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three-dimensional problems of the mathematical theory of elasticity and thermoelasticity

V.D. KUPRADZE
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PREFACE

In the best books on the theory of elasticity the investigation of three-dimensional boundary value problems has been so far limited to bodies of special shape (a half-space, a sphere, some cases of axially symmetrical bodies and so on). The greatest attention has been given to static problems, less attention to oscillation problems and still less to problems of general dynamics. Such a situation might be well expected – it reflects the historical background of the theory of elasticity which during the entire preceding period was concentrated on bodies of particular profiles and was above all interested in problems of static equilibrium.

It would be wrong to attribute this situation only to the importance of the above-mentioned problems for technology and engineering. The true reason is that the methods of classical elasticity were inadequate for developing a rigorous and sufficiently complete theory of three-dimensional boundary value problems.

Unlike the three-dimensional problems, the theory of the plane problem worked out mainly by the classical methods (the theory of analytic functions, Fredholm's theory of integral equations and, later, the theory of one-dimensional singular integral equations) has been extensively developed and found its perfect expression in I.N. Muskhelishvili's book "Some Basic Problems of the Mathematical Theory of Elasticity" the first edition of which appeared in 1933.

The situation is currently changing. The theory of three-dimensional problems may now be worked out by a variety of means. We shall just mention two of the possibilities: on the one hand, it is the modern theory of generalized solutions of differential equations (the method of Hilbert spaces, variational methods), on the other hand – the theory of multi-dimensional singular potentials and singular integral equations.

The first trend – based on the ideas of the modern functional analysis which are novel to the classical mechanics – is characterized by great generality involving the case of variable coefficients and boundary manifolds of the general type. Owing to such generality, it may be employed in the first place for proving theorems on the existence of non-classical solutions, requiring additional, sometimes essential, restrictions when used for classical solutions.

A fine, though concise, treatment of these topics may be found in G. Fichera's papers "Existence Theorems in Elasticity" and "Boundary Value Problems of Elasticity with Unilateral Constraints", Handbuch der Physik, VIa/2, Springer Verlag, 1972, and in C. Dafermos' paper "On the Existence of Asymptotic Stability of Solutions to the Equations of Thermoelasticity", Arch. Rat. Mech. Anal. 29, 4, 1968.

The second trend based on the rapidly developing theory of singular integrals and integral equations is a direct extension of the concepts of the theory of potentials and Fredholm equations which are, as known, the prevailing concepts of the classical mechanics. This approach, being not so general as the first one, allows to investigate in detail cases most important for the theory and application, retaining the efficiency of the methods of the classical mechanics of continua.

The present book has adopted the second trend. It is an attempt to develop – apparently for the first time with adequate completeness and at the modern level of mathematical rigour – the general theory of three-dimensional problems of statics, oscillation and dynamics for linear equations with constant and piecewise-constant coefficients of classical elasticity, thermoelasticity and couple-stress elasticity.

Much space in the book is assigned to general problems (existence and uniqueness theorems, an analysis of differential properties of solutions, the continuous dependence on the data of a problem etc.). A great deal of attention is also given to questions of the actual construction of solutions in a form allowing to express them numerically under very general conditions.

With this end in view the solutions are represented as generalized Fourier series to construct which there is no need to know the eigenfunctions and the eigenvalues of any auxiliary boundary value problems. New representations of solutions by quadratures have been found for some particular cases.

We think that the simple construction of solutions and the representation of elementary structures by explicit invertible operators, together with a detailed analysis of the smoothness of solutions, may serve in the conditions of modern computing facilities as the basis for obtaining convenient algorithms of numerical computations and for estimating approximations.

The book reproduces the monograph of the same authors published by the Tbilisi University Press in 1968. It was favourably received by readers and sold out within a short time. In 1971 the first edition was awarded the State Prize of the Georgian SSR.

When a second edition was called for, the book was extensively revised and enlarged. To make the book accessible to a wider circle of people the authors rewrote nearly all the chapters, simplified a number of proofs, corrected the noted errata.

The chapters of the book are divided into sections, the sections into articles; each section has its own numeration of formulas; the formula number is denoted by two figures enclosed in brackets; for example, (5.9) means the ninth formula in the fifth section. When reference is made to a formula, the number of a chapter is added to the number of the formula; thus, (VIII, 3.6) means the sixth formula in the third section of the eighth chapter. However, if reference is made to a formula within a given chapter, the chapter number is omitted.

Theorems, lemmas, definitions and notes are numerated in the same manner but without brackets. Theorem V, 2.10 therefore means the tenth theorem in the second section of the fifth chapter. Again, if reference is made within a given chapter, the chapter number is left out. All the chapters, except the first one, are supplemented with problems some of which may be used as a subject of independent research.

The bibliography consists of those titles which were available to the authors at the time of writing the book. It does not claim to bibliographic irreproachability and does not include the books published after 1972.

Tbilisi The Authors

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CHAPTER I

BASIC CONCEPTS AND AXIOMATIZATION

The first chapter is an introductory one. It deals with the basic concepts of the classical theory of elasticity, thermoelasticity and the couple-stress theory of elasticity. Recapitulating briefly the physical principles of these theories in §§1 to 10, we did not mean to substantiate the foundations of the theory of elasticity from the viewpoint of physics. Our intention was to throw a bridge between the theory of elasticity as a part of mechanics and the mathematical theory of elasticity and thus to facilitate the reading of the book for mechanicians who have not yet got accustomed to the axiomatic approach to the theory of elasticity and also to help mathematicians who studied the mathematical theory of elasticity to attach the definite physical meaning to some terms (stresses, displacements, isotropy, etc.) formally used in the axiomatic theory. Readers who have some knowledge of the physical foundations of the theory of elasticity may omit §§1 to 10, while those interested in them may refer to the bibliography listed in §15.

In the first chapter the basic assumptions of the classical theory of elasticity, thermoelasticity and the couple-stress theory are formulated first in the terms of mechanics and after that the axiomatization of these theories is carried out.

Throughout the book the classical theory of elasticity, the theory of thermoelasticity and the couple-stress theory of elasticity will be united under the common name of the theory of elasticity or simply elasticity.

The physical foundations of elasticity are worked out on the assumption that the stress-strain theory and Hooke's law are applicable to some media called elastic. Different interpretations of the stress-strain theory and Hooke's law have given rise to various theories, say, to the theory of elasticity for isotropic and anisotropic media, the theory of thermoelasticity, the couple-stress theory of elasticity, to mention just a few.

§1. Stresses

1. Internal and external forces

If no external force is applied to the body under consideration and the body is not deformed, then all its parts are in mechanical equilibrium. If the body is disturbed by some force, i.e., if it is deformed, then the initial equilibrium of molecules is changed; the parts of the body are no longer in mechanical equilibrium. In the deformed body internal forces are produced which struggle to return the body to its original state.

The internal forces are of the molecular nature and their radii of action are very small in comparison with distances considered in the theory of elasticity. It is therefore generally assumed that the internal forces acting on some part of the body from the side of the remainder of the body act only through the boundary of this part.

Forces acting from the environment are called external forces and are divided into mass and surface forces.

2. Mass and surface forces

If the body under consideration comes into contact with some external medium, then on the surface of the contact "short-range forces" arise. Such forces are called surface forces and are of the same nature as those described in Art. 1.

It is obvious that not every external action on the medium can be represented by surface forces. Gravity forces, magnetic forces and others may serve as an example of such action. In the theory of elasticity, in addition to surface forces, mass forces are introduced. It is assumed that the action of such forces on an elementary particle of the body is statically equivalent to a force applied to the centre of the particle mass and to a couple of forces. These forces and moments of couples are assumed to be proportional to the masses of the particles on which they act. They are called mass forces and mass moments.

Consider now a particle with the mass Δm. Let a point x be the centre of the particle mass. We have already said that the action of mass-dependent forces may be represented as a force acting on x and as a couple of forces. The resultant vector of this force is denoted by $\mathscr{F}(\Delta m)$ and the moment of a couple by $\mathscr{G}(\Delta m)$. Assume there exist the limits

$$\lim_{\Delta m \to 0} \frac{\mathscr{F}(\Delta m)}{\Delta m}, \qquad \lim_{\Delta m \to 0} \frac{\mathscr{G}(\Delta m)}{\Delta m},$$

which depend only on the point x and, in dynamics, on the time t. The limits are denoted by the vectors $\mathscr{F}(x, t)$ and $\mathscr{G}(x, t)$ corresponding to the concepts of the mass force and the mass moment, respectively.

By our assumption, the action of mass-dependent forces on the particle ΔD with the mass Δm can be represented as the mass force $\mathscr{F}(x, t)\Delta m$ and the mass moment $\mathscr{G}(x, t)\Delta m$ neglecting infinitesimal quantities relative to Δm. These vectors may be also represented in the form

$$\mathscr{F}(x, t)\rho(x)\,\mathrm{mes}(\Delta D), \quad \mathscr{G}(x, t)\rho(x)\,\mathrm{mes}(\Delta D),$$

where ρ is the density of the body and $\mathrm{mes}(\Delta D)$ the volume occupied by the particle ΔD.

The density of the medium is defined as a limit of the ratio $\Delta m/\mathrm{mes}(\Delta D)$, if $\mathrm{mes}(\Delta D)$ tends to zero so that the point x always remains within ΔD.

In classical elasticity, unlike the couple-stress theory, the influence of mass moments is neglected since they are assumed to be zero, $(\mathscr{G}(x, t) \equiv 0)$.

3. *Force- and couple-stresses*

The concept of stress is introduced in the theory of elasticity to characterize the internal forces. Let us choose a point x within the medium under consideration and speculatively draw through it a small surface ΔS. The internal forces produced by the interaction of the parts on the opposite sides of ΔS can be represented as forces applied to the points of ΔS.

The directions of these forces depend on the chosen part. The forces acting between the parts on the opposite sides of ΔS are equal and have opposite directions. To determine such forces proceed as follows: draw the normal n at the point x to the surface ΔS, give it a definite positive direction and consider the force which the part lying on the positive side of the normal n exerts on the part lying on the opposite side.

The forces acting on ΔS are assumed to be statically equivalent to a force and a couple. Denote the force vector by T and the moment of a couple by M and consider the ratios

$$T/\mathrm{mes}(\Delta S), \qquad M/\mathrm{mes}(\Delta S).$$

The limits of these ratios exist when the surface area mes(ΔS) tends to zero and depend on the point x and the normal n. The sign of the limit is changed if the opposite direction of n is taken as positive. Choosing another normal, i.e., taking another surface passing through x, we have a different situation brought about by the action of other parts of the medium. Accordingly, T and M are changed and so are the limits of the above ratios. It is also assumed that the limits do not depend on the surface shape or, in other words, the limits remain unchanged if they are calculated for another surface, passing through the same point x, with the same normal n.

Note that in dynamics the above limits will also depend on time. We introduce the notations

$$\tau^{(n)} \equiv \lim \frac{T}{\mathrm{mes}(\Delta S)}, \qquad \mu^{(n)} \equiv \lim \frac{M}{\mathrm{mes}(\Delta S)}.$$

According to the previous statement $\tau^{(n)}$ and $\mu^{(n)}$ depend, in addition to the direction of n, on the point x and the time t. $\tau^{(n)}(x, t)$ is called the force-stress vector and $\mu^{(n)}(x, t)$ the couple-stress vector directed along n at the instant of time t.

To avoid any misunderstanding note that the vectors $\tau^{(n)}$ and $\mu^{(n)}$ are not in general directed along n. In classical elasticity $\mu^{(n)}$ is assumed to be equal to zero, while in the couple-stress theory such an assumption is not made. In the classical theory there are no couple-stresses and therefore we shall simply use the term "stress" for "couple-stress".

We have thus seen two principal distinctions between classical elasticity and the couple-stress theory. The latter, in contrast to classical elasticity, considers mass forces and couple-stresses. It is to this circumstance that the couple-stress theory owes its name.[1]

The determination of stresses (force-stresses in classical elasticity and couple-stresses in the couple-stress theory) at each point in any direction and at any instant of time in the considered interval is one of the major problems of the elasticity theory.

[1]In the literature one may encounter quite a lot of names for the couple-stress theory. It is called the unsymmetrical elasticity theory, Cosserat's theory, the theory of elasticity with rotational interaction of particles, the micropolar theory, the nonlocal elasticity theory, the elasticity theory for the 2nd class medium etc. These theories, though dealing with different representations of the continuum model, take couple-stresses into consideration, which justifies the various names of the couple-stress theory (for relevant details see Ch. IX).

The foregoing assumption suggests that the surface forces (the force- and couple-stresses) acting on the small surface ΔS with the normal n are statically equivalent to the force $\tau^{(n)}\,\mathrm{mes}(\Delta S)$ and the moment $\mu^{(n)}\,\mathrm{mes}(\Delta S)$ neglecting higher order infinitesimal quantities relative to $\mathrm{mes}(\Delta S)$.

§2. Components of stress

1. Components of force- and couple-stress tensors[1]

One may have an infinite number of directions at each point of a medium. To have a complete idea of the stresses at a point it is necessary to know the stresses (force-stresses in classical elasticity and couple-stresses in the couple-stress theory) in all these directions.

However, if we know the stresses at the point in three mutually perpendicular directions, we may calculate with a certain accuracy the stresses at this point in any direction.

Let us take a Cartesian coordinate system $X_1X_2X_3$ and denote the stresses $\tau^{(n)}$ and $\mu^{(n)}$, when n coincides with the X_i-axis, by $\tau^{(i)}$ and $\mu^{(i)}$ and the coordinates of these vectors in the system $X_1X_2X_3$ by $\tau_{i1}, \tau_{i2}, \tau_{i3}$ and $\mu_{i1}, \mu_{i2}, \mu_{i3}$, respectively. Consider the matrix $\|\tau_{ij}(x, t)\|_{3\times 3}$. Later it will be shown that the force-stress vector $\tau^{(n)}(x, t)$ is expressed in any direction n (at the point x at the instant of time t) through the elements of the matrix $\|\tau_{ij}(x, t)\|$. These elements are called the force-stress components. It may be easily shown that these nine scalars form a second rank tensor which is called the force-stress tensor. Thus, the components of the force-stress are the components of the stress tensor.

The couple-stress components and the couple-stress tensor are defined similarly.

2. Expression of the force-stress vector in terms of components of the force-stress tensor

Let x be an arbitrary point in the medium under consideration and $n = (n_1, n_2, n_3)$ be an arbitrary unit vector whose direction neither coin-

[1]We do not employ tensor calculus here; throughout the book the word "tensor" will be used as a term (or rather as a constituent element of such phrases as "force-stress tensor", "couple-stress tensor", "tensor of strain" etc.) for denoting certain quantities. These quantities actually form tensors and this gives us the right to a free use of the term "tensor".

cides with nor is opposite to the coordinate axes. Draw through x three planes, parallel to the coordinate planes, and consider a small tetrahedron formed by them and by another plane, normal to n, drawn at a close distance to x. Denote by ΔS that face of the tetrahedron which is normal to n (Fig. 1).

It follows from the conditions of equilibrium that the sum of the resultant vectors of the external and inertial forces is equal to zero.

To calculate the resultant vector of the external forces acting on the tetrahedron we must take into account the force-stresses acting on its faces and the resultant vector of the mass forces acting on the tetrahedron mass.

We determine the force-stresses acting on the tetrahedron mass. The area of the face normal to the OX_i-axis is equal to $|n_i|\ \mathrm{mes}(\Delta S)$ and hence the force-stress with which the part of the medium external to the tetrahedron is acting on the latter through this face is equal to $-\tau^{(i)}(x, t)n_i\ \mathrm{mes}(\Delta S)$ neglecting higher order infinitesimal quantities relative to $\mathrm{mes}(\Delta S)$ (see §1, Art. 3). The force-stress acting through the face ΔS is equal to $\tau^{(n)}(x, t)\ \mathrm{mes}(\Delta S)$.

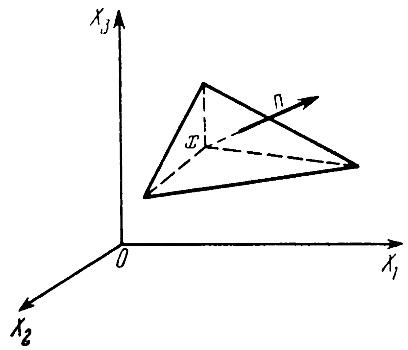

Figure 1.

The resultant vector of the mass and inertial forces acting on the tetrahedron is proportional to its mass and hence to its volume, i.e., it is a higher order infinitesimal quantity relative to $\mathrm{mes}(\Delta S)$. The sum of all these vectors which represents the resultant vector of the external forces is equated to zero. Dividing the obtained equation by $\mathrm{mes}(\Delta S)$ and passing to the limit as $\mathrm{mes}(\Delta S) \to 0$, the vector relation

$$\tau^{(n)}(x, t) - \sum_{i=1}^{3} \tau^{(i)}(x, t)n_i = 0$$

is obtained which in terms of the components becomes

$$\tau_j^{(n)}(x, t) = \sum_{i=1}^{3} \tau_{ij}(x, t)n_i, \quad j = 1, 2, 3. \tag{2.1}$$

If the direction of n coincides with some coordinate axis or is opposite to it, the validity of (2.1) is obvious. Relations (2.1) give the required force-stresses at the point in any direction in terms of the components of the force-stress tensor at the same point. These relations were first derived by Cauchy. They hold both for the classical and couple-stress theory of elasticity.

3. Expression of the couple-stress vector in terms of components of the couple-stress tensor

The resultant moments of the external and inertial forces acting on the tetrahedron are calculated and their sum is equated to zero. By the same reasoning as before we obtain the vector relation

$$\mu^{(n)}(x,t)-\sum_{i=1}^{3}\mu^{(i)}(x,t)n_i=0,$$

which in terms of the components is written as

$$\mu_j^{(n)}(x,t)=\sum_{i=1}^{3}\mu_{ij}(x,t)n_i,\quad j=1,2,3. \tag{2.2}$$

We have obtained the relations between the couple-stress acting at a point in any direction and the couple-stresses acting at the same point in three mutually perpendicular directions.

In classical elasticity μ_{ij} and $\mu_j^{(n)}$ are assumed to be zero and therefore relations (2.2) are not considered.

§3. Displacements and rotations

1. Displacement vector

It is supposed that at the initial instant of time t_0 the body is at rest, is not deformed and no force is applied to it. Subject the body to deformation and for a mathematical description of the deformed state introduce a fixed orthogonal coordinate system $OX_1X_2X_3$. Let the body occupy the domain D bounded by S at the time t_0 and the domain D^t bounded by S^t at the time t.

Consider some point $x\equiv(x_1,x_2,x_3)$ of the body at rest $(x\in\bar{D}=$

$D \cup S$). During the deformation the point x changes its position, i.e., it is displaced. Its position at the time t is denoted by $x^t = (x_1^t, x_2^t, x_3^t)$.

The difference[1] $x^t - x$ is called the displacement vector or simply the displacement or, more precisely, the value of the displacement vector at the point x at the time t. It is denoted by $u(x, t)$ and its components by $u_1(x, t)$, $u_2(x, t)$, $u_3(x, t)$.

Thus, to each point x of the undeformed state of the body ($x \in \bar{D}$) at any time t in the considered interval $[t_0, t_1]$, i.e., to each pair (x, t) from the set $\bar{D} \times [t_0, t_1]$, there corresponds the displacement vector $u(x, t) \equiv x^t - x$.

2. *Rotation vector*

The displacement vector $u(x, t)$ of each point x of the continuous medium under consideration completely determines the deformation at any time t. However, it is natural to conceive the material medium not as a continuous set of mathematical points of the three-dimensional Euclidean space but as a set of material particles. Such a conception has already been used in the earlier discussion while introducing stresses and deriving the basic relations for the stresses at a point. The elementary volume of the medium was then considered as a solid (rigid) body and the laws of statics were used.

If from the very beginning we had considered the medium as a set of mathematical points filling up some domain, we would have encountered serious difficulties, for example, in the derivation of the basic relations on which the theory of elasticity is based. Evidently, when one has to use the regularities of physics, the medium is to be considered as a set of material particles. On the other hand, when one wishes to employ the methods of the mathematical analysis, the medium should be considered as a continuous set of points.

The two conceptions are usually made compatible as follows: first, the medium is considered as a set of material particles and the required relations are derived; after that, the medium is idealized, i.e., it is represented as a continuous set, and the techniques of the mathematical analysis are applied.

If from the beginning the medium is treated as a continuous one, the

[1]The point and the corresponding radius-vector are denoted by the same symbol. Therefore, $x^t - x$ refers to the vector whose origin coincides with the point x and ends with the point x^t. Hence $x^t - x = (x_1^t - x_1, x_2^t - x_2, x_3^t - x^3)$.

picture of deformation will be completely determined by the displacement vector. If, however, it is taken as a set of material particles, the picture will be somewhat altered.

Let us consider an arbitrary particle with the gravity centre at the point x at rest. We introduce a new orthogonal system of coordinates with the origin at x which is rigidly fixed to the particle under consideration and align the axes of the new system with the corresponding axes of the fixed system.

It is assumed that the particle is a solid (rigid) body. Its motion is determined by six scalars, for example, by the displacement of the point x (which is determined by three coordinates of the displacement vector with respect to the fixed system) and by the rotation of the particle about its gravity centre (which is also determined by three scalars, i.e., by rotation angles of the mobile system with respect to the coordinates of the fixed system, e.g., by Euler angles).

At the time t the deformed medium will occupy a new position with respect to the fixed system. The point x will move to the position x^t. The displacement vector of x will be determined by $x^t - x = u(x, t)$. The mobile system will also take a new position with respect to the fixed one. It will rotate. Its rotation angles are denoted by $\omega_1(x, t)$, $\omega_2(x, t)$ and $\omega_3(x, t)$. The vector $\omega(x, t) = (\omega_1(x, t), \omega_2(x, t), \omega_3(x, t))$ will be called the vector of internal rotation or simply the internal rotation of the point x at the time t.

If now the medium is assumed to be a continuous one, the motion of each point will be specified not by three scalars (i.e., not by the components of the displacement vector) but by six scalars (i.e., by the components of the displacement and rotation vectors).

Such an approach is adopted in the couple-stress theory of elasticity. In classical elasticity the vector of internal rotation is considered not as independent of the displacement but as related to the latter by the formula[1] (see LOVE [1], MUSKHELISHVILI [1], LANDAU, LIFSHITZ [1], FILONENKO-BORODICH [1] et al.)

$$\omega_i = \frac{1}{2}(\operatorname{curl} u)_i = \frac{1}{2}\sum_{j,k}^{3} \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}.$$

[1]Though tensor calculus is not employed here, the unit vector (Kronecker's symbol δ_{ij}) and the so-called ε-tensor (Levi–Civita's symbol ε_{ijk}) are frequently used. $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$, $\varepsilon_{ijk} = 1$ or $\varepsilon_{ijk} = -1$ depending on whether i, j, k have an even or odd number of transpositions of the numbers 1, 2, 3; $\varepsilon_{ijk} = 0$ if at least two of the three indices i, j, k are equal.

Thus, each point of the medium possesses six degrees of freedom in the couple-stress theory of elasticity and three degrees in classical elasticity.

§4. Basic equations in terms of stress components

1. Equations of motion in classical elasticity

Detach some part from the body and denote its domain by Ω and the boundary by Γ. To write an equilibrium condition for this part sum up the external forces acting on it and equate the sum to the inertial forces with the sign reversed. The moments are treated similarly.

When calculating forces in the assumptions of classical elasticity the following must be taken into account:

1) Stresses. At the time t at every point y of the surface Γ the stress $\tau^{(n)}(y, t)$ is acting, where n is the outward normal to Γ at the point y. The sum of these forces is expressed by the integral

$$\int_\Gamma \tau^{(n)}(y, t)\, \mathrm{d}\Gamma,$$

where $\mathrm{d}\Gamma$ is an element of the area of Γ.

2) Mass forces. At the time t at each point x of the domain Ω the mass force $\mathscr{F}(x, t)$ is acting. The sum of these forces is expressed by the integral

$$\int_\Omega \rho\mathscr{F}(x, t)\, \mathrm{d}x,$$

where $\mathrm{d}x$ is an element of the volume, ρ is the body density which, unless stated otherwise, will always be assumed independent of a position of the point x and the time t.

3) Inertial forces. If $u(x, t)$ is the displacement of the point x, then the acceleration of this point at the time t is $\partial^2 u(x, t)/\partial t^2$ and the inertial forces applied to an element of the volume are

$$-\rho\, \frac{\partial^2 u(x, t)}{\partial t^2}\, \mathrm{d}x.$$

The sum of these forces is given by the integral

$$-\int_{\Omega} \rho \frac{\partial^2 u(x,t)}{\partial t^2}\, \mathrm{d}x.$$

It follows from the equilibrium condition (the vanishing sum of the resultant vector of active forces and the resultant vector of inertial forces) that

$$\int_{\Gamma} \tau^{(n)}(y,t)\, \mathrm{d}\Gamma + \int_{\Omega} \rho \mathscr{F}(x,t)\, \mathrm{d}x = \int_{\Omega} \rho \frac{\partial^2 u(x,t)}{\partial t^2}\, \mathrm{d}x, \tag{4.1}$$

Similar calculations of moments and the equilibrium condition (the vanishing sum of the resultant moment of active forces and the resultant moment of inertial forces) give the formula

$$\int_{\Gamma} y \times \tau^{(n)}(y,t)\, \mathrm{d}\Gamma + \int_{\Omega} \rho x \times \mathscr{F}(x,t)\, \mathrm{d}x = \int_{\Omega} \rho x \times \frac{\partial^2 u(x,t)}{\partial t^2}\, \mathrm{d}x. \tag{4.2}$$

The sign "×" between the vectors denotes their vector multiplication.

In view of formulas (2.1) the vector equation (4.1) assumes the form

$$\int_{\Gamma} \sum_{i=1}^{3} \tau_{ij}(y,t) n_i\, \mathrm{d}\Gamma + \int_{\Omega} \rho \mathscr{F}_j(x,t)\, \mathrm{d}x = \int_{\Omega} \rho \frac{\partial^2 u_j(x,t)}{\partial t^2}\, \mathrm{d}x$$

which due to the Gauss–Ostrogradski formula is rewritten as

$$\int_{\Omega} \left(\sum_i \frac{\partial \tau_{ij}(x,t)}{\partial x_i} + \rho \mathscr{F}_j(x,t) - \rho \frac{\partial^2 u_j(x,t)}{\partial t^2} \right) \mathrm{d}x = 0.$$

Hence, since Ω is arbitrary, we have

$$\sum_i \frac{\partial \tau_{ij}(x,t)}{\partial x_i} + \rho \mathscr{F}_j(x,t) = \rho \frac{\partial^2 u_j(x,t)}{\partial t^2} \quad (j = 1, 2, 3), \tag{4.3}$$

which is valid at any point of the body under consideration and at any time.

Taking (2.1) into account, the same manipulation with (4.2) results in

$$\int_\Gamma \sum_{l,k} \Big(\varepsilon_{jlk} y_l \sum_i \tau_{ik}(y,t) n_i\Big)\, \mathrm{d}\Gamma + \int_\Omega \rho \sum_{l,k} \varepsilon_{jlk} x_l \mathscr{F}_k(x,t)\, \mathrm{d}x$$

$$= \int_\Omega \rho \sum_{l,k} \varepsilon_{jlk} x_l \frac{\partial^2 u_k(x,t)}{\partial t^2}.$$

Using the Gauss–Ostrogradski formula, we obtain

$$\int_\Omega \sum_{l,k} \varepsilon_{jlk} x_l \Big(\sum_i \frac{\partial \tau_{ik}(x,t)}{\partial x_i} + \rho \mathscr{F}_k(x,t) - \rho \frac{\partial^2 u_k(x,t)}{\partial t^2}\Big)\, \mathrm{d}x$$

$$+ \int_\Omega \sum_{l,k} \varepsilon_{jlk} \tau_{lk}(x,t)\, \mathrm{d}x = 0.$$

The first term on the left-hand side vanishes by virtue of (4.3) and therefore

$$\sum_{l,k} \varepsilon_{jlk} \tau_{lk}(x,t) = 0.$$

Hence we finally have

$$\tau_{ij}(x,t) = \tau_{ji}(x,t). \tag{4.4}$$

Equation (4.4) shows that (in classical elasticity) the components of stress form a symmetrical matrix (i.e., the tensor of stress is symmetrical) and consequently only six out of nine scalars $\tau_{ij}(x,t)$ remain independent. These six scalars are related by three relations (4.3).

Relations (4.3) are called the basic equations of motion (in dynamics) of classical elasticity in terms of stress components.

If the stress components τ_{ij} and the displacement components u_i are known at each point and at any time, then the state of stress and strain of the body will be completely determined in classical elasticity. The determination of these nine scalars is the basic problem of the classical theory. It is recalled that they have so far been related only by three relations (4.3).

Assume now that the external forces do not depend on time. Then, naturally, the displacements and stresses are also independent of time and relations (4.3) assume the form

$$\sum_i \frac{\partial \tau_{ij}(x)}{\partial x_i} + \rho \mathscr{F}_j(x) = 0. \tag{4.5}$$

Equations (4.5) are called the equilibrium equations.

2. Equations of motion in the couple-stress theory

An arbitrary part of the body will be considered and the condition of equilibrium will be written for it.

Calculation of forces in the assumptions of the couple-stress theory is accomplished in the same way as in classical elasticity and leads to relation (4.3).

In calculating moments, in addition to force moments, it is necessary to take into account independent moments of couples. Let us discuss this point in detail.

Denote the isolated part by Ω and its boundary by Γ. To calculate the moments acting on Ω the following must be taken into account:

1) a moment of force-stresses

$$\int_\Gamma y \times \tau^{(n)}(y, t)\, \mathrm{d}\Gamma;$$

2) couple-stresses

$$\int_\Gamma \mu^{(n)}(y, t)\, \mathrm{d}\Gamma;$$

3) a moment of mass forces

$$\int_\Omega \rho x \times \mathscr{F}(x, t)\, \mathrm{d}x;$$

4) mass moments

$$\int_\Omega \rho \mathscr{G}(x, t)\, \mathrm{d}x;$$

5) a moment of inertial forces

$$-\int_\Omega \rho x \times \frac{\partial^2 u(x,t)}{\partial t^2}\,\mathrm{d}x;$$

6) a "spin" moment corresponding to internal rotations

$$-\int_\Omega \mathcal{J}\,\frac{\partial^2 \omega(x,t)}{\partial t^2}\,\mathrm{d}x,$$

where $\mathcal{J}$ is a specific dynamic characteristic[1] (see PALMOV [1], NOWACKI [8] et al.).

From the equilibrium condition follows

$$\int_\Gamma [y+\tau^{(n)}(y,t)+\mu^{(n)}(y,t)]\,\mathrm{d}\Gamma+\int_\Omega [\rho x\times \mathcal{F}(x,t)+\rho\mathcal{G}(x,t)]\,\mathrm{d}x$$

$$=\int_\Omega \left[\rho x\times\frac{\partial^2 u(x,t)}{\partial t^2}+\mathcal{J}\,\frac{\partial^2\omega(x,t)}{\partial t^2}\right]\mathrm{d}x.$$

Let us transform this formula. In view of (2.1) and (2.2) and the Gauss–Ostrogradski formula, the j-th component of the first term on the left-hand side takes the form

$$\int_\Gamma \left[\sum_{i,k}\varepsilon_{jik}y_i\tau_k^{(n)}(y,t)+\mu_j^{(n)}(y,t)\right]\mathrm{d}\Gamma$$

$$=\int_\Omega \sum_l \frac{\partial}{\partial x_l}\left[\sum_{i,k}\varepsilon_{jik}x_i\tau_{lk}(x,t)+\mu_{lj}(x,t)\right]\mathrm{d}x$$

$$=\int_\Omega \sum_l \left[\sum_{i,k}\left(\varepsilon_{jik}x_i\,\frac{\partial\tau_{lk}(x,t)}{\partial x_l}+\delta_{li}\varepsilon_{jik}\tau_{lk}(x,t)\right)\right.$$

$$\left.+\frac{\partial\mu_{lj}(x,t)}{\partial x_l}\right]\mathrm{d}x,$$

[1] Here, for convenience, a sphere with uniformly distributed masses (a symmetrical gyroscope) is taken as an elementary volume. If the gyroscope is not symmetrical, $\mathcal{J}$ is replaced by the tensor of inertia moment.

which allows to rewrite the previous formula as

$$\int\limits_{\Omega} \sum_{i,k} \varepsilon_{jik} x_i \left[\sum_l \frac{\partial \tau_{lk}(x,t)}{\partial x_l} + \rho \mathscr{F}_k(x,t) - \rho \, \frac{\partial^2 u_k(x,t)}{\partial t^2} \right] \mathrm{d}x$$

$$+ \int\limits_{\Omega} \left[\sum_l \frac{\partial \mu_{lj}(x,t)}{\partial x_l} + \sum_{i,k} \varepsilon_{jik} \tau_{ik}(x,t) + \rho \mathscr{G}_j(x,t) \right.$$

$$\left. - \mathscr{J} \, \frac{\partial^2 \omega_j(x,t)}{\partial t^2} \right] \mathrm{d}x = 0.$$

Using (4.3), which, as noted above, also holds for the couple-stress theory, we obtain from this relation in view of the arbitrariness of Ω

$$\sum_i \frac{\partial \mu_{ij}(x,t)}{\partial x_i} + \sum_{i,k} \varepsilon_{jik} \tau_{ik}(x,t) + \rho \mathscr{G}_j(x,t) = \mathscr{J} \, \frac{\partial^2 \omega_j(x,t)}{\partial t^2}. \tag{4.6}$$

Equations (4.3) and (4.6) are the basic equations of motion of the couple-stress theory in terms of the stress components.

(4.5) and

$$\sum_i \frac{\partial \mu_{ij}(x)}{\partial x_i} + \sum_{i,k} \varepsilon_{jik} \tau_{ik}(x) + \rho \mathscr{G}_j(x) = 0 \tag{4.7}$$

are the basic equilibrium equations of the couple-stress theory in terms of the stress components.

Note that in classical elasticity (4.6) is replaced by (4.4) which expresses the symmetry of the stress tensor. In the couple-stress theory the stress tensor is always assumed unsymmetrical, which accounts for the term "unsymmetrical theory" frequently encountered in the literature.

§5. Hooke's law in classical elasticity

1. Components of the strain tensor

What has been said above refers to any continuous medium for which the fundamental laws of mechanics are applicable and the concept of stress is meaningful. The theory of elasticity is concerned with elastic media. The elastic properties of the medium are described by a specific dependence (which is called Hooke's law) existing between stresses and strains or, more precisely, between the quantities characterizing the stressed and deformed states of the medium.

The state of stress, as mentioned above, is characterized by the stress components τ_{ij} (see §2).

The quantities describing the state of strain will now be introduced. The term "strain" refers to such a change in the positions of points of the medium that their relative distances are altered. It is obvious that not every change in the positions of the points is caused by strain. In the case of rigid displacement of the medium (rigid translational displacement and rigid rotation) the positions of the points are changed – the points are displaced – but the relative distances between them are not altered and, therefore, the medium remains undeformed.

Choose an arbitrary point $x = (x_1, x_2, x_3)$ in an undeformed medium (at the time t_0) and consider a point $x + \xi$ in a small neighbourhood of the chosen one. Calculate a change of the small vector $\xi = (\xi_1, \xi_2, \xi_3)$ caused by strain. Here x coincides with the origin of the vector and $x + \xi$ with its end. At the time t the point x will occupy the position $x + u(x, t)$ and the point $x + \xi$ the position $x + \xi + u(x + \xi, t)$. Therefore, the change of the vector ξ, which is denoted by $\Delta\xi(x, t)$ or simply by $\Delta\xi$, may be calculated by the formula

$$\Delta\xi(x, t) = u(x + \xi, t) - u(x, t).$$

Applying Taylor's expansion and neglecting, due to the smallness of the vector ξ, the terms of the order higher than $|\xi|$, we obtain

$$\Delta\xi_i(x, t) = \sum_i \frac{\partial u_i(x, t)}{\partial x_j} \xi_j, \tag{5.1}$$

where $\Delta\xi = (\Delta\xi_1, \Delta\xi_2, \Delta\xi_3)$.

Consider the representation

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right).$$

The quantities $\partial u_i/\partial x_j - \partial u_j/\partial x_i$ form an antisymmetrical matrix. They are the components of the curl u and describe a small rotation of the considered part of the medium taken as a whole. By introducing the notation

$$\vartheta_k(x, t) \equiv \frac{1}{2}(\text{curl } u(x, t))_k = \frac{1}{2}\sum_{p,q} \frac{\partial u_p(x, t)}{\partial x_q} \varepsilon_{kqp} \tag{5.2}$$

we have

$$\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right)=-\sum_k \varepsilon_{kij}\vartheta_k. \tag{5.3}$$

$\vartheta=(\vartheta_1,\vartheta_2,\vartheta_3)$ is called the vector of rigid rotation and ϑ_1, ϑ_2, ϑ_3 are its components. The vector of rigid rotation must not be confused with the vector of internal rotation discussed in §3.

The quantities

$$\frac{1}{2}\left(\frac{\partial u_i(x,t)}{\partial x_j}+\frac{\partial u_j(x,t)}{\partial x_i}\right)\equiv e_{ij}(x,t) \tag{5.4}$$

form a symmetrical matrix. They are called the components of the strain tensor or the components of strain (at the point x at the time t).

With this notation (5.1) takes the form

$$\Delta\xi_i=\sum_i e_{ij}\xi_j-\sum_{k,j}\varepsilon_{kij}\vartheta_k\xi_j, \tag{5.5}$$

whence it follows that the deformation in a small neighbourhood of every point may be considered as a linear and homogeneous function of the coordinates.

The deformed state is characterized by changes in the distances between the points. The deformed state at any point x may be therefore characterized by changes in length (or in the square of length) of every vector of the type ξ.

Now small displacements will be considered. The displacement vector and its derivatives with respect to the Cartesian coordinates are assumed so small that their product may be neglected. Calculate $|\xi+\Delta\xi|^2-|\xi|^2$. By (5.1) we have

$$\begin{aligned}(\xi+\Delta\xi)^2-\xi^2&=\sum_i\left(\xi_i+\sum_i\xi_j\frac{\partial u_i}{\partial x_j}\right)^2-\sum_i\xi_i^2\\&=2\sum_{i,j}\frac{\partial u_i}{\partial x_j}\xi_i\xi_j=2\sum_{i,j}e_{ij}\xi_i\xi_j.\end{aligned} \tag{5.6}$$

The square of length of the vector increment $|\Delta\xi|^2$ is expressed similarly,

$$|\Delta\xi(x,t)|^2=\sum_{i,j}e_{ij}(x,t)\xi_i\xi_j. \tag{5.7}$$

Formulas (5.6) and (5.7) show that changes in the distances between the points and hence the deformed state are characterized exclusively by the strain components.

If the strain components are zero, $e_{ij}(x,t)=0$, then a small neighbourhood of x will be in the same state as at the time t_0 (i.e., it will remain undeformed). The state at the time t may differ from that at the time t_0 only by rigid displacement. If, however, $e_{ij}(x,t)=0$ for any point x, then the foregoing statement holds for the whole medium.

Obviously, the converse statement is also true: if in some neighbourhood of x the distance between two points is not altered (in other words, the distances at the time t_0 and t coincide), then the strain components will be zero, $e_{ij}(x,t)=0$.

2. *Formulation of Hooke's law*

It is clear from the preceding article that there must exist some dependence between the stress components and the corresponding strain components. Hooke's law suggests the following simplest linear dependence between them:

$$\tau_{ij}(x,t)=\sum_{l,k} c_{ijlk}(x,t)e_{lk}(x,t), \tag{5.8}$$

where $c_{ijlk}(x,t)$ are certain quantities called elastic constants. They are constant in the sense of not being dependent on the strain components and hence on the stress components.

It will always be assumed in the sequel that the elastic constants are independent of time. Besides, the case of a specific and important dependence of the elastic constants on the position of x will be considered.

If the elastic constants do not depend on the position of a point in the medium, the medium is called homogeneous (in the sense of elastic properties but not in that of the mass distribution). If the elastic constants vary from one point to another, the medium is called inhomogeneous.

The symmetry of the matrices $\|\tau_{ij}\|$ and $\|e_{ij}\|$ implies

$$c_{ijlk}=c_{ijkl}=c_{jilk}=c_{jikl}. \tag{5.9}$$

The number of different elastic constants is now reduced from 81 to 36. It will be shown later that these constants, besides (5.9), also satisfy the condition

$$c_{ijlk}=c_{lkij}, \tag{5.10}$$

by virtue of which the above number is reduced to 21.

Hooke's law also assumes that the strain components are expressed (linearly and uniquely) through the stress components, i.e., system (5.8) admits a unique solution for the strain components,

$$e_{ij} = \sum_{l,k} c^{*}_{ijlk} \tau_{lk}. \tag{5.8'}$$

Formulas (5.8) and (5.8′) imply elasticity of the medium. By elasticity is meant an ability of the medium to restore its original shape after the forces that have caused deformation are removed. More precisely, elasticity is such a state of the continuous medium which is characterized by the one-to-one relation between stresses and strains; to zero stresses correspond zero strains.

The elastic constants are determined for each medium experimentally. A detailed account of the experimental methods used as well as tables with the numerical values of the elastic constants for many materials may be found in the references given in §15.

3. *Isotropic medium*

The stress and strain components and the elastic constants, related through Hooke's law, depend on the orientation of the coordinate axes. The medium is said to be isotropic, if its elastic constants c_{ijlk} do not depend on the orientation of the coordinate axes or, in other words, if the elastic properties of the medium are the same in all directions. If the medium is not isotropic, it is called anisotropic.

In an isotropic medium the number of different elastic constants is reduced to two. This can readily be shown by applying formula (5.8) and the properties of the stress and strain components (see, for example, LOVE [1], SNEDDON, BERRY [1], FILONENKO–BORODICH [1], LEKHNITSKI [1]). The following relations are obtained

$$c_{ijlk} = \lambda \delta_{ij} \delta_{lk} + \mu (\delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl}). \tag{5.11}$$

Substitution of (5.11) in (5.8) gives

$$\tau_{ij}(x, t) = \lambda \delta_{ij} \sum_{k} e_{kk}(x, t) + 2\mu e_{ij}(x, t). \tag{5.12}$$

The constants λ and μ are called the Lamé constants. They do not in general depend on the position of a point in the medium. We shall be concerned mostly with homogeneous media and always assume, unless stated otherwise, that the constants λ and μ do not depend on the position of the point x.

Formula (5.12) expresses Hooke's law for an isotropic medium. The notation

$$\Theta(x, t) = \sum_k e_{kk}(x, t) \tag{5.13}$$

is sometimes used (see, for example, MUSKHELISHVILI [1]) and Hooke's law is then written in the form

$$\tau_{ij}(x, t) = \lambda\delta_{ij}\Theta(x, t) + 2\mu e_{ij}(x, t), \tag{5.14}$$

or, using formulas (5.4), in the form

$$\tau_{ij}(x, t) = \lambda\delta_{ij}\operatorname{div} u(x, t) + \mu\left(\frac{\partial u_i(x, t)}{\partial x_j} + \frac{\partial u_j(x, t)}{\partial x_i}\right), \tag{5.15}$$

where $u = (u_1, u_2, u_3)$ is the displacement vector.

We have also included in Hooke's law an assumption that system (5.14) is solvable with respect to the strain components for any values of the stress components. This condition, as is easily verified, is reduced to the conditions

$$\mu \neq 0, \qquad 3\lambda + 2\mu \neq 0, \tag{5.16}$$

If these conditions are fulfilled, we obtain from (5.14)

$$e_{ij}(x, t) = \frac{1}{2\mu}\tau_{ij}(x, t) - \frac{\delta_{ij}\lambda}{2\mu(3\lambda + 2\mu)}\sum_k \tau_{kk}(x, t). \tag{5.14'}$$

Other elastic constants are sometimes introduced into consideration: the modulus of elasticity E (which is also referred to as Young's modulus), Poisson's ratio σ, the modulus of compressibility k, the Poisson number m. These quantities are related to the Lamé constants by

$$E = \frac{\mu(2\mu + 3\lambda)}{\lambda + \mu}, \qquad \sigma = \frac{\lambda}{2(\lambda + \mu)}, \qquad k = \frac{3\lambda + 2\mu}{3}, \qquad m = \frac{1}{\sigma}. \tag{5.17}$$

4. *Transversally isotropic medium*

An elastic medium is called transversally isotropic (see LOVE [1]) if there exists such an axis that in any plane perpendicular to it the elastic properties of the medium are the same in all directions or, in other words, all planes perpendicular to this axis are planes of isotropy.

Assume that the Cartesian coordinate system is chosen in such a way that the X_3-axis is directed perpendicular to the plane of isotropy. Then the elastic constants are not changed when the system rotates about the X_3-axis. Thus, the medium possesses transversal isotropy if and only if there exists such a Cartesian coordinate system $X_1X_2X_3$ at the rotation of which about X_3, i.e., at the transformation of the kind

$$x_1' = x_1 \cos\varphi + x_2 \sin\varphi, \qquad x_2' = x_2 \cos\varphi - x_1 \sin\varphi, \qquad x_3' = x_3, \tag{5.18}$$

the elastic constants remain unchanged. φ here is an arbitrary angle.

In this case the number of different elastic constants is reduced to five (see LOVE [1], LEKHNITSKI [1], SNEDDON, BERRY [1]) and Hooke's law assumes the form

$$\begin{aligned}
\tau_{11} &= c_1e_{11} + c_2e_{22} + c_3e_{33},\\
\tau_{22} &= c_2e_{11} + c_1e_{22} + c_3e_{33},\\
\tau_{33} &= c_3e_{11} + c_3e_{22} + c_4e_{33},\\
\tau_{23} &= c_5e_{23},\\
\tau_{13} &= c_5e_{13},\\
\tau_{12} &= \tfrac{1}{2}(c_1 - c_2)e_{12}.
\end{aligned} \tag{5.19}$$

We shall not go into a detailed study of transversally isotropic media. Some related topics will be discussed at the end of the book. Note only that relations (5.19), being a system of algebraic equations for the strain components, are solvable with respect to the latter for any τ_{ij}. This condition imposes certain restrictions on the constants

$$c_1, c_2, c_3, c_4, c_5.$$

The numerical values of the constants for many media have been calculated experimentally (see HUNTINGTON [1], LEKHNITSKI [1]).

Instead of the term "transversally isotropic medium" the term "hexagonal system" is also used because in this case the medium possesses hexagonal elastic symmetry.

§6. Strain energy in classical elasticity

1. Law of energy conservation

Let at the time t_0 an elastic medium be in its natural state and occupy a domain D bounded by S in the system $X_1X_2X_3$. We consider the state of the medium at the time t when by some external action it is brought from the state of rest into the state of strain.

Calculate the work done by the forces that have caused deformation in the interval from t_0 to t. In classical elasticity such forces are external stresses and mass forces. The effect of heat sources is ignored. Deformation is assumed to be a slow isothermic process such that at any instant of time the body is in thermodynamic equilibrium.

The work done by external stresses and mass forces in the time interval (t_0, t) is denoted by $\mathscr{R}(t)$ and the same work done in the interval $(t, t+\mathrm{d}t)$ by $\mathrm{d}\mathscr{R}(t)$. First, we calculate $\mathrm{d}\mathscr{R}(t)$.

The point occupying the position x in the medium at rest will move to the position $x + u(x, t)$ at the time t and to the position $x + u(x, t+\mathrm{d}t)$ at the time $t + \mathrm{d}t$. The displacement of the point x in the interval $\mathrm{d}t$ will therefore be $u(x, t+\mathrm{d}t) - u(x, t)$ which may be represented by

$$\frac{\partial u(x, t)}{\partial t}\,\mathrm{d}t,$$

neglecting higher order infinitesimal quantities relative to $\mathrm{d}t$.

A small area is isolated on the surface S in the neighbourhood of the point $y \in S$ and denoted by $\mathrm{d}S$. The external stresses acting on $\mathrm{d}S$ may be represented (see Art. 3, 1) in the form $\tau^{(n)}(y, t)\,\mathrm{d}S$, where $n = (n_1, n_2, n_3)$ is the unit outward normal to S at y. Thus, the work of the external stresses acting on $\mathrm{d}S$ in the interval $(t, t+\mathrm{d}t)$ is equal to

$$\frac{\partial u(y, t)}{\partial t}\,\mathrm{d}t\tau^{(n)}(y, t)\,\mathrm{d}S,$$

and the total amount of work performed by the external stresses is expressed by the integral

$$\int_S \mathrm{d}t \sum_i \tau_i^{(n)}(y, t)\frac{\partial u_i(y, t)}{\partial t}\,\mathrm{d}S.$$

The work done by the mass forces $\mathscr{F}$ in the interval $(t, t+\mathrm{d}t)$ is calculated in a likewise manner:

$$\int_D \mathrm{d}t \sum_i \rho \mathscr{F}_i(x,t) \frac{\partial u_i(x,t)}{\partial t} \mathrm{d}x.$$

Therefore (see (2.1)),

$$\frac{\mathrm{d}\mathscr{R}(t)}{\mathrm{d}t} = \int_S \sum_{i,j} \tau_{ji}(y,t) n_j \frac{\partial u_i(y,t)}{\partial t} \mathrm{d}S + \int_D \rho \sum_i \mathscr{F}_i(x,t) \frac{\partial u_i(x,t)}{\partial t} \mathrm{d}x.$$

Hence, applying the Gauss–Ostrogradski formula, we obtain

$$\frac{\mathrm{d}\mathscr{R}(t)}{\mathrm{d}t} = \int_D \sum_i \left(\sum_j \frac{\partial \tau_{ji}(x,t)}{\partial x_j} + \rho \mathscr{F}_i(x,t) \right) \frac{\partial u_i(x,t)}{\partial t} \mathrm{d}x$$

$$+ \int_D \sum_{i,j} \tau_{ji}(x,t) \frac{\partial^2 u_i(x,t)}{\partial x_j \partial t} \mathrm{d}x. \tag{6.1}$$

Using formula (4.3) and the notation

$$E^{(\mathrm{k})}(t) = \frac{1}{2} \int_D \rho \sum_i \left(\frac{\partial u_i(x,t)}{\partial t} \right)^2 \mathrm{d}x, \tag{6.2}$$

the first integral on the right-hand side of formula (6.1) assumes the form $\mathrm{d}E^{(\mathrm{k})}(t)/\mathrm{d}t$. $E^{(\mathrm{k})}(t)$ is the kinetic energy of the medium at the time t. The kinetic energy of a particle with the mass $\mathrm{d}m = \rho\, \mathrm{d}x$ at the time t is equal to $\frac{1}{2}\rho v^2(x,t)\, \mathrm{d}x$, where $v(x,t)$ is the velocity of the particle at the time t, i.e., $v(x,t) = \partial u(x,t)/\partial t$. The sum of all these quantities is the kinetic energy of the medium.

It follows from (6.2) that the kinetic energy does not depend on the state of strain at a given time.

The second term on the right-hand side of (6.1) will now be examined. Transforming the expression $\partial^2 u_i(x,t)/\partial x_j \partial t\, \mathrm{d}t$, we may obviously write it in the form

$$\frac{\partial^2 u_i(x,t)}{\partial x_j \partial t} \mathrm{d}t = \frac{\partial u_i(x, t+\mathrm{d}t)}{\partial x_j} - \frac{\partial u_i(x,t)}{\partial x_j} \equiv \mathrm{d}\, \frac{\partial u_i(x,t)}{\partial x_j}, \tag{6.3}$$

neglecting higher order infinitesimal quantities relative to $\mathrm{d}t$.

Therefore,

$$\sum_{i,j} \tau_{ji} \frac{\partial^2 u_i}{\partial x_j \partial t} \mathrm{d}t = \sum_{i,j} \tau_{ji} \,\mathrm{d} \frac{\partial u_i}{\partial x_j} = \sum_{i,j} \tau_{ji} \,\mathrm{d}e_{ij}, \tag{6.4}$$

where e_{ij} are the strain components (see (5.4)).

(6.1), (6.2), (6.4) imply

$$\mathrm{d}\mathcal{R} = \mathrm{d}E^{(\mathrm{k})} + \int_D \sum_{i,j} \tau_{ji} \,\mathrm{d}e_{ij} \,\mathrm{d}x. \tag{6.5}$$

It is now assumed that the medium under consideration is isotropic with the Lamé constants λ and μ. Introducing the function

$$E(x,t) = \frac{\lambda}{2}\Big(\sum_i e_{ii}(x,t)\Big)^2 + \mu \sum_{i,j} e_{ji}^2(x,t) \tag{6.6}$$

which is the quadratic form with respect to the strain components, we have by virtue of Hooke's law (see 5.12))

$$\frac{\partial E}{\partial e_{ji}} = \tau_{ji}, \qquad \frac{\partial E}{\partial t} = \sum_{i,j} \tau_{ji} \frac{\partial e_{ij}}{\partial t}. \tag{6.7}$$

From (6.5) and (6.7) it follows that

$$\mathrm{d}\mathcal{R} = \mathrm{d}E^{(\mathrm{k})} + \mathrm{d}E^{(\mathrm{p})}, \tag{6.8}$$

where

$$E^{(\mathrm{p})}(t) = \int_D E(x,t)\,\mathrm{d}x. \tag{6.9}$$

Integrating (6.8) from t_0 to t and taking into account that $E^{(\mathrm{k})}(t_0) = E^{(\mathrm{p})}(t_0) = 0$, we obtain

$$\mathcal{R}(t) = E^{(\mathrm{k})}(t) + E^{(\mathrm{p})}(t). \tag{6.10}$$

$E^{(\mathrm{p})}(t)$ is the potential energy of the body strain at the time t. In contrast to the kinetic energy $E^{(\mathrm{k})}(t)$, it depends essentially on the deformed state and represents the work to be done by external stresses and mass forces to produce a given deformed state.

Formula (6.10) expresses the law of energy conservation: the work of all the forces that have produced a deformed state is numerically equal to the sum of the kinetic energy of the medium and the potential energy of strain.

Formula (6.10) is also valid for an anisotropic medium. Consider some (not necessarily isotropic) elastic medium and introduce the function

$$E(x, t) = \frac{1}{2} \sum_{i,j,k,l} c_{ijkl} e_{ij}(x, t) e_{kl}(x, t), \tag{6.6'}$$

where c_{ijkl} are elastic constants defined by Hooke's law (see (5.8)).

Due to (5.10), (5.8) and the assumption that c_{ijkl} are time-independent we have

$$\frac{\partial E}{\partial e_{ij}} = \sum_{k,l} c_{ijkl} e_{kl} = \tau_{ji}, \qquad \frac{\partial E}{\partial t} = \sum_{i,j} \tau_{ji} \frac{\partial e_{ij}}{\partial t}. \tag{6.7'}$$

Hence, upon introducing function (6.9) in which $E(x, t)$ is determined by (6.6'), we obtain relation (6.10).

2. *Specific energy of strain*

As is seen from (6.9), $E(x, t)$ represents the potential energy calculated per unit volume (at the point x at the time t). In classical elasticity such energy is called the specific energy of strain.

For an arbitrary anisotropic medium the specific energy of strain is expressed by (6.6') and for an isotropic medium by (6.6). Applying formulas (5.19), the specific energy of strain may be calculated from (6.6') for transversally isotropic media and for other particular cases of anisotropy.

The expression for the specific energy of strain was obtained by using Hooke's law and assumption (5.10), the latter still remaining unsubstantiated.

Let us forget for a while what we know about the specific energy of strain and denote by $E(x, t)$ the work of strain or, what is the same thing, the potential energy of strain calculated per unit volume at the point x at the time t. This implies that the medium which is at rest at the time t_0 is considered and only a small part of it $\mathrm{d}D$ with the gravity centre at x is taken. The potential energy of strain of this part at the time t is calculated. The limit of the ratio of the energy to mes($\mathrm{d}D$) when mes($\mathrm{d}D$) tends to zero will be $E(x, t)$.

It is recalled that the kinetic energy is not taken into consideration in calculating the work of strain.

E will be called the specific energy of strain and the change of E in the interval $(t, t + \mathrm{d}t)$ will be denoted by $\mathrm{d}E$. From (6.5) it follows that

$$\mathrm{d}E(x, t) = \sum_{i,j} \tau_{ij}(x, t)\, \mathrm{d}e_{ij}(x, t). \tag{6.11}$$

The quantities e_{ij} characterize completely the deformed state of the medium. The potential energy, being dependent exclusively on the deformed state, will be a function of these quantities. Therefore, we shall sometimes write $E(e_{ij})(x, t)$ and $E(e_{ij})$ for $E(x, t)$ and E, respectively.

Expanding $E(e_{ij})$ in powers of e_{ij} near the state of rest ($e_{ij} = 0$) and neglecting the terms of order higher than two (small deformations are considered), we have

$$E(e_{ij}) = c_0 + \sum_{i,j} c_{ij}e_{ij} + \frac{1}{2} \sum_{i,j,k,l} c_{ijkl}e_{ij}e_{kl}, \tag{6.12}$$

where c_0, c_{ij}, c_{ijkl} are constants or rather quantities not depending on the strain components and hence on the stress components but depending, in general, on the position of a point in the medium.

Since the strain tensor is symmetrical, $e_{ij} = e_{ji}$, the product $e_{ij}e_{kj}$ is not changed when the indices in either of the pairs (i, j) and (k, l) are rearranged. Moreover, this product is not changed either if the pairs themselves are rearranged. Therefore, one may assume without loss in generality that the coefficients c_{ij} and c_{ijkl} satisfy the same symmetry conditions, i.e.,

$$c_{ij} = c_{ji}, \qquad c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}. \tag{6.13}$$

It follows from relation (6.11) that

$$\frac{\partial E}{\partial e_{ij}} = \tau_{ij}. \tag{6.14}$$

Thus, the partial derivative of the function E with respect to any one of the strain components is the corresponding stress component.

The derivative of E with respect to the strain components may be calculated by (6.12). We then have

$$\frac{\partial E}{\partial e_{ij}} = c_{ij} + \sum_{k,l} c_{ijkl}e_{kl}.$$

Therefore,

$$\tau_{ij} = c_{ij} + \sum_{k,l} c_{ijkl}e_{kl}.$$

Note that in expansion (6.12) $c_0 = 0$ because the condition $e_{ij} = 0$ implies $E = 0$ (at the initial time $t = t_0$ the medium is not deformed). Further, the absence of strains implies the absence of stresses; in other words, if all $e_{ij} = 0$, then all $\tau_{ij} = 0$. Therefore, from the above formula we conclude that $c_{ij} = 0$.

Now,

$$E(e_{ij}) = \frac{1}{2} \sum_{i,j,k,l} c_{ijkl} e_{ij} e_{kl}, \tag{6.15}$$

and we obtain for the stress components

$$\tau_{ij} = \sum_{k,l} c_{ijkl} e_{kl}. \tag{6.16}$$

Relations (6.16), establishing the relation between the components of stress and strain, express Hooke's law for any elastic medium within the framework of the classical theory: the deformations of the medium are proportional to the stresses applied to it or, more precisely, the deformations are linear combinations of the stresses. It should be noted that the elastic constants c_{ijkl} satisfy conditions (5.10) (see (6.13)) which in the earlier discussion were taken as assumptions.

The above arguments substantiate Hooke's law from the standpoint of physics.

From (6.15) and (6.16) one may derive other representations of the specific energy of strain, for example,

$$E = \frac{1}{2} \sum_{i,j} \tau_{ij} e_{ij}, \tag{6.17}$$

whence

$$\frac{\partial E}{\partial \tau_{ij}} = e_{ij}. \tag{6.18}$$

It may be shown that the constants c_{ijkl} form a fourth order tensor.

It follows from the physical meaning that the specific energy of strain is the positive definite quadratic form of six independent quantities e_{ij}. This condition imposes certain restrictions on the elastic constants. In the case of an isotropic medium these restrictions are reduced to

$$\mu > 0, \qquad 3\lambda + 2\mu > 0. \tag{6.19}$$

Though it has been shown experimentally that $\lambda > 0$, $\mu > 0$, (see Muskhelishvili [1], Landau, Lifshitz [1], Grammel [1], Huntington [1] et al.), we nevertheless subject these constants to weaker restrictions (6.19).