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Session XCI

Ultracold Gases and Quantum Information

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Ultracold Gases and Quantum Information

Edited by

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Preface

This volume contains the lecture notes of the courses that were given on ultracold gases and quantum information during the XCI session of the École de Physique des Houches. This extraordinary session was organized in Singapore from June 29th to July 24th 2009 at the Nanyang Technological University (NTU). The topics covered were degenerate quantum gases (bosons and fermions), weak and strong localization phenomena, quantum phase transitions, quantum Hall effects, Fermi and Luttinger liquids, quantum information, quantum computing and entanglement, quantum cryptography, and quantum information processing using ions, atoms, and optical devices.

P.1 Why Singapore?

It all began over a cup of coffee. Sometime in April 2007, a gang of four (Christian Miniatura, Leong-Chuan Kwek, Martial Ducloy and Berge Englert) gathered at the Spinelli Coffee outlet at the University Hall of the National University of Singapore (NUS) and came up with the idea of a Les Houches summer school session that would be organized in Singapore.

To many physicists around the world, the “École des Houches” is synonymous with the best advanced physics education that a young physicist could get. Since its establishment in 1951 by Cécile DeWitt-Morette, the École des Houches has maintained a high standard in the organization of physics summer schools, combining in-depth courses in the most advanced fields with appropriate pedagogy accessible to beginners in the field. The school has trained generations of high-level scientists, some of whom have since become Nobel prize-winners after their stint at the Les Houches summer schools, either as students or lecturers. With the fast-growing scientific and economic development of Asian countries, we felt that such a session outside France would significantly increase the visibility of the Les Houches school in East Asia and encourage greater participation from physics students from this part of the world. The hope was also to “export” the excellent Les Houches summer school structure to East Asia and to strengthen more scientific and academic links between France and Europe and East Asian countries. There were several reasons for proposing Singapore for this first Asian school: its central location in Asia; the large investments made by the Singapore Government in higher education as well as in science and technology research; the history of collaboration between Singapore and France, and in particular between the Centre for Quantum Technologies (CQT) at NUS and the Institut National de Physique (INP) du Centre National de la Recherche Scientifique (CNRS). It was in this spirit that we finally submitted our proposal for a Singapore school of Physics to the Les Houches Executive Committee.

After considerable deliberation, the Les Houches Summer School Executive Committee decided to accept this unique project and Singapore was chosen as the site for the first Les Houches summer school session ever organized outside of France since 1951. It bears the number XCI (91) in the long history of the Les Houches summer school sessions. Within Singapore, NTU was chosen as the most suitable location since it could offer good facilities with a sufficiently remote location, almost completely isolated from town. All lecturers and participants stayed at the Nanyang Executive Centre (NEC) as it is divided into two wings, one where the lectures were held and another one with hotel rooms for the lecturers and participants.

P.2 Our sponsors

This Singapore Les Houches session was directed by the Les Houches Summer School, in collaboration with the Institute of Advanced Studies (IAS) at NTU and the CQT at NUS and organized by the gang of four joined by Benoît Grémaud.

The organization of the school would not have been possible without generous financial sponsoring. We are particularly thankful to the Nanyang Technological University, the National University of Singapore, the Institute for Advanced Studies at NTU, The Centre for Quantum Technologies at NUS, the French Embassy in Singapore (Merlion programme), the Institut National de Physique du CNRS (PICS grant 4159) and the University of Nice Sophia (BQR funding).

P.3 Primary objective and scientific themes of the Singapore session

The primary objective of the Les Houches session in Singapore was to provide to the best students within the Asia–Pacific region an opportunity to attend top-level courses as typically provided by the Les Houches school. Indeed, it is a documented fact that Asian students seldom attend the sessions in France due to the distance and the lack of funding. We have the hope that the organization of this summer course will enhance closer scientific and technological cooperation between Asian and European research centers. In particular, we hope that Asian students will be keener to consider European and French universities and laboratories for their future studies.

With a bird's eye perspective, it seemed obvious to us that (at least) two fields in physics had seen an explosive growth in the past ten years, namely quantum gases and quantum information. Indeed, in 1994, Peter Shor proved that a quantum computer could, in principle, factor very large numbers into their prime factors much more efficiently than a conventional silicon-chip computer. An experimental realization of a quantum computer became a Holy Grail, though a formidable challenge. Almost concomitantly, in 1995, the first gaseous Bose–Einstein condensates were produced in labs, and a few years later degenerate Fermi gases, revolutionizing the field of atomic physics by letting it tread the condensed-matter turf. Both fields bubbled over with ideas and realizations: nuclear magnetic resonance based quantum bits (qubits), ion trap architecture, superconducting qubits, Mott–superfluid transition, BEC-BCS

crossover and so forth. This quantum realm has reached a stage where the current developments are now addressing technological aspects with the potential to impact deeply on our everyday lives, bringing with it important industrial, economic, as well as societal stakes. Many different scientific communities are involved and we anticipate that the synergy between them will spark many new ideas in science in the very near future. It is therefore not surprising to witness a growth in quantum science activities in the most developed Asian countries, and in Singapore in particular. We felt that Singapore, being a hub of education and research as well as a melting pot where Asian and Western cultures merge harmoniously, would offer an appropriate and fertile place to welcome students from all the surrounding countries and have them benefit from a school devoted to ultracold gases and quantum information.

However, the topics in both quantum degenerate gases and quantum information sciences are so vast that it was simply impossible to provide an exhaustive and comprehensive view of these subjects in one single school. We had to make a selection, sometimes driven by the haphazard acceptance or refusal of our invitations. The set of lectures that we have chosen are detailed in Section P.5 of this preface.

P.4 Demographics

When the applications for the school closed in February 2009, there were more than 110 postdoctoral fellows, graduates, and PhD students who had applied for the Singapore session. More than half of the applicants were from Asian countries. A total of 64 participants were eventually admitted to the school and we were pleased to welcome 19 female students. This ratio of one third is a nice indication that women in Asia do not hesitate to embrace a scientific path. Most of the selected students were graduate students rather than postdoctoral fellows (five only). In terms of demographics, there were 21 different nationalities at the school (see Figure P.1). These participants came from the universities and colleges of 16 different countries (see Figure P.3), with more than half of the participants (35) coming from Asian countries. 18 of the participants came from Singapore (NTU and NUS). Part of the reason for this strong representation from Singapore was due to the intensive research in cold atoms and quantum information in the country. It is important to note that amongst the 18 participants from Singapore, there were about six different nationalities, reflecting the cosmopolitan nature of the Singapore education system. There were also 23 participants from Europe with the majority (eight) from France.

As a net result, the school did indeed successfully increase the number of participants from Asian countries compared to sessions in France. However, what was particularly important was the active integration of postdoctoral fellows and students from French and other European universities with students from Asian countries.

P.5 Lecturers, courses, and content of this book

There were two principal themes for this school: ultracold atomic gases and quantum information. The session was organized around four fundamental courses supplemented by seven topical courses. To ensure a high level of pedagogical lectures, the lecturers

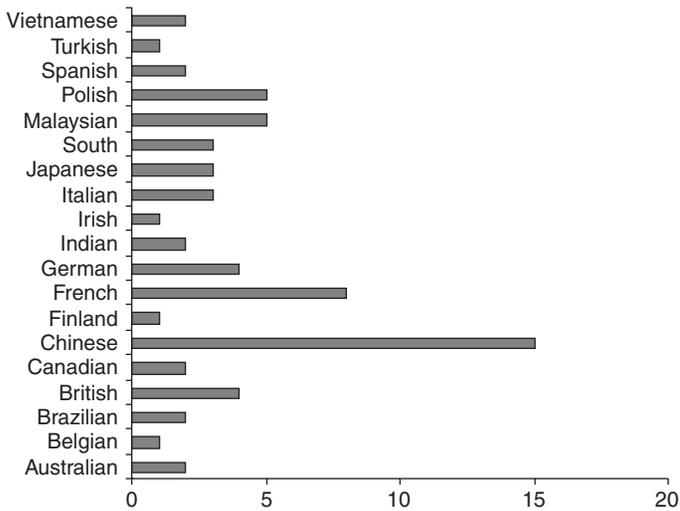


Fig. P.1 Bar chart showing the distribution of nationalities among the participants.

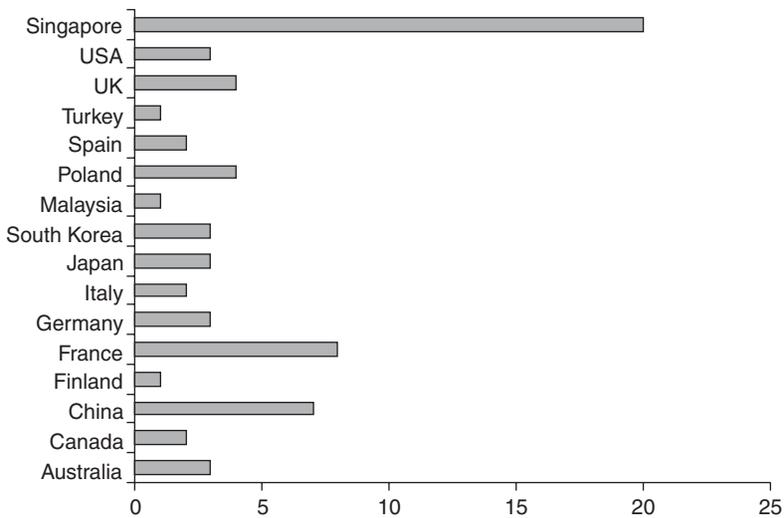


Fig. P.2 The bar chart shows the distribution of participants from different countries. NTU and NUS contributed 18 students to the total number of participants.

for the school were carefully selected for their expertise and their ability to deliver clear and succinct lectures at a graduate level. A total of 86 hours of lectures were delivered over four weeks, in addition to the special talks given by distinguished guests such as Anthony Leggett (2003 physics Nobel Prize winner), Frédéric Chevy (LKB, Paris), and Xing Zhizhong (IHEP, China).



Fig. P.3 Some participants chatting with Tony Leggett during the school.

The volume first starts with fundamental aspects of degenerate Bose and Fermi gases on the one hand and on foundational aspects of quantum theory and quantum information processing on the other hand. David Guéry-Odelin (LCAR, Toulouse, France) details the basic theory behind Bose–Einstein condensation, from the ideal gas to bimodal condensates, through mean-field theory and beyond. This introduction is then followed by an exposition by Patrizia Vignolo (INLN, Valbonne, France) on the fascinating realm of degenerate Fermi gases and the link between two worlds offered by the BEC–BCS crossover. To cover fundamental concepts in quantum theory, Valerio Scarani (CQT, NUS, Singapore) speaks about some of the intriguing aspects of quantum correlations, ranging from quantum cloning to quantum teleportation and ending with the power of Bell. This material is followed by lecture notes by Dagmar Bruss (Institut für Theoretische Physik, Düsseldorf, Germany) and Chiara Macchiavello (Istituto Nazionale di Fisicadella Materia, Pavia, Italy) on quantum networks, quantum algorithms, quantum error correction, and the one-way computing paradigm.

The second part of the book focuses on specific topics in both fields. Jürgen Eschner (ICFO, Barcelona, Spain) reviews the basic experimental techniques with trapped ions and explains how quantum bit encoding, logic gates, and quantum computation processing can be performed. Mark Goerbig (LPS, Orsay, France) presents

a comprehensive set of lectures on the quantum Hall effects. In recent years there have been proposals to implement topological codes on solid-state systems exhibiting quantum Hall effects. Mark's lecture notes therefore serve as a gentle introduction to those who would like to delve deeper into the subject. In the next lecture, George Batrouni (INLN, Valbonne, France) explains quantum phase transitions, i.e. the radical change that occurs in the topology of the ground state of a many-body system at zero temperature when the parameters of the system are varied. This is followed by Thierry Giamarchi (DPMC, Geneva, Switzerland) who presents fundamental tools for studying one-dimensional quantum fluids, a world where the Fermi-liquid description is doomed to fail and where collective-excitation physics is the rule. Then, Cord Müller (Physikalisches Institut, Bayreuth, Germany) and Dominique Delande (LKB, Paris, France) introduce the physics of weak and strong localization, i.e. the subtle interplay between disorder and interference that is observed when a wave propagates in a disordered medium and suffers many scattering events. These interference corrections to transport can lead, under suitable circumstances, to a subtle metal-insulator disorder-induced transition. Quantum cryptography has been deciphered by Norbert Lütkenhaus (Institute for Quantum Computing, Waterloo, Canada).¹ Finally, Christian Kurtsefer and Antía Lamas Linares (both at CQT, NUS, Singapore) close the volume by presenting how quantum information processes can be performed using quantum optical devices such as one-photon sources.

P.6 Entertainment and social events

The summer school was not all work and no play. The official opening for the school on Monday, June 29 2009 (first day of the school) was attended by the French Ambassador to Singapore, His Excellency Pierre Buhler. One of the enjoyable moments during the school was the celebration of the French National Day on July 14 with wine and cheese tasting, a primer for most of the Asian students.

Participants at the school had access to many sports facilities, including the swimming pool nearby at NTU, and enjoyed several sports events like basketball and football matches. The Staff Club next to NEC also provided an interesting venue at night where the participants could relax, mingle, and enjoy chats, guitar performances, and good laughs as well as play darts or billiards. Many students loved the place and it was not uncommon for the lecturers and students to exchange ideas over a mug of beer or a glass of wine at the club. During the weekends the students were free to arrange their own activities. However, special activities, such as trips to the Sungei Buloh natural reserve or Pulau Ubin island, were organized every Saturday for all participants. During these weekends, the more adventurous participants visited neighboring countries such as Malaysia, Indonesia, and Thailand. The students, especially those coming from Europe, were deeply impressed by what they saw during these getaways and shared their adventures with other participants at the school. We believe that it was a unique

¹ We regret that the quantum cryptography lecture notes are unfortunately missing from the book.



Fig. P.4 Opening ceremony on 29 June 2009. First row, from left to right: Dr Martial Ducloy (co-organizer of the Les Houches session in S'pore), Prof. Leticia Cugliandolo (Director of the Les Houches school), Prof. K-K Phua (Director of IAS), His Excellency Pierre Buhler (French Ambassador to Singapore) and Dr Guanng Su (President of NTU). Second row, from left to right: Marc Piton (French Counsellor for Culture, Science and Education), Walid Benzarti (French Attaché for Science and Higher Education), Tony Mayer, Monique van Donzel.

experience that they will not forget. It was surely an attractive flavor of the Singapore session. There were also other special “outings,” including a visit to CQT and tours of the research laboratories at the School of Physical and Mathematical Sciences at NTU.

P.7 Acknowledgments

This is always the most difficult part, as the fear of forgetting someone is always lurking in the back of your mind...

First, we would like to express our warmest gratitude to all the colleagues who believed in us and in this “crazy project.” In this respect we would particularly like to thank Antoine Mynard, from the French Embassy in Singapore, for his enthusiastic and effective constant support from the very start. In France, we benefited enormously from the decisive help of Christian Chardonnet and Patricio Leboeuf (INP, CNRS). Making this project fly would have been virtually impossible without them.

We are deeply thankful to the Les Houches Scientific Advisory Board for their bold decision, which allowed this extraordinary session to happen, the first one ever organized outside the small village in the French Alps, and which made this adventure possible. It is our great pleasure to acknowledge the tremendous help received from the staff at the French Embassy in Singapore, particularly from his Excellency Pierre Buhler (French Ambassador to Singapore), Olivier Guyonvarch (French Deputy Chief of Mission), Marc Piton (French Counsellor for Culture, Science and Education), and Walid Benzarti (French Attaché for Science and Higher Education).

Special thanks go to our sponsors whose generous fundings allowed us to organize a session with such high quality standards.

The hidden side of the iceberg is always the largest... We thank all the staff at CQT and IAS for their kind and immense help. We particularly wish to thank Professor Choy Sin Hew, Dr Jean Yong Wan Hong, Ms Chris Ong, Ms Corinne Lam, and Ms Ang Wuan Suan from IAS for their logistic and secretarial support and advice in the preparation and running of the session, and Ms Evon Tan, Ms Chan Chui Theng, and Mr Darwin Gosal from CQT who helped us organize some of the social activities. At this point we would like to point out that some social events benefited greatly from the Singapore branch of the French company Carrefour who generously donated a large number of bottles of French wine.

Special thanks to Toh-Miang Ng, our wizard webmaster, who crafted an efficient site for the conference. Special thoughts to Nathalie Hamel (INLN) for organizing the travel plans of the French participants and to Jorge Tredicce (INLN), a Singapore-lover, for his friendly and humorous support.

The daily organization of the school would not have been so smooth without the wonderful and dedicated assistance given by many student helpers: Ang Zhongzhi, Hu Jiazhong, Ni Xiaotong, Setiawan, Wu Peisan, Anton Peshkov, and Yenny Widjaya. They have provided relentless support for all the logistics during the school. Also, we would like to thank members of the Center for Excellence in Learning and Teaching (CELT), particularly Professor Daniel Tan, Goh Wee Sen, Andrew Chua, Paul Ang, Mohd Nazli bin Johar, and Ong Day Cheng for their help in various forms of technical support for the school.

Finally, we would like to thank all the lecturers for their dedication, the excellent quality of their performances, and the youthful spirit they showed when mingling with the students. And, above all, thank you to all the participants who created a magic, incredible, and unique atmosphere blended with different, amazing, and cross-fertilizing cultures. Thank you for the wonderful pictures that you have all taken. Thank you Thierry for your magic card tricks that left us all stunned and filled with wonder, thank you Cord and Dominique for setting up a CBS experiment on the spot, thank you Valerio and Han Rui for organizing football and basketball matches, thank you Jean-François for playing rock songs on your guitar – we almost broke chairs – and thank you Bess and Gabriel for your wonderful cello concerts!

Thank you all, we hope we met your expectations.

We'll miss every wonderful moment of this unique adventure!

Singapore, April 2010



Christian Miniatura



Leticia Cugliandolo



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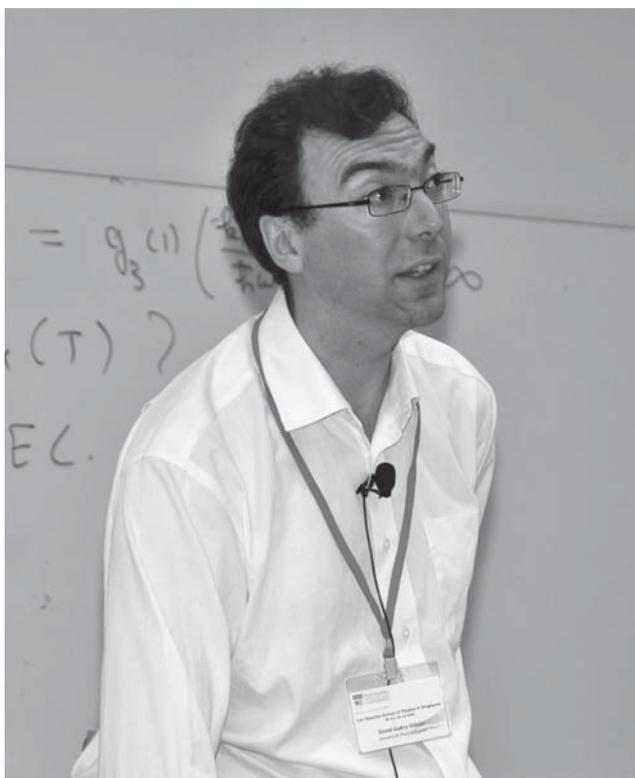
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Basics on Bose–Einstein condensation

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1.1 Introduction

These lecture notes provide some basic results on Bose–Einstein condensation. They are the written version of the set of lectures given by one of the authors (D. G.-O.) at the Les Houches School of Physics on Ultracold gases and Quantum Information held in Singapore from June, 29 to July, 24 2009.

Degenerate quantum fluids (i.e. fluids in which quantum statistical effects play a key role) are encountered in nature in very different systems, ranging from atomic nuclei, superfluid helium, conduction electrons in metals, to neutron stars, and give rise to spectacular physical properties. The systems just mentioned, although differing by orders of magnitude in density¹ for instance, have a common characteristics: the interactions between particles are strong and cannot be controlled easily. The achievement of Bose–Einstein condensation in dilute gases in 1995 (Anderson *et al.*, 1995; Davis *et al.*, 1995), shortly followed by Fermi degeneracy (DeMarco and Jin, 1999), has allowed physicists to study degenerate quantum *gases*, in which interactions can be weak, and, more importantly, controlled. The possibility to tailor almost at will the external potentials in which the particles evolve, as well as the interactions, has led to fascinating studies that are nowadays at the interface between atomic physics, condensed matter physics, and even high-energy physics.

The purpose of this set of lectures is to introduce basic notions about Bose–Einstein condensates (BECs). These notes are organized as follows. We first study ideal Bose gases, putting an emphasis on the role of the trapping geometry and of the dimensionality. We introduce the correlation functions characterizing coherence. The second chapter deals with weakly interacting BECs, which are described, in a mean-field approach, by the so-called Gross–Pitaevskii equation. We derive this equation and apply it to a variety of experimentally relevant situations. The third chapter deals with beyond-mean-field effects, and introduces in particular the Bogolubov approximation. The last chapter is devoted to the study of BECs in double-well potentials, a situation where beyond-mean-field effects can appear quite easily, and that remains simple enough to be studied theoretically in details.

Even restricted to the above-mentioned topics, the subject is vast and growing, and many interesting aspects of Bose–Einstein condensation will not be described in these notes. We thus give below a few general references dealing with topics not covered here:

- the lecture notes written by Yvan Castin for two previous sessions of the Les Houches School of Physics (Castin, 2001, 2004) are worth reading;
- an introduction to the important subject of ultracold collisions is given in (Dalibard, 1998);
- the two standard textbooks in the field are (Pethick and Smith, 2002) and (Pitaevskii and Stringari, 2003);
- for a recent review dealing with the many-body physics that can be explored with ultracold gases, the reader is referred to (Bloch *et al.*, 2008).

1.2 Ideal Bose–Einstein condensates

1.2.1 Indistinguishable particles and the Bose gas

The theoretical treatment of the Bose gas requires to account properly for the indistinguishability of the atoms of the gas. We therefore start by a short review on the second quantization formalism, which is a powerful framework to deal with identical particles. Its use in combination with statistical physics is summarized afterwards.

1.2.1.1 Second quantization

Let us consider the orthonormal basis of one-particle states $\{|\varphi_\alpha\rangle\}$. Any arbitrary N -particle state can be expanded in a basis that is the tensor product of those one-particle states, namely

$$|\Psi(1, \dots, N)\rangle = \sum_{\{n_\alpha\}} C(n_1, \dots, n_N) |\varphi_{n_1}\rangle \otimes \dots \otimes |\varphi_{n_N}\rangle. \quad (1.1)$$

For bosons, $|\Psi(1, \dots, N)\rangle$ is symmetric under an arbitrary exchange of particles ($j \longleftrightarrow k$), and therefore the coefficients $C(n_1, \dots, n_N)$ must be symmetric.

The second quantization procedure requires to consider an enlarged space of states in which the number of particles is not fixed. If we denote by \mathcal{H}_0 the Hilbert space with no particles, \mathcal{H}_1 the Hilbert space with only one particle and, in general, \mathcal{H}_N the Hilbert space for N particles, the direct sum of these spaces is called the Fock space: $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_N \oplus \dots$. An arbitrary state $|\Psi\rangle$ in Fock space is the sum over all the subspaces \mathcal{H}_N : $|\Psi\rangle = |\Psi^{(0)}\rangle + |\Psi^{(1)}\rangle + \dots + |\Psi^{(N)}\rangle + \dots$. The subspace with no particles is denoted $|0\rangle$ and is called the vacuum.

One can define creation and annihilation operators that act in the enlarged Hilbert space \mathcal{H} for bosons where \hat{a}_α^\dagger creates a particle in the state $|\varphi_\alpha\rangle$, $\hat{a}_\alpha^\dagger|0\rangle = |\varphi_\alpha\rangle$, and \hat{a}_α destroys a particle in $|\varphi_\alpha\rangle$, $\hat{a}_\alpha|\varphi_\alpha\rangle = |0\rangle$. It is thus convenient to express the states of \mathcal{H} using the orthonormal basis $\{|\dots n_\alpha \dots n_\beta \dots\rangle\}$, where n_α is the occupation number of state $|\varphi_\alpha\rangle$. In the case of bosons, the n_α s can be any non-negative integer. One can show that the symmetry of the states of systems made of many identical bosons is then simply expressed as commutation relations for the operators a_α and a_α^\dagger :

$$[\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha\beta}, \quad \text{and} \quad [\hat{a}_\alpha, \hat{a}_\beta] = [\hat{a}_\alpha^\dagger, \hat{a}_\beta^\dagger] = 0. \quad (1.2)$$

The action of the operators a_α and a_α^\dagger over a Fock state $|\dots n_\alpha \dots n_\beta \dots\rangle$ is given by

$$\begin{aligned} \hat{a}_\alpha^\dagger |\dots n_\alpha \dots n_\beta \dots\rangle &= \sqrt{n_\alpha + 1} |\dots n_\alpha + 1 \dots n_\beta \dots\rangle, \\ \hat{a}_\alpha |\dots n_\alpha \dots n_\beta \dots\rangle &= \sqrt{n_\alpha} |\dots n_\alpha - 1 \dots n_\beta \dots\rangle. \end{aligned} \quad (1.3)$$

Let $f(i)$ be a one-particle operator. For instance, $f(i)$ can represent the kinetic energy of particle i . In first quantization, the corresponding operator for a system made of N particles reads $F = \sum_{i=1}^N f(i)$. One can show (Landau and Lifshitz, 1958) that this operator becomes in second quantization

$$\hat{F} = \sum_{\alpha} \sum_{\beta} \langle \varphi_{\beta} | f | \varphi_{\alpha} \rangle \hat{a}_{\beta}^{\dagger} \hat{a}_{\alpha}. \quad (1.4)$$

Similarly, the two-particle operator acting on a system made of N particles is given, in first quantization, by $G = (1/2) \sum_{i=1}^N \sum_{j \neq i} g(i, j)$, where $g(i, j) = g(j, i)$ and, in second quantization by

$$\hat{G} = \frac{1}{2} \sum_{\alpha} \sum_{\beta} \sum_{\gamma} \sum_{\delta} \hat{a}_{\delta}^{\dagger} \hat{a}_{\gamma}^{\dagger} \hat{a}_{\beta} \hat{a}_{\alpha} \langle \varphi_{\delta}(1) \varphi_{\gamma}(2) | g(1, 2) | \varphi_{\alpha}(1) \varphi_{\beta}(2) \rangle. \quad (1.5)$$

1.2.1.2 Grand-canonical ensemble

The statistical ensemble well adapted to describe indistinguishable particles is the grand-canonical ensemble (Huang, 1963). Within this formalism, the system can exchange two extensive quantities with reservoirs: *energy* and *particles*. The equilibrium state is obtained by determining the density matrix $\hat{\rho}$ of the system. This is achieved by maximizing the missing information, or otherwise stated the statistical entropy $S(\hat{\rho}) = -k_B \text{Tr}[\hat{\rho} \log \hat{\rho}]$ with the two constraints

- of a fixed mean number of particles $\langle \hat{N} \rangle = N$;
- and of a fixed mean energy $\langle \hat{H} \rangle = E$, where \hat{H} is the hamiltonian of the system.

This maximization is readily carried out using Lagrange multipliers. One finds

$$\hat{\rho} = \frac{e^{-\alpha \hat{N} - \beta \hat{H}}}{Z_G}, \quad \text{where } Z_G = \text{Tr} \left(e^{-\alpha \hat{N} - \beta \hat{H}} \right) \quad (1.6)$$

is the grand-canonical partition function. α is the Lagrange multiplier associated with the constraint on the mean number of particles. The Lagrange multiplier β associated with the constraint on the mean energy can be related to the temperature T by $\beta = 1/k_B T$, and α to the chemical potential μ , that is the energy required to add one particle to the system, by $\alpha = \beta \mu$. It is convenient for calculations to introduce the *fugacity*, a dimensionless quantity defined by $z = e^{\beta \mu}$.

Let us consider that the system is in a three-dimensional box of volume V . We fix the mean number of particles N and the mean total energy E , so that all thermodynamical quantities depend on the three extensive parameters (V, N, E) . Once the expression for the partition function Z_G is known, the value of the fugacity $z(V, N, E)$ and of $\beta(V, N, E)$ is obtained by the relations

$$N = z \frac{\partial}{\partial z} \ln Z_G(V, z, \beta), \quad \text{and} \quad E = - \frac{\partial}{\partial \beta} \ln Z_G(V, z, \beta). \quad (1.7)$$

1.2.2 The ideal quantum gas

The explicit determination of Z_G is in general impossible for an interacting gas. However, the expression of Z_G can readily be derived for an ideal gas. The symmetrization principle in quantum mechanics, applied to an ideal gas at thermodynamical

equilibrium yields astonishing properties that are still essentially valid in the dilute limit. This is the reason why we begin these lecture notes by considering the case of an ideal gas.

For a system made of N particles that do not interact, the total hamiltonian \hat{H} is the sum of the individual one-body hamiltonians: $\hat{H} = \hat{h}(1) + \dots + \hat{h}(N)$. Let us introduce the eigenbasis $\{|\lambda\rangle\}$ of the one-body hamiltonian \hat{h} : $\hat{h}|\lambda\rangle = \varepsilon_\lambda|\lambda\rangle$.

If we denote, as in the previous section, the operator a_λ for destruction and a_λ^\dagger for creation of a particle in the individual state $|\lambda\rangle$, the hamiltonian operator and the total number of particles operator can then be recast in the form

$$\hat{H} = \sum_\lambda \varepsilon_\lambda a_\lambda^\dagger a_\lambda \quad \text{and} \quad \hat{N} = \sum_\lambda a_\lambda^\dagger a_\lambda. \quad (1.8)$$

These operators are obviously diagonal in the Fock basis $\{|N_\lambda, N_{\lambda'}, \dots\rangle\}$ where the N_λ are the occupation numbers of the individual quantum states. For a given microscopic configuration $|\ell\rangle = |N_\lambda, N_{\lambda'}, \dots\rangle$ one has:

$$\begin{aligned} \hat{N}|\ell\rangle &= N_\ell|\ell\rangle \quad \text{with} \quad N_\ell = \sum_\lambda N_\lambda, \\ \hat{H}|\ell\rangle &= E_\ell|\ell\rangle \quad \text{with} \quad E_\ell = \sum_\lambda N_\lambda \varepsilon_\lambda. \end{aligned}$$

The grand-canonical partition function takes a simple form in the $\{|\ell\rangle\}$ basis:

$$\begin{aligned} Z_G &= \sum_\ell e^{-\alpha N_\ell - \beta E_\ell} = \sum_{N_\lambda, N_{\lambda'}, \dots} e^{-(\alpha + \beta \varepsilon_\lambda) N_\lambda} \times e^{-(\alpha + \beta \varepsilon_{\lambda'}) N_{\lambda'}} \times \dots \\ &= \prod_\lambda \zeta_\lambda \quad \text{with} \quad \zeta_\lambda = \sum_{N_\lambda} e^{-(\alpha + \beta \varepsilon_\lambda) N_\lambda}. \end{aligned} \quad (1.9)$$

We therefore obtain a factorization of the partition function Z_G as a product of elementary partition functions, each of them related to an individual quantum state $|\lambda\rangle$. This is the major advantage of using the grand-canonical formalism. Let us derive the explicit form of the partition function for fermions and bosons:

- *The fermionic case*

$$\zeta_\lambda = 1 + e^{-(\alpha + \beta \varepsilon_\lambda)} \implies \log Z_G = \sum_\lambda \log \left(1 + e^{-(\alpha + \beta \varepsilon_\lambda)} \right), \quad (1.10)$$

since the occupation number of a given state λ can take only two values $N_\lambda = 0$ or $N_\lambda = 1$ due to the Pauli exclusion principle. We deduce the expression for the mean total number of particles:

$$N = \sum_\lambda N_\lambda \quad \text{with} \quad N_\lambda = \frac{1}{1 + e^{\beta(\varepsilon_\lambda - \mu)}}. \quad (1.11)$$

The chemical potential μ can take any real value. In the limit $|\mu| \gg k_B T$ and $\mu < 0$, $N_\lambda \simeq z e^{-\beta \varepsilon_\lambda}$ and we recover the Boltzmann result for classical statistics. In the opposite limit, $|\mu| \gg k_B T$ and $\mu > 0$, we find that $N_\lambda = 1$ for $\varepsilon_\lambda < \mu$ and 0 otherwise. Particles fully occupy the so-called Fermi sea. This regime for fermions is referred to as the degenerate regime, i.e. the regime where statistics play a key role at the macroscopic level for the whole gas.

- *The bosonic case*

$$\zeta_\lambda = \sum_{N_\lambda=0}^{\infty} e^{-(\alpha + \beta \varepsilon_\lambda) N_\lambda} = \frac{1}{1 - e^{\beta(\varepsilon_\lambda + \mu)}} \quad (1.12)$$

so that the mean total number of particles reads:

$$N = \sum_{\lambda} N_\lambda \quad \text{with} \quad N_\lambda = \frac{1}{e^{\beta(\varepsilon_\lambda - \mu)} - 1}. \quad (1.13)$$

The chemical potential cannot take any value. Indeed, μ must remain smaller than the minimum energy ε_{\min} to avoid an unphysical negative occupation number. The limit $|\mu| \gg k_B T$ and $\mu < \varepsilon_{\min}$ yields, as for fermions, the classical statistics result. The quantum degenerate regime, that will be extensively studied in the following, is reached when the chemical potential takes a value smaller than ε_{\min} but very close to this upper bound. For convenience, we usually set $\varepsilon_{\min} = 0$. Indeed, the value ε_{\min} can always be absorbed into the definition of the chemical potential.

In the grand-canonical formalism, $\beta = 1/k_B T$ and z are fixed, and the mean number of occupation N_λ of an individual energy state ε_λ is fixed and given by Eq. (1.13). Another related problem, closer to the experimental situation for dilute gases, consists in studying an isolated system made of N bosons at the microcanonical thermal equilibrium. One may wonder how the bosons will share on the individual energy levels. If we ignore the fluctuations of the number of particles about their average value, we can use the grand-canonical result with the constraint $N = \sum_{\lambda} N_\lambda(z, \beta)$. This equation gives the implicit value of the fugacity z as a function of N and T .

1.2.2.1 Bose–Einstein condensation in a harmonic trap

In the presence of an external confinement, the spectrum of the one-body hamiltonian \hat{h} is discrete. The number N' of particles in the excited levels has *a priori* an upper bound:

$$N' = \sum_{\lambda}' \frac{1}{e^{\beta(\varepsilon_\lambda - \mu)} - 1} < N'_{\max} = \sum_{\lambda}' \frac{1}{e^{\beta(\varepsilon_\lambda - \varepsilon_{\min})} - 1}, \quad (1.14)$$

where \sum_{λ}' denotes the sum over all eigenstates λ except the ground state. This upper bound N'_{\max} is called the saturation number and depends only on the temperature. Its precise value requires the knowledge of the type of confinement and of the dimensionality. Note that N'_{\max} may not be finite; an example of this situation is discussed in Section 1.2.2.4. In the following we assume that $N'_{\max}(T) < \infty$.

If, at a given temperature T , we put in the trap a number N of particles larger than $N'_{\max}(T)$, we are sure that $N - N'_{\max}$ particles are in the ground state. As a direct application of the saturation of the number of atoms in the excited states, let us consider a three-dimensional isotropic harmonic oscillator trap with an angular frequency ω . The eigenenergies of the individual states read:

$$\varepsilon_{n_x, n_y, n_z} = \hbar\omega \left(n_x + n_y + n_z + \frac{3}{2} \right). \quad (1.15)$$

The degeneracy of a level ε_n with $n = n_x + n_y + n_z$ is $g_n = (n+1)(n+2)/2$, and the saturation number is therefore given by

$$N'_{\max} = \sum_{(n_x, n_y, n_z) \neq (0,0,0)} \frac{1}{e^{\xi(n_x+n_y+n_z)} - 1} = \sum_{n=1}^{\infty} \frac{g_n}{e^{n\xi} - 1}, \quad (1.16)$$

where $\xi = \beta\hbar\omega$. In the limit $\xi \ll 1$, this discrete sum can be replaced by an integral²:

$$N'_{\max} \simeq \frac{1}{2\xi^3} \int_0^{\infty} \frac{x^2}{e^x - 1} dx = \frac{1}{2\xi^3} \int_0^{\infty} x^2 e^{-x} \sum_{n=0}^{\infty} e^{-nx} dx = \frac{g_3(1)}{\xi^3}, \quad (1.17)$$

where the $g_\alpha(z)$ are the Bose functions defined by

$$g_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha}. \quad (1.18)$$

From the expression of the saturation number we deduce the critical temperature below which a macroscopic fraction of particles occupies the ground state: $N'_{\max}(T_c) = N$. For the case of a 3D isotropic harmonic oscillator, we find $k_B T_c \simeq 0.94\hbar\omega N^{1/3}$. The non-trivial feature of Bose–Einstein condensation lies in the fact that the critical thermal energy $k_B T_c$ is very large compared to the energy-level spacing, as exemplified with the 3D harmonic oscillator for which the critical temperature is larger than $\hbar\omega$ by a factor $N^{1/3}$. Below the critical temperature, the number of atoms in the excited states of the 3D isotropic harmonic confinement is therefore $N'(T) = N(T/T_c)^3$ (see Figure 1.1):

$$N = N_0 + N'(T) \implies \frac{N_0}{N} = 1 - \left(\frac{T}{T_c} \right)^3. \quad (1.19)$$

1.2.2.2 Bose–Einstein condensation in the semi-classical limit

The semi-classical approximation, used in the previous section to estimate the critical temperature by replacing the discrete sums by continuous integrals, is justified when the energy difference between successive energy levels remains small compared to $k_B T$: $\delta\varepsilon \ll k_B T$. This is precisely the case, as illustrated in the previous section, for the thermodynamical description of the Bose–Einstein condensation phenomena since $k_B T_c \gg \delta\varepsilon$ if the system has a large enough number of particles. Within this

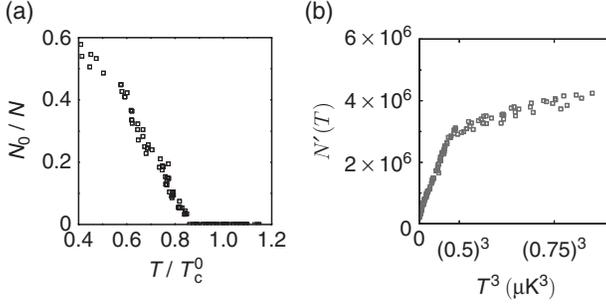


Fig. 1.1 (a) Condensate fraction N_0/N as a function of the temperature for a three-dimensional harmonic potential. (b) Number N' of atoms in the excited states as a function of temperature. One observes that below T_c , $N' \propto T^3$. The decrease in atom number above T_c for decreasing T is due to the evaporation of the most energetic particles. Figure taken from (Guéry-Odelin, 1998).

approximation, the calculation of all thermodynamic quantities requires only the knowledge of the density of states $\rho(\varepsilon)$, which depends on the dimensionality and on the type of confinement.

The general expression for the single-particle density of states is:

$$\rho(\varepsilon) = \int \frac{d^D r d^D p}{h^D} \delta[h(\vec{r}, \vec{p}) - \varepsilon], \quad (1.20)$$

where $h(\vec{r}, \vec{p})$ is the one-body hamiltonian, and D the dimensionality. The expressions of the density of states for a box or an isotropic harmonic potential in 2D and 3D are given in Table 1.1.

1.2.2.3 Bose–Einstein condensation in a 3D box

For a Bose gas confined in a 3D box and described by the semi-classical formalism, the expression for the fugacity as a function of the temperature T and the total number of particles N is given by

$$N = \int_0^\infty d\varepsilon \frac{\rho(\varepsilon)}{z^{-1}e^{\beta\varepsilon} - 1} = \frac{2}{\sqrt{\pi}} \frac{V}{\lambda_{\text{dB}}^3} I(z), \quad (1.21)$$

Table 1.1 Density of states for different types of confinement.

3D box	2D box	3D harmonic trap	2D harmonic trap
$\frac{Vm^{3/2}\sqrt{2\varepsilon}}{2\pi^2\hbar^3}$	$\frac{2\pi mL_x L_y}{h^2}$	$\frac{\varepsilon^2}{2(\hbar\omega)^3}$	$\frac{\varepsilon}{(\hbar\omega)^2}$

where V is the volume of the box, $\lambda_{\text{dB}} = h/\sqrt{2\pi mk_{\text{B}}T}$ the thermal de Broglie wavelength, and $I(z)$ a dimensionless integral whose expression in terms of Bose functions is:

$$I(z) = \int_0^\infty dx \frac{\sqrt{x}}{z^{-1}e^x - 1} = \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell^{3/2}} \int_0^\infty du \sqrt{u} e^{-u} = \frac{\sqrt{\pi}}{2} g_{3/2}(z). \quad (1.22)$$

We infer the expression for the saturation number of particles:

$$N'(T) = \frac{V}{\lambda_{\text{dB}}^3} g_{3/2}(1). \quad (1.23)$$

Figure 1.2 shows the variation of the Bose function $g_{3/2}(z)$ in the range $0 < z < 1$. It has an upper bound equal to $g_{3/2}(1) \simeq 2.612\dots$. As the density of states vanishes for $\varepsilon = 0$, one has to add explicitly the contribution of the number of particles in the ground state to the total number of particles:

$$N = \frac{z}{1-z} + \frac{V}{\lambda_{\text{dB}}^3} g_{3/2}(z) \quad \text{for } T > T_c, \quad (1.24)$$

$$N = \frac{z}{1-z} + \frac{V}{\lambda_{\text{dB}}^3} g_{3/2}(1) \quad \text{for } T < T_c, \quad (1.25)$$

where the critical temperature that enters the expression of λ_{dB} is defined by the equality $(N/V)\lambda_{\text{dB}}^3 = g_{3/2}(1)$. This relation shows explicitly that the degenerate regime is reached when the de Broglie wavelength becomes larger than the mean inter-particle distance. For a given density, it requires a sufficiently low temperature. This is the reason why laser cooling turned out to be a required step towards the production of dilute Bose–Einstein condensate with alkali atoms.

It is instructive to evaluate the number of particles in the first excited state of energy ε_1 . For the sake of simplicity, let us consider that half of the particles are in the

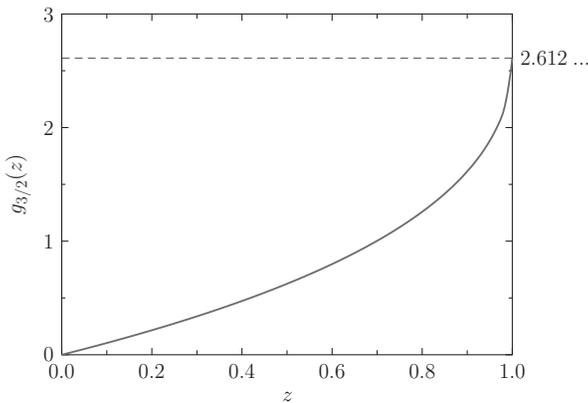


Fig. 1.2 Graph of the polylog function $g_{3/2}(z)$.

ground state: $N/2 = z/(1 - z)$ so that $z \simeq 1 - 2/N$. As we are dealing with the semi-classical approximation ($N \gg 1$), we have $\varepsilon_1 - \varepsilon_0 = \varepsilon_1 \ll k_B T$, and $e^{\beta\varepsilon_1} \simeq 1 - \beta\varepsilon_1$. The fraction of atoms in the first excited state can therefore be estimated:

$$\frac{N_1}{N} = \frac{1}{N} \frac{ze^{-\beta\varepsilon_1}}{1 - ze^{-\beta\varepsilon_1}} \simeq \frac{1}{N} \frac{1}{1 - (1 - 2/N)(1 - \beta\varepsilon_1)} = \frac{1}{2 + N^{1/3}} \simeq N^{-1/3} \ll 1. \quad (1.26)$$

We conclude that only the ground state is macroscopically occupied. We shall come back to this question when interactions are taken into account (see Section 1.4.2.2).

1.2.2.4 *The role of dimensionality*

In order to investigate the role played by the dimensionality, let us now consider a 2D box. The implicit expression for the fugacity is given by

$$N = \int_0^\infty \frac{\rho(\varepsilon)d\varepsilon}{e^{\beta(\varepsilon-\mu)} - 1} = \frac{L_x L_y}{\lambda_{\text{dB}}^2} \ln \left(\frac{e^{-\beta\mu}}{e^{-\beta\mu} - 1} \right), \quad (1.27)$$

where we have used the expression of the density of states given in Table 1.1. The fraction of atoms in the ground state is

$$\frac{N_0}{N} = \frac{1}{N} \frac{z}{1 - z} = \frac{e^{\sigma\lambda_{\text{dB}}^2} - 1}{N}, \quad (1.28)$$

where $\sigma = N/L_x L_y$ is the density of atoms per unit surface. Consider the thermodynamical limit for which the size of the system goes to infinity, while keeping constant the value of the intensive parameters such as the density σ and the temperature T . From Eq. (1.28), we find that N_0/N tends to zero when N tends to infinity. We conclude that there is no possible macroscopic occupation, and thus no Bose–Einstein condensation in a two-dimensional box in the thermodynamical limit. Otherwise stated, the saturation number tends to infinity faster than N when one takes the thermodynamical limit.

Let us emphasize that this result is intrinsically connected to the form of the density of states, which determines whether N'_{max} is finite or not. For a 2D box, the density of states does not depend on the energy ε . It is also the case for a one-dimensional harmonic oscillator for which $\rho(\varepsilon) = 1/\hbar\omega$. In this case also there is no possible condensation within the thermodynamical limit.

We conclude that confinement and dimensionality play a key role in the existence or not of Bose-Einstein condensation. Low-dimensional systems in the degenerate regime are particularly interesting in the presence of interactions. This topic is beyond the scope of this set of lectures (see (Pricoupenko *et al.*, 2004; Bloch *et al.*, 2008) for reviews).

1.2.2.5 *Bose–Einstein condensation in an arbitrary trap*

We consider now an ideal Bose gas confined in a general potential well $U(\vec{r})$. In the semi-classical approximation, one can show (Castin, 2001) using the Wigner distribution, that the density is given by:

$$n(\vec{r}) = \frac{1}{\lambda_{\text{dB}}^3} \sum_{\ell} \frac{z^{\ell}}{\ell^{3/2}} e^{-\beta \ell U(\vec{r})} = \frac{1}{\lambda_{\text{dB}}^3} g_{3/2} \left(z e^{-\beta U(\vec{r})} \right). \quad (1.29)$$

In the classical regime, $z \ll 1$, we recover the Boltzmann form for the atomic density $n(\vec{r}) \propto z e^{-\beta U(\vec{r})}$. If the trapping potential has its minimum of energy at $\vec{r} = \vec{0}$, the BEC transition condition is reached first at the center of the trap and the critical temperature is given by the relation: $n(\vec{0}) \lambda_{\text{dB}}^3 = g_{3/2}(1)$.

1.2.3 Coherence properties of a Bose–Einstein condensate

The second quantized formulation developed in the first section is a powerful tool for introducing field operators $\hat{\Psi}^{\dagger}(\vec{r})$ and $\hat{\Psi}(\vec{r})$ that create and destroy an atom at \vec{r} . They are in close analogy with the the electric-field operators $E^{(-)}(\vec{r})$ and $E^{(+)}(\vec{r})$ introduced in quantum optics to account for the coherence properties of light, and are useful for analyzing the coherence properties of Bose–Einstein condensates, as illustrated in the following with the calculation of the first- and second-order correlation functions.

1.2.3.1 Field operators

The field operators are linear combinations of the creation and annihilation operators where the coefficients are the single-particle wave functions:

$$\hat{\Psi}(\vec{r}) = \sum_{\alpha} \varphi_{\alpha}(\vec{r}) \hat{a}_{\alpha} \quad \text{and} \quad \hat{\Psi}^{\dagger}(\vec{r}) = \sum_{\alpha} \varphi_{\alpha}^{*}(\vec{r}) \hat{a}_{\alpha}^{\dagger}, \quad (1.30)$$

where the sum runs over the complete set of single-particle quantum numbers. Those operators are by construction defined at each space point \vec{r} . Their interpretation becomes clear when one calculates the action of the operator $\hat{\Psi}^{\dagger}(\vec{r})$ on the vacuum state $|0\rangle$:

$$\hat{\Psi}^{\dagger}(\vec{r}) |0\rangle = \sum_{\alpha} \varphi_{\alpha}^{*}(\vec{r}) \hat{a}_{\alpha}^{\dagger} |0\rangle = \sum_{\alpha} \varphi_{\alpha}^{*}(\vec{r}) |\varphi_{\alpha}\rangle. \quad (1.31)$$

Using the property $\varphi_{\alpha}^{*}(\vec{r}) = \langle \varphi_{\alpha} | \vec{r} \rangle$ and the fact that $\{|\varphi_{\alpha}\rangle\}$ forms a complete basis for the single-particle Hilbert space $\sum_{\alpha} |\varphi_{\alpha}\rangle \langle \varphi_{\alpha}| = 1$, one finds

$$\hat{\Psi}^{\dagger}(\vec{r}) |0\rangle = \sum_{\alpha} \varphi_{\alpha}^{*}(\vec{r}) |\varphi_{\alpha}\rangle = \sum_{\alpha} |\varphi_{\alpha}\rangle \langle \varphi_{\alpha} | \vec{r} \rangle = |\vec{r}\rangle. \quad (1.32)$$

The field operator $\hat{\Psi}^{\dagger}(\vec{r})$ is therefore an operator that creates an atom at \vec{r} . Similarly, the operator $\hat{\Psi}(\vec{r})$ annihilates an atom at \vec{r} . The field operators for bosons satisfy simple commutation relations

$$\left[\hat{\Psi}(\vec{r}), \hat{\Psi}^{\dagger}(\vec{r}') \right] = \delta(\vec{r} - \vec{r}') \quad \text{and} \quad \left[\hat{\Psi}(\vec{r}), \hat{\Psi}(\vec{r}') \right] = \left[\hat{\Psi}^{\dagger}(\vec{r}), \hat{\Psi}^{\dagger}(\vec{r}') \right] = 0. \quad (1.33)$$

The field operators $\hat{\Psi}^{\dagger}(\vec{r})$ and $\hat{\Psi}(\vec{r})$ play the same role in the basis $\{|\vec{r}\rangle\}$ as the operators $\hat{a}_{\alpha}^{\dagger}$ and \hat{a}_{α} in the basis $\{|\varphi_{\alpha}\rangle\}$.

1.2.3.2 First-order correlation function

The first-order correlation function $\langle \hat{\Psi}^\dagger(\vec{r})\hat{\Psi}(\vec{r}') \rangle$ is directly proportional to the visibility in an interference experiment. Let us choose the single-particle basis corresponding to a 3D box with periodic boundary conditions to work out the explicit expression of the first-order function correlation. The eigenstates are plane waves

$$\psi_{\vec{k}}(\vec{r}) = \frac{e^{i\vec{k}\cdot\vec{r}}}{L^{3/2}}, \quad \text{where} \quad k_i = \frac{2\pi}{L}n_i, \quad (1.34)$$

and n_i is any integer. The field operators are defined by

$$\hat{\Psi}(\vec{r}) = \frac{1}{L^{3/2}} \sum_{\vec{k}} \hat{a}_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} \quad \text{and} \quad \hat{\Psi}^\dagger(\vec{r}) = \frac{1}{L^{3/2}} \sum_{\vec{k}} \hat{a}_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{r}}. \quad (1.35)$$

The first-order correlation function $G^{(1)}(\vec{r}, \vec{r}')$ is defined by:

$$G^{(1)}(\vec{r}, \vec{r}') = \langle \hat{\Psi}^\dagger(\vec{r})\hat{\Psi}(\vec{r}') \rangle. \quad (1.36)$$

We consider an ideal Bose gas whose hamiltonian is thus

$$\hat{H} = \sum_{\vec{k}} \varepsilon_{\vec{k}} \hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}}, \quad \text{with} \quad \varepsilon_{\vec{k}} = \frac{\hbar^2 k^2}{2m}. \quad (1.37)$$

The calculation of $G^{(1)}(\vec{r}, \vec{r}')$ is performed using the density matrix whose expression in the grand-canonical formalism is $\hat{\rho}_{\text{eq}} = e^{-\beta(\hat{H} - \mu\hat{N})}/Z_G$:

$$G^{(1)}(\vec{r}, \vec{r}') = \frac{1}{L^3} \sum_{\vec{k}, \vec{k}'} e^{-i(\vec{k}\cdot\vec{r} - \vec{k}'\cdot\vec{r}')} \langle a_{\vec{k}}^\dagger a_{\vec{k}'} \rangle. \quad (1.38)$$

The invariance by translation implies $\langle a_{\vec{k}}^\dagger a_{\vec{k}'} \rangle = \langle n_{\vec{k}} \rangle \delta_{\vec{k}, \vec{k}'}$. The mean value $\langle n_{\vec{k}} \rangle$ obtained for an ideal Bose gas in the grand-canonical ensemble is

$$\langle n_{\vec{k}} \rangle = \text{Tr}(\hat{\rho}_{\text{eq}} a_{\vec{k}}^\dagger a_{\vec{k}}) = \frac{ze^{-\beta\varepsilon_k}}{1 - ze^{-\beta\varepsilon_k}} = \sum_{\ell=1}^{\infty} z^\ell e^{-\ell\beta\varepsilon_k}. \quad (1.39)$$

The first-order correlation function is therefore given by the Fourier transform of the momentum distribution. For a 3D ideal Bose gas, we therefore find

$$\begin{aligned} G^{(1)}(\vec{r}, \vec{r}') &= \frac{N_0}{L^3} + \frac{1}{(2\pi)^3} \int d^3\vec{k} e^{i\vec{k}\cdot(\vec{r}' - \vec{r})} \sum_{\ell=1}^{\infty} z^\ell e^{-\ell\beta\hbar^2 k^2/2m} \\ &= \frac{N_0}{L^3} + \frac{1}{\lambda_{\text{dB}}^3} \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell^{3/2}} \exp\left(-\frac{\pi(\vec{r}' - \vec{r})^2}{\ell\lambda_{\text{dB}}^2}\right). \end{aligned} \quad (1.40)$$

The fact that $G^{(1)}(\vec{r}, \vec{r}')$ depends only on the relative distance $|\vec{r}' - \vec{r}|$ is a direct consequence of the invariance by translation. The first term of the left-hand side of Eq. (1.40) accounts for the contribution of Bose condensed atoms, and the second term to that of thermal atoms.

The first-order correlation for $\vec{r} = \vec{r}'$ is simply the atomic density that is uniform in a box with periodic boundaries:

$$G^{(1)}(\vec{r}, \vec{r}) = n(\vec{r}) = \frac{N_0}{L^3} + \frac{1}{\lambda_{\text{dB}}^3} \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell^{3/2}} = \frac{N}{L^3}. \quad (1.41)$$

It is instructive to work out two limits:

- The limit where classical statistics is valid $n\lambda_{\text{dB}}^3 \ll 1$, $N_0 \ll N$ and $z \ll 1$,

$$G^{(1)}(\vec{r}, \vec{r}') \simeq \frac{N}{L^3} \exp^{-\pi(\vec{r}' - \vec{r})^2 / \lambda_{\text{dB}}^2}. \quad (1.42)$$

We find that for a classical gas the coherence length is $\lambda_{\text{dB}}/\sqrt{\pi}$.

- Below the critical temperature, the contribution of the condensed atoms yields an infinite coherence length. It reveals the presence of a long-range spatial order due to the condensate (see Figure 1.3a).

The previous results have been derived for an ideal gas in 3D. More generally, the first-order correlation function of the field operators allows one to identify the one-particle state having a macroscopic population that is at the origin of the long-range order. The wave function of this state appears to be the wave function of the condensate and is called the *order parameter*.

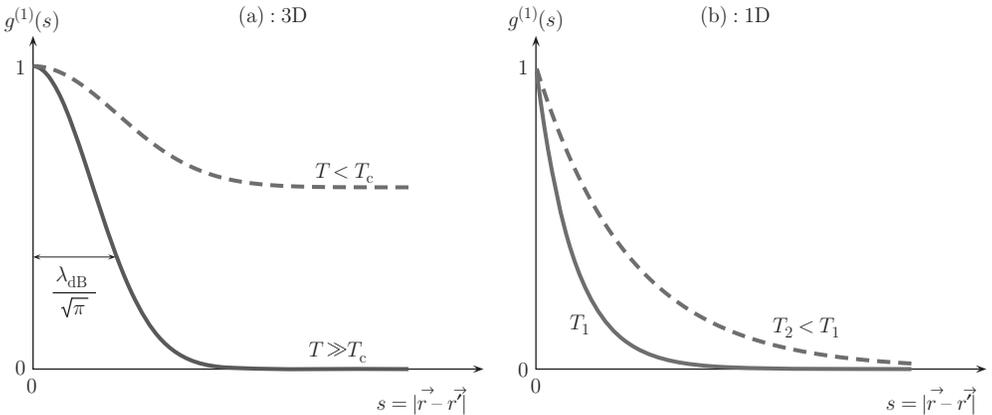


Fig. 1.3 Normalized first-order correlation function $g^{(1)}$ for a homogeneous Bose gas in 3D (a) in 1D (b). In 3D, below T_c , $g^{(1)}(s)$ goes to the constant value N_0/N when $s \rightarrow \infty$, clearly displaying off-diagonal long-range order. In contrast, in 1D, the correlation function always decays exponentially.

Coherence: the role of the dimensionality

For a 1D ideal Bose gas in the degenerate regime and within the validity range of the semi-classical description, $k_B T \gg |\mu| \gg h/L$, the first-order correlation function reads

$$G^{(1)}(x, 0) = \frac{1}{L} \int \frac{e^{-ipx/h}}{z^{-1}e^{-\beta p^2/2m} - 1} dp \propto \frac{1}{L} \int \frac{e^{-ipx/h}}{p^2 + p_c^2} dp \propto e^{-|x|/\xi},$$

where $p_c^2 = 2m|\mu|$ and $\xi = h/p_c$. From the expression of the total number of particle, and using the expansion for $n(p) \simeq (p^2 + p_c^2)^{-1}$, one finds $p_c = h/(n_{1D}\lambda_T^2)$. The first-order correlation exhibits an exponential decay with a coherence length $\xi = n_{1D}\lambda_T^2/2\pi$ and therefore there is no long-range order (see Figure 1.3b). We recover the fact that in the thermodynamical limit there is no Bose–Einstein condensation for a 1D Bose gas confined in a box. The first-order correlation function of a very elongated BEC has been investigated experimentally in (Dettmer *et al.*, 2001; Richard *et al.*, 2003).

Let us mention two experimental techniques that have been used to study the first-order coherence of Bose–Einstein condensates by interferometric means. The first one consists in putting a condensate, initially at rest, in a superposition of states with different momenta. This is realized using Bragg pulses. As the two condensates separate, one monitors the contrast of the matterwave interference fringes (Hagley *et al.*, 1999). The second method consists in outcoupling selectively atoms from two different (and adjustable) locations in the condensate; the visibility of the interference pattern observed on the outcoupled matterwaves then gives the value of the first-order correlation function (Bloch *et al.*, 2000).

1.2.3.3 Higher-order correlation function

The second-order correlation function is defined as

$$G^{(2)}(\vec{r}, \vec{r}') = \left\langle \hat{\Psi}^\dagger(\vec{r}) \hat{\Psi}^\dagger(\vec{r}') \hat{\Psi}(\vec{r}') \hat{\Psi}(\vec{r}) \right\rangle. \quad (1.43)$$

It is related to the conditional probability of presence of a particle at \vec{r} knowing that another one is at \vec{r}' . One often uses the normalized second-order correlation function:

$$g^{(2)}(\vec{r}, \vec{r}') = \frac{G^{(2)}(\vec{r}, \vec{r}')}{G^{(1)}(\vec{r}, \vec{r})G^{(1)}(\vec{r}', \vec{r}')}. \quad (1.44)$$

For the ideal Bose gas above the critical temperature, $G^{(2)}$ can be expressed in terms of $G^{(1)}$ in the grand canonical ensemble using Wick's theorem (Cohen-Tannoudji and Robillard, 2001):

$$g^{(2)}(\vec{r}, \vec{r}') = 1 + \frac{|G^{(1)}(\vec{r}, \vec{r}')|^2}{G^{(1)}(\vec{r}, \vec{r})G^{(1)}(\vec{r}', \vec{r}')}. \quad (1.45)$$

One finds $g^{(2)}(\vec{r}, \vec{r}) = 2$, a result that is referred to as the bosonic bunching effect. These quantum correlations are the atomic analog of the Hanbury-Brown–Twiss effect (Hanbury-Brown and Twiss, 1956). This correlation function can also be calculated for a condensate. Using a Fock state to describe the condensate, one finds $g^{(2)}(\vec{r}, \vec{r}) \simeq 1$ if the number of particles is sufficiently large. To measure this correlation

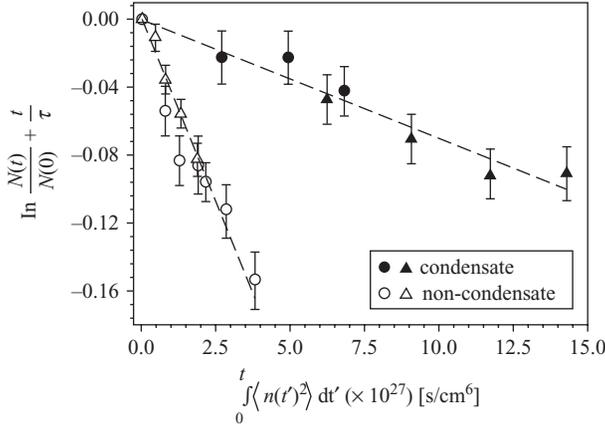


Fig. 1.4 Three-body losses in a cloud of ^{87}Rb . The slopes are proportional to $g^{(3)}(\vec{r}, \vec{r}, \vec{r})$. Figure taken from (Burt *et al.*, 1997), copyright American Physical Society.

function directly one needs to detect atoms one by one and to correlate their relative distances. This has been realized successfully using multichannel plate detectors for metastable helium atoms (Jeltes *et al.*, 2007), high-finesse optical cavities (Ritter *et al.*, 2007) or single-atom-sensitive fluorescence imaging (Manz *et al.*, 2009) for the alkalis, etc. The inhibition of the bosonic bunching for Bose–Einstein condensates has been observed, in good agreement with theory. Alternatively, $g^{(2)}(\vec{r}, \vec{r})$ can be inferred from the energy released during the ballistic expansion of a Bose–Einstein condensate (Ketterle and Miesner, 1997).

Finally, let us emphasize that third-order correlations can be investigated through the study of three-body losses (Burt *et al.*, 1997). The loss rate obeys the equation:

$$\frac{dN}{dt} = -\kappa \int n^3(\vec{r}, t) d^3r, \quad (1.46)$$

where κ is proportional to $g^{(3)}(\vec{r}, \vec{r}, \vec{r})$. One expects $g^{(3)}(\vec{r}, \vec{r}, \vec{r}) = 3! = 6$ for a thermal gas, and 1 for a condensate as a signature of its coherence. The loss rates in both situations are plotted in figure 1.4; the ratio of the slopes, proportional to $g^{(3)}(\vec{r}, \vec{r}, \vec{r})$, is measured to be 7.4 ± 2.6 . This experiment therefore clearly demonstrates the high coherence of a 3D Bose–Einstein condensate (in particular, the reduction of density fluctuations as compared to the thermal gas).

1.3 Mean-field theory

1.3.1 Introduction

In this section we study zero-temperature Bose–Einstein condensates in the presence of weak atom–atom interactions. Section 1.3.2 is devoted to the derivation, by a variational calculation, of the equation fulfilled by the condensate at equilibrium when

interactions are taken into account within the mean-field approximation. This equation is referred to as the Gross–Pitaevskii equation. We discuss the various physical quantities characterizing the condensate, and determine their dependence with the number N of condensed atoms (Sections 1.3.3 and 1.3.4). Another important issue is the role played by the sign of the scattering length, as it has dramatic consequences on the stability of the condensate, as discussed in Section 1.3.5.

In several experiments performed with Bose–Einstein condensates, the trapping potential is varied in time. To describe the condensate dynamics in those contexts, we derive the time-dependent Gross–Pitaevskii equation from a least-action principle (Section 1.3.6). We also recast this equation as a set of hydrodynamic equations (Section 1.3.7). This formalism is used to describe the ballistic expansion of the condensate and the low-lying excitations for a trapped condensate in the Thomas–Fermi regime (Section 1.3.8). The study of such elementary excitations provides us with a powerful tool for probing the fundamental properties of quantum many-body systems. They have been, for instance, extensively studied in the context of solid-state physics (Pines, 1999; Leggett, 2006), superfluid helium (Pines and Nozieres, 1966) and nuclear physics (Mottelson, 1976). In the context of trapped Bose–Einstein condensates, the measurement of the excitation frequencies helped in establishing the time-dependent Gross–Pitaevskii equation as an excellent description of condensate dynamics at low temperatures. The collective modes are finally discussed for a harmonically trapped Bose–Einstein condensate in Section 1.3.9.

1.3.2 Mean-field description of the condensate

In this section, we consider N identical bosons trapped in an external potential $V_{\text{trap}}(\vec{r})$ at equilibrium and at temperature $T = 0$. In the absence of interactions, all atoms are in the ground state of the trap, and the N -body wave function of the condensate reads $|\psi\rangle = |\varphi_0(1)\rangle \otimes |\varphi_0(2)\rangle \otimes \dots \otimes |\varphi_0(N)\rangle$, where $|\varphi_0\rangle$ denotes the wave function of the ground state determined by the confinement.

In the presence of interactions, the ground-state wave function that describes the structure of the condensate is that of the following N -body hamiltonian:

$$H = \sum_{i=1}^N \left[\frac{\vec{p}_i^2}{2m} + V_{\text{trap}}(\vec{r}_i) \right] + \frac{1}{2} \sum_i \sum_{j \neq i} V(\vec{r}_i - \vec{r}_j), \quad (1.47)$$

where the terms $V(\vec{r}_i - \vec{r}_j)$ account for two-body interactions. In most cases, it is impossible to determine from this hamiltonian the exact expression for the ground-state energy and the associated N -body wave function. Alternatively, one can resort to an approximate determination of the ground state using the variational approach.

1.3.2.1 Variational calculation of the condensate wave function

Such a calculation is performed within a given family of functions. By extension of the exact N -body wave function in the absence of interactions, we restrict ourselves to the family of tensor products of N single-particle identical states:

$$|\psi\rangle = |\varphi(1)\rangle \otimes |\varphi(2)\rangle \otimes \dots \otimes |\varphi(N)\rangle. \quad (1.48)$$

Those states are by definition symmetric in particle permutations, as required for identical bosons. Being tensor products of N states, they cannot describe the quantum correlations between the N atoms.

In the subspace generated by the vectors (1.48), the best state to approximate the ground state minimizes the energy functional $E_{\text{tot}}[\varphi]$ defined by:

$$E_{\text{tot}}[\varphi, N] = \langle H \rangle = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle}, \quad (1.49)$$

with the constraint $\langle \psi | \psi \rangle = 1$, or equivalently $\langle \varphi | \varphi \rangle = 1$. This best state will describe how the state of each atom is modified by the mean-field of the $N - 1$ other atoms.

The method of Lagrange multipliers permits one to recast the problem into the minimization of $\langle \psi | H | \psi \rangle - \mu \langle \psi | \psi \rangle$, where μ is the Lagrange multiplier associated with the conservation of the norm of the wave function. The functional differentiation $\delta(\langle \psi | H | \psi \rangle - \mu \langle \psi | \psi \rangle) = 0$ gives

$$\begin{aligned} N \int d^3r \delta\varphi^*(\vec{r}) \left\{ -\frac{\hbar^2}{2m} \Delta\varphi(\vec{r}) + V_{\text{ext}}(\vec{r})\varphi(\vec{r}) \right. \\ \left. + (N-1) \left[\int d^3r' V(\vec{r} - \vec{r}') |\varphi(\vec{r}')|^2 \right] \varphi(\vec{r}) - \mu\varphi(\vec{r}) \right\} + \text{c.c.} = 0. \end{aligned} \quad (1.50)$$

Since the variations of $\delta\varphi^*$ and $\delta\varphi$ can be considered as independent, the coefficient of $\delta\varphi^*$ must vanish, yielding:

$$\begin{aligned} -\frac{\hbar^2}{2m} \Delta\varphi(\vec{r}) + V_{\text{trap}}(\vec{r})\varphi(\vec{r}) \\ + (N-1) \left[\int d^3r' V(\vec{r} - \vec{r}') |\varphi(\vec{r}')|^2 \right] \varphi(\vec{r}) = \mu\varphi(\vec{r}). \end{aligned} \quad (1.51)$$

This equation, which resembles the Schrödinger equation, gives the evolution of each atom in the trapping potential and in the mean-field created at its position by the $(N - 1)$ other atoms.³

1.3.2.2 Stationary Gross-Pitaevskii equation

The variational method neglects the correlations between atoms at short distances. The gas is therefore supposed to be dilute. In this approximation, atoms are essentially far away one from another, and the interactions are governed by the large-distance asymptotic behavior of the wave function. Under this assumption, one can replace the true interacting potential by the corresponding pseudo-potential $V_{\text{pseudo}}(\vec{r} - \vec{r}') = g \delta(\vec{r} - \vec{r}') = (4\pi\hbar^2 a/m) \delta(\vec{r} - \vec{r}')$, where a is the scattering length of the real potential Dalibard (1998); Castin (2001). With such a contact potential, Eq. (1.51) takes the simple form

$$-\frac{\hbar^2}{2m}\Delta\varphi(\vec{r}) + V_{\text{trap}}(\vec{r})\varphi(\vec{r}) + (N-1)g|\varphi(\vec{r})|^2\varphi(\vec{r}) = \mu\varphi(\vec{r}). \quad (1.52)$$

Usually, we are dealing with a sufficiently large number of atoms ($N \gg 1$) so that we can replace in the previous equation $N-1$ by N . Equation (1.52), referred to as the stationary Gross–Pitaevskii equation, plays a central role in the study of the static properties of Bose–Einstein condensation in the dilute limit.

In order to relate the Lagrange multiplier μ to a known physical quantity, we substitute into the energy functional (1.49) the real interaction potential by V_{pseudo} , and get:

$$E_{\text{tot}}[\varphi, N] = N \int d^3r \varphi^*(\vec{r}) \left[-\frac{\hbar^2}{2m}\Delta + V_{\text{trap}}(\vec{r}) + \frac{(N-1)g}{2}|\varphi(\vec{r})|^2 \right] \varphi(\vec{r}). \quad (1.53)$$

$E_{\text{tot}}[\varphi, N]$ depends explicitly on the number of atoms N , and also implicitly through the N dependence of φ so that:

$$\begin{aligned} \frac{dE_{\text{tot}}[\varphi, N]}{dN} &= \frac{\partial E_{\text{tot}}[\varphi, N]}{\partial N} + \frac{\delta E_{\text{tot}}[\varphi, N]}{\delta\varphi} \frac{\partial\varphi}{\partial N} = \frac{\partial E_{\text{tot}}[\varphi, N]}{\partial N} + 0 \\ &= \int d^3r \varphi^*(\vec{r}) \left[-\frac{\hbar^2}{2m}\Delta + V_{\text{trap}}(\vec{r}) + \left(N - \frac{1}{2}\right)g|\varphi(\vec{r})|^2 \right] \varphi(\vec{r}), \end{aligned} \quad (1.54)$$

where we have used explicitly the fact that the functional derivative $\delta E_{\text{tot}}[\varphi, N]/\delta\varphi$ vanishes since φ is such that $E_{\text{tot}}[\varphi, N]$ is extremal for any variation of φ . The Gross–Pitaevskii equation (1.52) gives an integral expression for the Lagrangian multiplier μ :

$$\mu = \int d^3r \varphi^*(\vec{r}) \left[-\frac{\hbar^2}{2m}\Delta + V_{\text{trap}}(\vec{r}) + (N-1)g|\varphi(\vec{r})|^2 \right] \varphi(\vec{r}), \quad (1.55)$$

where we have used the normalization property $\langle\varphi|\varphi\rangle = 1$. If we compare Eq. (1.54) with Eq. (1.55), in the limit of large N , we deduce that

$$\mu = \frac{\partial E_{\text{tot}}[\varphi]}{\partial N} = E_{\text{tot}}[\varphi, N] - E_{\text{tot}}[\varphi, N-1].$$

The Lagrange multiplier μ therefore corresponds to the variation of the total mean energy when N varies by one unit, which is simply the definition of the chemical potential.

1.3.2.3 *Expression of the various quantities in terms of the spatial density*

From Eq. (1.53), we can write the total energy E_{tot} as a sum of three terms $E_{\text{tot}} = E_{\text{kin}} + E_{\text{trap}} + E_{\text{int}}$ that can be expressed in terms of the spatial density $n(\vec{r}) = N|\varphi(\vec{r})|^2$:

- the kinetic energy due to the confinement⁴:

$$E_{\text{kin}} = N \frac{\hbar^2}{2m} \int d^3r \left| \vec{\nabla} \varphi(\vec{r}) \right|^2 = \frac{\hbar^2}{2m} \int d^3r \left[\vec{\nabla} \sqrt{n(\vec{r})} \right]^2, \quad (1.56)$$

- the trapping energy:

$$E_{\text{trap}} = N \int d^3r V_{\text{trap}}(\vec{r}) |\varphi(\vec{r})|^2 = \int d^3r V_{\text{trap}}(\vec{r}) n(\vec{r}), \quad (1.57)$$

- and the interaction energy:

$$E_{\text{int}} = \frac{N(N-1)}{2} g \int d^3r |\varphi(\vec{r})|^4 \simeq \frac{g}{2} \int d^3r [n(\vec{r})]^2. \quad (1.58)$$

It is instructive to rewrite the chemical potential in terms of those three energies. Multiplying the Gross–Pitaevskii equation (1.52) by $\varphi^*(\vec{r})$ and integrating over r gives

$$\mu = \frac{1}{N} (E_{\text{kin}} + E_{\text{trap}} + 2E_{\text{int}}). \quad (1.59)$$

We conclude that the chemical potential is not equal to the mean total energy per atom ($\mu \neq E_{\text{tot}}/N$). This is due to the fact that, contrary to E_{trap} and E_{kin} , E_{int} does not increase linearly with N .

Finally, we give an extra relation between the three energies that enter the expression of the total energy and that is valid for a harmonic trapping potential:

$$2E_{\text{kin}} - 2E_{\text{trap}} + 3E_{\text{int}} = 0. \quad (1.60)$$

This equation results from the virial theorem (Dalfovo *et al.*, 1999).

1.3.3 Condensate in a box and healing length

1.3.3.1 Condensate in a 1D box

The atom–atom interactions yield a new characteristic length, the healing length. Its physical meaning appears clearly by considering a 3D condensate in a box of volume L^3 with periodic boundary conditions along two axes and strict boundary conditions along the planes $z = 0$ and $z = L$. In the absence of interactions, all the atoms are in the ground state of the trap. Their wave function is thus given by

$$\Psi(x, y, z) = \frac{1}{L} \varphi_0(z) \quad \text{with} \quad \varphi_0(z) = \frac{2}{\sqrt{L}} \sin\left(\frac{\pi z}{L}\right), \quad (1.61)$$

and the corresponding atomic linear density $N|\varphi_0|^2$ is inhomogeneous. In the presence of interactions, the wave function $\varphi(z)$ still has to vanish at $z = 0$ and $z = L$, but tends to be homogeneous far from the walls since this minimizes the interaction energy for repulsive interactions. This behavior can be shown in the following manner (Cohen-Tannoudji, 1998). Let us calculate the interaction energy for a homogeneous density $n_0 = N/L$ over a distance L :

$$E_{\text{int}}^{\text{hom}} = \frac{g}{2} \int dz n_0^2 = \frac{g n_0^2 L}{2}. \quad (1.62)$$

If we consider another state with the same total number N of atoms but with an inhomogeneous density $n(z)$, it has an interaction energy:

$$E_{\text{int}}^{\text{inh}} = \frac{g}{2} \int dz n^2(z). \quad (1.63)$$

The comparison between those two interaction energies is obtained by calculating the difference of energies:

$$E_{\text{int}}^{\text{inh}} - E_{\text{int}}^{\text{hom}} = \frac{g}{2} \int dz [n^2(z) - n_0^2]. \quad (1.64)$$

Using the normalization relation $\int n(z) dz = \int n_0 dz$, we can recast the previous equation in the following form:

$$E_{\text{int}}^{\text{inh}} - E_{\text{int}}^{\text{hom}} = \frac{g}{2} \int dz [n(z) - n_0]^2 \geq 0. \quad (1.65)$$

This inequality shows that the homogeneous distribution of atoms is the one giving the smallest interaction energy, it also gives the smallest kinetic (or confinement) energy since $d\varphi/dz = 0$, and it is thus the one privileged by the system.

1.3.3.2 *Healing length*

One may wonder what is the characteristic length scale ξ over which the wave function, in the presence of interactions, varies from 0 at a wall position to its constant value \tilde{n}_0 (see Figure 1.5). The total number of particles N being fixed, the removal of atoms near the walls increases the spatial density \tilde{n}_0 far from the walls and thus the interaction energy E_{int} . We deduce that when ξ increases, the interaction energy E_{int} increases, and correlatively, the kinetic energy E_{kin} decreases since the gradient of density is weaker. The equilibrium shape of the condensate corresponds to the

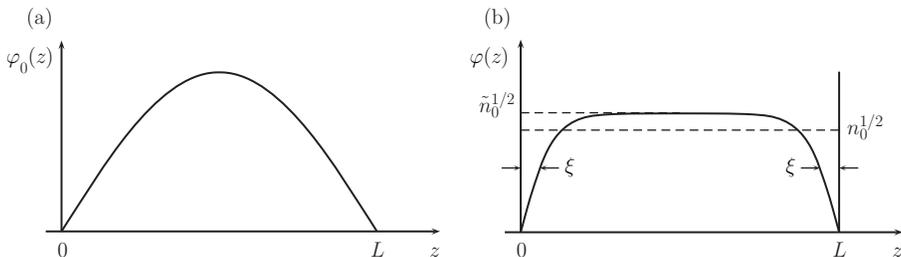


Fig. 1.5 (a) Ground-state wave function in a one-dimensional box of size L with strict boundary conditions. (b) Ground-state wave function with the same confinement but in the presence of interactions that tend to flatten the wave function. The healing length ξ is the distance over which the boundary condition no longer affects the wave function.

value of ξ for which the sum of these two energies is minimum. This characteristic length ξ is called the healing length, and more generally, represents the length after which the condensate recovers from a local perturbation (which, here, is due to the walls).

The order of magnitude of the healing length is readily obtained by the scaling of both the kinetic and interaction energies. The order of magnitude of E_{kin} is given by

$$E_{\text{kin}}(\xi) = \frac{\hbar^2}{2m} \int dz \left(\frac{\partial \sqrt{n}}{\partial z} \right)^2 \simeq \frac{\hbar^2}{2m} 2\xi \frac{n}{\xi^2} \simeq \frac{\hbar^2 n_0}{m\xi}. \quad (1.66)$$

In order to estimate the interaction energy, one needs the expression for the density plateau \tilde{n}_0 . From Figure 1.5, one finds approximately $\tilde{n}_0 - n_0 \simeq (2\xi/L) n_0$, which implies $\tilde{n}_0 \simeq (1 + 2\xi/L) n_0$ assuming $\xi \ll L$. The order of magnitude of E_{int} is given by

$$E_{\text{int}}(\xi) = \frac{g}{2} \int dz n^2 \simeq \frac{g}{2} (L - 2\xi) n_0^2 (1 + 2\xi/L)^2 \simeq \frac{gn_0^2 L}{2} + gn_0^2 \xi. \quad (1.67)$$

As expected intuitively, $E_{\text{kin}}(\xi)$ is a decreasing function of ξ and $E_{\text{int}}(\xi)$ an increasing function of ξ . The order of magnitude of the healing length is obtained by minimizing $E_{\text{kin}}(\xi) + E_{\text{int}}(\xi)$:

$$\frac{\partial}{\partial \xi} \left(\frac{\hbar^2}{m\xi} + gn_0\xi \right) = 0 \quad \Rightarrow \quad \xi \simeq \frac{\hbar}{\sqrt{mgn_0}} = \frac{1}{\sqrt{4\pi a n_0}}. \quad (1.68)$$

Alternatively, one can solve the Gross–Pitaevskii equation for a one-dimensional box:

$$-\frac{\hbar^2}{2m} \frac{d^2\varphi}{dz^2} + Ng\varphi^3(z) = \mu\varphi(z), \quad (1.69)$$

with $\varphi(z)$ a real function that obeys the boundary conditions $\varphi(z=0) = \varphi(z=L) = 0$. Far from the walls ($z \sim L/2$), one can neglect $d^2\varphi/dz^2$ and deduce the approximate value for the chemical potential $\mu \simeq Ng\varphi^2(z) = gn(z) \simeq g\tilde{n}_0$. Let us introduce the *standard definition of the healing length*,

$$\xi_0 = \left(\frac{\hbar^2}{2mgn_0} \right)^{1/2} = \frac{1}{(8\pi a \tilde{n}_0)^{1/2}}. \quad (1.70)$$

Using the dimensionless variable $\zeta = z/\xi_0$, the stationary Gross–Pitaevskii equation (1.69) can be rewritten in the form:

$$\frac{d^2\varphi(\zeta)}{d\zeta^2} - \frac{N}{n_0}\varphi^3(\zeta) + \varphi(\zeta) = 0. \quad (1.71)$$

The solution of Eq. (1.71) reads $\varphi(\zeta) = \sqrt{n_0/N} \text{th}(\zeta/\sqrt{2})$. Starting from 0 at $z = 0$, the wave function reaches a constant value after a few healing lengths ξ_0 .

1.3.4 Condensate in a harmonic trap

Experimentally, one uses either a magnetic trap (Pritchard, 1983) or a far-off resonance dipole trap (Grimm *et al.*, 2000) to confine the atoms, and the condensate experiences a harmonic confinement. Let us first consider an isotropic harmonic trap of angular frequency ω_0 . In the absence of interactions, the spatial extent of the ground state is given by the oscillator length $a_{\text{ho}} = (\hbar/m\omega_0)^{1/2}$. We work out in the following the scalings of the different contributions to the total energy for a non-ideal Bose–Einstein condensate held in a harmonic trap. These straightforward estimates are useful to classify the different interacting regimes.

1.3.4.1 Scaling

Let us denote by R the typical radius of the condensate. In the absence of interactions, R is on the order of a_{ho} . We estimate in the following how R is modified by interactions. To answer this question it is convenient to express R in units of a_{ho} : $w = R/a_{\text{ho}}$, and to use the following gaussian ansatz for the wave function of atoms in an isotropic harmonic trap:

$$\varphi(\vec{r}) = \frac{1}{\pi^{3/4} (w^3 a_{\text{ho}}^3)^{1/2}} \exp\left[-\frac{r^2}{2w^2 a_{\text{ho}}^2}\right]. \quad (1.72)$$

The three energies that contribute to the total energy can be readily calculated analytically, and one finds:

$$E_{\text{tot}}[w] = E_{\text{kin}} + E_{\text{trap}} + E_{\text{int}} = N\hbar\omega_0 \left[\frac{3}{4} \frac{1}{w^2} + \frac{3}{4} w^2 + \frac{1}{\sqrt{2\pi}} \frac{aN}{a_{\text{ho}}} \frac{1}{w^3} \right]. \quad (1.73)$$

1.3.4.2 Different interacting regimes

In the absence of interactions ($a = 0$), the last term of Eq. (1.73) vanishes, and the minimum of the sum of kinetic and trapping energy is obtained for $w = 1$. In this limit, we recover the well-known expression for the ground-state wave function of an harmonic oscillator. If the scattering length is non-zero, the order of magnitude of the interaction energy compared to the kinetic and trapping energies for $w = 1$ is determined by the dimensionless parameter $\chi = Na/a_{\text{ho}}$. If $\chi \ll 1$, interactions can be ignored. In the opposite limit ($\chi \gg 1$), the last term of Eq. (1.73) plays a crucial role and one must determine the new value of w that minimizes the total energy. The result depends on the sign of the scattering length: if $a > 0$, the effective interactions are repulsive and w scales as $N^{1/5}$, i.e. the size of the ground-state wave function increases with the number of condensed atoms, if $a < 0$, the effective interactions are attractive and one finds $w < 1$ when a solution exists (see Section 1.3.5).

1.3.4.3 Condensate with a positive scattering length and the Thomas–Fermi limit

For a positive scattering length, both the kinetic and interaction energies are decreasing functions of w , whereas the trapping energy increases with w . There is always

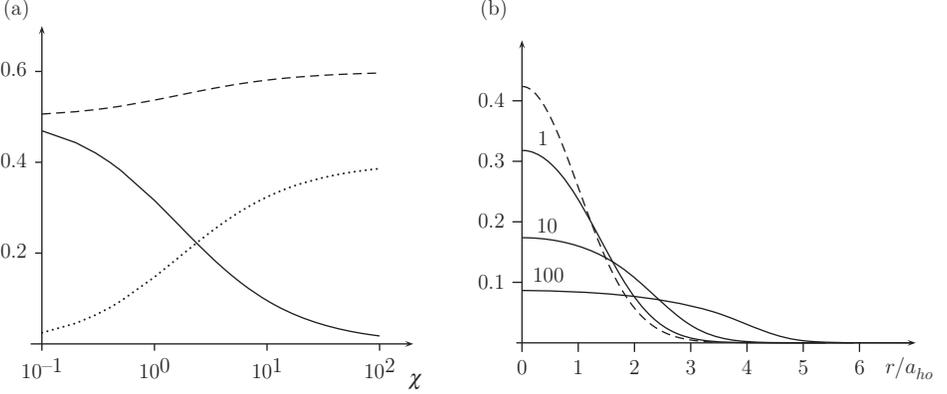


Fig. 1.6 (a) Relative contributions of the different energy terms to the total energy as a function of the dimensionless parameter $\chi = Na/a_{ho}$: $E_{\text{trap}}/E_{\text{tot}}$ (dashed line), $E_{\text{kin}}/E_{\text{tot}}$ (solid line), $E_{\text{int}}/E_{\text{tot}}$ (dotted line). (b) Condensate wave function, at $T = 0$, obtained by solving numerically the stationary Gross–Pitaevskii equation Eq. (1.52) in a spherical trap and with repulsive interactions. The dashed line corresponds to the ideal gas ($a = 0$); the solid lines to $\chi = 1, 10, 100$. Figure courtesy of S. Giorgini.

a value of w that minimizes the total energy and that corresponds to a stable condensate. The radius of the condensate increases when the strength of the repulsive interactions increases as illustrated in Figure 1.6 where different numerical solutions of the stationary Gross–Pitaevskii equation for increasing values of the χ parameter are represented.

We have plotted in Figure 1.6.a the relative contribution to the total energy of the trapping, kinetic and interaction energies as a function of the interacting parameter χ . In the limit $\chi \gg 1$, referred to as the Thomas–Fermi limit, one can neglect the kinetic energy term, and the Gross–Pitaevskii equation becomes a simple algebraic equation

$$V_{\text{trap}}(\vec{r}) + Ng|\varphi(\vec{r})|^2 = V_{\text{trap}} + gn_0(\vec{r}) = \mu. \quad (1.74)$$

For a harmonic confinement, $V_{\text{trap}}(\vec{r}) = m\omega_0^2 r^2/2$, and the spatial density $n_0(\vec{r}) = N|\varphi(\vec{r})|^2$ has the shape of an inverted parabola: $n(\vec{r}) = [\mu - m\omega_0^2 r^2/2]/g$ that starts from the value μ/g for $r = 0$ and that vanishes for $r \geq r_{\text{max}} = (2\mu/m\omega_0^2)^{1/2}$. The expression for the chemical potential is obtained from the normalization condition. Integrating the density profile over r , we find

$$\mu(N) = \frac{\hbar\omega_0}{2} \left(15 \frac{Na}{a_{ho}} \right)^{2/5}. \quad (1.75)$$

The total energy is obtained by integrating Eq. (1.55):

$$E_{\text{tot}}(N) = \int_0^N \mu(N') dN' = \frac{\hbar\omega_0}{2} \left(15 \frac{a}{a_{ho}} \right)^{2/5} \frac{5N^{7/5}}{7}, \quad (1.76)$$

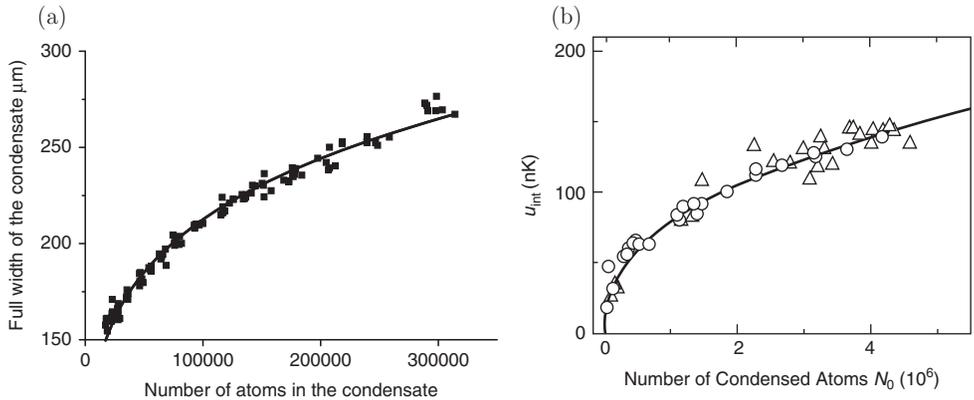


Fig. 1.7 (a) Variations of the full width of the condensate along one axis, as a function of the number of condensed atoms N . The solid line is a fit proportional to $N^{1/5}$, figure taken from (Söding *et al.*, 1999). (b) Mean-field energy per condensed atom versus the number of atoms in the condensate. The solid line is a fit proportional to $N^{2/5}$. Figure from (Mewes *et al.*, 1996). Copyright American Physical Society.

and the total energy per particle is equal to $E_{\text{tot}}(N)/N = 5\mu/7$. Using the relations (1.59) and (1.60), one deduces the expression for the interaction energy per particle in the Thomas–Fermi limit for which the kinetic energy⁵ is negligible:

$$\frac{E_{\text{int}}(N)}{N} = \frac{2}{7}\mu(N) = \frac{\hbar\omega_0}{7} \left(15\frac{Na}{a_{\text{ho}}}\right)^{2/5}. \quad (1.77)$$

The size of the condensate depends on the number of atoms through the chemical potential:

$$r_{\text{max}}(N) = \sqrt{\frac{2\mu}{m\omega_0^2}} = a_{\text{ho}} \left(\frac{15Na}{a_{\text{ho}}}\right)^{1/5}. \quad (1.78)$$

This atom number dependence of the size is illustrated in Figure 1.7.a where the fit proportional to $N^{1/5}$ is in good agreement with the size observed experimentally for different condensed atoms number.

The interaction energy can be measured by removing abruptly the confinement (Mewes *et al.*, 1996). Just after the switch off, the total energy is equal to the interaction energy (which has not changed) plus the kinetic energy, which is negligible in the Thomas–Fermi limit. The interaction energy is converted into kinetic energy during the expansion of the condensate (see Section 1.3.9.2). The measurement of this energy shows that it varies with the number of condensed atoms as $N^{2/5}$ (see Figure 1.7.b), as expected from the Thomas–Fermi limit result (1.77).

Most experiments use cylindrically symmetric harmonic traps:

$$V(z, r) = (m/2) [\omega_z^2 z^2 + \omega_\perp^2 r^2].$$

The inverted parabola shape of the density profile is limited axially to $\pm z_{\max}$ and radially to $\pm r_{\max}$ defined, respectively, by $m\omega_z^2 z_{\max}^2 = 2\mu$ and $m\omega_\perp^2 r_{\max}^2 = 2\mu$. The aspect ratio of the condensate is therefore given by $z_{\max}/r_{\max} = \omega_\perp/\omega_z$. In the absence of interactions, this aspect ratio is equal to the ratio of the oscillator lengths of each oscillator:

$$\frac{z_{\max}}{r_{\max}} = \frac{a_{\text{ho}}^z}{a_{\text{ho}}^\perp} = \sqrt{\frac{\omega_\perp}{\omega_z}} < \frac{\omega_\perp}{\omega_z}, \quad \text{if } \frac{\omega_\perp}{\omega_z} > 1. \quad (1.79)$$

We deduce that interactions tend to magnify the aspect ratio of the condensate with respect to the ideal Bose gas.

A remarkable feature of Bose–Einstein condensates in the Thomas–Fermi limit is that interactions can be important whilst the gas is dilute. Consider a condensate contained in a volume R^3 . It can be considered as dilute as soon as the mean distance $d = (N/R^3)^{-1/3}$ between atoms is large compared to the scattering length a . Since R is always larger than a_{ho} , one has:

$$\frac{a}{d} < \frac{a}{a_{\text{ho}}} N^{1/3}. \quad (1.80)$$

The ratio a/d increases at most as $N^{1/3}$ only, whereas $\chi = Na/a_{\text{ho}}$, which characterizes the importance of interactions increases as N . One can therefore have a condensate in the Thomas–Fermi regime ($\chi \gg 1$) while remaining in the dilute regime for which $a \ll d$. For example, let us consider the case of ^{87}Rb atoms, whose scattering length is $a = 5 \text{ nm}$, in a harmonic trap of frequency $\omega/2\pi = 250 \text{ Hz}$ yielding an oscillator length $a_{\text{ho}} = 0.68 \text{ }\mu\text{m}$. For a condensate of $N = 10^6$ atoms, one finds for the interaction parameter $\chi = Na/a_{\text{ho}} \simeq 7400 \gg 1$. The Thomas–fermi radius is then $r_{\max} \simeq 6.9 \text{ }\mu\text{m}$ and the parameter that characterizes the diluteness of the gas $a/d \sim (a/r_{\max}) N^{1/3} \simeq 7.2 \times 10^{-2} \ll 1$.

1.3.5 Condensate with a negative scattering length

1.3.5.1 Condition of stability

Theoretical studies predict that a homogeneous Bose–Einstein condensate with attractive interactions is unstable (Stoof, 1994). This can be understood simply from the spectrum of elementary excitations (see Section 1.3.6.4): for $a < 0$, the frequency of small momenta excitations is imaginary, yielding the so-called *phonon instability*. In the presence of a confinement, a condensate may form if the atom number is not too large. Physically, the stability originates from the tradeoff between the attractive interaction energy, that tends to contract the cloud, and the kinetic energy term resulting from the position–momentum uncertainty in the presence of confinement.⁶

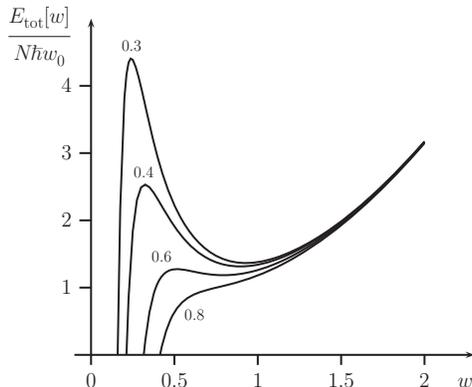


Fig. 1.8 Total energy, normalized to $N\hbar\omega_0$, as a function of the width parameter w for the Gaussian model of Section 1.3.4.1, for different values of the interaction parameter $|\chi| = 0.3, 0.4, 0.6, 0.8$.

This balance is illustrated in the case of an isotropic harmonic confinement in Figure 1.8.(i) where we have plotted the total energy per particle normalized to $\hbar\omega_0$ as a function of the effective width w for the Gaussian model of Eq. (1.73), for several values of the interaction parameter $|\chi| = N|a|/a_{\text{ho}}$. There exists a critical value χ_c such that if $|\chi|$ is larger than χ_c , there is no longer any local minimum of the total energy E_{tot} . In other words, for a given negative value of the scattering length a , the condensate can accommodate only a finite number of atoms $N < N_c = \chi_c a_{\text{ho}}/|a|$ (Bradley *et al.*, 1997). The critical value of the interaction parameter is equal to $\chi_c \simeq 0.671$ within the gaussian ansatz approximation (see Section 1.3.4.1). A more refined theoretical analysis yields $\chi_c = 0.574$ (see (Dalfovo *et al.*, 1999) and references therein), in rather good agreement with experimental observations (Roberts *et al.*, 2001).

1.3.5.2 Collapse and explosion of a condensate with a negative scattering length

Beyond the stability limit, self-attraction overwhelms the repulsion due to the quantum pressure and causes the condensate to collapse. During the collapse, the density rises yielding a dramatic increase of the inelastic collision rate such as three-body recombination and therefore induces atom losses. One may wonder what happens after the collapse.

The group of R. Hulet was able to observe a ${}^7\text{Li}$ atoms condensate regrowing after the collapse (Gerton *et al.*, 2000). Indeed, just after the collapse the gas is out of equilibrium and a condensate may form filled through elastic collisions between thermal atoms in the gas. The condensate is therefore found to undergo many cycles of growth and collapses before reaching a stationary regime (Sackett *et al.*, 1998).

The JILA group has explored the dynamics of collapse and its subsequent explosion when the balance of forces governing the Bose–Einstein condensate size and shape is

suddenly altered. The collapse is induced by tuning abruptly the interactions from repulsive to attractive using an externally applied magnetic field close to a Feshbach resonance. This technique has allowed for the observation of an exploding atomic ejection from the collapsing Bose–Einstein condensate (Donley *et al.*, 2001).

Surprisingly, after this dramatic event, a remnant and highly excited condensate with a number of atoms greater than N_c was sometimes observed. This apparent contradiction with the stability criterium was recently solved. Indeed, it turns out that the remnant is composed of multiple solitons that have relative phases such that they repel each other and oscillate in the trapping potential for a long time without degradation (Cornish *et al.*, 2006; Strecker *et al.*, 2002).

1.3.6 Time-dependent Gross–Pitaevskii equation

In several experiments performed with Bose–Einstein condensates, the trapping potential is varied in time. For instance, time-of-flight experiments where one switches off suddenly the confinement to observe the ballistic expansion of the condensate provide a way to investigate the properties of the many-body ground state and in particular the role played by atom–atom interactions. To describe the dynamics of the condensate in those contexts, one needs to extend (using a least-action principle) to time-dependent phenomena the Gross–Pitaevskii equation introduced in Section 1.3.2 for analyzing static properties of condensates.

1.3.6.1 Derivation of Schrödinger equation from a principle of least action

As a starting point, we recall (Cohen-Tannoudji, 1998) that the Schrödinger equation for a particle in a confining potential $V_{\text{trap}}(\vec{r}, t)$,

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = -\frac{\hbar^2}{2m} \Delta \psi(\vec{r}, t) + V_{\text{trap}}(\vec{r}, t) \psi(\vec{r}, t), \quad (1.81)$$

can be obtained by minimizing the action $S = \int_{t_1}^{t_2} dt \int d^3r \mathcal{L}$ related to the Lagrangian density

$$\mathcal{L}(\psi, \psi^*, \vec{\nabla} \psi, \vec{\nabla} \psi^*, \dot{\psi}, \dot{\psi}^*) = i\frac{\hbar}{2} [\dot{\psi}^* \psi - \dot{\psi} \psi^*] - \frac{\hbar^2}{2m} \vec{\nabla} \psi^* \cdot \vec{\nabla} \psi - V_{\text{trap}}(\vec{r}, t) \psi^* \psi, \quad (1.82)$$

where $\dot{\psi}$ and $\dot{\psi}^*$ refer to the time derivatives of ψ and ψ^* , respectively.

Similarly, the Schrödinger equation for N interacting bosons

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r}_1, \dots, \vec{r}_N, t) = -\frac{\hbar^2}{2m} \sum_{i=1}^N \Delta_i \psi(\vec{r}_1, \dots, \vec{r}_N, t) + \sum_{i=1}^N V_{\text{trap}}(\vec{r}_i, t) \psi(\vec{r}_1, \dots, \vec{r}_N, t) + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N V(\vec{r}_i - \vec{r}_j) \psi(\vec{r}_1, \dots, \vec{r}_N, t) \quad (1.83)$$

can be derived by minimizing the Lagrangian density

$$\begin{aligned} \mathcal{L} = & i \frac{\hbar}{2} \left[\psi^* (\vec{r}_1, \dots, \vec{r}_N, t) \dot{\psi} (\vec{r}_1, \dots, \vec{r}_N, t) - \dot{\psi}^* (\vec{r}_1, \dots, \vec{r}_N, t) \psi (\vec{r}_1, \dots, \vec{r}_N, t) \right] \\ & - \frac{\hbar^2}{2m} \sum_{i=1}^N \left(\vec{\nabla}_{\vec{r}_i} \psi^* (\vec{r}_1, \dots, \vec{r}_N, t) \right) \cdot \left(\vec{\nabla}_{\vec{r}_i} \psi (\vec{r}_1, \dots, \vec{r}_N, t) \right) \\ & - \left[\sum_{i=1}^N V_{\text{trap}} (\vec{r}_i, t) + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N V (\vec{r}_i - \vec{r}_j) \right] \psi^* (\vec{r}_1, \dots, \vec{r}_N, t) \psi (\vec{r}_1, \dots, \vec{r}_N, t). \end{aligned} \quad (1.84)$$

1.3.6.2 Determination of the best time-dependent N -particle state

As for the time-independent case, we look for a N -particle wave function equal to a product of N identical single-particle functions:

$$\psi (\vec{r}_1, \dots, \vec{r}_N, t) = \varphi (\vec{r}_1, t) \varphi (\vec{r}_2, t) \dots \varphi (\vec{r}_N, t). \quad (1.85)$$

Such a fully symmetric state describes a situation where all N bosons evolve in the same way, and neglects quantum correlations between the atoms.

Inserting the ansatz (1.85) into the Lagrangian density (1.84) and expressing that the variation δS of the corresponding action S vanishes to first order in $\delta\varphi$ for any variation $\delta\varphi$ of φ leads to the following time-dependent Gross–Pitaevskii equation:

$$i \hbar \frac{\partial}{\partial t} \varphi (\vec{r}, t) = - \frac{\hbar^2}{2m} \Delta \varphi (\vec{r}, t) + V_{\text{trap}} (\vec{r}, t) \varphi (\vec{r}, t) + N g |\varphi (\vec{r}, t)|^2 \varphi (\vec{r}, t), \quad (1.86)$$

where we have replaced the interaction potential V by the contact potential V_{pseudo} (see Section 1.3.2.2).

If $V_{\text{trap}} (\vec{r}, t) = V_0 (\vec{r})$ does not depend on time, one can look for stationary solutions of Eq. (1.86) of the form $\varphi (\vec{r}, t) = \varphi_0 (\vec{r}) \exp (-i\mu t/\hbar)$. In this way, one exactly recovers the time-independent Gross–Pitaevskii equation (1.52).

1.3.6.3 Response of a condensate to a time-dependent perturbation

The time-dependent Gross–Pitaevskii equation allows one to infer the response of the condensate to a small perturbation. To address this problem, we consider a small time-dependent perturbation $\delta V (\vec{r}, t)$ that is added to the confining potential $V_0 (\vec{r})$: $V_{\text{trap}} (\vec{r}, t) = V_0 (\vec{r}) + \delta V (\vec{r}, t)$. As an example, the perturbation δV can account for a slight modulation of the strength of the trapping potential at a controlled frequency. The calculation of the response of the condensate to this excitation and to first order in δV is obtained by searching a solution of Eq. (1.86) in the form:

$$\varphi (\vec{r}, t) = [\varphi_0 (\vec{r}) + \delta\varphi (\vec{r}, t)] e^{-i\mu t/\hbar}, \quad (1.87)$$

where $\delta\varphi (\vec{r}, t)$ accounts for the time-dependent modifications of the wave function driven by $\delta V (\vec{r}, t)$. In this way, one readily obtains the following set of coupled equations:

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \delta\varphi \\ \delta\varphi^* \end{pmatrix} = \mathcal{L}_{\text{GP}} \begin{pmatrix} \delta\varphi \\ \delta\varphi^* \end{pmatrix} + \begin{pmatrix} \varphi_0 \delta V \\ -\varphi_0^* \delta V \end{pmatrix}, \quad (1.88)$$

where \mathcal{L}_{GP} denotes the following time-independent 2×2 matrix:

$$\mathcal{L}_{\text{GP}} = \begin{pmatrix} H_0 - \mu + 2Ng|\varphi_0|^2 & Ng\varphi_0^2 \\ Ng(\varphi_0^*)^2 & -(H_0 - \mu + 2Ng|\varphi_0|^2) \end{pmatrix} \quad (1.89)$$

and

$$H_0 = -\frac{\hbar^2}{2m} \Delta + V_0(\vec{r}). \quad (1.90)$$

The second term in the r.h.s. of Eq. (1.88) is proportional to δV and acts as a source term. Equation (1.88) is simply the linearization of the time-dependent Gross–Pitaevskii equation (1.86), and is therefore valid for $|\delta V| \ll |V_0|$.

1.3.6.4 Frequencies of the small-amplitude oscillations

Let us suppose that one applies a small perturbation δV for a given duration. Just after the application of the perturbation, the condensate is out of equilibrium, and, as a result, oscillates with a small amplitude at its eigenfrequencies ω . To determine these frequencies, one must diagonalize the 2×2 matrix \mathcal{L}_{GP} , and solve the two coupled equations:

$$\begin{aligned} [H_0 - \mu + 2Ng n_0(\vec{r})] u(\vec{r}) + Ng n_0(\vec{r}) v(\vec{r}) &= \hbar\omega u(\vec{r}), \\ -Ng n_0(\vec{r}) u(\vec{r}) - [H_0 - \mu + 2Ng n_0(\vec{r})] v(\vec{r}) &= \hbar\omega v(\vec{r}), \end{aligned} \quad (1.91)$$

where u and v are defined by $\delta\varphi(\vec{r}, t) = u(\vec{r})e^{-i\omega t} + v^*(\vec{r})e^{i\omega t}$. This set of equations is commonly referred to as the Bogolubov–de Gennes equations, and is generally solved numerically to infer the frequencies ω for any value of the spatial density n_0 .

Consider as an example a condensate in a box with cyclic boundary conditions. The density is constant and equal to $n_0 = N/L^3$. The spectrum $\omega(k)$ for solutions of $u(\vec{r})$ and $v(\vec{r})$ in form of plane waves $\exp(i\vec{k} \cdot \vec{r})$ is given by the determinant deduced from \mathcal{L}_{GP} :

$$\begin{vmatrix} \frac{\hbar^2 k^2}{2m} + gn_0 - \hbar\omega & gn_0 \\ -gn_0 & -\frac{\hbar^2 k^2}{2m} - gn_0 - \hbar\omega \end{vmatrix} = 0. \quad (1.92)$$

One infers from Eq. (1.92) the dispersion law of the elementary excitations of a condensate in a box, known as the Bogolubov spectrum:

$$\hbar\omega = \left[\frac{\hbar^2 k^2}{2m} \left(\frac{\hbar^2 k^2}{2m} + 2gn_0 \right) \right]^{1/2}. \quad (1.93)$$

The dispersion law varies linearly with the wave vector k in the large-wavelength regime: $\omega = kc$, where $c = (gn_0/m)^{1/2}$ is the sound velocity and plays a key role for the superfluidity properties of a Bose–Einstein condensate (see Section 1.4.3.4).

1.3.7 Analogy with hydrodynamic equations

The Gross–Pitaevskii equation can be rewritten in the form of hydrodynamic equations. Their expression is particularly simple in the strong-interaction regime, and turns out to be very useful for the interpretation of several physical effects.

1.3.7.1 Density field and velocity field

In order to derive the set of hydrodynamic equations that is equivalent to the time-dependent Gross–Pitaevskii equation, it is convenient to normalize the wave function to the number of particles: $\int d^3r |\varphi(\vec{r}, t)|^2 = N$. The spatial density is then given by the modulus of the wave function: $n(\vec{r}, t) = |\varphi(\vec{r}, t)|^2$. With such a choice for the normalization of the wave function, there is no longer an explicit dependence of the time-dependent Gross–Pitaevskii with the number of atoms N :

$$i\hbar \frac{\partial}{\partial t} \varphi(\vec{r}, t) = -\frac{\hbar^2}{2m} \Delta \varphi(\vec{r}, t) + V_{\text{trap}}(\vec{r}, t) \varphi(\vec{r}, t) + g |\varphi(\vec{r}, t)|^2 \varphi(\vec{r}, t). \quad (1.94)$$

It is instructive to rewrite the wave function $\varphi(\vec{r}, t)$ in terms of its phase $S(\vec{r}, t)$ and its modulus: $\varphi(\vec{r}, t) = \sqrt{n(\vec{r}, t)} \exp[iS(\vec{r}, t)]$.

1.3.7.2 Continuity equation. Evolution equation of the velocity field

Using Eq. (1.94) to derive the equation fulfilled by the modulus $n(\vec{r}, t)$, one obtains:

$$\frac{\partial}{\partial t} n(\vec{r}, t) + \vec{\nabla} \cdot [n(\vec{r}, t) \vec{v}(\vec{r}, t)] = 0, \quad (1.95)$$

where \vec{v} denotes the velocity field and is proportional to the gradient of the phase $S(\vec{r}, t)$:

$$\vec{v}(\vec{r}, t) = \frac{\hbar}{m} \vec{\nabla} S(\vec{r}, t). \quad (1.96)$$

Equation (1.95) is the continuity equation. It shows that the integral of $n(\vec{r}, t)$ over space does not change in time, or otherwise stated, that, if φ is normalized at $t = 0$, it remains normalized for all $t > 0$. The quantity $n(\vec{r}, t) \vec{v}(\vec{r}, t)$ is the current density. We emphasize that the velocity field obeys the equation $\vec{\nabla} \times \vec{v}(\vec{r}, t) = \vec{0}$ and is therefore irrotational.⁷

Similarly, the equation for the evolution of the phase $S(\vec{r}, t)$, and thus of the velocity field $\vec{v}(\vec{r}, t)$ is readily inferred from Eq. (1.94):

$$m \frac{\partial \vec{v}}{\partial t} = \vec{\nabla} \left[\frac{\hbar^2}{2m} \frac{1}{\sqrt{n}} \Delta \sqrt{n} - \frac{1}{2} m \vec{v}^2 - V_{\text{trap}} - gn \right]. \quad (1.97)$$

The continuity equation (1.95) for the density and the Euler-like equation (1.97) for the velocity field are often referred to as hydrodynamics equations, and are strictly equivalent to the time-dependent Gross–Pitaevskii equation (1.94) as soon as the solutions do not exhibit singularities.

1.3.7.3 Quantum pressure

The first term of the right-hand side of the Euler equation (1.97) is the only one that contains explicitly \hbar , and is called the quantum pressure term. It originates from the kinetic-energy term arising from density gradients, and is a direct consequence of the Heisenberg uncertainty principle.

In order to determine in which circumstances this term plays a role, let us denote by R the characteristic length of the spatial variations of the atomic density $n(\vec{r})$. The quantum pressure term scales as $\hbar^2/2mR^2$ and is negligible compared to the interaction term gn when

$$R \gg \left(\frac{\hbar^2}{2mgn} \right)^{1/2} = \xi, \quad (1.98)$$

where ξ is the healing length. The healing length thus appears as the characteristic length ξ such that the energy of confinement in a volume ξ^3 is equal to the interaction energy.

1.3.7.4 Small-amplitude oscillations of an homogeneous condensate. Bogolubov dispersion law

The hydrodynamic equations can be used to find the frequencies of the small-amplitude oscillations of the condensate around equilibrium in the absence of confining potential $V_{\text{trap}} = 0$ (homogeneous condensate). To perform this calculation, we start by expanding the density and the velocity field with respect to their equilibrium values in the form

$$n(\vec{r}, t) = n_0 + \delta n(\vec{r}, t) \quad \text{and} \quad \vec{v}(\vec{r}, t) = \vec{0} + \delta \vec{v}(\vec{r}, t). \quad (1.99)$$

By inserting these relations into the hydrodynamics equations and restricting the expansion to the first order in δn and δv , one obtains:

$$\frac{\partial^2 \delta n}{\partial t^2} + n_0 \vec{\nabla} \cdot \left(\frac{\partial \delta \vec{v}}{\partial t} \right) = 0 \quad \text{and} \quad \frac{\partial \delta \vec{v}}{\partial t} = \vec{\nabla} \cdot \left(\frac{\hbar^2}{4m^2 n_0} \Delta \delta n - g \delta n \right). \quad (1.100)$$

Combining those two equations, we get the equation obeyed by the density perturbation $\delta n(\vec{r}, t)$:

$$\frac{\partial^2 \delta n}{\partial t^2} + \frac{\hbar^2}{4m} \Delta (\Delta \delta n) - \frac{gn_0}{m} \Delta \delta n = 0. \quad (1.101)$$

By inserting in Eq. (1.101) solutions having the form of a plane wave propagating through the homogeneous condensate, $\delta n(\vec{r}, t) = \delta n_0 \exp[i(kx - \omega t)]$, one recovers the Bogolubov dispersion law:

$$-\omega^2 + \frac{\hbar^2}{4m^2}k^4 + \frac{gm_0}{m}k^2 = 0 \Leftrightarrow \omega = ck\sqrt{1 + \frac{1}{k^2c^2}\left(\frac{\hbar k^2}{2m}\right)^2}. \quad (1.102)$$

1.3.8 Thomas–Fermi approximation for time-dependent problems

In the stationary case, the time-independent Gross–Pitaevskii equation can be simplified in the limit of strong interactions ($\chi \gg 1$) by neglecting the kinetic-energy term.

In problems involving the dynamics of the condensate, it is not correct, in the limit $\chi \gg 1$, to neglect the kinetic-energy term in the time-dependent Gross–Pitaevskii equation (1.94). For example, in the ballistic expansion of a condensate, after switching off suddenly the confinement, the kinetic energy term is small at the beginning of the expansion compared to the interaction energy, but the interaction energy is converted in the course of the expansion into kinetic energy that therefore becomes very large and even dominant for long expansion times.

However, the Thomas–Fermi approximation takes a simple form on the Euler hydrodynamic equation where the contributions of the amplitude gradient and phase gradient to the kinetic energy are clearly separated, being represented, respectively, by the first two terms of the right-hand side of Eq. (1.97). When $\chi \gg 1$, the amplitude gradient (appearing in the quantum pressure term) remains small at all times, whereas the second term, coming from phase gradients, can become very large.

The Thomas–Fermi limit, in the time-dependent case, thus corresponds to a situation where the quantum pressure term can be neglected, so that the set of hydrodynamic equations is, in this limit, equivalent to the following two equations

$$\begin{aligned} \frac{\partial}{\partial t}n + \vec{\nabla} \cdot [n\vec{v}] &= 0, \\ m\frac{\partial\vec{v}}{\partial t} &= \vec{\nabla} \left[-\frac{1}{2}m\vec{v}^2 - V_{\text{trap}} - gn \right]. \end{aligned} \quad (1.103)$$

It is worth noticing that, in the regime $\chi \gg 1$, \hbar no longer appears in the equation for the velocity field, which consequently appears as a classical Euler equation describing the motion of a fluid in the trapping potential and in the pressure field due to the density of other particles. The motion of a condensate, in the Thomas–Fermi limit, and in the mean-field approximation, can thus be described by classical, irrotational hydrodynamics (since by definition the velocity field obeys the relation $\vec{\nabla} \times \vec{v}(\vec{r}, t) = \vec{0}$).

In the static case, for which there is no global motion of the condensate, the velocity field \vec{v} is equal to $\vec{0}$, and Eq. (1.103) gives:

$$\vec{\nabla} [V_{\text{trap}}(\vec{r}) + gn(\vec{r})] = \vec{0}. \quad (1.104)$$

We recover here the equilibrium shape in the Thomas–Fermi limit derived in Section 1.3.4.3.

The hydrodynamic equations (1.103) also give access to the motion of the small-amplitude oscillations of the condensate in the presence of a confinement ($V_{\text{trap}} \neq 0$). The calculation is similar to the one performed in Section 1.3.7.4 to derive the Bogolubov dispersion law. One linearizes the hydrodynamic equations around the equilibrium state defined by

$$n_0(\vec{r}) = [\mu - V_{\text{trap}}(\vec{r})]/g, \quad \text{and} \quad \vec{v}_0(\vec{r}) = \vec{0}. \quad (1.105)$$

One finds that the first-order corrections δn to the density and $\delta \vec{v}$ to the velocity field obey the linear set of equations:

$$\frac{\partial \delta n}{\partial t} = -\vec{\nabla} \cdot (n_0 \delta \vec{v}), \quad (1.106)$$

$$m \frac{\partial \delta \vec{v}}{\partial t} = -\vec{\nabla} (V_{\text{trap}} + g n_0 + g \delta n) = -g \vec{\nabla} \delta n. \quad (1.107)$$

The equation governing the density perturbation $\delta n(\vec{r}, t)$ therefore reads:

$$\frac{\partial^2 \delta n(\vec{r}, t)}{\partial t^2} = \vec{\nabla} \cdot [c^2(\vec{r}) \vec{\nabla} \delta n(\vec{r}, t)], \quad \text{with} \quad c^2(\vec{r}) = \frac{g}{m} n_0(\vec{r}). \quad (1.108)$$

The quantity $c(\vec{r})$ is a local sound velocity. Sound waves can propagate in a non-uniform medium. For a cylindrical geometry with a transverse harmonic confinement, the sound velocity in the longitudinal direction is $(\mu/2M)^{1/2}$ (Zaremba, 1998; Kavoulakis and Pethick, 1998; Stringari, 1998*b*). This result differs from the one obtained in a box by a factor of $1/\sqrt{2}$ since it is the average density over the radial direction, and not the peak density, that dictates the value of the sound velocity in such an elongated geometry. The propagation of sound waves in such a geometry has been studied experimentally in (Andrews *et al.*, 1998).

1.3.9 Thomas–Fermi dynamics for harmonic confinement

We suppose in this section that the trapping potential is harmonic but not necessarily isotropic:

$$V_{\text{trap}}(\vec{r}, t) = \frac{1}{2} \sum_{i=x,y,z} m \omega_i^2(t) r_i^2, \quad (1.109)$$

with $(r_1, r_2, r_3) = (x, y, z)$, and that the condensate is in the Thomas–Fermi regime ($\chi \gg 1$), so that we can use the hydrodynamic equations (1.103) without the quantum pressure term. The time dependence of the trapping frequencies $\omega_i(t)$ will allow us to analyze several problems that can be readily investigated experimentally.

1.3.9.1 Scaling transformation

When the strength of the confining potential is changed as a function of time, the time-dependent Gross–Pitaevskii can be solved using a scaling transformation. In the following, we denote by $\omega_i(0)$ the angular frequency along the $i = x, y, z$ axis for a time

$t \leq 0$, and $\omega_i(t)$ for $t \geq 0$. The time-dependent density resulting from the excitation is searched in the form (Kagan *et al.*, 1997):

$$n(r_i, t) = \frac{1}{b_x(t) b_y(t) b_z(t)} n_0 \left(\frac{r_i}{b_i(t)} \right) = \frac{1}{\Pi_j b_j(t)} n_0 \left(\frac{r_i}{b_i(t)} \right), \quad (1.110)$$

where n_0 is the initial equilibrium density distribution. The prefactor $\Pi_j b_j^{-1}(t)$ ensures the normalization of the density to the number of atoms. The ansatz (1.110) inserted in the continuity equation gives the expression for the velocity field:

$$v_j(\vec{r}, t) = \frac{\dot{b}_j(t)}{b_j(t)} r_j. \quad (1.111)$$

The Euler equation (1.103) yields:

$$m \frac{\partial v_j}{\partial t} + \frac{\partial}{\partial r_j} \left(\frac{1}{2} m v^2 + V_{\text{trap}} \right) = \frac{\ddot{b}_j(t)}{b_j(t)} r_j + m \omega_j^2(t) r_j - g \frac{\partial n}{\partial r_j}. \quad (1.112)$$

The calculation of the last term requires the knowledge of the equilibrium Thomas–Fermi profile (1.74), from which one deduces

$$n(r_i, t) = \frac{1}{\Pi_j b_j(t)} \left[\frac{\mu}{g} - \frac{m}{2g} \right] \sum_i \omega_i^2(0) \frac{r_i^2}{b_i^2(t)}. \quad (1.113)$$

Combining this expression with Eq. (1.112), we find the set of non-linear coupled equations fulfilled by the dilation factors b_j :

$$\ddot{b}_j(t) + \omega_j^2(t) b_j(t) - \frac{\omega_j^2(0)}{b_j(t)} \frac{1}{\Pi_i b_i(t)} = 0. \quad (1.114)$$

This set of equations for the scaling factors means that it is possible to account for large-amplitude oscillations, and to investigate non-linear features associated, for example, with the dynamics of the expansion of the gas, by simply solving a set of three non-linear coupled ordinary differential equations. It is worth noticing, however, that such an approach is restricted to quadratic potentials.

1.3.9.2 *Ballistic expansion*

To analyze the properties of the condensate, the standard method consists in monitoring the evolution of the shape of the condensate after having switched off suddenly the trapping potential (Anderson *et al.*, 1995). Such a ballistic expansion is usually unavoidable for an optical detection since the in-trap transverse size of the condensate is on the order of 1 μm , and, in addition its optical density is very large.

For an ideal gas, the *in situ* position dispersion $\Delta x_i(0)$ along a given axis gives rise, through the Heisenberg principle, to a velocity dispersion $\Delta v_i = \hbar/(2m\Delta x_i(0))$. After switching off the trapping, the cloud expands and the position dispersion evolves according to $\Delta x_i(t) = [(\Delta x_i(0))^2 + (\Delta v_i)^2 t^2]^{1/2}$. For long expansion times,