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INTRODUCTION

In scattering theory one investigates the link between the asymptotic behaviour of a system as the time t tends to $-\infty$ and $+\infty$, respectively. Usually one compares the solutions of a perturbed system with the solutions of a free system whose solutions can be described easily. The evolution of the perturbed system is given by a contraction semigroup (unitary group) of operators $V(t)$, $t \geq 0$, acting in a Hilbert space H, called the energy space. Similarly, the free system is described by a unitary group $U_0(t)$ of operators acting in the free energy space H_0.

The main problem in the construction of scattering theory is the existence of the wave operators

$$W_- f = \lim_{t \to \infty} V^*(t) J U_0(-t) f \, , \qquad f \in H_0 \, ,$$

$$W f = \lim_{t \to \infty} U_0(-t) J^* V(t) f \, , \qquad f \in \mathcal{H} \, ,$$

where $J \ : \ H_0 \to H$ is a projection, $V^*(t)$ is the adjoint semigroup and $J^* \ : \ H \to H_0$ is the adjoint of J. Here $\mathcal{H}$ is a subspace of H with the property that the local energy of $V(t)f$ with $f \in \mathcal{H}$ decreases as $t \to \infty$. This property is necessary for the existence of Wf and it is natural to find a maximal subspace $\mathcal{H}$ of this type and, also, to establish the inclusion

$$(1) \qquad \operatorname{Ran} W_- \subset \mathcal{H} \, .$$

Once this is done, one is able to introduce the scattering operator $S = W \circ W_-$ following the diagram

$$\begin{array}{ccccc} & & \mathcal{H} & & \\ & W & & W_- & \\ H_0 & & & & H_0 \\ & & S & & \end{array}$$

In the same way one investigates perturbed systems with time-dependent perturbations (time-dependent potentials, moving obstacles). In this situation the evolution is given by a family of operators acting from the energy space $H(s)$ to the energy space $H(t)$, and the solution with initial data $f \in H(s)$ has the form $U(t,s)f$. The wave operators connecting $U(t,0)$ and $U_0(t)$ become

$$W_- = \lim_{t\to\infty} U(0,-t)J(-t)U_0(-t)f\,, \qquad f \in H_0\,,$$

$$Wf = \lim_{t\to\infty} U_0(-t)J^*(t)U(t,0)f\,, \qquad f \in \mathcal{H} \subset H(0)$$

with $J(t) : H_0 \to H(t)$ being a bounded operator which plays the role of a restriction. Also in this case the description of a maximal subspace $\mathcal{H}$ of $H(0)$ such that Wf exists for $f \in \mathcal{H}$ is an important problem.

The existence of W_-, W and S has been established for a large class of perturbed systems described by a unitary group of operators which arise in mathematical physics. In this direction the reader should consult the classical books of Lax and Phillips [2] and of Reed and Simon [3], where an excellent exposition of this subject is presented. In their analysis the spectral properties of the generator of the unitary group plays an essential role.

The situation changes considerably when one studies systems related to a contraction semigroup $V(t)$ or to a propagator $U(t,s)$. In the first case one has dissipation of the energy, while, in the second one, the global or even the local energy might increase exponentially when the time t goes to infinity. Another difficulty is connected with the fact that the spectral theory for generators G of contraction semigroups is not so well developed as the spectral theory for self-adjoint operators. A similar difficulty arises for $U(t,s)$, and, in general, one cannot find an operator playing the role of G. Some simplification is possible when one examines time periodic perturbations. Then one introduces the monodromy operator $V = U(T,0)$, $T > 0$ being the period of the perturbation. Nevertheless, in general a suitable spectral theory for V is not available.

During the last fifteen years, scattering theory for dissipative and time-dependent systems has been intensively studied. The results in this field are exposed in many papers based on various tools and techniques which are scattered in the scientific literature. In this book we present an approach to these problems founded on a few ideas, which can be applied simultaneously to the cases of even and of odd space dimension. These ideas are connected with the RAGE type theorem, with Enss' decomposition of the phase space and with a time-dependent proof of the existence of the operator W which exploits the decay of the local energy of the perturbed and free systems. Moreover, we treat some inverse scattering problems for time-dependent potentials and moving obstacles with an arbitrary geometry. These problems are connected with the so-called generalized scattering kernels $K^{\#}$ which can be defined even if the scattering operator S cannot be determined in the sense mentioned above. On the other hand, if S exists, then $K^{\#}$ coincides with the kernel of S–Id considered in a suitable representation. Thus we obtain a natural generalization of the situation of stationary (time-independent) potentials and obstacles, where our methods imply the results previously obtained by other means.

In Chapter I, we expose some results from functional analysis which are necessary for the monograph. We restrict our attention to the so-called RAGE type theorems for contraction semigroups and power bounded operators. These theorems play a crucial role in the analysis of the decay of local energy.

In Chapter II, basic facts are collected for the unitary group $U_0(t)$ related to the Cauchy problem for the wave equation. We treat both cases when the space dimension n is odd or even. Also, we construct the translation representations of $U_0(t)$ associated with the Lax-Phillips spaces $D_{\pm}$. We describe the link between these translation representations and the asymptotic wave profiles. Following the approach of Cooper and Strauss [2], [4], we introduce outgoing (incoming) solutions for time-dependent problems and we show, that these solutions admit asymptotic wave profiles. This fact is important for the definitions of the generalized scattering kernels $K^{\#}$ in Chapters VI and VIII.

Chapter III is devoted to first order symmetric systems with characteristics of variable multiplicity in the exterior of a domain with uniformly characteristic boundary. Such systems arise in many problems of mathematical physics, a typical example being Maxwell's system in electrodynamics. We apply an adaption of Enss' method proposed by Georgiev and Stefanov [1] and developed by Vodev [3]. The existence of zero speeds as well as the lack of smoothness of the characteristic roots lead to some difficulties in the construction of a suitable Enss' decomposition. We treat dissipative boundary conditions and short range perturbations.

For some dissipative systems it is possible to construct a solution $V(t)f$ with $f \neq 0$ which vanishes for $t \geq T_0$. Such solutions are called disappearing and they might perturb the inverse scattering problem and the controllability of the system. Moreover, if there exists an $f \in H$ with the mentioned property, we can find an infinite dimensional space of initial data with the same property. In Chapter IV we study this phenomenon and obtain a link between the existence of a disappearing solution and the controllability of the system. The systems described by unitary grouips do not admit disappearing solutions and such solutions are typical for dissipative systems.

In Chapter V we begin the examination of time-dependent perturbations. Here we study the equation $(\Box + q(t,x))u = 0$ with a potential $q(t,x)$ which is periodic in time. Assuming the global energy is bounded as $t \to \infty$, we obtain a decay of the local energy for initial data f orthogonal to a space H_b generated by the eigenfunctions of the adjoint operator V^* of V with eigenvalues on the unit circle S^1. If the dimension is odd and the data f have compact support, we establish an exponential rate of the local energy decay and show that H_b is finite dimensional. We discuss the case when the global energy is not bounded and describe the spectrum of the monodromy operator V. In this analysis the local evolution operator $Z^a(t,0) = P_+^a U(t,0) P_-^a$ plays an essential role. Here $P_\pm^a$ denote the orthogonal projections on the orthogonal complements of the spaces $D_\pm^a$. Finally, the results of this chapter are applied to stationary potentials.

In Chapter VI we examine the inverse scattering problem for time-dependent potentials and define the generalized scattering kernel $K^\#$. We prove the uniqueness of the inverse scattering problem showing that, if the kernels $K_i^\#$, $i = 1,2$, related to the potentials $q_i(t,x)$, $i = 1,2$, coincide, then the potentials q_1 and q_2 coincide, too. Moreover, we find a procedure for recovering a stationary potential. In the exposition we follow a time-dependent approach developed by Stefanov [8].

Chapter VII is devoted to the scattering for the wave equation in the exterior of a moving obstacle which may change its form and position with a speed less than the propagation speed of the solutions of the wave equations. Supposing that the global energy is bounded as $t \to \infty$, we obtain the existence of the operators W_-, W, assuming a decay of the local energy. For periodically moving non-trapping obstacles we establish such a local energy decay by using the propagation of the singularities of the solutions of the wave equation along the generalized bicharacteristics. We also treat some generalizations concerning Neumann and Robin boundary conditions.

In Chapter VIII we investigate an inverse scattering problem related to the leading singularity of the generalized scattering kernel $K^{\#}$. After a localization of this singularity we construct a microlocal parametrix in a domain sufficiently close to the place and the time of the first reflection of a plane wave with incident direction ω. We cover both the generic and the non-generic cases using a result of Soga [4] for the asymptotics of oscillatory integrals with degenerate phase functions. For completeness we discuss this result in Appendix II. Finally, in this chapter we show that for all $t \in \mathbb{R}$ the convex hull of the obstacle $\mathcal{K}(t)$ can be recovered from the back scattering data related to the singularities of $K^{\#}$.

In the exposition we assume some knowledge of the theory of contraction semigroups and some background in partial differential equations and functional analysis in Hilbert spaces. Furthermore, we use some standard notations and facts from microlocal analysis. However, in Sections 7.4 and 8.3 we need a deep result of Melrose and Sjöstrand [1]. The reader may choose to admit this result and go with the arguments in these sections.

In the list of references at the end we have tried to be complete with respect to recent publications on scattering theory for dissipative systems and time-dependent perturbations, but we have made no efforts to include an exhaustive coverage of the immense literature in all aspects of scattering theory.

Some chapters of this book have been exposed in two courses on scattering theory given by the author at the Federal University of Pernambuco in Recife, Brazil and at the University of Sofia, Bulgaria. An extended version of the first course has been published as Notas de Curso by the Federal University of Pernambuco in 1987.

In the preparation of the book I have profited from numerous discussions with my colleagues V. Georgiev, G. Popov, Tz. Rangelov, Pl. Stefanov and G. Vodev in Sofia. They gave permission to include some of their yet unpublished results. Moreover, they read and criticized the manuscript, helping me to correct some errors and to improve the exposition. For all this I am very grateful to them.

I am also grateful to J. Cooper and W. Strauss for the possibility to include some of their results given in an unpublished manuscript and to J. Fresnel for the proof of Theorem A.1.1 in Appendix I.

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CHAPTER I
CONTRACTION SEMIGROUPS AND POWER BOUNDED OPERATORS

In this chapter we collect some facts from functional analysis which are important for the present monograph. In Section 1.1 we prove a theorem characterizing the generators of contraction semigroups and we establish some technical lemmas. Section 1.2 is devoted to the so-called RAGE type theorem for contraction semigroups. In Section 1.3 we deal with power bounded operators V defined by the property that $\sup_{m \in \boldsymbol{N}} \|V^m\| \leq C_0$. We obtain a representation of V including a partial isometry. We apply this result in Section 1.4 to establish a RAGE type theorem for power bounded operators. The RAGE type theorems play an essential role in the analysis of the decay of local energy discussed in Chapters III, V, VII.

1.1. Contraction semigroups

Let H be a Hilbert space with inner product $(\ , \ )$ and norm $\| \cdot \|$. Also the norm of the bounded operators in H will be denoted by $\| \cdot \|$. A one-parameter family $\{V(t) \ : \ t \geq 0\}$ of operators in H will be called a contraction semigroup if $V(t)$ satisfy the following properties:

(i) $V(0) = \mathrm{Id}$,

(ii) $V(t+s) = V(t)V(s)$ for all $t, s \geq 0$,

(iii) For all $f \in H$ the function $t \to V(t)f$ is continuous on H,

(iv) $\|V(t)\| \leq 1$ for all $t \geq 0$.

The basic properties of contraction semigroups can be found in Hille and Phillips [1]. We discuss here only the properties which are necessary for our exposition.

For every contraction semigroup there exists a closed linear operator G with dense domain $D(G) \subset H$ such that

$$\lim_{h\to 0} \frac{V(t+h)f - V(t)f}{h} = V(t)Gf = GV(t)f$$

for all $f \in D(G)$ and all $t \geq 0$. The operator G is called the generator of $V(t)$ and so we write $V(t) = e^{tG}$. For $\operatorname{Re}\lambda > 0$ consider the operator

$$R_\lambda f = -\int_0^\infty e^{-\lambda t} e^{tG} f\, dt \ .$$

We have

$$\left(\tfrac{e^{hG}-1}{h}\right)(R_\lambda f) = \tfrac{1-e^{\lambda h}}{h} \int_0^\infty e^{-\lambda t} e^{tG} f\, dt + \\ + h^{-1} e^{\lambda h} \int_0^h e^{-\lambda t} e^{tG} f\, dt \ .$$

Letting $h \to 0$, the right-hand side tends to $\lambda R_\lambda f + f$, hence $R_\lambda f \in D(G)$ and $(G-\lambda)R_\lambda f = f$. On the other hand, for $f \in D(G)$ we obtain

$$G \int_0^\infty e^{-\lambda t} e^{tG} f\, dt = \int_0^\infty e^{-\lambda t} G e^{tG} f\, dt = \\ = \int_0^\infty e^{-\lambda t} e^{tG} G f\, dt \ .$$

Thus for $f \in D(G)$ we have $(G-\lambda)R_\lambda f = R_\lambda(G-\lambda)f = f$ and we obtain the resolvent formula

$$(G-\lambda)^{-1} f = -\int_0^\infty e^{-\lambda t} e^{tG} f\, dt\,, \qquad \operatorname{Re}\lambda > 0 \ . \tag{1.1.1}$$

Moreover, it is clear that

$$\|(G-\lambda)^{-1}\| \leq 1/\lambda\,, \qquad \lambda > 0 \ .$$

A sufficient condition for a closed operator to be a generator of a contraction semigroup is given by the following.

Theorem 1.1.1 (Hille-Yosida).
A linear densely defined closed operator G generates a contraction semigroup in H if and only if

(1) $(0,\infty) \subset \rho(G)$,

(2) $\|(G-\lambda)^{-1}\| \leq 1/\lambda$ for all $\lambda > 0$,

$\rho(G)$ being the resolvent set of G.

We refer to Hille and Phillips [1] for the proof of this theorem.

If $V(t)$ is a contraction semigroup, the family of adjoint operators $\{V^*(t) \ : \ t \geq 0\}$ is again a contraction semigroup. Denoting by J the generator of $V^*(t)$, we obtain from (1.1.1) the equality

$$[(J-1)^{-1}]^* = -\int\limits_0^\infty e^{-t}e^{tG}dt = (G-1)^{-1}$$

yielding $J = G^*$.

For applications it is important to have a simple condition on G which guarantees that G generates a contraction semigroup. For this purpose we introduce the notion of accretive operator.

Definition 1.1.2.
An operator A with dense domain $D(A) \subset H$ is called accretive if for all $f \in D(A)$ we have

(1.1.2) $\qquad \mathrm{Re}(Af, f) \leq 0$.

The operator A is called maximal accretive if A is accretive and there are no proper accretive extentions of A.

The following theorem explains the role of the accretive operators.

Theorem 1.1.3.
A closed operator G in H is the generator of a contraction semigroup if and only if G is accretive and $\mathrm{Ran}(G-\lambda_0) = H$ for some $\lambda_0 > 0$.

Proof. Assume $V(t) = e^{tG}$ is a contraction semigroup. Then for $f \in D(G)$ we have

$$\frac{d}{dt}\,\mathrm{Re}(e^{tG}f, f)\Big|_{t=0} = \mathrm{Re}(Gf, f) \ .$$

On the other hand,

$$\mathrm{Re}(V(t)f, f) \leq |(V(t)f, f)| \leq \|f\|^2$$

and

$$\frac{\mathrm{Re}(V(t)f, f) - \|f\|^2}{t} \leq 0 \quad \text{for } t > 0$$

leads to (1.1.2). Therefore, it follows from condition (1) of Theorem 1.1.1 that $\mathrm{Ran}(G - \lambda_0) = H$.

Now let G be an accretive closed operator with $\mathrm{Ran}(G - \lambda_0) = H$ for $\lambda_0 > 0$. For $f \in D(G)$ and $\lambda > 0$ we obtain

$$\text{(1.1.3)} \qquad \begin{aligned} \lambda\|f\|^2 \leq \ & \lambda\|f\|^2 - \mathrm{Re}(Gf, f) = \mathrm{Re}((\lambda - G)f, f) \ \leq \\ \leq \ & \|(G - \lambda)f\| \, \|f\| \ . \end{aligned}$$

Consequently, $\mathrm{Ran}(G - \lambda)$ is closed and $(G - \lambda)^{-1}$ is defined and bounded on $\mathrm{Ran}(G - \lambda)$ with norm less or equal to λ^{-1}. It remains to show that $\mathrm{Ran}(G - \lambda)$ is dense in H. Since

$$\text{(1.1.4)} \qquad [\mathrm{Ran}(G - \lambda)]^{\perp} = \mathrm{Ker}(G^* - \lambda) \ ,$$

it is sufficient to prove that $\dim \mathrm{Ker}(G^* - \lambda)$ is locally constant. Indeed, then $\mathrm{Ker}(G^* - \lambda_0) = \{0\}$ implies $\overline{\mathrm{Ran}(G - \lambda)} = H$.

Take $\eta \in C$, $|\eta|$ small enough, and assume

$$G^* u = (\lambda + \eta)u \ , \quad \|u\| = 1 \ , \quad u \in D(G^*) \ .$$

Assuming $u \in [\mathrm{Ker}(G^* - \lambda)]^{\perp}$, by (1.1.4) we get $u \in \mathrm{Ran}(G - \lambda)$, that is $u = (G - \lambda)f$ with some $f \in D(G)$. Therefore

$$\begin{aligned} 0 = ((G^* - (\lambda + \eta))u, f) = \ & (u, (G - \lambda)f) - \eta(u, f) \ = \\ = \ & \|u\|^2 - \eta(u, f) \ . \end{aligned}$$

On the other hand, from (1.1.3) we deduce $\|f\| \leq \|u\|/\lambda$. So for $|\eta| < \lambda$ we obtain a contradiction and there are no elements $u \in \mathrm{Ker}(G^* - (\lambda + \eta))$ which are in $[\mathrm{Ker}(G^* - \lambda)]^{\perp}$. This implies easily

$$\dim \mathrm{Ker}(G^* - (\lambda + \eta)) \leq \dim \mathrm{Ker}(G^* - \lambda) \ .$$

Repeating this argument, we get for $|\eta| < \lambda$ the inequality

$$\dim \operatorname{Ker}(G^* - \lambda) \leq \dim \operatorname{Ker}(G^* - (\lambda + \eta)) .$$

Finally, $\dim\operatorname{Ker}(G^* - \lambda)$ is locally constant and the proof of Theorem 1.1.3 is complete.

□

Corollary 1.1.4.
Let G be a closed operator in H. If both G and its adjoint G^* are accretive, then G generates a contraction semigroup.

Proof. Assume there exists $g \in H$, $g \neq 0$ such that

$$((G-1)f, g) = 0 \quad \text{for all } f \in D(G) .$$

Then $g \in D(G^*)$, $G^*g = g$ and $\operatorname{Re}(G^*g, g) = \|g\|^2$ which contradicts the assumption that G^* is accretive. This shows that $\operatorname{Ran}(G-1)$ is dense. As in the proof of Theorem 1.1.3, $\operatorname{Ran}(G-1)$ is closed. Then $\operatorname{Ran}(G-1) = H$ and we apply Theorem 1.1.3.

□

If G is a generator of a contraction semigroup, then G is maximal accretive. In fact, let $\tilde{G}$ be an accretive extension of G. For each $f \in D(\tilde{G})$ we can find $g \in D(G)$ such that

$$(\tilde{G} - 1)f = (G-1)g = (\tilde{G} - 1)g .$$

Applying to both sides $(\tilde{G} - 1)^{-1}$, we obtain $f = g$ and $D(\tilde{G}) = D(G)$. The converse assertion is also true: every maximal accretive operator generates a contraction semigroup. Here we do not apply this result.

Next we prove some technical lemmas concerning generators G of contraction semigroups $V(t)$. First, notice that if λ is an eigenvalue of G, then $e^{\lambda t}$ is an eigenvalue of $V(t)$ for all $t \geq 0$. Indeed, consider the equation

$$\frac{d}{dt}\left(e^{tG} f\right) = e^{tG} G f = \lambda e^{tG} f ,$$