalence* is defined exactly as in Theorem 2.10.

2.2. Handle Decompositions and Kirby Diagrams

Next, we discuss how one can recover boundary Lefschetz fibrations from their cycle systems. Along the way, we exhibit useful handle decompositions of total spaces of boundary Lefschetz fibrations.

Proposition 2.14 *Any abstract cycle system $(a; b_1, \ldots, b_\ell)$ is the cycle system of some boundary Lefschetz fibration over the disc.*

Proof We will build a 4-manifold obtained by attaching a handle to $T^2 \times D^2$. We choose points $\theta_0, \ldots, \theta_\ell \in \partial D^2$ which appear in anticlockwise order and consider a copy of a in $T^2 \times \{\theta_0\}$ and of b_i in $T^2 \times \{\theta_i\}$ for $i > 0$. Note that, for all these curves, there is a natural choice of framing determined by parallel push-offs inside the fibres of $T^2 \times S^1 \to S^1$. We first attach a 2-handle along the copies of b_i for $i > 0$ with respect to the fibre framing -1 and call the resulting manifold Z. It is well known that the projection $T^2 \times D^2$ extends to a Lefschetz fibration on Z over a slightly larger disc, which we immediately rescale to D^2, such that the Lefschetz vanishing cycles along the straight line from θ_i to zero is b_i. By construction, the boundary fibres over S^1 and the anticlockwise monodromy measured in $T^2 \times \{\theta_0\}$ is $B_\ell \circ \cdots \circ B_1 = \pm A^k$. In particular, ∂Z is diffeomorphic as an oriented manifold to the circle bundle with Euler number k over the torus or the Klein bottle. Let $N^{\pm}_{-k}$ be the corresponding disc bundle with Euler number $-k$. Then $\partial N^{\pm}_{-k}$ is diffeomorphic to ∂Z with the orientation reversed so that we can form a closed manifold X by glueing Z and $N^{\pm}_{-k}$ together, and the orientation of Z extends. Moreover, it was shown in [5] that $N^{\pm}_{-k}$ admits a boundary fibration over the annulus which can be used to extend the Lefschetz fibration on Z to a boundary Lefschetz fibration on X, again over a larger disc which we rescale to D^2, in such a way that the boundary vanishing cycle along the straight line from θ_0 to zero is a.

Thus we have found a boundary Lefschetz fibration together with a Hurwitz system which produces the desired cycle system. □

Remark 2.15 (Construction of the Kirby diagram) Observe that the glueing of $N^{\pm}_{-k}$ also has an interpretation in terms of handles. It is well known that $N^{\pm}_{-k}$ has a handle decomposition with one 0-handle, two 1-handles, and a single 2-handle. Turning this decomposition upside down gives a relative handle decomposition on $-\partial N^{\pm}_{-k} \cong \partial Z$ with a single 2-handle, two 3-handles, and a 4-handle. Moreover, the 2-handle can be chosen such that its core disc is a fibre. In particular, since the glueing of $N^{\pm}_{-k}$ to Z preserves the circle fibration, we can arrange that the 2-handle of $N^{\pm}_{-k}$ is attached along the copy of a in the fibre of ∂Z over θ_0. However, in contrast to the Lefschetz handles, this time the framing is actually the fibre framing.

To summarize, the closed 4-manifold X is obtained from $T^2 \times D^2$ by attaching, in order, a 2-handle along the boundary vanishing cycle with the fibre framing, and then 2-handles along the Lefschetz vanishing cycles $b_i \subset T^2 \times \{\theta_i\}$ with fibre framing -1. The two 3-handles as well as the 4-handle attach uniquely, as explained in [9, Section 4.4].

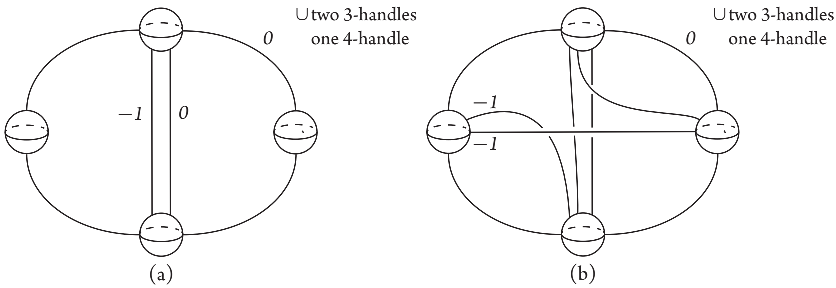

Figure 2 Kirby diagrams corresponding to the abstract cycle systems $(a; a)$ (Panel (a)) and $(a; b + 2a, b)$ (Panel (b)). The numbers indicate the blackboard framing of the corresponding 2-handles.

As an illustration of this procedure, Figure 2 shows the Kirby diagrams corresponding to the abstract cycle systems $(a; a)$ and $(a; b + 2a, b)$, where $\{a, b\}$ is a dual pair of curves.

Next we show that the topology of the total space of a boundary Lefschetz fibration can be recovered from the cycle system.

Proposition 2.16 *If two boundary Lefschetz fibrations over $(D^2, \partial D^2)$ have equivalent cycle systems, then their total spaces are diffeomorphic.*

Proof Elaborating on the proof of Theorem 2.14, one can show that, if a Hurwitz system and identification of the reference fibre with T^2 for a given boundary Lefschetz fibration produces the cycle system $(a; b_1, \dots, b_\ell)$, then its total space is diffeomorphic to the manifold constructed by attaching handles to $T^2 \times D^2$, as explained above. Similarly, one can then argue that the manifolds constructed from equivalent cycle systems are diffeomorphic. The details are somewhat tedious but straightforward and we leave them to the inclined reader. □

As a consequence, in order to classify closed 4-manifolds admitting boundary Lefschetz fibrations over $(D^2, \partial D^2)$, it is enough to identify all 4-manifolds obtained from abstract cycle systems as in the proof of Theorem 2.14. Moreover, as we argued in Theorem 2.15, this problem is naturally accessible to the methods of Kirby calculus via the handle decompositions. For the relevant background about Kirby calculus, we refer to [9] (Chapter 8, in particular).

3. Boundary Lefschetz Fibrations over $(D^2, \partial D^2)$

As a warm-up for our main theorem, it is worth considering the following more basic question: Which oriented 4-manifolds are boundary fibrations over $(D^2, \partial D^2)$? The answer is simple:

Lemma 3.1 *Let X be a compact, orientable manifold and let $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ be a boundary fibration. Then X is diffeomorphic to $S^1 \times S^3$, and $\mathcal{D}$ is co-orientable.*

Proof Note that a boundary fibration is a boundary Lefschetz fibration without Lefschetz singularities. As such, its cycle systems consist of a single curve $a \subset T^2$ corresponding to the boundary vanishing cycle. Thus, a is essential and we can therefore assume that $a = \{1\} \times S^1$. According to the discussion in Subsection 2.2, X is obtained from glueing $T^2 \times D^2$ together with a suitable disc bundle over a torus or Klein bottle, such that the boundary of a disc fibre is identified with a. Obviously, the only possibility is $N_0^+ = D^2 \times T^2$, the trivial disc bundle over the torus, and the glueing can be arranged such that $\partial D^2 \times \{(1,1)\} \subset N_0^+$ is identified with $a \times \{1\} \subset T^2 \times D^2$. Since this is achieved by the diffeomorphism of T^3 which flips the first two factors, we see that

$$\begin{aligned} X &\cong S^1 \times S^1 \times D^2 \cup_\varphi D^2 \times S^1 \times S^1 \\ &\cong S^1 \times S^1 \times D^2 \cup_{\mathrm{id}} S^1 \times D^2 \times S^1 \\ &\cong S^1 \times \left(S^1 \times D^2 \cup_{\mathrm{id}} D^2 \times S^1\right) \cong S^1 \times S^3, \end{aligned}$$

where the last diffeomorphism comes from the standard decomposition of S^3 considered as sitting in $\mathbb{C}^2$ and split into two solid tori by $S^1 \times S^1 \subset \mathbb{C}^2$. □

Now we move on to study honest boundary Lefschetz fibrations over the disc and eventually prove our classification theorem, Theorem 3.13. The proof of the theorem itself is done by induction on the number of singular fibres. So, in order to achieve our aim, we need to study the base cases, i.e. boundary Lefschetz fibrations with only a few singular fibres, and explain how to systematically reduce the number of singular fibres to bring us back to the base cases. It turns out that there is a step that appears frequently, namely, the blow-down of certain (-1)-spheres, which is interesting on its own as it gives the notion of a relatively minimal boundary Lefschetz fibration. In the rest of this section, we will first study blow-downs and relatively minimal fibrations. We then move on to study the cases with one and two singular fibres and finally prove Theorem 3.13.

3.1. *The Blow-Down Process and Relative Minimality*

Given a usual Lefschetz fibration $f\colon X^4 \to \Sigma^2$, we can perform the blow-up in a regular point $x \in X$ with respect to a local complex structure compatible with the orientation of X. The result is a manifold $\widetilde{X}$ together with a blow-down map $\sigma : \widetilde{X} \to X$ and it turns out that the composition $\widetilde{f} = f \circ \sigma : \widetilde{X} \to \Sigma$ is a Lefschetz fibration with one more critical point than f in the fibre over $y = f(x)$. Moreover, the exceptional divisor sits inside the (singular) fibre $\widetilde{f}^{-1}(y)$ as a sphere with self-intersection -1. Conversely, given any (-1)-sphere in a singular fibre of a Lefschetz fibration, this process can be reversed: the (-1)-sphere can be blown down, producing a Lefschetz fibration with one critical point less. For that reason, it is enough to study *relatively minimal* Lefschetz fibrations: fibrations whose fibres do not contain any (-1)-spheres. Equivalently, a Lefschetz fibration is relatively minimal if no vanishing cycle bounds a disc in the reference fibre; and, on the level of cycle systems, the blow-up and blow-down procedures simply amount to adding or removing null-homotopic vanishing cycles.

For a boundary Lefschetz fibration $f\colon (X^4, \mathcal{D}^2) \to (\Sigma^2, Z^1)$, there is another way a (-1)-sphere can occur in relation to the fibration. These spheres arise if there is a simple path connecting a Lefschetz singular value of f to a component of Z with the property that the Lefschetz vanishing cycle in one end of the path agrees with the boundary vanishing cycle. In this case, we can form the corresponding Lefschetz thimble from the Lefschetz singularity which then closes up at the other end of the path to give rise to a (-1)-sphere, E, which intersects the divisor $\mathcal{D}$ at one point, as observed in [7]. Note that, in the case where $(\Sigma, Z) = (D^2, \partial D^2)$, this is equivalent to a cycle system $(a; b_1, \ldots, b_\ell)$ such that some b_i agrees with a. From this description, it is clear that we can blow E down to obtain a new manifold, X'. What is not immediately clear is that X' admits the structure of a boundary Lefschetz fibration.

Proposition 3.2 *Let $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ be a boundary Lefschetz fibration which has a cycle system $(a; b_1, \ldots, b_\ell)$ such that $b_i = a$ for some i. Then there exists a boundary Lefschetz fibration $f'\colon (X', \mathcal{D}') \to (D^2, \partial D^2)$ with the cycle system*

$$(a; A(b_1), \ldots, A(b_{i-1}), b_{i+1}, \ldots, b_\ell).$$

Moreover, we have $X \cong X' \# \overline{\mathbb{C}P^2}$, and $\mathcal{D}'$ has the same co-orientability as $\mathcal{D}$.

Proof This is our first exercise in Kirby calculus. Using Hurwitz moves, we have the equivalence of cycle systems:

$$\big(a; b_1, \ldots, b_{i-1}, a, b_{i+1}, \ldots, b_\ell\big) \cong \big(a; a, A(b_1), \ldots, A(b_{i-1}), b_{i+1}, \ldots, b_\ell\big),$$

so we may assume without loss of generality that $b_1 = a$. Further, we can take a to be the first cycle of a dual pair $\{a, b\}$, that is, we may assume that $a = S^1 \times \{1\} \subset T^2$. We now compare the Kirby diagrams obtained from the cycle systems $(a; a, b_2, \ldots, b_\ell)$ and $(a; b_2, \ldots, b_\ell)$.

As we mentioned in Theorem 2.15, to draw a Kirby diagram for a boundary Lefschetz fibration corresponding to a cycle system, we start with the Kirby diagram of $D^2 \times T^2$ and add cells corresponding to the boundary vanishing cycle followed by the Lefschetz vanishing cycles ordered anticlockwise. Therefore, the Kirby diagram for $(a; a, b_2, \ldots, b_\ell)$ is the Kirby diagram for $(a; a)$ with a number of 2-handles on top of it representing the cycles $b_2, \ldots, b_\ell$. The Kirby move we use next does not interact with these last $(l - 1)$ 2-handles; therefore, we will not represent them in the diagram. With this in mind, the relevant part of the Kirby diagram of $(a; a, b_2, \ldots, b_\ell)$ is the Kirby diagram of $(a; a)$ as drawn in Figure 2(a). Sliding the -1-framed 2-handle corresponding to the first Lefschetz singularity over the 0-framed 2-handle corresponding to the boundary vanishing cycle produces a -1-framed unknot which is unlinked from the rest (see Figure 3). The remaining Kirby diagram is precisely that corresponding to the cycle system $(a; b_2, \ldots, b_\ell)$. Since an isolated -1-framed unknot represents a connected sum with $\overline{\mathbb{C}P^2}$, the result follows. □

The previous proof is prototypical for much of what follows from now on. In light of Theorem 3.2, we make the following definition.

Definition 3.3 (Relative minimality) Let $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ be a boundary Lefschetz fibration. Then f is called *relatively minimal* if it does not have any cycle system $(a; b_1, \ldots, b_\ell)$ in which some Lefschetz vanishing cycle b_i is either null-homotopic or isotopic to a.

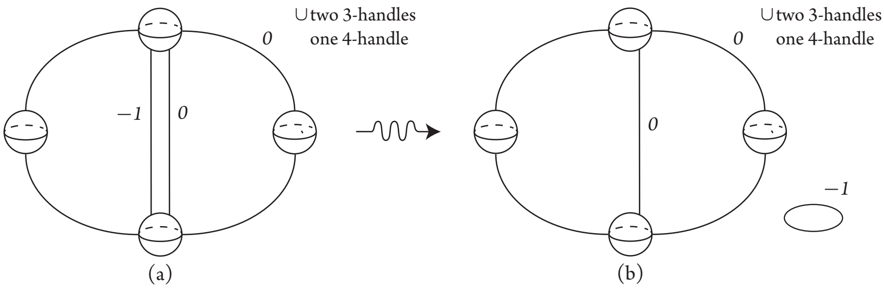

Figure 3 Figure showing the relevant part of the Kirby diagram of the cycle system $(a; a, b_2, \ldots, b_\ell)$, in Panel (a), and the result of sliding the -1-framed 2-handle over the 0-framed one, in Panel (b).

3.2. Boundary Lefschetz Fibrations over $(D^2, \partial D^2)$ with Few Lefschetz Singularities

The next step is to determine which manifolds admit boundary Lefschetz fibrations with only one or two Lefschetz singularities.

Lemma 3.4 *Let $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ be a boundary Lefschetz fibration with one Lefschetz singularity. Then f is not relatively minimal, we have $X \cong (S^1 \times S^3)\#\overline{\mathbb{C}P^2}$, and $\mathcal{D}$ is co-orientable.*

Proof This is [5, Example 8.4] but, in light of our discussion about blow-ups in terms of cycle systems, we can determine it directly. Indeed, any cycle system of f has the form $(a; b_1)$ such that $B_1 = \pm A^k$. Clearly, this is only possible when b_1 is either null-homotopic or parallel to a. In either case, f is not relatively minimal and can be blown down to a boundary fibration, which, by Theorem 3.1, is diffeomorphic to $S^1 \times S^3$. □

Lemma 3.5 *Let $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ be a relatively minimal boundary Lefschetz fibration with two Lefschetz singularities. Then $X \cong S^4$, and $\mathcal{D}$ is not co-orientable.*

Proof All cycle systems of f have the form $(a; b_1, b_2)$ with b_1 and b_2 essential and not parallel to a. At this level of difficulty, one can still perform direct computations. This was done by Hayano in [12]. The outcome is that, for suitable orientations, we must have that $b_1 = A^2(b_2) = b_2 + 2a$, and $\langle a, b_1\rangle = \langle a, b_2\rangle = 1$. Using the relations $AB_2A = B_2AB_2$, and $(AB_2)^3 = -1$, in $\mathcal{M}(T^2)$ we find that

$$\mu(f) = B_2B_1 = B_2A^2B_2A^{-2} = B_2A(AB_2A)AA^{-4} = B_2A(B_2AB_2)AA^{-4} = -A^{-4}.$$

The corresponding Kirby diagram for X is given in Figure 2(b). This particular type of Kirby diagram will appear repeatedly in this paper, so we deal with it in a separate claim.

Just as we mentioned in the proof of Theorem 3.2, when drawing the Kirby diagram for a boundary Lefschetz fibration, we must draw, from bottom to top, a 0-framed 2-handle corresponding to the boundary vanishing cycle and then -1-framed 2-handles for each Lefschetz singularity ordered counterclockwise. We will often want to make simplifications to the diagram which involve only the bottom two or three 2-handles.

Lemma 3.6 *Let $a, b \subset T^2$ be a dual pair of curves. Then the Kirby diagram associated to a cycle system of the form $(a; A^k(b), b, \dots) = (a; b + ka, b, \dots)$ is equivalent to that in Figure 4.* □

Proof The proof is a simple exercise: slide the 2-handle corresponding to $b + ka$ k times over the 0-framed 2-handle representing a, and once over the 2-handle corresponding to b. None of these manoeuvres interacts with the other handles. □

Proof of Lemma 3.5 continued. Using Theorem 3.6, we see that the boundary Lefschetz fibration is equivalent to the one depicted in Figure 4 with $k = 2$. If we slide the outer 2-handle over the 'a-handle' twice, we get the diagram depicted in Figure 5. There, a few things happen: the outer 0-framed 2-handle can be pushed out of the 1-handle and cancels a 3-handle. The 'a-handle' cancels one of the 1-handles, and the 'b-handle' cancels the other, so we are left with a 0-framed unknot which cancels the remaining 3-handle. After all this cancellation, we are left only with the 0-handle and the 4-handle; hence, X is S^4. □

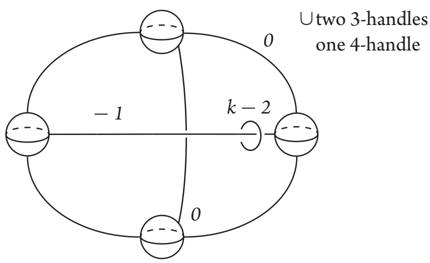

Figure 4 A Kirby diagram for cycle systems $(a; b + ka, b, \dots)$ after handle slides. Only the first three 2-handles are shown; the other handles appear above the diagram in their standard form. In particular, they are unlinked from the $(k-2)$-framed unknot.

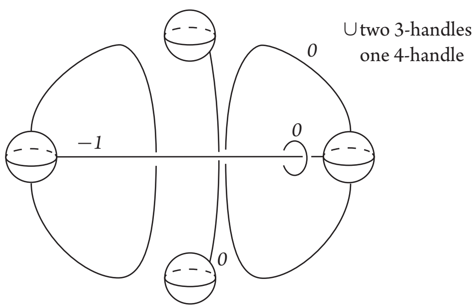

Figure 5 Kirby diagram for X after two handle slides. Now everything cancels.