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Preface

Loop quantum gravity has emerged as a possible avenue towards the quantization of general relativity. Looking at the top 50 cited papers of all time of the arXiv.org:gr-qc preprint repository (which includes papers on gravity as a whole, not only quantum gravity), according to the SPIRES database in SLAC in its latest edition (2006) one finds that 13 papers are on loop quantum gravity. Although a smaller field than string theory, the other main approach to quantum gravity pursued today, the number of researchers working in loop quantum gravity is significant. At the moment both string theory and loop quantum gravity remain incomplete paradigms and as a consequence controversies about which is a more promising approach naturally arise. We will address some of these in this book.

Many people, including of course physics undergraduates, are interested in learning about loop quantum gravity. There are indeed excellent recent textbooks by Rovelli (2007) and Thiemann (2008) for those who want to pursue the topic in depth, roughly speaking at the level of an advanced US graduate school course. This type of treatment however, is largely inadequate for physics undergraduates and for others who want to get some minimal grasp of the subject in a relatively short period of time and without the depth expected for someone that is to do research in the field. An additional complication is that these "graduate-level" treatments of the subject have knowledge of general relativity as a prerequisite. This becomes a barrier for many readers. Although recent books by Hartle (2003) and Schutz (2009) make teaching general relativity at an undergraduate level possible, most undergraduate students have curricula that place general relativity at the very end of their careers (if their institution offers the course at all), which makes such courses unsuitable as prerequisites for a course in loop quantum gravity. Something that is not always appreciated is that undergraduates, particularly in the last semester of their career, are busy individuals. Most are taking several courses, perhaps conducting some undergraduate research work, preparing for the graduate record examinations (GRE), and applying to graduate schools, sometimes on top of holding a job. The time available to a specific course is very limited.

In these notes we will attempt to introduce loop quantum gravity without assuming previous knowledge of general relativity. The only background we will assume is knowledge of Maxwell's electromagnetism, a minimal knowledge of Lagrangian and Hamiltonian mechanics, and special relativity and quantum mechanics. This will inevitably imply we will be taking *many* shortcuts in the coverage of loop quantum gravity, but this will be the price we pay to introduce the topic in a way that is widely accessible to undergraduates at most physics programs in the US within the confines of a one-semester three-credit course. *Some* undergraduates and other readers may feel slightly disappointed in not getting more details and a more complete picture, but we believe the majority will welcome a book that is light and nimble as a good introduction to a subject that may otherwise appear intimidating. Some experts may feel we are short-changing readers by oversimplifying several issues for the sake of expediency. We will try to be careful to warn readers when we are doing so. Another goal we had in mind was to create a *short book*. Given that we are only introducing people to the topics, being deliberately superficial, and not attempting a full discussion, it is not worth trying to be exhaustive and discussing all issues in full detail. Long books tend to be intimidating to the reader and we think we will serve a larger audience with a compact book.

The organization of this book is as follows: in Chapter 1 we address the question of why one should quantize gravity. Chapter 2 will review Maxwell's electromagnetism and in particular its relativistic formulation. Chapter 3 will introduce some minimal elements of general relativity. Chapter 4 will deal with the Hamiltonian formulation of mechanics and field theories, including constraints. Chapter 5 will discuss Yang–Mills theories. Chapter 6 will cover quantum mechanics and some elements of quantum field theory. Chapter 7 introduces Ashtekar's new variables for general relativity. Chapter 8 develops the loop representation for general relativity. Chapter 9 presents quantum cosmology as an application. Chapter 10 discusses several miscellaneous applications including black hole entropy, the master constraint program and uniform discretizations, spin foams, possible experimental signatures, and the problem of time. The book ends with a chapter on the controversies surrounding loop quantum gravity.

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A Gabriela y Martha

Contents

1
Why quantize gravity?

In our current understanding, there exist four fundamental interactions in nature: electromagnetism, weak interactions, strong interactions, and gravity. Everyone is familiar with electromagnetism. Weak interactions are involved in the decay of nuclei. Strong interactions keep nuclei together. The rules of quantum mechanics have been applied to electromagnetism, the weak and strong interactions. It is sort of natural to apply the rules of quantum mechanics to such interactions since they play key roles in the dynamics of atoms and nuclei and one knows that at such scales classical mechanics does not give correct predictions. The rules of quantum mechanics have not been applied to gravity in a satisfactory manner up to now. Loop quantum gravity is an attempt to do so, but it is an incomplete theory.

Before continuing we should clarify that from now on "gravity" is meant to be described not by the theory of Newton that one learns in elementary physics courses, but by Einstein's general theory of relativity. Numerous experimental tests of high accuracy agree with the predictions of general relativity (Will 2005). In such description, gravity is not really an "interaction" but rather a deformation of space-time. The latter is not flat and therefore objects do not follow naturally straight trajectories in space-time. This accounts for what one normally perceives as a gravitational "force," which in reality does not exist as such. In everyday parlance, we reinterpret the curved space-time around us as generating a gravitational force.

As we will discuss in this book, the fact that gravity is not a force but a deformation of space-time will make its quantization harder. The standard computational techniques of quantum field theories all assume one works on a given background space-time, even though in the end the goal is to create an S-matrix assuming that space-time is flat in the asymptotic past, future, and spatial infinity. But in gravity space-time is a field and therefore the object to be quantized without the presence of any background space-time. The lack of a background structure technically translates itself into the theory being naturally invariant under diffeomorphisms[1] of the space-time points, since there is nothing to distinguish one point from another. There is little experience in applying quantum field theory techniques to diffeomorphism invariant theories, except for certain topological theories with no local degrees of freedom. Moreover, gravity is not a very important force in the microscopic realm, where one expects quantum effects to be dominant. To understand this, consider the ratio of gravitational to electromagnetic forces between, say, a proton and an electron; gravity is about 10^{-40} times weaker. This is the root cause of why even today we do not have a single experiment that clearly requires quantum gravity for its explanation. It is perhaps the first time in the history of physics that one is trying to build a theory without experimental guidance. If gravity is harder to quantize, and in domains where it really is important quantum effects are expected to be small, why bother quantizing it at all? Could we not keep it as a classical interaction? This is not a moot question: attempts to quantize gravity have been made since the 1930s. If in nearly 80 years we have not succeeded, why keep trying?

To begin with, even though there are no calls from experimentally accessible situations where one needs to quantize gravity for their description, there are many physical processes one can imagine that require a quantum theory of gravity for their description. A simple example would be to study the collision of two particles at energies so high that gravity becomes relevant. Another example we will discuss later in the book would be to consider a black hole that evaporates via Hawking radiation until its mass is comparable to Planck's mass (10^{-5}g). Or another point we will cover later in the book: what happened to the universe close to the Big Bang? There is also the issue of conceptual clarity and unity in physics that suggests that quantizing three of the four fundamental interactions while keeping gravity classical is unsatisfactory. It is to be noted that the point of view of unification of theories (or more precisely unification of frameworks underlying theories) has been highly successful in history. For instance, putting Newton's mechanics and Maxwell's electromagnetism on the same footing led to special relativity. Incorporating gravity into

[1] A diffeomorphism is a map that assigns to each point of the manifold another point and that it is differentiable. It essentially "moves the points of the manifold around."

this framework led to general relativity. Incorporating special relativity into quantum mechanics led to quantum field theory. Similarly, unifying electromagnetism and the weak interactions led to the first satisfactory theory of the latter. In all instances putting theories on the same footing has led to the prediction of new physics, some of which have had dramatic implications. For instance, incorporating gravity, an apparently weak force, into special relativity leads to the notion of black holes and the Big Bang. Quantum field theory led to the notion of particles and antiparticles being created all the time in the vacuum.

Moreover, since we do not have a complete theory of quantum gravity it is hard to argue precisely that there are no experimental consequences of such a potential theory. There certainly are unexplained phenomena related to gravity out there, for instance those associated with dark energy and dark matter in cosmology, that—some conjecture—may eventually require a modification of the theory of gravity. That quantum gravity is in any way responsible for those effects remains to be seen.

But if we ignore theoretical considerations, is there a practical need to quantize gravity? As we argued before, there are no outstanding experiments that we know of that require a theory of quantum gravity for their explanation. There is no conclusive answer to this point, but it can be argued that it will be hard to have a consistent theory of classical gravity interacting with quantum fields. One quickly runs into problems with the uncertainty principle. Eppley and Hannah (1977) and Page and Geilker (1981) have devised thought experiments that illustrate this point (see however Mattingly (2006) for criticisms). For instance, consider a quantum object in a state with very small uncertainty in momentum and therefore a large uncertainty in position. We measure its position with great accuracy using a very sharp package of (classical) gravitational waves. That will require a superposition involving waves of very high frequency, which, however, in classical gravity can have arbitrarily small momentum. Through the measurement of the position with great accuracy, the uncertainty in momentum in the quantum system has suddenly become very large. We have therefore potentially produced a large change in the momentum of the total system, suggesting conservation of momentum may be violated.

Such experiments are not conclusive proof, since they cannot be carried out in practice (the above one has the problem that gravitational waves are very difficult to generate and control due to the weakness of the gravitational interaction). Carlip (2008) has argued that to consistently couple a classical gravitational field to a quantum system will require non-linear modifications of quantum mechanics that could be experimentally tested in some future.

In addition to this the two main paradigms of physics, general relativity and quantum field theories, have problems of their own. In general relativity powerful mathematical theorems proved in the 1960s and 1970s indicate that under generic conditions space-times become singular. Examples of such singularities are the Big Bang we believe to be present at the origin of the universe and the singularities that arise inside black holes. A singularity generically is associated with a divergence in quantities that indicates the theory has been pushed beyond its realm of applicability. Near such singularities one usually encounters energy densities that are not compatible with a completely classical treatment anymore. The expectation is therefore that a theory unifying quantum field theory with general relativity could offer a new perspective, and perhaps eliminate the singularities altogether.

Similarly, the quantum theory of fields has the problem that many quantum operators are in the mathematical sense not functions but distributions, like the Dirac delta. When one studies interactions one has to consider products of these operators and such products are usually not well defined. Some of the divergences in quantum field theories can be eliminated, redefining the coupling constants through a procedure known as renormalization, and one can use the theories to formulate physical predictions. In spite of this, as freestanding mathematical theories quantum field theories are usually poorly defined, and have to be treated using perturbative series that are not really convergent (they are what is technically known as asymptotic expansions). Singularities in quantum field theory arise due to the distributional nature of the fields and operators. It is clear that if one changes the underlying picture of space and time the distributional nature of fields and operators may change. This may open the possibility of eliminating the singularities of field theory.

Summarizing the last two points: the current physical paradigms for gravity and for field theories are incomplete and include singularities. There appears the distinct possibility that by unifying these paradigms, singularities could be eliminated. We will take the point of view that aesthetics, the previously mentioned thought experiments, and the attractive possibility that unification may cure the problems of the standalone paradigms of gravity and quantum field theory, are enough motivation to say that gravity needs to be quantized.

The previous discussion also highlights some of the open problems of the field that people are attempting to address in contemporary research. Is there a singularity at the Big Bang or did our current universe evolve from a previous universe? If so, are there remnants of information coming from the previous universe? Does such a potential modification of the Big Bang influence the rest of the cosmological evolution, in particular how inflation developed, nucleosynthesis, and the formation of structure in the universe? What happens inside a black hole when curvatures become large? Does one again travel into another space-time region? We now know that black holes radiate like black bodies and could eventually evaporate. How is such evaporation described in detail? Since the final product of the evaporation is purely thermal radiation with no distinctive features apart from its temperature, what happened to the information included in the matter that formed the black holes? Are there any phenomenological consequences of quantizing gravity that we could observe? We will touch upon all these topics in the applications chapter towards the end of the book.

Let us now turn to a bit of history of the field. We will not attempt a detailed history here, just give some minimal background. A good concise treatment of the history of quantum gravity is in the article by Rovelli (2002). Although one already encounters mention of the quantization of gravity in papers by Einstein in 1916 and Rosenfeld and Bronstein then wrote the first detailed papers on the subject in the 1930s, significant attempts to quantize gravity started only in the early 1960s. Three different approaches emerged. One approach was canonical quantization, which we will largely follow in this book, since it is the one that resembles the elementary treatments of quantum mechanics of undergraduate textbooks that students may be familiar with. In this approach one has to separate space-time into space and time and as we will see, this will add complications. It took quite a while to understand how to work out the Hamiltonian formulation of theories like the one that describes gravity. We will get a flavor in this book of why it took some effort. Another approach was to study the theory perturbatively, by assuming that space-time is flat plus small perturbations. Such perturbative approaches worked well for electromagnetism and the weak and strong interactions (in the latter case in certain particular regimes). In gravity this approach ran into trouble. In electromagnetism and the weak and strong interactions one can formulate the theory in terms of a coupling constant that is dimensionless (in electromagnetism, for instance, it is the fine structure constant). In gravity one cannot do that. Having a coupling constant with dimensions implies that if one makes expansions in powers of the coupling constant, as one does in perturbation theory, extra powers of momentum have to be introduced at each order to keep the expression dimensionally uniform. The extra powers of momentum make the integrals arising in the interaction terms divergent. One can correct such divergences by modifying the action, but it requires an infinite number of modifications to cure all divergences. To have to specify an infinite number of terms by hand implies that the theory does not have predictive power. This problem is known as non-renormalizability. Stelle (1977) showed that one could cure the problem by adding some higher order terms to the action, but the resulting theory of gravity has unphysical properties. A good review of the perturbative approach is that of Woodard (2009). We will present a highly simplified discussion in this book. The third approach that was tried is the one known as Feynman path integral. Such an approach requires summing probability amplitudes over all classical trajectories, which in the case of gravity requires summing over all possible space-times. This has proved formidably difficult to do. Remarkably, loop quantum gravity techniques are helping define the path integral in a rigorous way, in an approach called *spin foams*. We will cover spin foams only briefly in this book.

At the same time as these approaches were encountering difficulties, a parallel line of thought was being pursued, namely that of unification of the elementary interactions into a single theory. A motivation for this comes from the weak interactions: it turns out that one cannot quantize the weak interactions by themselves, but only when they are integrated into a theory that unifies them with electromagnetism. Could the situation be similar in gravity? Could it be that integrating gravity with the other interactions into a single theory would help with its quantization? This point of view tends to be favored by most physicists with backgrounds in particle physics. Over the years it has led to a series of theories that attempt to unify gravity with the other interactions and at the same time provide a theory of quantum gravity. The various approaches included the Kaluza–Klein theories, supergravity, and lately, string theory and M-theory. We will not attempt a discussion of these approaches here. A textbook for undergraduates on string theory is available by Zwiebach (2009).

In addition to the approaches described above, there are other ideas that are pursued by smaller groups of researchers. These include causal dynamical triangulations, causal sets, matrix models, Regge calculus, twistors, noncommutatiave geometries, and the asymptotic safety scenario. We will not discuss them in this book. A good introductory overview is presented in the book by Smolin (2002).

In the mid 1980s, Ashtekar noted that one could rewrite the equations of gravity in terms of variables that made the theory resemble the theories of particle physics. This raised hopes that techniques from particle physics could be imported to the quantization of gravity. The resulting approach to quantizing gravity is called "loop quantum gravity" and is the one we will cover in this book. It is an attempt to understand the quantization of gravity by itself, without the need to unify it with other interactions.

Currently, both string theory and loop quantum gravity are incomplete theories. Some people view them as competing theories and imply that if one ends up being correct the other will not be. Our point of view is more conservative. It could end up being that string theory and loop quantum gravity both provide quantum theories of gravity cast in different language and highlighting different aspects of the problem in more natural ways in each approach. At the moment it is still unclear if this is the case.

2
Special relativity and electromagnetism

Newton's laws of mechanics take their simplest form in certain reference frames called inertial frames. One cannot distinguish a preferred inertial frame, they are all equivalent, and in all of them Newton's laws take the same form. This is the principle of Galilean relativity. However, when one considers electromagnetism as formulated by Maxwell there do appear to exist preferred frames of reference. This caught the attention of Einstein, who found the situation unsatisfactory. He was particularly troubled that in order to describe a magnet moving close to a wire, or a wire moving close to a magnet, one needed to use two different physical laws even though the end result, the production of a current in the wire, was exactly the same in both cases. This was only one of the many apparent conflicts that arose when trying to understand electromagnetic behavior in mechanical terms.

Einstein's observation was that all physical laws should be subject to the principle of relativity, they should take the same form in all inertial frames. In particular, if Maxwell's equations take the same form in all inertial frames, the speed of light should take the same value in all inertial frames. This last observation indicates that inertial frames cannot be related by Galilean transformations, since the latter do not keep the speed of light constant. If one accepts this point of view, Galilean transformations have to be abandoned along with the concomitant hypotheses about space and time that were used to build Newton's theory. In their place one needs to introduce new transformation laws to relate events viewed from different inertial reference frames. Such transformations are the Lorentz transformations. Galilean relativity was built on daily life observations that seemed rooted in common sense, but in reality were only observations made in a relatively narrow range of relative speeds. It turns out that Galilean relativity is only a slow speed approximation of a more fundamental (and more importantly, physically correct) relativity principle. The latter has many implications involving our ideas of space, time, and simultaneity. In this chapter we will explore some of these ideas and in particular introduce the mathematical notation for it that we will later use to describe general relativity.

2.1 Space and space-time

Let us start with some elementary vector notation in ordinary three-dimensional space. We start by setting up Cartesian coordinates in space x^i with $i = 1, 2, 3$. The distance between two points in space is given by Δs with,

$$\Delta s^2 = \left(\Delta x^1\right)^2 + \left(\Delta x^2\right)^2 + \left(\Delta x^3\right)^2 = \sum_{i=1}^{3} \left(\Delta x^i\right)^2 \qquad (2.1)$$

with Δx^i the separation of the two points along the i-th coordinate. One can choose a new set of Cartesian axes, like those shown in Fig. 2.1 (for simplicity we draw a two-dimensional example), and the distance between two points remains unchanged,

$$\Delta s^2 = \sum_{i=1}^{3} \left(\Delta x^{i'}\right)^2 \qquad (2.2)$$

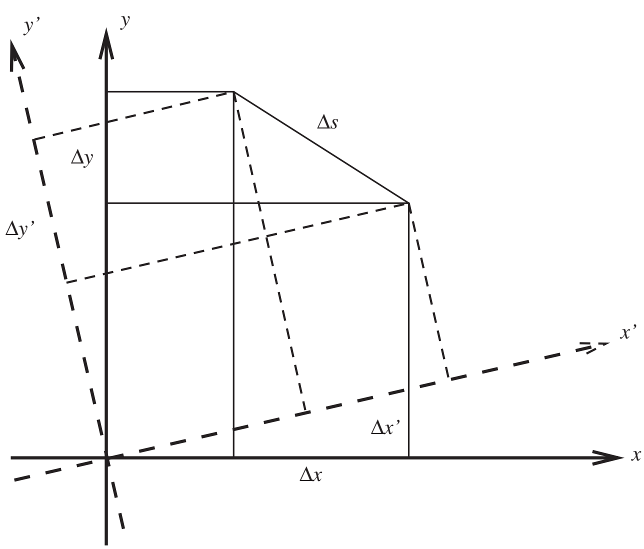

Fig. 2.1 *Two sets of Cartesian axes.*

where we have denoted by $x^{i'}$ the new set of axis. This notation of using a prime in the index rather than in the name of the coordinate will prove useful later on. If the rotation between the axes shown in the figure is given by an angle θ then one can relate the values of x^i and $x^{i'}$ by,

$$x^{i'} = \Lambda^{i'}{}_i x^i \equiv \sum_{i=1}^{3} \Lambda^{i'}{}_i x^i, \tag{2.3}$$

and also

$$\Delta x^{i'} = \Lambda^{i'}{}_i \Delta x^i, \tag{2.4}$$

where we have introduced the *Einstein summation convention*, in which any index that is repeated is summed over from one to three. At the moment the use of subscripts or superscripts in a given quantity makes no difference, but it will later on. The matrix Λ is given by

$$\Lambda^{i'}{}_i = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.5}$$

A vector is a collection of three numbers A^i that transform under coordinate transformations like the coordinates themselves, that is,

$$A^{i'} = \Lambda^{i'}{}_i A^i. \tag{2.6}$$

You are probably familiar with all this, we are just fixing notation here. Something you are perhaps not so familiar with is with the notion of *tensor*. A tensor is a multi-index generalization of a vector. The key idea is that each index in a tensor transforms as if it were sitting on a vector, without any regard to what happens to the other indices. For instance, if we consider a tensor with two indices S^{ij}, it will transform as,

$$S^{i'j'} = \Lambda^{i'}{}_i \Lambda^{j'}{}_j S^{ij} \tag{2.7}$$

where again we are assuming two summations, one on i and one on j.

In ordinary Newtonian physics, if one uses Cartesian coordinates, these are all the vector transformations one needs. In particular the time variable t remains unchanged. In special relativity the situation is different. Special relativity refers to space-time and involves transformations that mix space and time together. Remarkably, we will see that the language for coordinate transformations that we have described up to now applies almost unchanged to special relativity.

Let us now consider a space-time, that is, a four-dimensional space with coordinates x^μ with $\mu = 0, 1, 2, 3$, with the $1, 2, 3$ components coinciding with x^i, and the zeroth component will be given by ct where c is the speed of light. The reason we need c is to have the same units in all the components. In theoretical physics it is common to choose units in which $c = 1$. In this text we will choose this convention unless we want to emphasize the role of the speed of light, in which case we will state it explicitly. A choice of $c = 1$ requires measuring time in units of distance. A "point" in space-time is an "event," it is an assignment of a point in space and an instant in time.

Up to now there is nothing special being done from the point of view of physics. We could have set up a similar notation in Newtonian mechanics, bundling space and time into four-dimensional vectors. But it would not have been very useful. Since in Newtonian mechanics time is unchanging, we would only be doing transformations that involve the components $1, 2, 3$ of the four-vectors and leaving the zeroth unchanged. There would be nothing gained.

For something physically new to happen we need to introduce the idea of "distance" in space-time that is physically useful in special relativity. Such "distance" between two events is given by,

$$\Delta s^2 = -(c\Delta t)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2. \tag{2.8}$$

Notice that this is not a true distance in that it can be positive, negative, or even zero (even for points that do not coincide). This distance is kept invariant under the *Lorentz transformations*, that is[1], $x^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^\mu$,

$$\Delta s^2 = -(c\Delta t')^2 + (\Delta x^{1'})^2 + (\Delta x^{2'})^2 + (\Delta x^{3'})^2. \tag{2.9}$$

Lorentz transformations come in various kinds. First of all, ordinary spatial rotations are included, for instance the θ rotation we considered in Fig. 2.1 yields the following Lorentz transformation,

[1]When we are considering space-time quantities repeated indices are summed from 0 to 3.