AN INTRODUCTION TO NONSTANDARD REAL ANALYSIS

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An Introduction to Nonstandard Real Analysis

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Dedicated to the memory of
ABRAHAM ROBINSON

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Contents

Chapter III
Nonstandard Theory of Topological Spaces

Chapter IV
Nonstandard Integration Theory

Appendix

Preface

The notion of an infinitesimal has appeared off and on in mathematics since the time of Archimedes. In his formulation of the calculus in the 1670s, the German mathematician Wilhelm Gottfried Leibniz treated infinitesimals as ideal numbers, rather like imaginary numbers, which were smaller in absolute value than any ordinary real number but which nevertheless obeyed all of the usual laws of arithmetic. Leibniz regarded infinitesimals as a useful fiction which facilitated mathematical computation and invention. Although it gained rapid acceptance on the continent of Europe, Leibniz's method was not without its detractors. In commenting on the foundations of calculus as developed both by Leibniz and Newton, Bishop George Berkeley wrote, "And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?" The question was, How can there be a positive number which is smaller than any real number without being zero? Despite this unanswered question, the infinitesimal calculus was developed by Euler and others during the eighteenth and nineteenth centuries into an impressive body of work. It was not until the late nineteenth century that an adequate definition of limit replaced the calculus of infinitesimals and provided a rigorous foundation for analysis. Following this development, the use of infinitesimals gradually faded, persisting only as an intuitive aid to conceptualization.

There the matter stood until 1960 when Abraham Robinson gave a rigorous foundation for the use of infinitesimals in analysis. More specifically, Robinson showed that the set of real numbers can be regarded as a subset of a larger set of "numbers" (called hyperreal numbers) which contains infinitesimals and also, with appropriately defined artithmetic operations, satisfies all of the arithmetic rules obeyed by the ordinary real numbers. Even more, he demonstrated that the relational structure over the reals (sets, relations, etc.) can be extended to a similar structure over the hyperreals in such a way that all statements true in the real structure remain true, with a suitable interpretation, in the hyperreal structure. This latter property, known as the transfer principle, is the pivotal result of Robinson's discovery.

Robinson's invention, called nonstandard analysis, is more than a justification of the method of infinitesimals. It is a powerful new tool for mathematical research. Rather quickly it became apparent that every mathematical structure has a nonstandard model from which knowledge of the original structure can be gained by applications of the appropriate transfer principle. In the twenty-five years since Robinson's discovery, the use of nonstandard models has led to many new insights into traditional mathematics, and to solutions of unsolved problems in areas as diverse as functional analysis, probability theory, complex function theory, potential theory, number theory, mathematical physics, and mathematical economics.

Robinson's first proof of the existence of hyperreal structures was based on a result in mathematical logic (the compactness theorem). It was perhaps this aspect of his work, more than any other, which made it difficult to understand for those not adept at mathematical logic. At present, the most common demonstration of the existence of nonstandard models uses an "ultrapower" construction. But the use of ultrapowers is not restricted to nonstandard analysis. Indeed, the construction of ultrapower extensions of the real numbers dates back to the 1940s with the work of Edwin Hewitt [17] and others, and the use of ultrapowers to study Banach spaces [10,16] has become an important tool in modern functional analysis. Nonstandard analysis is a far-reaching generalization of these applications of ultrapowers. One essential difference between the method of ultrapowers and the method of nonstandard analysis is the consistent use of the transfer principle in the latter. To present this principle one needs a certain amount of mathematical logic, but the logic is used in an essential way only in stating and proving the transfer principle, and not in applying nonstandard analysis. We hope to demonstrate that the amount of logic needed is minimal, and that the advantages gained in the use of the transfer principle are substantial.

The aim of this book is to make Robinson's discovery, and some of the subsequent research, available to students with a background in undergraduate mathematics. In its various forms, the manuscript was used by the second author in several graduate courses at the University of Illinois at Urbana–Champaign. The first chapter and parts of the rest of the book can be used in an advanced undergraduate course. Research mathematicians who want a quick introduction to nonstandard analysis will also find it useful. The main addition of this book to the contributions of previous textbooks on nonstandard analysis [12, 37, 42, 46] is the first chapter, which eases the reader into the subject with an elementary model suitable for the calculus, and the fourth chapter on measure theory in nonstandard models.

A more complete discussion of this book's four chapters must begin by noting H. Jerome Keisler's major contribution to nonstandard analysis in the form of his 1976 textbook, "Elementary Calculus" [23] together with the instructor's volume, "Foundations of Infinitesimal Calculus" [24]. Keisler's book is an excellent calculus text (see the second author's review [30]) which makes that part of nonstandard analysis needed for the calculus available to freshman students. Keisler's approach uses equalities and inequalities to transfer properties from the real number system to the hyperreal numbers. In our first chapter, we have modified that approach to an equivalent one by formulating a simple transfer principle based on a restricted language.

The first chapter begins by using ultrafilters on the set of natural numbers to construct a simple ultrapower model of the hyperreal numbers. A formal language is then developed in which only two kinds of sentences are used to transfer properties from the real number system to the larger, hyperreal number system. The rest of the chapter is devoted to extensive applications of this simple transfer principle to the calculus and to more-advanced real analysis including differential equations. By working through these applications, the reader should acquire a good feeling for the basics of nonstandard analysis by the end of the chapter. Anyone who begins this book with no background in mathematical logic should have no problem with the logic in the first chapter and hence should easily pick up the background needed to proceed. Indeed, it is our hope that such a reader will grow quite impatient with the restrictions on the language we impose in the first chapter, and thus be more than ready for the general language introduced in Chapter II and used in the rest of the book. We will not comment on what might be in the mind of a logician at that point.

Chapter II extends the context of Chapter I to "higher-order" models appropriate to the discussion of sets of sets, sets of functions, etc., and covers the notions of internal and external sets and saturation. These topics, together with a general language and transfer principle, are held in abeyance until the second chapter so that the beginner can master the subject in reasonably easy steps. They are, however, essential to the applications of nonstandard analysis in modern mathematics. External constructions, such as the nonstandard hulls discussed in Chapter III and the standard measure spaces on nonstandard models described in Chapter IV, have been the principal tools through which new results in standard mathematics have been obtained using nonstandard analysis.

The general theory of Chapter II is applied in Chapter III to topological spaces. These are sets with an additional structure giving the notion of nearness. The presentation assumes no familiarity with topology but is rather brisk, so that acquaintance with elementary topological ideas would be useful. The chapter includes discussions of compactness and of metric, normed, and Hilbert spaces. We present a brief discussion of nonstandard hulls of metric spaces, which are important in nonstandard technique. Some of the more advanced topics in Kelley's "General Topology," such as function spaces and compactifications, are also included.

Finally, in Chapter IV, we introduce the reader to nonstandard measure theory, certainly one of the most active and fruitful areas of present-day research in nonstandard analysis. With measure theory one extends the notion of the Riemann integral. We shall take a "functional" approach to the integral on nonstandard spaces. This approach will produce both classical results in standard integration theory and some new results which have already proved quite useful in probability theory, mathematical physics, and mathematical economics. The development in this chapter does not assume familiarity with measure theory beyond the Riemann integral. Most of the results in [27, 29, 32, 33] are presented without further reference. We note here that the measures and measure spaces constructed on nonstandard models in Chapter IV are often referred to in the literature as Loeb measures and Loeb spaces.

With one exception (Section I.15), every section of the book has exercises. In designing the text, we have assumed the active participation of the reader, so some of the exercises are details of proofs in the text. At the back of the book there is a list of the notation used, together with the page where the notation is introduced. Of course, we freely use the symbols $\in$, $\cup$, and $\cap$ for set membership, union, and intersection. We have starred sections that can be skipped at the first reading. Every item in the book has three numbers, the number of the chapter (I, II, III or IV), the number of the section, and the number of the item in the section. Thus, Theorem IV.2.3 is the third item in the second section of the fourth chapter. In referring to an item, we shall omit the chapter number for items in the same chapter as the reference, and the section number for items in the same section as the reference.