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Real Analysis

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Great Clarendon Street, Oxford, OX2 6DP,
United Kingdom

Oxford University Press is a department of the University of Oxford.
It furthers the University's objective of excellence in research, scholarship,
and education by publishing worldwide. Oxford is a registered trade mark of
Oxford University Press in the UK and in certain other countries

First Edition published in 2016
Impression: 1

Published in the United States of America by Oxford University Press
198 Madison Avenue, New York, NY 10016, United States of America

British Library Cataloguing in Publication Data
Data available

Library of Congress Control Number: 2016939650

ISBN 978–0–19–879042–6 (hbk.)
ISBN 978–0–19–879043–3 (pbk.)

Printed and bound by
CPI Group (UK) Ltd, Croydon, CR0 4YY

Contents

Preface vii

1 Introduction and Preliminaries 1

1.1 Summability of systems of real numbers 2
1.2 Double series 4
1.3 Coin tossing 6
1.4 Metric spaces and normed vector spaces 9
1.5 Semi-continuities 17
1.6 The space $\ell^p(\mathbb{Z})$ 19
1.7 Compactness 24
1.8 Extension of continuous functions 35
1.9 Connectedness 36
1.10 Locally compact spaces 37

2 A Glimpse of Measure and Integration 40

2.1 Families of sets and set functions 40
2.2 Measurable spaces and measurable functions 43
2.3 Measure space and integration 47
2.4 Egoroff theorem and monotone convergence theorem 49
2.5 Concepts related to sets of measure zero 52
2.6 Fatou lemma and Lebesgue dominated convergence theorem 55
2.7 The space $L^p(\Omega, \Sigma, \mu)$ 57
2.8 Miscellaneous remarks 61

3 Construction of Measures 65

3.1 Outer measures 65
3.2 Lebesgue outer measure on $\mathbb{R}$ 67
3.3 Σ-algebra of measurable sets 70
3.4 Premeasures and outer measures 72
3.5 Carathéodory measures 80
3.6 Construction of Carathéodory measures 82
3.7 Lebesgue–Stieltjes measures 84
3.8 Borel regularity and Radon measures 88
3.9 Measure-theoretical approximation of sets in $\mathbb{R}^n$ 89
3.10 Riesz measures 94

3.11 Existence of nonmeasurable sets 99
3.12 The axiom of choice and maximality principles 100

4 Functions of Real Variables 104

4.1 Lusin theorem 104
4.2 Riemann and Lebesgue integral 106
4.3 Push-forward of measures and distribution of functions 110
4.4 Functions of bounded variation 114
4.5 Riemann–Stieltjes integral 119
4.6 Covering theorems and differentiation 126
4.7 Differentiability of functions of a real variable and related functions 140
4.8 Product measures and Fubini theorem 150
4.9 Smoothing of functions 156
4.10 Change of variables for multiple integrals 160
4.11 Polar coordinates and potential integrals 168

5 Basic Principles of Linear Analysis 179

5.1 The Baire category theorem 179
5.2 The open mapping theorem 184
5.3 The closed graph theorem 185
5.4 Separation principles 187
5.5 Complex form of Hahn–Banach theorem 196
5.6 Hilbert space 198
5.7 Lebesgue–Nikodym theorem 204
5.8 Orthonormal families and separability 207
5.9 The space $L^2[-\pi,\pi]$ 211
5.10 Weak convergence 221

6 L^p Spaces 226

6.1 Some inequalities 226
6.2 Signed and complex measures 229
6.3 Linear functionals on L^p 242
6.4 Modular distribution function and Hardy–Littlewood maximal function 247
6.5 Convolution 252
6.6 The Sobolev space $W^{k,p}(\Omega)$ 258

7 Fourier Integral and Sobolev Space H^s 265

7.1 Fourier integral for L^1 functions 265
7.2 Fourier integral on L^2 274
7.3 The Sobolev space H^s 277
7.4 Weak solutions of the Poisson equation 280
7.5 Fourier integral of probability distributions 285

Postscript 299

Bibliography *301*
List of Symbols *303*
Index *307*

Preface

This text is based on lecture notes I prepared for a first-year graduate analysis course that I taught at National Taiwan University from time to time over a span of more than 30 years. The choice and presentation of the materials have been strongly influenced by the experience I gained from offering courses such as Functional Analysis, Calculus of Variations, Applied Mathematics, Potential Theory, and Probability Theory.

Introduction of the Lebesgue measure and the corresponding theory of integration at the beginning of the twentieth century changed in a fundamental way the general view of the theory of functions of a real variable; due in essence to the fact that certain properties of sets and functions in terms of the Lebesgue measure (dubbed metric properties) were found to be very useful in resolving certain problems that had puzzled analysts for some time; some of these problems being connected to expansion of functions in trigonometric series. Also, the metric properties and the then new theory of integration provided appropriate means for classifying sets and functions for further in-depth studies. It is this view of the theory of functions of a real variable, framed in the setting of general measure and integration, that defines the core of the subject matter of real analysis that is adopted in this book. Needless to say, strong emphasis is placed on measures and functions defined in Euclidean n-space; in particular, function spaces defined in terms of Lebesgue measure on $\mathbb{R}^n$ are treated in some detail, including introduction of useful operations on these spaces, since this part of real analysis plays a fundamental role in many fields of mathematical discipline and lends a helping hand to the analysis of various problems in mathematical physics and engineering.

It is assumed that readers are familiar with the basic concepts and operations in linear algebra and have a fair acquaintance with the real number system. We also assume that they have had solid training in rigorous analysis as is usually offered in a course on advanced calculus. But for the reader's convenience, a concise treatment of analysis in metric spaces is included in Chapter 1. The first two sections of Chapter 1 consider a simple example, illustrating the key points of general measure and integration, while the third section brings out the necessity of constructing a suitable measure in order to model a simple random phenomenon—coin tossing. The reader is advised to proceed to Chapter 2 after studying the first three sections of Chapter 1, returning to the remaining part of the chapter for reference if required. Readers may start immediately at Chapter 2 if they are comfortable with abstract thinking.

We particularly stress the universality of the method of outer measure in constructing measures in the hope of conveying to readers the salient role played by regularity of measures in the study of sets and functions.

A chapter on the basic principles of linear analysis is also included, because some of these principles are applied in later chapters; and also because the concept of orthonormal basis in Hilbert space is an important interpretation of the Fourier expansion of functions in trigonometric series which provides a consensus on introduction of Lebesgue measure and integration.

Exercises are interposed within the main body of the text. Many of these are fairly easy, just meant to help to fix ideas that have been introduced. Statements that follow quite directly from propositions that are already established are usually left to be verified as exercises. Some of the exercises are more difficult, in that familiarity and insight into the methods presented in the text are required, together with some degree of ingenuity in order to resolve them; for such exercises, hints are usually provided.

The 慈澤 Foundation for Education has provided funds for classroom testing of the text in a core curriculum course on real analysis, offered by the Institute of Mathematics, Academia Sinica-Taipei, as well as for secretarial assistance during the preparation of the text. This is gratefully acknowledged.

Taipei, 2015 Fon-Che Liu

1 Introduction and Preliminaries

This chapter serves two purposes. The first purpose is to prepare the reader for a more systematic development in later chapters of the methods of real analysis through some introductory accounts of a few specific topics. The second purpose is, in view of the possible situation where some readers might not be conversant with basic concepts in elementary abstract analysis, to acquaint them with the fundamentals of abstract analysis. Nevertheless, readers are assumed to have some basic training in rigorous analysis as usually offered by courses in advanced calculus, and to have some acquaintance with the rudiments of linear algebra.

Throughout the book, the field of real numbers and that of complex numbers are denoted, respectively, by $\mathbb{R}$ and $\mathbb{C}$, while the set of all positive integers and the set of all integers are denoted by $\mathbb{N}$ and $\mathbb{Z}$ respectively.

The standard set-theoretical terminology is assumed; but terminology and notations regarding mappings will now be briefly recalled. If T is a mapping from a set A into a set B (expressed by $T : A \to B$), $T(a)$ denotes the element in B which is associated with $a \in A$ under the mapping T; for a subset **S** of **A**, the set $\{T(x) : x \in S\}$ is denoted by TS and is called the **image of *S* under *T***; thus $T\{a\} = \{T(a)\}$. $T(a)$ is sometimes simply written as Ta if no confusion is possible, and at times, an element a of a set and the set $\{a\}$ consisting of an element are not clearly distinguished as different objects. For example, Ta and $T\{a\}$ may not be distinguished and Ta is also called the image of a under T. A mapping $T : A \to B$ is said to be **one-to-one** or **injective** if $Ta = Ta'$ leads to $a = a'$, and is said to be **surjective** if $TA = B$; T is **bijective** if it is both injective and surjective. If $TA = B$, T is also referred to as a mapping from A onto B. Mappings are also called maps. Synonyms for maps are operators and transformations. As usual, a map from a set into $\mathbb{R}$ or $\mathbb{C}$ is called a function.

Some convenient notations for operations on sets are now introduced. Regarding a family $\mathcal{F} = \{A_\alpha\}_{\alpha \in I}$ of sets indexed by an index set I, the union $\bigcup_{\alpha \in I} A_\alpha$ is also expressed by $\bigcup \mathcal{F}$; if A and B are sets in a vector space and α a scalar, the set $\{x + y : x \in A, y \in B\}$ is denoted by $A + B$, and the set $\{\alpha x : x \in A\}$ by αA.

1.1 Summability of systems of real numbers

Summability of systems of real numbers is a special case in the theory of integration, to be treated in Chapter 2, but it reveals many essential points of the theory.

For a set S, the family of all nonempty finite subsets of S will be denoted by $F(S)$. Consider now a system $\{c_\alpha\}_{\alpha\in I}$ of real numbers indexed by an index set I. The system $\{c_\alpha\}_{\alpha\in I}$ will be denoted simply by $\{c_\alpha\}$ if the index set I is assumed either explicitly or implicitly. The system is said to be **summable** if there is $\ell \in \mathbb{R}$, such that for any $\varepsilon > 0$ there is $A \in F(I)$, with the property that whenever $B \in F(I)$ and $B \supset A$, then

$$\left|\sum_{\alpha\in B} c_\alpha - \ell\right| < \varepsilon. \tag{1.1}$$

Exercise 1.1.1 Show that if ℓ in the preceding definition exists, then it is unique.

If $\{c_\alpha\}$ is summable, the uniquely determined ℓ in the above definition is called the sum of $\{c_\alpha\}$ and is denoted by $\sum_{\alpha\in I} c_\alpha$.

Before we go further it is worthwhile remarking that the convergence of the series $\sum_{n=1}^{\infty} c_n$ depends on the order $1 < 2 < 3 < \cdots$ and $\sum_{n\in\mathbb{N}} c_n$, if it exists, does not depend on how $\mathbb{N}$ is ordered. Hence $\sum_{n\in\mathbb{N}} c_n$ may not exist while $\sum_{n=1}^{\infty} c_n$ exists. We will come back to this remark in Exercise 1.1.5.

Theorem 1.1.1 *If $\{c_\alpha^{(1)}\}_{\alpha\in I}$ and $\{c_\alpha^{(2)}\}_{\alpha\in I}$ are summable, then so is $\{ac_\alpha^{(1)} + bc_\alpha^{(2)}\}_{\alpha\in I}$ for fixed real numbers a and b, and*

$$\sum_{\alpha\in I}\left(ac_\alpha^{(1)} + bc_\alpha^{(2)}\right) = a\sum_{\alpha\in I} c_\alpha^{(1)} + b\sum_{\alpha\in I} c_\alpha^{(2)}.$$

Proof We may assume that $|a| + |b| > 0$, and for convenience put $\sum_{\alpha\in I} c_\alpha^{(1)} = l_1$, $\sum_{\alpha\in I} c_\alpha^{(2)} = l_2$. Let $\varepsilon > 0$ be given, there are A_1 and A_2 in $F(I)$ such that when B_1, B_2 are in $F(I)$ with $B_1 \supset A_1$, $B_2 \supset A_2$, we have $\left|\sum_{\alpha\in B_1} c_\alpha^{(1)} - l_1\right| < \frac{\varepsilon}{|a|+|b|}$ and $\left|\sum_{\alpha\in B_2} c_\alpha^{(2)} - l_2\right| < \frac{\varepsilon}{|a|+|b|}$. Choose now $A = A_1 \cup A_2$, then for $B \in F(I)$ with $B \supset A$, we have $\left|\sum_{\alpha\in B}(ac_\alpha^{(1)} + bc_\alpha^{(2)}) - (al_1 + bl_2)\right| \le |a|\left|\sum_{\alpha\in B} c_\alpha^{(1)} - l_1\right| + |b|\left|\sum_{\alpha\in B} c_\alpha^{(2)} - l_2\right| < \frac{|a|\varepsilon}{|a|+|b|} + \frac{|b|\varepsilon}{|a|+|b|} = \varepsilon$. This shows that $\{ac_\alpha^{(1)} + bc_\alpha^{(2)}\}$ is summable and $\sum_{\alpha\in I}(ac_\alpha^{(1)} + bc_\alpha^{(2)}) = al_1 + bl_2$. ∎

Theorem 1.1.2 *If $c_\alpha \ge 0\ \forall\alpha \in I$, then $\{c_\alpha\}$ is summable if and only if*

$$\left\{\sum_{\alpha\in A} c_\alpha : A \in F(I)\right\} \tag{1.2}$$

is bounded.

Proof That boundedness of (1.2) is necessary for $\{c_\alpha\}$ to be summable is left as an exercise. Now we show that boundedness of (1.2) is sufficient for $\{c_\alpha\}$ to be summable. Let ℓ be the least upper bound of $\{\sum_{\alpha\in A} c_\alpha : A \in F(I)\}$; for any $\varepsilon > 0$ there is $A \in F(I)$ such that

$$0 \leq \ell - \sum_{\alpha\in A} c_\alpha < \varepsilon. \tag{1.3}$$

Let now $B \in F(I)$ and $B \supset A$, then

$$\left|\sum_{\alpha\in B} c_\alpha - \ell\right| = \ell - \sum_{\alpha\in B} c_\alpha \leq \ell - \sum_{\alpha\in A} c_\alpha < \varepsilon.$$ ■

We note before moving on that if a subset S of $\mathbb{R}$ is bounded from above, then the least upper bound of S exists uniquely and is denoted by $\sup S$; similarly, if S is bounded from below, then the greatest lower bound exists uniquely and is denoted by $\inf S$. If $S = \{s_\alpha : \alpha \in I\}$, then $\inf S$ and $\sup S$ are also expressed, respectively, by $\inf_{\alpha\in I} s_\alpha$ and $\sup_{\alpha\in I} s_\alpha$.

Exercise 1.1.2 Show that boundedness of (1.2) is necessary for $\{c_\alpha\}$ to be summable.

Because of Theorem 1.1.2, if $\{c_\alpha\}$ is a system of nonnegative real numbers and is not summable, then we write $\sum_{\alpha\in I} c_\alpha = +\infty$. Hence, $\sum_{\alpha\in I} c_\alpha$ always has a meaning if $\{c_\alpha\}$ is a system of nonnegative numbers.

Theorem 1.1.3 (Cauchy criterion) *A system $\{c_\alpha\}$ is summable if and only if for any $\varepsilon > 0$ there is $A \in F(I)$, such that $|\sum_{\alpha\in B} c_\alpha| < \varepsilon$ whenever $B \in F(I)$ and $A \cap B = \emptyset$.*

Proof Sufficiency: Choose $A \in F(I)$ such that $|\sum_{\alpha\in B} c_\alpha| < 1$ for $B \in F(I)$, satisfying $A \cap B = \emptyset$, then obviously if $B \in F(I)$ with $B \cap A = \emptyset$, we have $\sum_{\alpha\in B} c_\alpha^+ < 1$, where $c_\alpha^+ = c_\alpha$ or 0 according to whether $c_\alpha \geq 0$ or < 0. Now, for $B \in F(I)$, we have

$$\sum_{\alpha\in B} c_\alpha^+ = \sum_{\alpha\in B\cap A} c_\alpha^+ + \sum_{\alpha\in B\setminus A} c_\alpha^+ < \sum_{\alpha\in A} c_\alpha^+ + 1,$$

i.e., $\{\sum_{\alpha\in B} c_\alpha^+ : B \in F(I)\}$ is bounded; hence by Theorem 1.1.2 $\{c_\alpha^+\}$ is summable.

Similarly $\{c_\alpha^-\}$ is summable, where $c_\alpha^- = -c_\alpha$ or 0 according to whether $c_\alpha \leq 0$ or > 0. Now $c_\alpha = c_\alpha^+ - c_\alpha^-$, hence $\{c_\alpha\}$ is summable by Theorem (1.1).

The necessary part is left for the reader to verify. ■

Exercise 1.1.3 Suppose that $\{c_\alpha\}_{\alpha\in I}$ is summable and that J is a nonempty subset of I. Show that (i) $\{c_\alpha\}_{\alpha\in J}$ is summable, and (ii) $\sum_{\alpha\in I} c_\alpha = \sum_{\alpha\in J} c_\alpha + \sum_{\alpha\in I\setminus J} c_\alpha$.

Exercise 1.1.4 Show that $\{c_\alpha\}$ is summable if and only if $\{|c_\alpha|\}$ is summable; show also that $\{c_\alpha\}$ is summable if and only if

$$\left\{\left|\sum_{\alpha\in A} c_\alpha\right| : A \in F(I)\right\}$$

is bounded.

Exercise 1.1.5 Show that $\{c_\alpha\}_{\alpha\in\mathbb{N}}$ is summable if and only if the series $\sum_{\alpha=1}^{\infty} c_\alpha$ is absolutely convergent. Show also that $\sum_{\alpha\in\mathbb{N}} c_\alpha = \sum_{\alpha=1}^{\infty} c_\alpha$ if $\{c_\alpha\}_{\alpha\in\mathbb{N}}$ is summable.

Exercise 1.1.6 Show that $\{c_\alpha\}_{\alpha\in I}$ is summable if and and only if (i) $\{\alpha \in I : c_\alpha \neq 0\}$ is finite or countable; and (ii) if $\{\alpha \in I : c_\alpha \neq 0\} = \{\alpha_1, \alpha_2, \ldots\}$ is infinite; then the series $\sum_{k=1}^{\infty} c_{\alpha_k}$ converges absolutely.

Exercise 1.1.7 Suppose that for each $n = 1, 2, 3, \ldots$, there is $A_n \in F(I)$, with the property that for each $A \in F(I)$, there is a positive integer N such that $A \subset A_n$ for all $n \geq N$. Show that if $\{c_\alpha\}_{\alpha\in I}$ is summable, then

$$\sum_{\alpha\in I} c_\alpha = \lim_{n\to\infty} \sum_{\alpha\in A_n} c_\alpha.$$

Give an example to show that it is possible that $\lim_{n\to\infty} \sum_{\alpha\in A_n} c_\alpha$ exists and is finite, but $\{c_\alpha\}$ is not summable.

Example 1.1.1 Suppose that $I = \bigcup_{n\in\mathbb{N}} I_n$, where I_n's are pairwise disjoint. Let $\{c_\alpha\}_{\alpha\in I}$ be summable, then $\sum_{\alpha\in I} c_\alpha = \sum_{n\in\mathbb{N}} (\sum_{\alpha\in I_n} c_\alpha)$. By Exercise 1.1.4, we may assume that $c_\alpha \geq 0$ for all $\alpha \in I$. It follows from $\sum_{\alpha\in I} c_\alpha = \sup\{\sum_{\alpha\in A} c_\alpha : A \in F(I)\}$ that $\sum_{\alpha\in I} c_\alpha \leq \sum_{n\in\mathbb{N}} (\sum_{\alpha\in I_n} c_\alpha)$. It remains to be seen that $\sum_{\alpha\in I} c_\alpha \geq \sum_{n\in\mathbb{N}} (\sum_{\alpha\in I_n} c_\alpha)$. Let $k \in \mathbb{N}$ and $\varepsilon > 0$. For each $n = 1, \ldots, k$, there is a finite set $A_n \subset I_n$ such that $\sum_{\alpha\in I_n} c_\alpha < \sum_{\alpha\in A_n} c_\alpha + \frac{\varepsilon}{k}$. Then, if we put $B_k = \bigcup_{n=1}^{k} A_n$, we have $\sum_{\alpha\in I} c_\alpha \geq \sum_{\alpha\in B_k} c_\alpha > \sum_{n=1}^{k} (\sum_{\alpha\in I_n} c_\alpha - \frac{\varepsilon}{k}) = \sum_{n=1}^{k} (\sum_{\alpha\in I_n} c_\alpha) - \varepsilon$; since $\varepsilon > 0$ is arbitrary, $\sum_{\alpha\in I} c_\alpha \geq \sum_{n=1}^{k} (\sum_{\alpha\in I_n} c_\alpha)$ for each $k \in \mathbb{N}$. Now let $k \to \infty$ to obtain $\sum_{\alpha\in I} c_\alpha \geq \sum_{n\in\mathbb{N}} (\sum_{\alpha\in I_n} c_\alpha)$. Observe from the proof that $\{\sum_{\alpha\in I_n} c_\alpha\}_{n\in\mathbb{N}}$ is summable.

We shall recognize in Example 2.3.3 that summability considered in this section is the integrability with respect to the counting measure on I.

1.2 Double series

Let $I = \mathbb{N} \times \mathbb{N} = \{(i,j) : i, j = 1, 2, \ldots\}$ and write c_{ij} for $c_{(i,j)}$. When the summability of the system $\{c_{ij}\}$ is in question, the system $\{c_{ij}\}$ is referred to as a **double series** and is denoted by $\sum c_{ij}$. Hence the double series $\sum c_{ij}$ is summable if $\{c_{ij}\} = \{c_{(i,j)}\}$ is summable, and $\sum_{(i,j)\in I} c_{ij}$ is called the sum of the double series $\sum c_{ij}$.

For a double sequence $\{a_{mn}\}$, we say that $\lim_{m,n\to\infty} a_{mn} = \ell$, if for any $\varepsilon > 0$ there is a positive integer N such that $|a_{mn} - \ell| < \varepsilon$ whenever $m, n \geq N$.

Theorem 1.2.1 *If the double series $\sum c_{ij}$ is summable, then*

$$\sum_{(i,j)\in I} c_{ij} = \lim_{m,n\to\infty} \sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} c_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{ij}.$$

Proof We show first that $\sum_{(i,j)\in I} c_{ij} = \lim_{n,m\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij}$. Let $\ell = \sum_{(i,j)\in I} c_{ij}$. Given $\varepsilon > 0$, there is $A \in F(I)$ such that

$$\left| \sum_{(i,j)\in B} c_{ij} - \ell \right| < \varepsilon$$

whenever $B \in F(I)$ and $B \supset A$. Let $N = \max\{i \vee j : (i,j) \in A\}$, where $i \vee j$ is the larger of i and j. For $n, m \geq N$, let $B_{mn} = \{(i,j) \in I : 1 \leq i \leq m,\ 1 \leq j \leq n\}$, then $B_{mn} \in F(I)$ and $B_{mn} \supset A$, hence

$$\left| \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} - \ell \right| = \left| \sum_{(i,j)\in B_{mn}} c_{ij} - \ell \right| < \varepsilon.$$

This means that $\ell = \lim_{m,n\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij}$.

Since $\sum_{(i,j)\in I} c_{ij} = \sum_{(i,j)\in I} c_{ij}^{+} - \sum_{(i,j)\in I} c_{ij}^{-}$, in the remaining part of the proof, we may assume that $c_{ij} \geq 0$ for all $(i,j) \in I$. Observe then that

$$\ell = \sup_{n,m\geq 1} \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij}.$$

Hence,

$$\ell \geq \lim_{m\to\infty} \left(\sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} \right) = \sum_{j=1}^{n} \sum_{i=1}^{\infty} c_{ij}$$

for each n and consequently

$$\ell \geq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} c_{ij}.$$

On the other hand,

$$\begin{aligned} \ell &= \sup_{n,m\geq 1} \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} \leq \sup_{n\geq 1} \left(\sum_{j=1}^{n} \sum_{i=1}^{\infty} c_{ij} \right) = \lim_{n\to\infty} \left(\sum_{j=1}^{n} \sum_{i=1}^{\infty} c_{ij} \right) \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} c_{ij}. \end{aligned}$$

We have shown that $\ell = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} c_{ij}$; similarly,

$$\ell = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_{ij}.$$ ■

Example 1.2.1 If $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are summable, then the double series $\sum a_n b_m$ is summable and $\sum_{(n,m)\in\mathbb{N}\times\mathbb{N}} a_n b_m = (\sum_{n\in\mathbb{N}} a_n)(\sum_{m\in\mathbb{N}} b_m)$. That $\sum a_n b_m$ is summable follows from Exercise 1.1.4 and the observation that $\{\sum_{(n,m)\in A} |a_n b_m| : A \in F(\mathbb{N}\times\mathbb{N})\}$ is bounded from above by $(\sum_{n\in\mathbb{N}} |a_n|)\cdot(\sum_{m\in\mathbb{N}} |b_m|)$. Then, by Theorem 1.2.1, $\sum_{(n,m)\in\mathbb{N}\times\mathbb{N}} a_n b_m = \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} a_n b_m = (\sum_{n\in\mathbb{N}} a_n)(\sum_{m\in\mathbb{N}} b_m)$. For $k \geq 2$ in $\mathbb{N}$, put $A_k = \{(n,m)\in\mathbb{N}\times\mathbb{N} : n+m=k\}$; then $\sum_{(n,m)\in\mathbb{N}\times\mathbb{N}} a_n b_m = \sum_{\substack{k\in\mathbb{N}\\ k\geq 2}} (\sum_{(n,m)\in A_k} a_{nm})$ from Example 1.1.1. The system $\{\sum_{(n,m)\in A_k} a_n b_m\}_{k\geq 2}$ is called the product of $\{a_n\}$ and $\{b_n\}$; we have shown that the sum of the product is the product of the sums.

The following exercise complements Theorem 1.2.1.

Exercise 1.2.1 Copy the proof of Theorem 1.2.1 to show that if $c_{ij} \geq 0$ for all i and j in $\mathbb{N}$, then the conclusion of Theorem 1.2.1 still holds, even if $\sum_{(i,j)\in I} c_{ij} = \infty$ (recall that for a system $\{c_\alpha\}$ of nonnegative numbers, $\sum_\alpha c_\alpha = \infty$ means that $\{c_\alpha\}$ is not summable).

Remark For i, j in $\mathbb{N}$, let

$$c_{ij} = \begin{cases} 1 & \text{if } i = j; \\ -1 & \text{if } j = i+1; \\ 0 & \text{otherwise,} \end{cases}$$

then $\sum c_{ij}$ is not summable and $0 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{ij} \neq \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} c_{ij} = 1$.

1.3 Coin tossing

A pair of symbols H and T, associated, respectively, with nonnegative numbers p and q such that $p + q = 1$ is called a **Bernoulli trial** and is denoted by $B(p, q)$. A Bernoulli trial $B(p, q)$ is a mathematical model for the tossing of a coin, of which heads occur with probability p and tails turn out with probability q; this explains the symbols H and T. In particular, $B(\frac{1}{2}, \frac{1}{2})$ models the tossing of a fair coin.

In this section, we consider the first step towards construction of a mathematical model for a sequence of tossing of a fair coin. For convenience, we replace H and T by 1 and 0 in this order; then an infinite sequence $\omega = (\omega_1, \omega_2, \ldots, \omega_k, \ldots)$ of 0's and 1's represents a realization of a sequence of coin tossing. Let

$$\Omega = \{0,1\}^\infty := \{\omega = (\omega_k),\ \omega_k = 0 \text{ or } 1 \text{ for each } k\},$$

where we adopt the usual convention of expressing an infinite sequence $(\omega_1, \ldots, \omega_k, \ldots)$ by (ω_k) with the understanding that ω_k is the entry at the k-th position of the sequence. In terminology of probability theory, elements in Ω are called **sample points** of a sequence of coin tossings and Ω is called the **sample space** of the sequence of tossings. Subsets of Ω will often be referred to as **events**. Now for $n \in \mathbb{N}$, let

$$\Omega_n = \{0, 1\}^n := \{(\varepsilon_1, \ldots, \varepsilon_n) : \varepsilon_j \in \{0, 1\},\ j = 1, \ldots, n\},$$

and for $(\varepsilon_1, \ldots, \varepsilon_n) \in \Omega_n$, call the set

$$E(\varepsilon_1, \ldots, \varepsilon_n) = \{\omega = (\omega_k) \in \Omega : \omega_k = \varepsilon_k,\ k = 1, \ldots, n\}$$

an **elementary cylinder**; but if n is to be emphasized, it is called an **elementary cylinder of rank n**. A finite union of elementary cylinders is called a **cylinder** in Ω. Since intersection of two elementary cylinders is either empty or an elementary cylinder, every cylinder in Ω can be expressed as a disjoint union of elementary cylinders; in fact, if Z is a cylinder in Ω, there is $n \in \mathbb{N}$ and $H \subset \Omega_n$ such that

$$Z = \bigcup\{E(\varepsilon_1, \ldots, \varepsilon_n) : (\varepsilon_1, \ldots, \varepsilon_n) \in H\},$$

of which one notes that $E(\varepsilon_1, \ldots, \varepsilon_n)$'s are mutually disjoint. Of course, a cylinder Z can be expressed as above in many ways. We denote by $\mathcal{Q}$ the family of all cylinders in Ω. Since $\Omega = E(0) \cup E(1)$, $\Omega \in \mathcal{Q}$; $\emptyset$ is also in $\mathcal{Q}$, because it is the union of an empty family of elementary cylinders.

Exercise 1.3.1 Show that $\mathcal{Q}$ is an algebra of subsets of Ω, in the sense that $\mathcal{Q}$ satisfies the following conditions: (i) $\Omega \in \mathcal{Q}$; (ii) if $Z \in \mathcal{Q}$, then $Z^c = \Omega \backslash Z$ is in $\mathcal{Q}$; and (iii) if Z_1, Z_2 are in $\mathcal{Q}$, then $Z_1 \cup Z_2$ is in $\mathcal{Q}$.

For an event Z in $\mathcal{Q}$, we define its probability $P(Z)$ as follows. First, for an elementary cylinder $C = E(\varepsilon_1, \ldots, \varepsilon_n)$, define $P(C) = (\frac{1}{2})^n$; intuitively, this definition of $P(C)$ means that we consider the modeling of a sequence of independent tossing of a fair coin. Now if $Z \in \mathcal{Q}$ is given by

$$Z = \bigcup\{E(\varepsilon_1, \ldots, \varepsilon_n) : (\varepsilon_1, \ldots, \varepsilon_n) \in H\},$$

where $H \subset \Omega_n$, then define

$$P(Z) = \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in H} P(E(\varepsilon_1, \ldots, \varepsilon_n)) = \#H \cdot 2^{-n},$$

where $\#H$ is the number of elements in H. We claim that $P(Z)$ is well defined. Actually if Z is also given by

$$Z = \bigcup\{E(\varepsilon_1, \ldots, \varepsilon_m) : (\varepsilon_1, \ldots, \varepsilon_m) \in H'\},$$

where $H' \subset \Omega_m$, then (assuming $m \geq n$) $H' = \{(\varepsilon_1, \ldots, \varepsilon_m) \in \Omega_m : (\varepsilon_1, \ldots, \varepsilon_n) \in H\}$ and therefore $\#H' = \#H \cdot 2^{m-n}$; consequently

$$\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in H'} P(E(\varepsilon_1, \ldots, \varepsilon_m)) = \#H' \cdot 2^{-m} = \#H \cdot 2^{m-n} \cdot 2^{-m}$$
$$= \#H \cdot 2^{-n} = \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in H} P(E(\varepsilon_1, \ldots, \varepsilon_n)),$$

implying that the definition of $P(Z)$ is independent of how Z is expressed as a finite disjoint union of elementary cylinders of a given rank. We complete the definition of P by letting $P(\emptyset) = 0$. Note that $P(\Omega) = 1$.

Exercise 1.3.2

(i) Show that P is additive on $\mathcal{Q}$, i.e. $P(Z_1 \cup Z_2) = P(Z_1) + P(Z_2)$ if Z_1, Z_2 are disjoint elements of $\mathcal{Q}$.

(ii) For $k \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}$, put $E^k_\varepsilon = \{\omega \in \Omega : \omega_k = \varepsilon\}$. Show that

$$P(E^{k_1}_{\varepsilon_1} \cap \cdots \cap E^{k_n}_{\varepsilon_n}) = \prod_{j=1}^{n} P(E^{k_j}_{\varepsilon_j}) = 2^{-n}$$

for any finite sequence $k_1 < k_2 < \cdots < k_n$ in $\mathbb{N}$.

From now on we write $d_j(\omega) = \omega_j, j = 1, 2, \ldots,$ if $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$; and for each n define a function S_n on Ω by

$$S_n(\omega) = \sum_{j=1}^{n} d_j(\omega).$$

Exercise 1.3.3 Show that, for each $k = 0, 1, 2, \ldots, n$, the set $\{S_n = k\} := \{\omega \in \Omega : S_n(\omega) = k\}$ is in $\mathcal{Q}$ and

$$P(\{S_n = k\}) = \binom{n}{k} \frac{1}{2^n},$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

For a given realization ω of a sequence of independent coin tossing, $S_n(\omega)$ is the number of heads that appear in the first n tosses and $\frac{S_n(\omega)}{n}$ measures the relative frequency of appearance of heads in the first n tosses. Let

$$E = \left\{\omega \in \Omega : \lim_{n\to\infty} \frac{S_n(\omega)}{n} = \frac{1}{2}\right\};$$

E is easily seen to be not in $\mathcal{Q}$. Nevertheless, we expect that P can be extended to be defined on a larger family of sets than $\mathcal{Q}$ in such a way that $P(A)$ can be interpreted as the probability of event A, and such that $P(E)$ is defined with value 1. We expect $P(E) = 1$, because this is what a fair coin is accounted for intuitively. Discussion of the subject matter of this section will be continued in Example 1.7.1, Example 2.1.1, Example 3.4.6, and Example 7.5.2; and eventually we shall answer positively to this expectation in the paragraph following Corollary 7.5.3.

1.4 Metric spaces and normed vector spaces

The usefulness of the concept of continuity has already surfaced in elementary analysis of functions defined on an interval. This section considers a structure on a set which allows one to speak of "nearness" for elements in the set, so that a concept of continuity can be defined for functions defined on the set, parallel to that for functions defined on an interval of the real line. We shall not treat the most general situation; instead, we consider the situation where an abstract concept of distance can be defined between elements of the set, because this situation abounds sufficiently for our purposes later. When the set considered is a vector space, it is natural to consider the case where the distance defined and the linear structure of the set mingle well, as in the case of a real line or Euclidean plane. This leads to the concept of normed vector spaces.

Let M be a nonempty set and let $\rho : M \times M \to [0, +\infty)$ satisfy (i) $\rho(x, y) = \rho(y, x) \geq 0$ for all $x, y \in M$ and $\rho(x, y) = 0$ if and only if $x = y$; (ii) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all x, y, and z in M. Such a ρ is then called a **metric** on M, and (M, ρ) is called a **metric space**. Usually we say that M is a metric space with metric ρ, or simply that M is a metric space when a certain metric ρ is explicitly or implicitly implied. For a nonempty subset S of M the restriction of ρ to $S \times S$ is a metric on S which will also be denoted by ρ. The metric space (S, ρ) is called a subspace of (M, ρ) and ρ is called the metric on S inherited from M. Unless stated otherwise, if S is a subset of a metric space M, S is equipped with the metric inherited from M. For a nonempty subset A of M, the **diameter** of A, denoted diam A, is defined by

$$\operatorname{diam} A := \sup_{x,y \in A} \rho(x, y);$$

while $\operatorname{diam} A = 0$ if $A = \emptyset$.

A subset A of M is said to be **bounded** if $\operatorname{diam} A < \infty$. In other words, A is bounded if $\{\rho(x, x_0) : x \in A\}$ is a bounded set in $\mathbb{R}$ for every $x_0 \in M$.

Elements of a metric space are often called **points** of the space.

Example 1.4.1 Let $M = \mathbb{R}^n$ and for $x, y \in \mathbb{R}^n$ let $\rho(x, y) = |x - y|$, where $|x| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$ if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. To show that ρ is a metric on $\mathbb{R}^n$ we first establish the well-known **Schwarz inequality**: $|x \cdot y| \leq |x||y|$ if $x, y \in \mathbb{R}^n$, where, for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, $x \cdot y := \sum_{i=1}^n x_i y_i$ is called the inner product of x and y. For this purpose we note first that for $x \in \mathbb{R}^n$, $|x|^2 = x \cdot x$ and that we may assume that $x \neq 0$ and $y \neq 0$, hence $|x| > 0$ and $|y| > 0$. For $t \in \mathbb{R}$, we have

$$\begin{aligned} 0 \leq |x + ty|^2 &= (x + ty) \cdot (x + ty) = |x|^2 + 2t(x \cdot y) + t^2|y|^2 \\ &= \left(|x| + t|y|\right)^2 + 2t\left(x \cdot y - |x||y|\right), \end{aligned}$$

from which by taking $t = -|x|/|y|$ we obtain $x \cdot y \leq |x||y|$. Then $|x \cdot y| \leq |x||y|$ follows, because $-(x \cdot y) \leq |x|| - y| = |x||y|$. Now for x, y, and z in $\mathbb{R}^n$, we have

$$\begin{aligned} \rho(x,z)^2 &= |x - z|^2 = |x - y + y - z|^2 = |x - y|^2 + 2(x - y) \cdot (y - z) + |y - z|^2 \\ &\leq |x - y|^2 + 2|x - y||y - z| + |y - z|^2 = \left(|x - y| + |y - z|\right)^2 \\ &= \left[\rho(x,y) + \rho(y,z)\right]^2, \end{aligned}$$

i.e.

$$\rho(x,z) \leq \rho(x,y) + \rho(y,z).$$

Hence $\mathbb{R}^n$ is a metric space with metric ρ defined above. This metric is called the **Euclidean metric** on $\mathbb{R}^n$. Unless stated otherwise, $\mathbb{R}^n$ is considered as a metric space with this metric, then $\mathbb{R}^n$ is called the n-dimensional Euclidean space.

Similarly, $\mathbb{C}^n$ is a metric space, with the metric ρ defined by $\rho(\zeta,\eta) = (\sum_{j=1}^{n} |\zeta_j - \eta_j|^2)^{1/2}$ for $\zeta = (\zeta_1, \ldots, \zeta_n)$ and $\eta = (\eta_1, \ldots, \eta_2)$ in $\mathbb{C}^n$. $\mathbb{C}^n$ with this metric is called the **n-dimensional unitary space**. This follows, as in the case of the Euclidean metric for $\mathbb{R}^n$, from the Schwarz inequality $|\zeta \cdot \eta| \leq |\zeta||\eta|$ for ζ, η in $\mathbb{C}^n$, where $\zeta \cdot \eta = \sum_{j=1}^{n} \zeta_j \bar{\eta}_j$ and $|\zeta| = (\sum_{j=1}^{n} |\zeta_j|^2)^{\frac{1}{2}}$. As before, if $t \in \mathbb{R}$, we have

$$\begin{aligned} 0 \leq |\zeta + t\eta|^2 &= (\zeta + t\eta) \cdot (\zeta + t\eta) = |\zeta|^2 + 2t\,\mathrm{Re}\zeta \cdot \eta + t^2|\eta|^2 \\ &= (|\zeta| + t|\eta|)^2 + 2t\{\mathrm{Re}\,\zeta \cdot \eta - |\zeta||\eta|\}, \end{aligned}$$

from which we infer that $\mathrm{Re}\,\zeta \cdot \eta \leq |\zeta||\eta|$ by choosing $t = -|\zeta||\eta|^{-1}$ if $\eta \neq 0$. Then, $|\zeta \cdot \eta| \leq |\zeta||\eta|$ follows from replacing ζ by $e^{-i\theta}\zeta$ if $\zeta \cdot \eta = |\zeta \cdot \eta|e^{i\theta}$. Note that for a complex number α, $\bar{\alpha}$ denotes the conjugate of α, while $\mathrm{Re}\,\alpha$ denotes the real part of α.

Example 1.4.2 For a closed finite interval $[a,b]$ in $\mathbb{R}$, let $C[a,b]$ denote the space of all real-valued continuous functions defined on $[a,b]$. For $f, g \in C[a,b]$, let $\rho(f,g) = \max_{a \leq t \leq b} |f(t) - g(t)|$. It is easily verified that $C[a,b]$ is a metric space with metric ρ so defined. Unless stated otherwise, $C[a,b]$ is equipped with this metric, which is often referred to as the **uniform metric** on $C[a,b]$. $C[a,b]$ is also used to denote the space of all complex-valued continuous functions on $[a,b]$ with metric defined similarly. When $C[a,b]$ denotes the latter space, it shall be explicitly indicated.

Exercise 1.4.1 Show that $\mathbb{R}^n$ is also a metric space, with metric ρ defined by $\rho(x,y) = \max_{1 \leq i \leq n} |x_i - y_i|$ if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

A map from $\mathbb{N}$, the set of all positive integers, to a set M is called a **sequence** in M or a **sequence of elements** of M. Such a sequence will be denoted by $\{x_n\}$, where x_n

is the image of the positive integer n under the mapping. If $\{x_n\}$ is a sequence in M, then $\{x_{n_k}\}$ is called a subsequence of $\{x_n\}$ if $n_1 < n_2 < \cdots < n_k < \cdots$ is a subsequence of $\{n\}$. A sequence $\{x_n\}$ in a metric space M is said to converge to $x \in M$ if for any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $\rho(x_n, x) < \varepsilon$ whenever $n \geq n_0$. Since x is uniquely determined, x is called the **limit** of $\{x_n\}$ and is denoted by $\lim_{n\to\infty} x_n$. That $x = \lim_{n\to\infty} x_n$ is often expressed by $x_n \to x$. If $\lim_{n\to\infty} x_n$ exists, then we say that $\{x_n\}$ converges in M and $\{x_n\}$ is referred to as a **convergent sequence**. A sequence $\{x_n\}$ in M is usually expressed by $\{x_n\} \subset M$ by abuse of notation, and therefore $\{x_n\}$ also denotes the range of the sequence $\{x_n\}$. A sequence in M is said to be bounded if its range is bounded.

Example 1.4.3 $\{f_n\} \subset C[a, b]$ converges if and only if $f_n(x)$ converges uniformly for $x \in [a, b]$.

A sequence $\{x_n\} \subset M$ is called a **Cauchy sequence** if for any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\rho(x_n, x_m) < \varepsilon$ whenever $n, m \geq n_0$. Clearly, a Cauchy sequence is bounded.

Exercise 1.4.2 Show that if $\{x_n\} \subset M$ converges, then $\{x_n\}$ is a Cauchy sequence.

Exercise 1.4.3 Let $\{x_n\}$ be a Cauchy sequence. Show that if $\{x_n\}$ has a convergent subsequence, then $\{x_n\}$ converges.

A metric space M is called ***complete*** if every Cauchy sequence in M converges in M.

Exercise 1.4.4 Show that both $\mathbb{R}^n$ and $C[a, b]$ are complete.

Exercise 1.4.5 If instead of the uniform metric we equip $C[a, b]$ with a new metric ρ', defined by

$$\rho'(f, g) = \int_a^b |f(t) - g(t)|dt$$

for f, g in $C[a, b]$, show that $C[a, b]$ is not complete when considered as a metric space with metric ρ'.

Exercise 1.4.6 Show that any nonempty set M can be considered as a complete metric space by defining $\rho(x, y) = 0$ or 1 depending on $x = y$ or $x \neq y$. Such a metric ρ is said to be discrete.

Let M_1, M_2 be metric spaces with metrics ρ_1 and ρ_2 respectively. A map $T : M_1 \to M_2$ is said to be **continuous at** $x \in M_1$ if for any $\varepsilon > 0$, there is $\delta > 0$ such that $\rho_2(T(x), T(y)) < \varepsilon$ whenever $\rho_1(x, y) < \delta$. If T is continuous at every point of M_1, then T is said to be continuous on M_1 and is called a **continuous map** from M_1 into M_2. A continuous map from a metric space M into $\mathbb{R}$ or $\mathbb{C}$ is called a **continuous function** on M and is generically denoted by f. The space of all continuous real(complex)-valued functions on a metric space M is denoted by $C(M)$; $C(M)$ is a real- or complex vector space depending on whether the functions in question are real- or complex-valued.

A point x of a set A in a metric space is called an **interior** point of A if there is $\varepsilon > 0$ such that $y \in A$ whenever $\rho(x, y) < \varepsilon$; the set of all interior points of A is denoted by $\mathring{A}$. A set G in a metric space M is said to be **open** if $\mathring{G} = G$. The complement of an open set is called a **closed** set. For $x \in M$ and $r > 0$, let $B_r(x) = \{y \in M : \rho(y,x) < r\}$ and $C_r(x) = \{y \in M : \rho(y,x) \leq r\}$. It is easily verified that $B_r(x)$ is an open set and $C_r(x)$ is a closed set. $B_r(x)$ ($C_r(x)$) is usually referred to as the **open (closed)** ball centered at x and with radius r. A point $x \in M$ is said to be **isolated** if $B_r(x) = \{x\}$ for some $r > 0$. A set $N \subset M$ is called a **neighborhood** of $x \in M$ if N contains an open set which contains x; similarly, if N contains an open set which contains a set A, then N is called a **neighborhood** of A. It is clear that a sequence $\{x_n\}$ in M converges to $x \in M$ if and only if, for any neighborhood N of x, there is $n_0 \in \mathbb{N}$ such that $x_n \in N$ whenever $n \geq n_0$. One notes that if x_0 is an isolated point of M, then any map T from M into any metric space is continuous at x_0.

Note that open sets depend on the metric ρ, and when ρ is to be emphasized, an open set in a metric space with metric ρ is more precisely said to be open w.r.t. ρ.

Exercise 1.4.7 Let M_1, M_2 be metric spaces and let $T : M_1 \to M_2$.

(i) Show that T is continuous at $x \in M_1$ if and only if, for any sequence $\{x_n\} \subset M_1$ with $\lim_{n\to\infty} x_n = x$, it holds that $\lim_{n\to\infty} T(x_n) = T(x)$ in M_2; also show that T is continuous at $x \in M_1$ if and only if, for every sequence $\{x_n\} \subset M_1$ with $\lim_{n\to\infty} x_n = x$, it holds that $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} T(x_{n_k}) = T(x)$.

(ii) Show that T is continuous at $x \in M_1$ if and only if, for any neighborhood N of $T(x)$ in M_2, the set $T^{-1}N = \{y \in M_1 : T(y) \in N\}$ is a neighborhood of x in M_1.

(iii) Show that T is continuous on M_1 if and only if for any open set $G_2 \subset M_2$, $T^{-1}G_2$ is an open subset of M_1.

Exercise 1.4.8 Let $\mathcal{T}$ be the family of all open subsets of a metric space M. Show that:

(i) $\emptyset$ and M are in $\mathcal{T}$;

(ii) $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$;

(iii) if $\{A_i\}_{i\in I} \subset \mathcal{T}$, then $\bigcup_{i\in I} A_i \in \mathcal{T}$, where I is any index set.

Suppose that (M_1, ρ_1) and (M_2, ρ_2) are metric spaces. Let $M_1 \times M_2 := \{(x,y) : x \in M_1, y \in M_2\}$ be the **Cartesian product** of M_1 and M_2; define a metric ρ on $M_1 \times M_2$ by

$$\rho((x,y),(x',y')) = \rho_1(x,x') + \rho_2(y,y')$$

for (x,y) and (x',y') in $M_1 \times M_2$. It is easily verified that ρ is actually a metric on $M_1 \times M_2$. With this metric ρ, $M_1 \times M_2$ is called the **product space** of M_1 and M_2 as metric space.

Exercise 1.4.9 Let $M_1 \times M_2$ be the product space of metric spaces M_1 and M_2.

(i) For $A \subset M_1$ and $B \subset M_2$, show that $A \times B$ is open in $M_1 \times M_2$ if and only if both A and B are open in M_1 and M_2 respectively.

(ii) Let G be an open set in $M_1 \times M_2$; show that $G_1 := \{x \in M_1 : (x,y) \in G$ for some y in $M_2\}$ and $G_2 := \{y \in M_2 : (x,y) \in G$ for some x in $M_1\}$ are open in M_1 and M_2 respectively.

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let E be a vector space over $\mathbb{K}$. Elements of $\mathbb{K}$ are called scalars. Suppose that for each $x \in E$, there is a nonnegative number $\|x\|$ associated with it so that:

(i) $\|x\| = 0$ if and only if x is the zero element of E;

(ii) $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{K}$ and $x \in E$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all x, y in E (**triangle inequality**).

Then E is called a **normed vector space** (abbreviated as **n.v.s.**) with **norm** $\|\cdot\|$, and $\|\cdot\|$ is called a **norm on** E.

If E is a n.v.s., for x, y in E, let

$$\rho(x, y) = \|x - y\|,$$

then ρ is a metric on E and is called the metric associated with norm $\|\cdot\|$. Unless stated otherwise, we always consider this metric for a n.v.s.. The n.v.s. E with norm $\|\cdot\|$ is denoted by $(E, \|\cdot\|)$ if the norm $\|\cdot\|$ is to be emphasized.

Lemma 1.4.1 *Suppose that E is a n.v.s. and $x_n \to x$ in E, then $\|x\| = \lim_{n\to\infty} \|x_n\|$. In other words, $\|\cdot\|$ is a continuous function on E.*

Proof The lemma follows from the following sequence of triangle inequalities:

$$\|x_n\| - \|x_n - x\| \leq \|x\| \leq \|x_n\| + \|x_n - x\|. \qquad \blacksquare$$

A *normed vector space* is called a **Banach space** if it is a complete metric space.

Both $\mathbb{R}^n$ and $C[a, b]$ are Banach spaces, with norms given respectively by $\|x\| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\|f\| = \max_{a\leq t\leq b} |f(t)|$ for $f \in C[a, b]$. Similarly, the unitary space $\mathbb{C}^n$ is a Banach space with norm $\|z\| = (\sum_{j=1}^n |z_j|^2)^{\frac{1}{2}}$ for $z = (z_1, \ldots, z_n)$ in $\mathbb{C}^n$. The norms defined above for $\mathbb{R}^n$ and $\mathbb{C}^n$ are called respectively the **Euclidean norm** and the **unitary norm** and are denoted by $|\cdot|$ in both cases, in accordance with the notations introduced in Example 1.4.1; note that their associated metrics are the metrics introduced for $\mathbb{R}^n$ and C^n in Example 1.4.1. The norm defined for $C[a, b]$ is called the **uniform norm**; its associated metric is the uniform metric defined in Example 1.4.2.

A class of well-known Banach spaces, the l^p spaces, will be introduced in §1.6. This class of Banach spaces anticipates the important and more general class of L^p spaces treated in Section 2.7 and in Chapter 6.

In the remaining part of this section, linear maps from a normed vector space E into a normed vector space F over the same field $\mathbb{R}$ or $\mathbb{C}$ are considered. Recall that a map T from a vector space E into a vector space F over the same field is said to be **linear** if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$, for all x, y in E and all scalars α, β. Linear maps are more often called **linear transformations** or **linear operators**.

Exercise 1.4.10 Suppose that T is a linear transformation from E into F. Show that T is continuous on E if and only if it is continuous at one point.

Theorem 1.4.1 *Let T be a linear transformation from E into F, then T is continuous if and only if there is $C \geq 0$ such that*

$$\|Tx\| \leq C\|x\|$$

for all $x \in E$.

Proof If there is $C \geq 0$ such that $\|Tx\| \leq C\|x\|$ holds for all $x \in E$, then T is obviously continuous at $x = 0$ and hence by Exercise 1.4.10 is continuous on E.

Conversely, suppose that T is continuous on E, and is hence continuous at $x = 0$. There is then $\delta > 0$ such that if $\|x\| \leq \delta$, then $\|Tx\| \leq 1$. Let now $x \in E$ and $x \neq 0$, then $\left\| \frac{\delta}{\|x\|} x \right\| = \delta$, so $\left\| T\left(\frac{\delta}{\|x\|} x\right) \right\| \leq 1$. Thus $\|Tx\| \leq \frac{1}{\delta}\|x\|$. If we choose $C = \frac{1}{\delta}$, then $\|Tx\| \leq C\|x\|$ for all $x \in E$. ■

From this theorem it follows that if T is a continuous linear transformation from E into F, then

$$\|T\| := \sup_{x \in E,\, x \neq 0} \frac{\|Tx\|}{\|x\|} < +\infty,$$

and is the smallest C for which $\|Tx\| \leq C\|x\|$ for all $x \in E$. $\|T\|$ is called the norm of T. Of course, $\|T\|$ can be defined for any linear transformation T from E into F; then $\|Tx\| \leq \|T\|\|x\|$ holds always and T is continuous if and only if $\|T\| < +\infty$. Hence a continuous linear transformation is also called a **bounded** linear transformation.

Exercise 1.4.11 Show that $\|T\| = \sup_{x \in E, \|x\|=1} \|Tx\|$.

Exercise 1.4.12 Let $L(E, F)$ be the space of all bounded linear transformations from E into F. Show that it is a normed vector space with norm $\|T\|$ for $T \in L(E, F)$ as previously defined.

Remark Any linear map T from a Euclidean space $\mathbb{R}^n$ into a Euclidean space $\mathbb{R}^m$ is continuous. This follows from the representation of T by a matrix (a_{jk}), $1 \leq j \leq m$, $1 \leq k \leq n$, of real entries, in the sense that if $y = Tx$, then $y_j = \sum_{k=1}^{n} a_{jk} x_k$, $j = 1, \ldots, m$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$, by observing that

$$|y|^2 = \sum_{j=1}^{m} \left(\sum_{k=1}^{n} a_{jk} x_k \right)^2 \leq \left(\sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk}^2 \right) |x|^2.$$

Theorem 1.4.2 *If F is a Banach space, then $L(E, F)$ is a Banach space.*

Proof Let $\{T_n\}$ be a Cauchy sequence in $L(E,F)$. Since

$$\|T_nx - T_mx\| = \|(T_n - T_m)x\| \leq \|T_n - T_m\| \cdot \|x\|,$$

$\{T_nx\}$ is a Cauchy sequence in F for each $x \in E$. Since F is complete, $\lim_{n\to\infty} T_nx$ exists. Put $Tx = \lim_{n\to\infty} T_nx$. T is obviously a linear transformation from E into F.

We claim now $T \in L(E,F)$. Since $\{T_n\}$ is Cauchy, $\|T_n\| \leq C$ for some $C > 0$, and for all n. Now, from Lemma 1.4.1,

$$\|Tx\| = \lim_{n\to\infty} \|T_nx\| \leq \left(\sup_n \|T_n\|\right)\|x\| \leq C\|x\|$$

for each $x \in E$. Hence T is a bounded linear transformation.

We show next, $\lim_{n\to\infty} \|T_n - T\| = 0$. Given $\varepsilon > 0$, there is n_0 such that $\|T_n - T_m\| < \varepsilon$ if $n, m \geq n_0$. Let $n \geq n_0$, we have

$$\begin{aligned}
\|T_n - T\| &= \sup_{x\in E, \|x\|=1} \|T_nx - Tx\| \\
&= \sup_{x\in E, \|x\|=1} \lim_{m\to\infty} \|T_nx - T_mx\| \\
&\leq \sup_{x\in E, \|x\|=1} \left(\sup_{m\geq n_0} \|T_n - T_m\|\right)\|x\| \\
&\leq \sup_{x\in E, \|x\|=1} \varepsilon\|x\| = \varepsilon;
\end{aligned}$$

this shows that $\lim_{n\to\infty} \|T_n - T\| = 0$, or $\lim_{n\to\infty} T_n = T$. Thus the sequence $\{T_n\}$ has a limit in $L(E,F)$. Therefore $L(E,F)$ is complete. ■

$L(E,\mathbb{C})$, or $L(E,\mathbb{R})$, depending on whether E is a complex or a real vector space, is called the topological dual of E and is denoted by E^*; E^* is a Banach space. Elements of E^* are called bounded linear functionals on E.

When $E = F$, $L(E,F)$ is usually abbreviated to $L(E)$. For S, T in $L(E)$, $S \circ T$ is in $L(E)$ and $\|S \circ T\| \leq \|S\| \cdot \|T\|$, as follows directly from definitions. Usually, we shall denote $S \circ T$ by ST; then for S, T, and U in $L(E)$, $(ST)U = S(TU)$, and we may therefore denote TT by T^2, $(TT)T$ by T^3, ... etc. for $T \in L(E)$ free of misinterpretation. Note that $\|T^k\| \leq \|T\|^k$ for $T \in L(E)$ and $k \in \mathbb{N}$. For convenience, we put $T^\circ = 1$, the identity map on E.

Exercise 1.4.13 Let S be a nonempty set and consider the vector space $B(S)$ of all bounded real(complex)-valued functions on S. Addition and multiplication by scalar in $B(S)$ are usual for functions. For $f \in B(S)$, let $\|f\| = \sup_{s\in S} |f(s)|$.

(i) Show that $(B(S), \|\cdot\|)$ is a Banach space.

(ii) For $a \in B(S)$, define $A : B(S) \to B(S)$ by $(Af)(s) = a(s)f(s)$, $s \in S$. Show that A is a bounded linear transformation from $B(S)$ into itself and that $\|A\| = \|a\|$.

Exercise 1.4.14 Consider $C[0,1]$ and let $g \in C[0,1]$. Define a linear functional ℓ on $C[0,1]$ by

$$\ell(f) = \int_0^1 f(x)g(x)dx.$$

Show that $\ell \in C[0,1]^*$ and $\|\ell\| = \int_0^1 |g(x)|dx$.

Exercise 1.4.15 Let g be a continuous function on $[0,1] \times [0,1]$ and for $f \in C[0,1]$, let the function Tf be defined by $Tf(x) = \int_0^1 g(x,y)f(y)dy$. Show that $T \in L(C[0,1])$ and $\|T\| = \max_{x\in[0,1]} \int_0^1 |g(x,y)|dy$.

We now consider a series of elements in a n.v.s. E. A symbol of the form $\sum_{k=1}^{\infty} x_k$ with each x_k in E is called a **series**. For each $n \in \mathbb{N}$, $\sum_{k=1}^{n} x_k$ is called the n-th partial **sum** of the series $\sum_{k=1}^{\infty} x_k$. If it happens that $\lim_{n\to\infty} \sum_{k=1}^{n} x_k$ exists in E, say x, then the series $\sum_{k=1}^{\infty} x_k$ is said to be convergent in E and x is called the **sum** of the series, $\sum_{k=1}^{\infty} x_k$, symbolically expressed by $x = \sum_{k=1}^{\infty} x_k$, i.e. when $\sum_{k=1}^{\infty} x_k$ converges, we attach a meaning to the symbol $\sum_{k=1}^{\infty} x_k$ by referring to it as $\lim_{n\to\infty} \sum_{k=1}^{n} x_k$, or the sum of the series.

Theorem 1.4.3 *Let $\{x_k\}$ be a sequence in a Banach space E such that $\sum_{k=1}^{\infty} \|x_k\| < \infty$. Then $\sum_{k=1}^{\infty} x_k$ converges in E.*

Proof For $n \in \mathbb{N}$, let $y_n = \sum_{k=1}^{n} x_k$. Then for $m > n$ in $\mathbb{N}$,

$$\|y_m - y_n\| = \left\| \sum_{k=n+1}^{m} x_k \right\| \le \sum_{k=n+1}^{m} \|x_k\| \to 0$$

as $n \to \infty$. This means that $\{y_n\}$ is a Cauchy sequence in E, but the fact that E is complete implies that $\{y_n\}$ converges in E, i.e. $\lim_{n\to\infty} \sum_{k=1}^{n} x_k$ exists in E. ∎

Exercise 1.4.16 Suppose that $\sum_{k=1}^{\infty} x_k$ is a convergent series in a n.v.s. E. Show that

$$\left\| \sum_{k=1}^{\infty} x_k \right\| \le \sum_{k=1}^{\infty} \|x_k\|.$$

Exercise 1.4.17 Suppose that $\sum_{k=1}^{\infty} \alpha_k$ is a convergent series in $\mathbb{R}$.

(i) If x is an element of a n.v.s. E, show that $\sum_{k=1}^{\infty} \alpha_k x$ converges in E.

(ii) If $\{x_k\}$ is a bounded sequence in a Banach space E and $\sum_{k=1}^{\infty} \alpha_k$ is absolutely convergent, show that $\sum_{k=1}^{\infty} \alpha_k x_k$ converges in E.

The following example, which complements Theorem 1.4.3, illustrates a method to extract a convergent subsequence from a given sequence.

Example 1.4.4 If a series $\sum_{n=1}^{\infty} x_n$ in a n.v.s. E converges whenever $\sum_{n=1}^{\infty} \|x_n\| < \infty$, then E is a Banach space. To show this, let $\{y_n\}$ be a Cauchy sequence in E. Since $\{y_n\}$ is Cauchy, there is an increasing sequence $n_1 < n_2 < \cdots < n_k < \cdots$ in $\mathbb{N}$ such that $\|y_{n_{k+1}} - y_{n_k}\| < \frac{1}{k^2}$ for each k. Then $\sum_{k=1}^{\infty} \|y_{n_{k+1}} - y_{n_k}\| < \infty$ and hence

$\sum_{k=1}^{\infty}(y_{n_{k+1}} - y_{n_k})$ converges, which is equivalent to $\{y_{n_k}\}$ being a convergent sequence. We have shown that $\{y_n\}$ has a convergent subsequence; thus $\{y_n\}$ converges by Exercise 1.4.3 and E is therefore complete.

Remark We conclude this section with a remark on norms on a vector space E. Suppose that $\|\cdot\|'$ and $\|\cdot\|''$ are different norms on a vector space E, in general, $\|\cdot\|'$ and $\|\cdot\|''$ will generate different families of open sets; but a moment's reflection convinces us that $\|\cdot\|'$ and $\|\cdot\|''$generate the same family of open sets if and only if there is $c > 0$ such that

$$c\|x\|'' \leq \|x\|' \leq \frac{1}{c}\|x\|''$$

for all x in E (in this case $\|\cdot\|'$ and $\|\cdot\|''$ are said to be equivalent). We shall see in Proposition 1.7.2 that all norms on a finite-dimensional vector space are equivalent.

1.5 Semi-continuities

For real-valued functions, the fact that the real field $\mathbb{R}$ is ordered plays an important role in the analysis of functions. In particular, for real-valued functions defined on a metric space, lower semi-continuity and upper semi-continuity are useful concepts that owe their existence to $\mathbb{R}$ being ordered. Semi-continuities are our concern in this section. For a subset S of $\mathbb{R}$ we shall adopt the convention that $\inf S = \infty$ and $\sup S = -\infty$ if S is empty; and that $\inf S = -\infty$ if S is not bounded from below, while $\sup S = \infty$ if S is not bounded from above.

For a sequence x_n, $n = 1, 2, \ldots$, of real numbers, let

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right), \tag{1.4}$$

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right). \tag{1.5}$$

Notice that $\inf_{k\geq n} x_k$ is increasing and $\sup_{k\geq n} x_k$ is decreasing as n increases, hence both limits on the right-hand sides of (1.4) and (1.5) exist, although they may not be finite. Thus $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$ always exist, and are called respectively the **inferior limit** and the **superior limit** of $\{x_n\}$. Clearly, $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

Exercise 1.5.1

(i) Show that $\lim_{n\to\infty} x_n$ exists if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$, and $\lim_{n\to\infty} x_n$ is the common value $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$ if it exists.

(ii) Show that $\liminf_{n\to\infty}(x_n + y_n) \geq \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$ ($\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$), if $\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$ ($\limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$) is meaningful. Note that $\alpha + \beta$ is meaningful if at least one of α and β is finite, or if both α and β are either ∞ or $-\infty$.

(iii) Show that $\liminf_{n\to\infty}(x_n + y_n) \le \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$ if the right-hand side is meaningful and that $\limsup_{n\to\infty}(x_n + y_n) \ge \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$ if the right-hand side is meaningful.

A real-valued function f defined on a metric space M with metric ρ is said to be **lower semi-continuous (upper semi-continuous)** at $x \in M$ if, for every sequence $\{x_n\}$ in M with $x = \lim_{n\to\infty} x_n$, $f(x) \le \liminf_{n\to\infty} f(x_n)$ $(f(x) \ge \limsup_{n\to\infty} f(x_n))$ holds. Lower semi-continuity and upper semi-continuity will often be abbreviated as l.s.c. and u.s.c. respectively. It is clear that a function f is l.s.c. (u.s.c.) at x if and only if for any given $\varepsilon > 0$ there is $\delta > 0$ such that $f(y) > f(x) - \varepsilon$ $(f(y) < f(x) + \varepsilon)$ if $\rho(y, x) < \delta$.

Exercise 1.5.2

(i) Show that f is lower semi-continuous (upper semi-continuous) at x if and only if

$$f(x) = \lim_{\delta\searrow 0}\left[\inf_{y\in M,\ \rho(x,y)<\delta} f(y)\right]\left(f(x) = \lim_{\delta\searrow 0}\left[\sup_{y\in M,\ \rho(x,y)<\delta} f(y)\right]\right);$$

(ii) show that f is continuous at x if and only if f is both lower semi-continuous and upper semi-continuous at x.

Because of the assertions of Exercise 1.5.2, if x is not an isolated point of M, we define $\liminf_{y\to x} f(y)$ and $\limsup_{y\to x} f(y)$ by

$$\liminf_{y\to x} f(y) = \lim_{\delta\searrow 0}\left[\inf_{y\in M,\ 0<\rho(x,y)<\delta} f(y)\right];$$

$$\limsup_{y\to x} f(y) = \lim_{\delta\searrow 0}\left[\sup_{y\in M,\ 0<\rho(x,y)<\delta} f(y)\right],$$

since $\inf_{y\in M,\ 0<\rho(x,y)<\delta} f(y)$ increases as δ decreases and $\sup_{y\in M,\ 0<\rho(x,y)<\delta} f(y)$ decreases as δ decreases, both $\liminf_{y\to x} f(y)$ and $\limsup_{y\to x} f(y)$ exist, although they may not be finite. If $\liminf_{y\to x} f(y) = \limsup_{y\to x} f(y)$, the common value is called the limit of $f(y)$ as $y \to x$ and is denoted by $\lim_{y\to x} f(y)$. Usually, $\lim_{y\to x} f(y)$ is simply called the limit of the function f at x. Note that $\liminf_{y\to x} f(y)$ and $\limsup_{y\to x} f(y)$ are defined if f is defined on a neighborhood of x with x excluded. If x is an isolated point of M and f is defined at x, then $\liminf_{y\to x} f(y) = \limsup_{y\to x} f(y) = \lim_{y\to x} f(y) = f(x)$ by definition.

Exercise 1.5.3

(i) Show that $\liminf_{y\to x} f(y) \le \limsup_{y\to x} f(y)$ and that f is continuous at x if and only if $\lim_{y\to x} f(y) = f(x)$.

(ii) Show that f is l.s.c. (u.s.c.) at x if and only if $f(x) \le \liminf_{y\to x} f(y)$ $(f(x) \ge \limsup_{y\to x} f(y))$.

If f is lower semi-continuous (upper semi-continuous) at every point of M, then f is said to be lower semi-continuous (upper semi-continuous) on M.

Exercise 1.5.4 Show that f is lower semi-continuous (upper semi-continuous) on M if and only if $\{x \in M : f(x) > \alpha\}$ ($\{x \in M : f(x) < \alpha\}$) is open for every $\alpha \in \mathbb{R}$.

Exercise 1.5.5 Let f_α, $\alpha \in I$, be a family of real-valued continuous functions defined on M and assume that $\sup_{\alpha\in I} f_\alpha(x)$ $(\inf_{\alpha\in I} f_\alpha(x))$ is finite for each $x \in M$; show that $\sup_{\alpha\in I} f_\alpha(x)$ $(\inf_{\alpha\in I} f(x))$ is lower (upper) semi-continuous on M.

Exercise 1.5.6 Suppose that f is a real-valued function defined on a metric space and assume that f is bounded from below on M, i.e. there is $c \in \mathbb{R}$ such that $f(z) \geq c$ for all $z \in M$. For each $k \in \mathbb{N}$ is defined a function f_k on M by

$$f_k(x) = \inf_{z\in M}\{f(z) + k\rho(x, z)\}, \quad x \in M.$$

(i) Show that $f_k(x)$ is finite for all $x \in M$ and

$$|f_k(x) - f_k(y)| \leq k\rho(x, y)$$

for all x, y in M.

(ii) Suppose that f is l.s.c. on M. Show that

$$f(x) = \lim_{k\to\infty} f_k(x), \quad x \in M.$$

(iii) Show that f is l.s.c. on M if and only if there is an increasing sequence $\{f_k\}$ of continuous functions on M such that

$$f(x) = \lim_{k\to\infty} f_k(x)$$

for all $x \in M$.

Exercise 1.5.7 A metric space M is called a compact space if every sequence in M has a subsequence which converges in M. Show that if f is lower semi–continuous (upper semi-continuous) on a compact metric space M, then f assumes its minimum (maximum) on M. (Hint: There is a sequence $\{x_n\}$ in M such that $\lim_{n\to\infty} f(x_n) = \inf_{x\in M} f(x)$)

1.6 The space $\ell^p(\mathbb{Z})$

The Banach spaces considered in this section are included in the more general class of L^p spaces, to be introduced in Section 2.7; but it is expedient to give a separate and direct treatment here without recourse to general theory of measure and integration.

Let $\mathbb{Z}$ be the set of all integers and consider the space L of all real-valued functions defined on $\mathbb{Z}$. With the usual definition of addition of functions and multiplication of a function by a scalar, L is a real vector space. For $f \in L$ and $j \in \mathbb{Z}$, if we denote $f(j)$ by f_j, then f can be identified with the two-way sequence $(f_j)_{j\in\mathbb{Z}}$ of real numbers and L is the space of all sequences $(a_j)_{j\in\mathbb{Z}}$ of real numbers. For $f \in L$ and $1 \leq p \leq \infty$, let

$$\|f\|_p = \begin{cases} \left(\sum_{j\in\mathbb{Z}} |f(j)|^p\right)^{\frac{1}{p}} & \text{if } p < \infty; \\ \sup_{j\in\mathbb{Z}} |f(j)| & \text{if } p = \infty. \end{cases}$$

Now consider the space $\ell^p(\mathbb{Z})$, $1 \leq p \leq \infty$, defined by

$$\ell^p(\mathbb{Z}) = \{f \in L : \|f\|_p < \infty\}.$$

Presently we shall prove that $\ell^p(\mathbb{Z})$ is a vector space and $\|\cdot\|_p$ is a norm on $\ell^p(\mathbb{Z})$, but for this purpose we first show an inequality which is a generalization of the Schwarz inequality and is called Hölder's inequality. Two extended real numbers $p, q \geq 1$ are called **conjugate** exponents if $\frac{1}{p} + \frac{1}{q} = 1$ ($\frac{1}{\infty} = 0$; for further arithmetic conventions regarding ∞ and $-\infty$, see the first paragraph of Section 2.2), while two nonnegative numbers α and β will be called a **convex** pair if $\alpha + \beta = 1$.

Lemma 1.6.1 *If α and β is a convex pair, then for any $0 \leq \zeta, \eta < \infty$ the following inequality holds:*

$$\zeta^\alpha \eta^\beta \leq \alpha\zeta + \beta\eta. \tag{1.6}$$

Proof We may assume that $0 < \alpha, \beta < 1$ and $\zeta, \eta > 0$.

Since $(1 + x)^\alpha \leq \alpha x + 1$, for $x \geq 0$, we have

$$y^\alpha \leq \alpha y + \beta, \ \ y \geq 1. \tag{1.7}$$

Now either $\zeta\eta^{-1} \geq 1$ or $\zeta^{-1}\eta \geq 1$; if $\zeta\eta^{-1} \geq 1$, take $y = \zeta\eta^{-1}$ in (1.7), while if $\zeta^{-1}\eta \geq 1$, take $y = \zeta^{-1}\eta$ in (1.7) with α and β interchanged, then proceed to (1.6). ■

Lemma 1.6.2 (**Hölder's inequality**) *If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are in $\mathbb{R}^n$, then for conjugate exponents p and q we have*

$$\sum_{j=1}^{n} |x_j y_j| \leq \|x\|_p \|y\|_q.$$

Remark Since an element x of $\mathbb{R}^n$ can be identified with an element f of L by $f(1) = x_1, \ldots, f(n) = x_n$, and $f(j) = 0$ for other j, $\|x\|_p$ is defined.

Proof of Lemma 1.6.2 It is clear that if one of p and q is ∞, the lemma is trivial, hence we suppose that $1 < p, q < \infty$. Since $\|x\|_p = 0$ if and only if $x = 0$, we may assume that $\|x\|_p > 0$ and $\|y\|_p > 0$. For $1 \le j \le n$, choose $\zeta = \left(\frac{|x_j|}{\|x\|_p}\right)^p$ and $\eta = \left(\frac{|y_j|}{\|y\|_q}\right)^q$ in Lemma 1.6.1. with $\alpha = \frac{1}{p}$ and $\beta = \frac{1}{q}$, then

$$\frac{|x_j y_j|}{\|x\|_p \|y\|_q} \le \frac{1}{p}\frac{|x_j|^p}{\|x\|_p^p} + \frac{1}{q}\frac{|y_j|^q}{\|y\|_q^q},$$

and consequently

$$\sum_{j=1}^{n} |x_j y_j| \le \|x\|_p \|y\|_q \left(\frac{1}{p} + \frac{1}{q}\right) = \|x\|_p \|y\|_q. \qquad \blacksquare$$

Exercise 1.6.1 Suppose that $\alpha > 0$ and $\beta > 0$ is a convex pair. Show that

$$\zeta^\alpha \eta^\beta = \alpha\zeta + \beta\eta,\ \zeta \ge 0,\ \eta \ge 0$$

if and only if $\zeta = \eta$.

We are now in a position to prove that $\ell^p(\mathbb{Z})$ is a vector space and $\|\cdot\|_p$ is a norm on $\ell^p(\mathbb{Z})$. That $\|f\|_p = 0$ if and only if $f = 0$ and that $\lambda f \in \ell^p(\mathbb{Z})$ and $\|\lambda f\|_p = |\lambda|\|f\|_p$ for $\lambda \in \mathbb{R}$ and $f \in \ell^p(\mathbb{Z})$ are obvious. It only remains to show that $\|f+g\|_p \le \|f\|_p + \|g\|_p$ for f, g in $\ell^p(\mathbb{Z})$. For this purpose, we may assume that $1 < p < \infty$ and $\|f+g\|_p > 0$. Under this assumption, there is $A \in F(\mathbb{Z})$ such that $\sum_{j\in A} |f(j)+g(j)|^p > 0$. For such A, we have

$$0 < \sum_{j\in A} |f(j)+g(j)|^p \le \sum_{j\in A} |f(j)+g(j)|^{p-1}\left(|f(j)| + |g(j)|\right),$$

from which, by using Hölder's inequality (see Lemma 1.6.2.), we have

$$\begin{aligned} 0 &< \sum_{j\in A} |f(j)+g(j)|^p \\ &\le \left(\sum_{j\in A} |f(j)+g(j)|^{(p-1)q}\right)^{\frac{1}{q}} \left\{\left(\sum_{j\in A} |f(j)|^p\right)^{\frac{1}{p}} + \left(\sum_{j\in A} |g(j)|^p\right)^{\frac{1}{p}}\right\} \\ &\le \left(\sum_{j\in A} |f(j)+g(j)|^p\right)^{\frac{1}{q}} \left(\|f\|_p + \|q\|_p\right), \end{aligned}$$

and thus, on dividing the last sequence of inequalities by $\left(\sum_{j\in A} |f(j)+g(j)|^p\right)^{\frac{1}{q}}$, we obtain

$$\left(\sum_{j\in A} |f(j)+g(j)|^p\right)^{\frac{1}{p}} \le \|f\|_p + \|g\|_p. \tag{1.8}$$

Now observe that (1.8) holds for any $A \in F(\mathbb{Z})$. Taking the supremum on the left-hand side of (1.8) over $A \in F(\mathbb{Z})$, we see that $\|f+g\|_p \leq \|f\|_p + \|g\|_p$. Therefore, $\ell^p(\mathbb{Z})$ is a vector space and $\|\cdot\|_p$ is a norm on $\ell^p(\mathbb{Z})$. We shall always refer to $\ell^p(\mathbb{Z})$ as a normed vector space with this norm.

Exercise 1.6.2 Let $k_1 < \cdots < k_n$ be a finite sequence in $\mathbb{Z}$ of length n; define a map T from $\ell^p(\mathbb{Z})$ to the n-dimensional Euclidean $\mathbb{R}^n$ by

$$T(f) = \big(f(k_1), \ldots, f(k_n)\big), \ f \in \ell^p(\mathbb{Z}).$$

Show that T is continuous from $\ell^p(\mathbb{Z})$ onto $\mathbb{R}^n$ and that the image under T of any open set in $\ell^p(\mathbb{Z})$ is an open set in $\mathbb{R}^n$.

Exercise 1.6.3 Suppose $1 \leq p < \infty$; show that $|a_1 + \cdots + a_n|^p \leq n^{p-1} \sum_{j=1}^n |a_j|^p$ for $a_1, \ldots, a_n$ in $\mathbb{R}$.

Exercise 1.6.4 Let $f_1, f_2, \ldots, f_n, \ldots$ be a Cauchy sequence in $\ell^p(\mathbb{Z})$; show that $\lim_{n\to\infty} f_n(j)$ exists and is finite for every $j \in \mathbb{Z}$.

Exercise 1.6.5 Show that $\ell^\infty(\mathbb{Z})$ is a Banach space.

Theorem 1.6.1 *$\ell^p(\mathbb{Z})$ is a Banach space for $1 \leq p \leq \infty$.*

Proof The case $p = \infty$ is relatively easy and is left as an exercise (see Exercise 1.6.5). Consider now the case $1 \leq p < \infty$. Let $f_1, f_2, \ldots, f_n, \ldots$ be a Cauchy sequence in $\ell^p(\mathbb{Z})$, then $\lim_{n\to\infty} f_n(j)$ exists and is finite for each $j \in \mathbb{Z}$ (see Exercise 1.6.4), say $f(j) = \lim_{n\to\infty} f_n(j)$. We show first that $f \in \ell^p(\mathbb{Z})$. Since $f_1, f_2, \ldots, f_n, \ldots$ is a Cauchy sequence, it is necessarily bounded. Let $\|f_n\|_p \leq M$ for all n. There is $n_0 \in \mathbb{N}$ such that

$$\|f_n - f_m\|_p < 1, \ \ n, m \geq n_0.$$

Now fix $m \geq n_0$ and let $A \in F(\mathbb{Z})$, then

$$\begin{aligned}
\sum_{j\in A} |f(j)|^p &= \lim_{n\to\infty} \sum_{j\in A} |f_n(j)|^p = \lim_{n\to\infty} \sum_{j\in A} |f_n(j) - f_m(j) + f_m(j)|^p \\
&\leq \limsup_{n\to\infty} \sum_{j\in A} \big\{|f_n(j) - f_m(j)| + |f_m(j)|\big\}^p,
\end{aligned}$$

from which, by Exercise 1.6.3, we have

$$\begin{aligned}
\sum_{j\in A} |f(j)|^p &\leq \limsup_{n\to\infty} 2^{p-1} \left\{ \sum_{j\in A} |f_n(j) - f_m(j)|^p + \sum_{j\in A} |f_m(j)|^p \right\} \\
&\leq 2^{p-1} \left\{ \limsup_{n\to\infty} \|f_n - f_m\|_p^p + \|f_m\|_p^p \right\} \\
&\leq 2^{p-1}\{1 + M^p\}.
\end{aligned}$$

Thus,

$$\sum_{j\in\mathbb{Z}}|f(j)|^p = \sup_{A\in F(\mathbb{Z})}\sum_{j\in A}|f(j)|^p \leq 2^{p-1}(1+M^p) < \infty,$$

which shows $f \in \ell^p(\mathbb{Z})$. We now claim $\lim_{n\to\infty} f_n = f$ in $\ell^p(\mathbb{Z})$. Actually, given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$\|f_n - f_m\|_p < \varepsilon, \ \ n, m \geq N.$$

Now, for $n \geq N$ and $A \in F(\mathbb{Z})$,

$$\begin{aligned}\sum_{j\in A}|f(j) - f_n(j)|^p &= \lim_{m\to\infty}\sum_{j\in A}|f_m(j) - f_n(j)|^p \\ &\leq \liminf_{m\to\infty}\|f_m - f_n\|_p^p \leq \varepsilon^p,\end{aligned}$$

which implies

$$\|f - f_n\|_p^p = \sup_{A\in F(\mathbb{Z})}\sum_{j\in A}|f(j) - f_n(j)|^p \leq \varepsilon^p,$$

or

$$\|f - f_n\|_p \leq \varepsilon, \ \ n \geq N.$$

In other words, $\lim_{n\to\infty} f_n = f$ in $\ell^p(\mathbb{Z})$. This shows that $\ell^p(\mathbb{Z})$ is complete and hence is a Banach space. ■

Exercise 1.6.6 Let f, g be in $\ell^1(\mathbb{Z})$.

(i) Show that $\{f(n-m)g(m)\}_{(n,m)\in\mathbb{Z}\times\mathbb{Z}}$ is summable and

$$\sum_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} f(n-m)g(m) = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} f(n-m)g(m).$$

(ii) Define $f * g(n) = \sum_{m\in\mathbb{Z}} f(n-m)g(m)$, $n \in \mathbb{Z}$. Show that $f * g \in \ell^1(\mathbb{Z})$, $f * g = g * f$, and $\|f * g\|_1 \leq \|f\|_1\|g\|_1$.

Exercise 1.6.7 Suppose that $f \in l^p(\mathbb{Z})$ and $g \in l^1(\mathbb{Z})$. Show that $f * g$ can be defined similarly as in Exercise 1.6.6 (ii); then show that $f * g = g * f$, and

$$\|f * g\|_p \leq \|f\|_p\|g\|_1.$$

Remark For any nonempty set S and $1 \leq p \leq \infty$, the Banach space $\ell^p(S)$ can be defined in the same way that $\ell^p(\mathbb{Z})$ is defined. The first such space is the space $\ell^2(\mathbb{N})$ introduced by D. Hilbert in his study of the Fredholm theory of integral equations.