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Integrable Systems

Twistors, Loop Groups, and Riemann Surfaces

N. J. Hitchin, G. B. Segal, R. S. Ward

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Twistors, Loop Groups, and Riemann Surfaces

Based on lectures given at a conference on integrable systems organized by N.M.J. Woodhouse and held at the Mathematical Institute, University of Oxford, in September 1997.

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Preface

This book is based on lectures given by the authors at an instructional conference on integrable systems held at the Mathematical Institute in Oxford in September 1997. Most of the participants were graduate students from the United Kingdom and other European countries. The lectures emphasized geometric aspects of the theory of integrable systems, particularly connections with algebraic geometry, twistor theory, loop groups, and the Grassmannian picture.

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N. M. J. Woodhouse
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1
Introduction

Nigel Hitchin

Integrable systems, what are they? It's not easy to answer precisely. The question can occupy a whole book (Zakharov 1991), or be dismissed as Louis Armstrong is reputed to have done once when asked what jazz was—'If you gotta ask, you'll never know!'

If we steer a course between these two extremes, we can say that integrability of a system of differential equations should manifest itself through some generally recognizable features:

- the existence of many conserved quantities;
- the presence of algebraic geometry;
- the ability to give explicit solutions.

These guidelines should be interpreted in a very broad sense: the algebraic geometry is often transcendental in nature, and explicitness *doesn't* mean solvability in terms of sines, exponentials or rational functions.

The most classical example of integrable systems shows all these properties: the motion of a rigid body about its centre of mass. If $\mathbf{\Omega}$ is the angular velocity vector in the body and I_1, I_2, I_3 the principal moments of inertia, then these equations take the form

$$\begin{aligned} I_1\dot{\Omega}_1 &= (I_2 - I_3)\Omega_2\Omega_3 \\ I_2\dot{\Omega}_2 &= (I_3 - I_1)\Omega_3\Omega_1 \\ I_3\dot{\Omega}_3 &= (I_1 - I_2)\Omega_1\Omega_2 \,. \end{aligned}$$

To analyse them it is easier to rescale and obtain the simpler equations

$$\begin{aligned} \dot{u}_1 &= u_2u_3 \\ \dot{u}_2 &= u_3u_1 \\ \dot{u}_3 &= u_1u_2 \,. \end{aligned}$$

So what do we look for first? *Conserved quantities.* Note that differentiating $u_1^2 - u_2^2$ gives $2u_1(u_2u_3) - 2u_2(u_3u_1) = 0$ and so $u_1^2 - u_2^2$ is constant. We similarly get

$$\begin{aligned} u_1^2 - u_2^2 &= A \\ u_1^2 - u_3^2 &= B\,. \end{aligned}$$

So A and B are two conserved quantities as (u_1, u_2, u_3) evolves.

What about *algebraic geometry*? Take the first equation $\dot{u}_1 = u_2u_3$ and substitute for u_2 and u_3 given by the expressions above, then we obtain

$$\dot{u}_1^2 = (u_1^2 - A)(u_1^2 - B)\,.$$

Putting $y = \dot{u}_1$ and $x = u_1$, we can rewrite this as

$$y^2 = (x^2 - A)(x^2 - B)$$

which is the equation of an algebraic curve, in fact an elliptic curve, and

$$\mathrm{d}t = \mathrm{d}x/y$$

is a regular differential form on the curve.

Finally how about *explicit solutions*? Any elliptic curve can be written in a standard form

$$y^2 = 4x^3 - g_2x - g_3$$

and there is a meromorphic function, the Weierstrass $\wp$-function, which is doubly periodic:

$$\wp(u + 2m\omega_1 + 2n\omega_2) = \wp(u)$$

and satisfies

$$\wp(u)'^2 = 4\wp(u)^3 - g_2\wp(u) - g_3\,.$$

Using the $\wp$-function, the solution becomes

$$\mathrm{d}t = \mathrm{d}\wp/\wp' = \mathrm{d}u$$

This means not only that if we are prepared to use elliptic functions, we can solve the equation, but also that time in the original equation is linear in the natural parameter u: we have achieved in some sense a linearization of the non-linear differential equation for the rigid body.

The study of integrable systems is not just about cunning methods of solving isolated special equations. Each equation is slightly different, and indeed there are many of them: a trawl through a couple of standard books on the subject gives at least the following list of equations which are seriously considered to be related to integrability:

Calogero-Moser system, Calogero–Sutherland system, Euler–Arnold rigid body, Clebsch rigid body, Euler–Poinsot top, Garnier system, Gaudin system, Goryachev–Chaplyagin top, Henon–Heiles system, Kepler problem, Kirchoff rigid body, Kowalewski top, Lagrange top, Neumann problem, Toda lattice, Ruijsenaars system, Steklov rigid body, Nahm's equation, Boussinesq equation, Burger's equation, Davey–Stewartson equation, Drinfeld–Sokolov construction, Ernst equation, Painlevé equation, Euler–Arnold–Manakov equation, Gelfand–Levitan–Marchenko equation, Heisenberg ferromagnet equation, Korteweg–de Vries equation, Kadomtsev–Pietviashvili equation, Krichever construction, Landau–Lifschitz equation, Hasimoto equation, Lax equation, Liouville equation, Manakov–Zakharov model, modified KdV equation, nonlinear Schrödinger equation, Riccati equation, Schlesinger equation, sine–Gordon equation, Zakharov–Shabat equation, Benjamin–Ono equation, Calogero–Degasperis–Fokas equation, Harry–Dym equation, Fermi–Pasta–Ulam problem, massive Thirring model, Melnikov equation, Benjamin–Bona–Mahoney equation, Maxwell–Bloch equation ...,

Another task of the mathematician, apart from solving individual equations, is to put some order into a universe like this. Is there some overarching structure of which all these are special cases which explains integrability?

The point where most discussions of integrability begin is with the idea of a system of differential equations which can be put in *Lax pair* form. Let's begin with a finite-dimensional system

$$\frac{\mathrm{d}A}{\mathrm{d}t} = [A, B]$$

where

$$A(\mathbf{z}) = A_0 + \mathbf{z}A_1 + \cdots + \mathbf{z}^n A_n \qquad B(\mathbf{z}) = B_0 + \mathbf{z}B_1 + \cdots + \mathbf{z}^m B_m$$

are polynomials of $k \times k$ matrices. Because of the differential equation, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{tr}(A^p) &= \operatorname{tr}(p[A,B]A^{p-1}) \\ &= p\operatorname{tr}(ABA^{p-1} - BA^p) \\ &= p\operatorname{tr}(BA^p - BA^p) \\ &= 0\end{aligned}$$

so all the coefficients of the polynomials $\operatorname{tr} A(\mathbf{z})^p$ for all p are conserved quantities. Since the components of the characteristic polynomial are expressible in terms of these traces, it is the whole spectrum of $A(\mathbf{z})$ which is preserved. Clearly equations of Lax pair type satisfy the first criterion for integrability that there should be many conserved quantities. In fact, algebraic geometry appears again very naturally.

The characteristic equation

$$\det(\mathbf{y} - A(\mathbf{z})) = 0$$

defines an algebraic curve, called the spectral curve, which is preserved by the flow. For each point $(\mathbf{y}, \mathbf{z})$ on this curve we have a one-dimensional space

$$L_{(\mathbf{y},\mathbf{z})} = \ker(\mathbf{y} - A(\mathbf{z}))$$

and this varies with time—it forms a line bundle over the spectral curve. To study equations of this type, then one must study the algebraic geometry of algebraic curves and line bundles over them.

It is a well-known fact that the space of line bundles is a complex torus—the quotient space of a vector space $\mathbb{C}^g$ by a lattice subgroup, and it is here that the final criterion of explicitness of solutions is fulfilled. We regard the equation as integrable if the line bundle L moves in a linear fashion with t in this vector space. Under these circumstances the solutions can in principle be written down in terms of theta-functions. It requires a specific form for the matrix $B(\mathbf{z})$ to be able to do this, however. An arbitrary matrix B would give a non-linear isospectral deformation of $A(\mathbf{z})$.