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F. KÁRTESZI
Editor

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The Theory of Space

With Introduction, Comments, and Addenda

Edited by

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Supplement by

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CONTENTS

Part IV

PREFACE TO THE ENGLISH EDITION

This book is a revised edition of the memorial volume I wrote in 1952, by invitation, to the 150th anniversary of János Bolyai's birth. That time, I could spend only two months with writing the text and drawing the illustrations. Therefore in the second edition I have somewhat revised and corrected the original.

Encouraged by people abroad interested in the subject, I gave consent to publish my book in English. However, for the better information of these readers I stipulated that the book should be supplemented with a brief historical survey. The task was taken on by Professor Barna Szénássy. Using the latest documents, he wrote a concise historical supplement. I believe that learning some facts of Hungarian political and science history will help the less informed reader get acquainted with the miserable fate of János Bolyai and with his intellectual world.

Initially, at several suggestions, I thought that the book written a quarter of a century ago should be completely renewed and made more conforming to the demands and style of today. In fact, recent efforts have more and more aimed at a definitive showdown of the intuitive elements of knowledge still to be found. The excessive freedom of traditional scientific style and language should be eliminated through the systematic use of a modern, strictly formalized language. This is a remarkable point of view, indeed. Accordingly, I ought to present non-Euclidean geometry in the most up-to-date manner and comment on Bolyai's work in that connection.

Doing so, however, I could not make evident what an epoch-making discovery the geometry of János Bolyai was in its own time; it should be emphasized how natural Bolyai's ideas, his revolutionary aspect of mathematical space theory seem to be today and what it has since grown into. On the other hand, the very thing the reader interested in the history of science shall clearly see is that contemporaries, excepting perhaps Gauss and Lobachevsky, were averse to Bolyai's thoughts and considered them an artificial, or even obscure, intellectual construction.

For these reasons, the commentator of a classical work that has strongly influenced the development of science must present his subject by putting it back to its own time. He must sketch the antecedents as well as describe the difficulties rooted in the relative primitiveness of contemporary scientific opinion and hindering the evolution of new ideas. Also the triumphal spreading of the new ideas, especially in its initial period, should be described. I tried to write my book so as to serve this purpose.

In our country, Bolyai's work is generally called the *Appendix*. In Bolyai's Latin manuscript the title *Scientia Spatii* (*Raumlehre* in the German one) occurs. For the sake of historical fidelity, I gave my book the title *Appendix, the science of space*. To bring out the historical point of view, the facsimile of a copy of Bolyai's work printed in June 1831 is also included, though I have prepared — in cooperation with György Hajós and Imre Trencsényi Waldapfel, late professor of classical philology — a careful translation of the text into modern Hungarian and added it to the Latin original.

Part I is a historical introduction which makes use of the latest literary sources. The Supplement, written for the English edition by Professor Barna Szénássy, completes this part and helps the foreign reader.

Part II contains the Latin original and its translation into present-day language. Though the translation follows the requirements of modern language and style, it accurately reflects the concise Latin text. Dissection into chapters not occurring in the original, changing notation to that used today, setting the illustrations at suitable places of the text, application of an up-to-date drawing technique and, finally, the presence of some new illustrations help to avoid the unnecessary difficulties usually encountered when reading old prints and texts.

Part III is a series of informal short remarks divided into sections corresponding to those of the original work. Actually, with these remarks we try to make easier the comprehension of the text which, because of its conciseness, can only be read with close attention, thinking the material over and over again. This happens once by completion, once by reformulation and more detailed explanation in order to dissolve conciseness, yet another time by addition and further argument.

In Part IV, by picking out and sketching some important topics, we attempt to indicate the effect that may be traced in the development of modern mathematics after János Bolyai's space theory had become generally known.

Ferenc Kárteszi

PART I

EVOLUTION OF THE SPACE CONCEPT UP TO THE DISCOVERY OF NON-EUCLIDEAN GEOMETRY

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1. FROM THE EMPIRICAL STUDY OF SPACE TO DEDUCTIVE GEOMETRY

As far as we nowadays know, in pre-Greek times a great deal of empirical knowledge had already accumulated, and this collection of practical facts served for Greek genius as a source in creating the deductive science called geometry. We also know that speculative logic had initially developed independently of mathematics and reached a high level before its application to empirical facts concerning space began. From that time onward, it was not only logic which assisted the development of geometry, but geometry has also reacted on the evolution of logic. In pre-Greek mathematics the concepts of theorem, proof, definition, axiom and postulate had not yet occurred; all of them are creations of Greek intellect.

According to the history of science it was THALES (624–548 B. C.) who, on the visual level, began to arrange the facts gained by experience and to search for explanations which reduce the complicated to the simple ("demonstration" in a "perceptible" way); the evolution in that direction started in his times.

One century later the application of the methods of speculative logic for proving geometrical assertions was begun. In this period (450—325 B. C.) the following circumstances deserve special attention. From the very beginning, extremely rigorous, exact proofs were produced. The method of indirect proof was used remarkably often. The validity of many geometrical statements which had been known and obvious for a long time was proved.

The first textbook of geometry, entitled *Elements*, was written by HIPPOCRATES (about 450—430 B. C.), who attempted to put contemporary geometrical knowledge in a strict logical order. Hence one concludes that geometry began to turn into a deductive science in the period before HIPPOCRATES.

The book of EUCLID (about 325 B. C.), also entitled *Elements*, partly rests on former works and is a synthesis of deductive geometry, as created by the Greeks, in a perfect system (here the word "perfect" refers to the level of science reached at that time, and not to demands emerging in the course of later progress).

EUCLID's work is a textbook in the best sense of the word. It teaches us what kind of requirements should be raised against scientific knowledge, in which way facts should be treated, and how to pass on the results obtained. All this is done in the highly purified manner which had evolved as a fruit of long-lasting meditations of the Greeks.

Elements attains these aims indirectly, by providing a model. It arranges the material in groups: definitions, postulates and axioms are coming first, succeeded by the statements and proofs of the theorems already known (actually, for the most part, known for a very long time).

EUCLID himself does not at all explain what the point of this grouping is. He may have assumed that the intelligent reader would find out the motive of the scheme by getting acquainted with the work in its entirety and thinking it over and over again. Probably the fundamental principles of the order revealed in the book had already crystallised and become current in science to such an extent that they required an exact and possibly complete realisation rather than a mere exposition.

We know that Greek scholars, as early as in the days of PLATON, had recognized the following: in the chain of mathematical proofs there is no "regressus ad infinitum": mathematics must have foundations which cannot be proved any longer. The formation of these principles had required more than a century of immense intellectual effort. In the possession of these principles, Euclid could start composing his work from a high level of scientific thought.

Elements begins with 35 definitions, 5 postulates, and 5 axioms. They together have been called *foundations* (principles), for EUCLID endeavoured to deduce all the rest of his work by a logical process starting from these foundations and relying only on them.

It should be noted that in *Elements* definition, postulate and axiom mean something else than they mean today. We will not make a linguistic analysis or an appraisal of the foundations by modern standards. Instead, we are going to state the foundations in an up-to-date language.

We cite the first three definitions for clarifying the sense, different from present-day usage, of the word "definition", and the last one.

1. Point is the thing that has no parts.
2. Line is length without breadth.
3. The ends of a line are points.

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35. Those straight lines are parallel which are in one plane and which, produced to any length on both sides, do not meet.

The postulates are the following.

1. A straight line may be drawn from any point to any other point.
2. The straight line may be produced to any length.
3. Around any point as a centre, a circle of any radius may be described.
4. Any two right angles are equal.
5. If a straight line meets two other straight lines so as to make the sum of the two interior angles on one side of it less than two right angles, then the other straight lines,